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Distributed and Finite-Time Estimation in Networked Systems

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Dissertation presented in partial fulfillment of the requirements for the degree of Doctor in Engineering Science (PhD): Computer Science at KU Leuven and Docteur en Specialité : Automatique, Génie informatique, Traitment du Signal et des Images at Centrale Lille

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To my son Levon

Preface

Applying for this PhD position was one of those defining moments in my life. Little did I expect that I will be working on something as theoretical but fundamental. At times it has been confusing and not clear to me what the sense in all of this is. Looking back, I would do a few things differently. However, in the end everything turned out right and the spark of research is what I will forever remember.

In this sense, I am very grateful to the European Union for funding this opportunity. I truly wish that the knowledge gained during these three and a half years spent in France and Belgium will be of great service to future research pursuits. The present dissertation should hopefully provide ample motivation for that.

At this point, I would like to thank first and foremost my thesis supervisors. I dearly appreciate Jean-Pierre Richards relentless support. I will always be indebted to Wim Michiels for holding me to high standards in my research undertaking. It has also been a great pleasure to work with Denis Efimov and Rosane Ushirobira. Thank you all for your numerous pieces of advice!

Next to my supervisors, I would like to thank the jury members for their time spent reviewing the manuscript and their feedback. I am glad that they all agreed to take part in the defense in times of the corona crisis.

I have had the luck to work in different groups among many helpful people, too many to name them all. Undoubtedly, the best part of this PhD were my colleagues and friends from the Valse team (formerly Non-A) at Inria, SHOC (formerly SyNeR) at CRIStAL laboratory, NUMA unit at the Department of Computer Science and the members of the UCoCoS project. I really hope that we will stay in contact.

Finally, my biggest gratitude goes to Kadri for her endless love and commitment.

Haik
June 2020

Abstract

This thesis is a broad treatment of the distributed state estimation problem for linear systems. In this setting, a network of observer nodes collectively estimates the state of a dynamical system, since individually they are not able to do so. The proposed solution consists of a distributed observer which uses diffusive coupling and leads to three complementary contributions.

The first one considers exponential convergence with arbitrary rates. Various design approaches are put into a unified framework to facilitate their comparison. Differences in the continuous-time and discrete-time case are related with synchronization. The designs are feasible under the necessary condition of distributed observability with respect to the graph of the network, which is akin to observability in centralized state estimation. Furthermore, diagonal stability of the compressed Laplacian matrix and block-diagonal stability of the error dynamics in original coordinates form crucial building blocks for the other contributions of the thesis. It is concluded that a more general design procedure is desirable to reduce the size of exchanged information and to account for delays.

The second contribution is the design of distributed observers where the estimates reach the state of the system exactly in a finite time, in contrast to the asymptotic convergence of the preceding linear designs. Taking advantage of local observability decompositions, centralized finite-time observers are used for the observable substate while a nonlinear consensus-term is employed for the unobservable substate. Sufficient bounds on the gain parameters are obtained using the concept of homogeneity.

As a third contribution, an advantage of distributed observers is demonstrated by taking into account the specific effects of communications, which are modeled using time-varying delays. With the Lyapunov-Krasovskii functional approach, the selection of gains which guarantee a certain exponential convergence rate in the face of communication are cast into solving a linear matrix inequality. In a numerical example, diffusively coupled observer nodes achieve a better performance compared to the direct transmission of partial outputs. The exchange of artificial outputs with decreasing size shows a gradual decline in performance to the level of output exchange, which is balanced by weaker connectivity requirements. Finally, the vector Lyapunov approach is extended to cope with time-varying delays.

Résumé

Cette thèse est consacrée aux problèmes d'estimation d'état distribuée pour les systèmes linéaires en réseau. Dans ce contexte, un réseau de nœuds d'observateurs estime collectivement l'état d'un système dynamique, alors que chacun des noeuds n'en serait pas capable individuellement. La solution proposée consiste en un observateur distribué qui utilise un couplage diffusif. Ce point de vue donne lieu à trois contributions distinctes et complémentaires.

La première contribution considère une convergence exponentielle avec des taux garantis. Plusieurs approches de conception sont rassemblées en un cadre conceptuel unifié afin de faciliter leur comparaison. Les différences entre temps continu et temps discret sont ramenées à des questions de synchronisation. La faisabilité est liée à la propriété d'observabilité par rapport au graphe du réseau, qui est un concept similaire à l'observabilité dans l'estimation d'état centralisée. De plus, la stabilité diagonale de la matrice laplacienne ainsi que la stabilité bloc-diagonale des dynamiques de l'erreur donnent les bases essentielles pour les deux autres contributions de cette thèse. Il est conclu qu'une procédure de conception plus générale est souhaitable afin de réduire la taille de l'information échangée et de pouvoir prendre en compte les retards.

La deuxième contribution porte sur la conception d'observateurs distribués qui atteignent l'état du système observé en un temps fini, en contraste avec la convergence asymptotique des observateurs linéaires précédents. En tirant parti des décompositions locales d'observabilité, un observateur en temps fini centralisé est utilisé pour chaque sous-état observable tandis qu'un terme de consensus non-linéaire est utilisé pour les sous-états inobservables. Des limites sur les gains sont obtenues (en termes de conditions suffisantes) grace à l'utilisation du concept d'homogénéité.

Avec la troisième contribution, l'avantage des observateurs distribués est démontré en tenant en compte les effets des communications, qui sont modélisées en utilisant des retards temporels variables. En utilisant l'approche des fonctionnelles de Lyapunov-Krasovskii, la sélection des gains qui garantissent une certaine convergence exponentielle malgré les communications est transformée en résolution d'une inégalité de matrice linéaire. Dans un exemple numérique, on montre que des nœuds d'observation communiquant selon un couplage diffusif peuvent conduire à une performance supérieure à celle d'une solution de transmission directe des sorties. L'échange de sorties artificielles avec des tailles décroissantes montrent un déclin graduel de la performance par rapport au niveau d'échange de la sortie, ce qui est équilibré par une nécessité de connectivité plus faible. Finalement, l'approche Lyapunov est étendue en vectoriel afin de gérer les délais temporels variables.

Samenvatting

Deze doctoraatsthesis betreft een brede studie van het probleem van gedistribueerd schatten voor lineaire systemen. In dit raamwerk wordt de toestand van een dynamisch systeem collectief geschat door een netwerk van lokale schatters, die op zichzelf daartoe niet in staat zijn. Er worden oplossingen voorgesteld die gebruik maken van gedistribueerde schatters met diffuse koppeling, wat geleid heeft tot drie complementaire bijdragen.

De eerste bijdrage betreft het realiseren van exponentiële convergentie met een vrij te kiezen snelheid. Verschillende ontwerpmethodologieën worden samengebracht in een unificerend raamwerk, wat het vergelijken gemakkelijker maakt. De verschillen in de oplossingen voor continue tijd en discrete tijd worden gerelateerd aan het mechanisme van synchronisatie. De doestelling kan gerealiseerd worden onder de noodzakelijke voorwaarde van gedistribueerde observeerbaarheid met betrekking tot de graaf van het netwerk, dewelke de notie van observeerbaarheid voor gecentraliseerd toestandsschatting op een natuurlijke manier veralgemeent. Bovendien vormen de zogenaamde diagonale stabiliteit van de gereduceerde Laplaciaan en de blokstabiliteit van de dynamica van de schattingsfouten belangrijke bouwstenen voor de andere bijdragen in deze thesis. Er wordt geconcludeerd dat een meer algemene ontwerpprocedure wenselijk is, waarbij enerzijds de uit te wisselen informatie gereduceerd wordt, en anderzijds rekening gehouden wordt met tijdsvertragingen in de koppeling tussen de schatters.

De tweede bijdrage bestaat uit het ontwerp van gedistribueerde schatters waarbij de ge-schatte toestand en de eigenlijke toestand van het systeem samenvallen vanaf een eindige tijd, wat contrasteert met de asymptotische convergentie voor lineaire schatters. Daarbij worden lokale observeerbaarheidsdecomposities uitgebuit. Meer bepaald worden gecentraliseerde eindige-tijd schatters gebruikt voor het lokaal observeerbaar deel van de toestand, terwijl een niet-lineaire consensusterm gebruikt wordt voor de reconstructie van het niet-observeerbaar deel van de toestand. Grenzen op de parameters van de versterkingen die de eindige-tijd convergentie garanderen worden bekomen op basis van het concept van homogeniteit.

Als derde bijdragen worden voordelen van het gedistribueerd schatten aangetoond, door specifieke effecten van communicatie, gemodelleerd door middel van variërende tijdsvertragingen, in rekening te brengen. Met behulp van Lyapunov Krasovskii functionalen wordt het probleem van het selecteren van de versterkingen, die een gegeven exponentiële convergentiesnelheid garanderen voor alle communicatiekanalen die aan de specificaties voldoen, omgezet naar het oplossen van

een lineaire matrixongelijkheid. In een numeriek voorbeeld bereiken lokale schatters met diffuse koppeling betere prestaties in vergelijking to het rechtstreeks doorsturen ven een deel van de uitgangen. Het uitwisselen van artificiële uitgangen met afnemende dimensie leidt tot een graduele vermindering van de prestaties, wat echter gebalanceerd wordt met minder vereisten voor de communicatiekanalen. Tenslotte wordt de vector Lyapunov aanpak uitgebreid naar systemen met tijdsvariërende tijdsvertragingen.

Notations

1_n	$n \times 1$ matrix of ones
I_n	$n \times n$ identity matrix, index sometimes omitted
e	Euler's number
$\mathrm{diag}(x)$	Diagonal matrix formed by the ordered entries of vector x
$\mathrm{blkdiag}(A, B, C)$	Block-diagonal matrix formed by ordered matrices A, B and C
$\boldsymbol{A}$, $\mathrm{blkdiag}(A_i)_{i \in J}$	Block-diagonal matrix formed by ordered set of matrices A_i with index in set J, bold-typeset used for $J = \{1, \ldots, N\}$
$[A_i]_{i \in J}$, $[A_i^\mathsf{T}]_{i \in J}^\mathsf{T}$	Row-wise and column-wise concatenation of ordered set of matrices A_i with index in set J
$A \otimes B$	Kronecker product of matrices A and B
$\mathcal{L}$	Laplacian matrix of a graph (see section 2.2.1)
$\mathcal{N}_i$	Set of adjacent vertices of vertex i
$\mathcal{O}(A, C)$	Observability matrix of the pair (A, C)
$\ker(A)$, $\mathrm{Im}(A)$	Null space and image of matrix A
$U \cap V$	Intersection of the subspaces U and V
$P \succ Q$, $(P \succeq Q)$	Matrix $P - Q$ is positive (semi-)definite
$\lambda(A)_{\min}$, $\lambda(A)_{\max}$	Smallest and largest eigenvalues of the real symmetric matrix A
$\lceil a \rfloor^b$	Sign preserving element-wise exponention of vector a with vector b (see section 3.2)

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Chapter 1

Introduction

Recent decades have witnessed an unprecedented progress in information technology and its influence on every aspect of life. Things we used to do in a certain way have changed drastically, since vast amounts of real-time information and the necessary computing power to process it is readily available. However, how to exploit these new possibilities and how to cope with the associated challenges is still an outstanding and urgent research question. We can witness this by the ever-increasing prominence of topics like Industry 4.0, Internet of Things and the challenges of Big Data.

A manifestation of the technological advancement is the presence of low-cost radio connected sensors (Chong and Kumar, 2003), forming wireless sensor networks. These small embedded devices with computational power and communication capability can be deployed virtually everywhere: in homes, buildings, streets and factories, constantly exchanging information with each other. Moreover, they can be mounted on mobile robots or drones to enhance further their operative range. The aim is to efficiently apply this infrastructure in applications like intelligent transportation, environmental monitoring or health care, while copping with the associated challenges as limited bandwidth and energy.

There is another side of the coin. From the connectivity more complexity arises by having systems which are distributed over large areas and are increasingly heterogeneous, consisting of many subcomponents working on different time-scales (e.g. traffic, the power-grid, the banking network). Each agent in these networks is operating autonomously, but at the same time they are also interdependent. There is

huge potential in this so-called hyper-connectivity, but it carries along an increased risk of cascading failures, congestion collapses and cyber-threats, to name a few.

For control systems, this means that existing centralized approaches are doomed to fail, since they do not provide the necessary scalability, flexibility and adaptivity. Moreover, while global security and performance remain crucial objectives, centralizing the corresponding algorithms on one or few nodes is not a solution anymore in this large-scale, heterogeneous and networked environment. The key is to have distributed solutions, that is to say, local algorithms for each agent which in cooperation are able to attain a global goal (Lamnabhi-Lagarrigue et al., 2017).

1.1 Distributed state estimation

The above solution also applies to the fundamental task of state estimation. The *state* of a dynamical system represents important information, since it describes not only its current situation but says how the system will behave in response to a future input (e.g. the position and velocity of a robot, the temperature distribution in a room, the voltages and currents in the power grid). Thus, having an approximate knowledge of the state is at the root of many control algorithms. In most instances, the full state is not directly accessible to measurements but reveals itself through the system's output. The goal of a state estimator, or observer, is to reconstruct the state from the measured output using a model description of the system. The design of observers for estimating a dynamical system's state is an engineering problem with long history and has been addressed by many centralized solutions (most notable is the Kalman filter).

In this thesis we are interested in estimating the state of a large scale system in the context of a network of nodes, or agents, acting as sensors. In particular, each node obtains just a partial output. The centralized approach of state estimation would be to collect all the outputs at one or more central entities and to use a classical estimation technique. Though it might be the preferred approach in some cases, it puts very high demands on the network and the fusion centers, especially in a situation where the complete output is necessary to reconstruct the state. The situation is even more so exacerbated, if the goal is to provide every node with an estimate of the full state

(e.g. to decide on its action as suggested by Zhao et al., 2002).

The preferred approach in this case is to have local estimators at each node which through local information exchange are able to accomplish the estimation task. Specifically, we do not want to rely on some nodes to receive the complete output, nor that a partial output is sufficient to reconstruct the state. A distributed estimator should therefore be as capable as a centralized one by utilizing the complete information, but require communication solely over a sparse network. In the present thesis such a solution is investigated.

1.1.1 Distributed Kalman-filtering

The search for distributed versions of the Kalman filter has attracted a wide interest on the onset of the deployment of multi-sensor systems. A native approach is only suited to the case when the sensors are all directly connected to each other (Grime and Durrant-Whyte, 1994), but with rise of wireless sensor networks the demand for solutions which require only local interactions grew. Simultaneously, there has been an ever-increasing interest in autonomous cooperative mobile robots, leading to a surge in tasks which require distributed algorithms. A prominent example is the tracking of targets (Hu and Hu, 2010). Notably, average consensus methods, which are useful for the coordination of agents, can also be applied to fuse the estimates of a sensor network (Blondel et al., 2005).

Inspired by the use of consensus algorithms, Olfati-Saber (2007) proposed to use diffusive coupling of local state-estimates to design estimation filters for dynamical systems in the distributed setting. The motivation lies mainly in overcoming the need of all-to-all communication in previous methods. One the side it addresses the difficulty that partial outputs are not sufficient to estimate the state completely (i.e. the reason why connections are indispensable in the first place; similar ideas were expressed earlier by Julier and Uhlmann, 1997).

The work sparked an interest in distributed Kalman-filtering (Zampieri, 2008), such that it is impossible to give a complete overview of the topic.[1] However, most early

[1] A Zotero Group for related literature is available at
https://www.zotero.org/groups/distributed_state_estimation

proposals and also many later ones rely on partial outputs that are sufficient to reconstruct the state, assume decoupled systems or look at static problems (e.g. Carli et al., 2008; Khan and Moura, 2008; Cattivelli and Sayed, 2010), among other limitations. Therefore, they suggest a consequent need for more general techniques of distributed estimation, as the ones laid out in this thesis. In fact, a solution without such restrictive conditions was already presented by Battistelli et al. (2011; see also Casbeer and Beard, 2009; Arambel et al., 2001).

The proposed distributed Kalman-filtering approach uses an additional step between local measurement updates, where two convex combinations of local information are taken over the neighboring nodes. One is of the state estimates weighted by information matrices, the other one is of the matrices themselves, whose inverse gives an upper bound on the covariance matrix of the respective estimation error. The applicability of the design to general connected communication networks was proven by noticing the analogy with consensus problems (Battistelli and Chisci, 2014). Subsequently, considerable effort has been undertaken to improve this approach by looking for optimal consensus weights, additionally exchanging measurements or doing multiple consensus steps in between updates (see Wang and Ren (2018) for an overview).

Kalman filtering works in the stochastic setting where the performance of an estimator is assessed by the convergence of the estimation error variance. However, since for the distributed Kalman-filter local information matrices are obtained which provide only bounds on the actual covariances, their performance is hard to quantify, especially since the optimality in the sense of classical Kalman-filtering is not clear (Cacace, 2019). In this thesis we focus on distributed estimators for linear systems in the deterministic setting and design each networked estimator as stationary observer. The convergence properties can then be simply investigated on the resulting dynamics of the estimation error. This line of research has gained popularity only in recent years.

1.1.2 Challenges

In summary, the main challenge of distributed state estimation lies in the local reconstruction of the state at a node which requires global information, yet the communication is constrained to neighboring nodes. Furthermore, any estimation algorithm,

centralized or distributed alike, should be fast in producing accurate results, have low design complexity and robustness with respect to various sorts of perturbations. Various pioneering solutions have been proposed to distributed state estimation that employ diffusive coupling and thereby succeeded in addressing the first challenge. However, since they have been published shortly before or in parallel with the current thesis a more general perspective has been lacking. In establishing such a unified framework we will gain important insights on the design criteria of such distributed observers.

The interest in distributed solutions is additionally driven by the appropriation of networked control systems. In this context, point to point links are replaced in favor of packet switched networks. This facilities remote as well as large-scale operations with advantages for example in reduced costs and added flexibility. However, the accompanying delays and limited rates in the communication pose severe constraints on the estimation performance. The intriguing question is whether distributed state estimation provides a means to tackle this issue. We aim to answer this by including the effects of communication in our analysis.

Lately in the research on distributed estimation also topics are considered, which go beyond typical problems of networked control systems, like time-varying network topologies (Mitra et al., 2019; Wang et al., 2020) and cybersecurity (Mitra and Sundaram, 2019; Ugrinovskii, 2020; Lee et al., 2020). Ultimately, the distributed estimation context raises issues which have not been on the horizon of centralized approaches and in turn may be not adequately addressed by classical estimation algorithms. In this sense, a design grounded in the notion of homogeneity is an interessting extension to conventional approaches as it permits to impose convergence in a finite time (in contrast to asymptotic convergence in linear designs; see the recent monograph of Polyakov, 2020). Both practical and theoretical results which have been obtained so far have motivated the extension of the finite-time estimation paradigm to the distributed scenario in this thesis.

1.2 Outline of the thesis

The thesis is structured into three main chapters. In the following we summarize the scope of the main chapters and list the publications they have put forth. Figure 1.1 depicts their relation.

Chapter 2 We introduce the necessary background in observer and graph theory and formally define the distributed estimation problem. The distributed observer is presented as a solution and designs are given which can impose any exponential convergence rate in the delay-free case. Both continuous-time and discrete-time perspectives are taken and the designs are put into a unified framework. The insights gained in this chapter will form the basis for the remaining thesis.

H. Silm, D. Efimov, W. Michiels, R. Ushirobira, J.-P. Richard (2020a). A simple finite-time distributed observer design for linear time-invariant systems. *Systems & Control Letters*, *141*, pp. 1–7.

——— (2019b). Design of a distributed finite-time observer using observability decompositions. *2019 18th European Control Conference (ECC)*. Naples, Italy, pp. 1816–1821.

Chapter 3 This chapter deals with finite-time observers for distributed estimation which are based on the homogeneity concept. A centralized finite-time observer forms the basis of a distributed finite-time observer for a system of integrator chains and subsequently for general linear systems. For the general case we exploit a certain diagonal stability property established before in Chapter 2.

H. Silm, R. Ushirobira, D. Efimov, J.-P. Richard, W. Michiels (2019a). A note on distributed finite-time observers. *IEEE Transactions on Automatic Control*, *64*(2), pp. 759–766.

——— (2017). A distributed finite-time observer for linear systems. *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*. Melbourne, Australia, pp. 1731–1736.

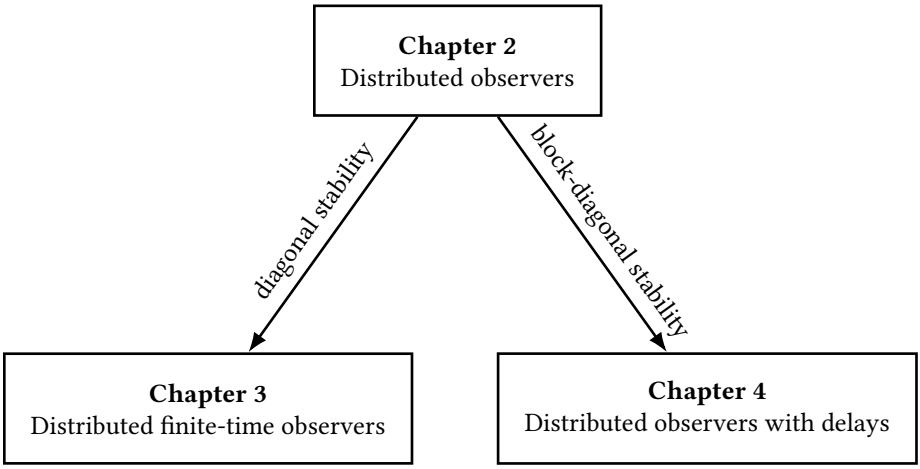


Figure 1.1: Overview of the main chapters

Chapter 4 The effects of communication between the observer nodes are introduced and a design to handle these is derived using the time-varying delay approach. The feasibility of the design hinges on the block-diagonal stability property established in Chapter 2. The design allows for a comparison of the centralized and the distributed case on an example in terms of achievable performance. Further the exchange of artificial outputs and the vector Lyapunov approach are explored.

H. Silm, R. Ushirobira, D. Efimov, E. Fridman, J.-P. Richard, W. Michiels (2020b). Distributed observers with time-varying delays. *IEEE Transactions on Automatic Control* (provisionally accepted), pp. 1–8.

——— (2018). Comparison of the time-delay margin of a distributed and centralized observer. *2018 European Control Conference (ECC)*. Limassol, Cyprus, pp. 1963–1968.

In the concluding chapter the results of the thesis and directions for future research are discussed.

Chapter 2

Distributed Observers

As far as one can tell, Mandić et al. (2010; see also Açıkmeşe et al., 2014) were the first to prove a distributed estimation method for discrete-time systems, where in between system time-steps a node does multiple intermediate steps of consensus seeking with neighboring estimates. Its derivation is noteworthy, since the authors used a suitable coordinate transformation together with algebraic properties of the graph of the communication network, which are at the basis of many later designs too. Ironically, only superficial interest has been paid to this work (cf. a recent survey from Rego et al., 2019).

The above design, works if all links in the network are bidirectional and has been extended to directional connectivity (e.g. a directed cycle). Nevertheless, it does not cover more general cases; for example, if there is a node which can receive from other nodes, but its partial output is redundant and therefore it does not need to transmit to any other node. The most universal situation was formulated by Park and Martins (2017; we will call it later *distributed observability*). There also a method was proposed which does not use multiple interactions between the time steps, however one node in the network has to house a coordinator. The method of Mitra and Sundaram (2018) does not require intermediate steps nor a central coordinator, but divides the estimation of the state vector into non-overlapping parts among the nodes. Consequently, complete information has to travel over the whole network and thus for both designs essentially global interactions are necessary.

Kim et al. (2016b) made a major contribution by pointing to the intuitive idea that each node can already estimate a part of the state from its partial output; therefore,

it has to a reach a consensus with its neighbors only on the part of the state which it cannot recover on its own (see also del Nozal et al., 2017). Exploiting the well known observability decomposition they presented a simple design for continuous-time systems which was also the starting point of the present thesis. An analogous approach applied to the discrete-time setting was presented by Wang et al. (2019b). Despite having the focus on continuous-time systems, in this chapter there will be a short discussion on the difference of the two settings for distributed estimation.

By passing from local observability decompositions to the networked case del Nozal et al. (2019) establish the possibility of placing the poles of the estimation error dynamics. However, what are favorable locations of the poles in the distributed setting is hard to tell, as is open the question of finding Kalman-like designs for continuous-time systems (for a start see Wu et al., 2016; Kim et al., 2016a). The design criteria of classical approaches might be not well suited, for example due to the inevitable delays in the communication (one main driver behind the search for distributed solutions in the first place). Note an early work of Ugrinovskii (2011) for continuous-time systems which pays attention to the robustness of the agreement between the nodes with respect to perturbations.

We begin this chapter by very briefly revising the general definitions and notions of classical state observers as they will serve for the distributed estimation problem. In particular, we highlight the role of the observability matrix in the solvability of the estimation problem, the role of output feedback in the construction of an observer, and how the convergence of an observer can be recast to the stability of the error dynamics. We continue with the formal introduction of the problem of state estimation in the setting of multiple observer nodes for linear systems, where the observer nodes aim at reconstructing the full state, but having only access to a part of the output, they compensate the possible lack of observability from their partial output by exchanging information with neighboring nodes.

In this context, preceded by some graph theoretic terms, we define distributed observability with respect to a graph as a useful extension of the notion of observability to the distributed state estimation setting. Then we introduce the class of diffusively coupled observers nodes as a solution to the distributed state estimation problem and present different design methods in the continuous-time and the discrete-time setting that have been established so far since 2016. To facilitate their comparison, we strive

for a unified framework. We close the chapter with an outlook on the possibility of reducing the amount of information exchange between observer nodes.

2.1 State observers for linear systems

The task of state estimation is concerned with dynamical system models in state-space representation, especially linear time-invariant systems of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input, $y \in \mathbb{R}^m$ is the output, and A, B, C and D are the system matrices. The question is whether the (initial) state of the system can be determined from the input $u(t)$ and output $y(t)$.

Definition 2.1. *System* (2.1) *is called observable if knowing A, B, C and D the initial state x_0 can be determined from any input $u(t)$ and corresponding output $y(t)$ on some time interval $t \in [0, T]$.*

For a sufficiently smooth input, the ith derivative of the output can be expressed as

$$y^{(i)} = CA^i x + \sum_{k=1}^{i} CA^{k-1} B u^{(i-k)} + Du^{(i)}.$$

Therefore, it is apparent that the system (2.1) is observable if the observability matrix of the pair (A, C)

$$O(A, C) := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full rank. Due to the Cayley-Hamilton theorem the condition is also necessary and we can directly call the pair (A, C) observable.

2.1.1 Full-order Luenberger observer

A constructive way to obtain the state from the inputs and outputs is by a Luenberger observer which employs a copy of the system model together with an output feedback to correct the estimate $\tilde{x}(t)$ such that it converges towards the true state $x(t)$ with a desired rate. It is described by the equation

$$\dot{\tilde{x}} = A\tilde{x} - L(C\tilde{x} - y + Du) + Bu$$

where L is the observer gain to be designed.

Introducing the error $e = \tilde{x} - x$ we arrive to the error dynamics

$$\dot{e} = (A - LC)\, e.$$

If system (2.1) is observable then the spectrum of the matrix $A - LC$ can be freely assigned by an appropriate choice of L. Therefore the observer can be designed such that the error converges to zero with any desired decay rate. Otherwise there are eigenvalues of A that are invariant with respect to LC, so-called fixed modes.

For an observable system with a scalar output, the observer gain L can be obtained from the desired location of the spectrum of the error dynamics using Ackermann's formula. In the multiple-output case the choice of L for a given spectrum is generally not unique and rather than a closed-form expression, iterative methods have been proposed, for example to use the additional degrees of freedom to reduce to the sensitivity of the poles with respect to perturbations (Kautsky et al., 1985).

2.1.2 Observability decomposition

Even if the pair (A, C) is not observable, it might still be possible to select L such that the resulting error dynamics is asymptotically stable. This is the case if all the fixed-modes are located in the open left half of the complex plane. However, there will be a limit on the obtainable decay rate. Systems with pairs (A, C) for which this is the case are called detectable and can be determined by looking at the system dynamics in some special coordinate system, the observability decomposition.

Let the columns of the matrix T_u be an orthonormal basis of the largest A-invariant subspace that is in the kernel of C. By the Cayley-Hamilton theorem this subspace is equal to the kernel of $\mathcal{O}(A, C)$, the unobservable subspace. Let the columns of T_o be an orthonormal basis of the orthogonal complement of the kernel of $\mathcal{O}(A, C)$. The orthogonal complement is often called the observable subspace (although generally it is not A-invariant).

Matrix $T = \begin{bmatrix} T_\mathrm{o} & T_\mathrm{u} \end{bmatrix}$ is then an orthogonal transformation matrix, such that in new coordinates the dynamics matrix is block-triangular

$$T^\mathsf{T} A T = \begin{bmatrix} A_\mathrm{o} & 0 \\ A_\mathrm{r} & A_\mathrm{u} \end{bmatrix}$$

and

$$CT = \begin{bmatrix} C_\mathrm{o} & 0 \end{bmatrix}.$$

It can be checked that the pair $(A_\mathrm{o}, C_\mathrm{o})$ is observable, therefore there exists an $L = T \begin{bmatrix} L_\mathrm{o} \\ 0 \end{bmatrix}$ such that $A_\mathrm{o} - L_\mathrm{o} C_\mathrm{o}$ is stable. Consequently, detectability of the pair (A, C) is equivalent to A_u being a Hurwitz matrix.

Pole placement is one way of design. In practical applications we deal with disturbances and placing the poles for fast convergence might be not a good idea if robustness is important, since there is generally a trade off. Various other methods e.g. the stationary Kalman-filter or $\mathcal{H}_\infty$-filter provide optimal gains while taking disturbances into account.

2.2 Distributed state estimation problem

The theory on state estimation for linear systems is mature and its extension to nonlinear problems has been very useful to tackle real-world applications. Nonetheless, as was exposed in Chapter 1, a fundamentally new outlook is necessary to be able to cope with the challenges of modern use-cases dealing with large-scale heterogeneous networked systems (and for the understanding of complex phenomena in nature and society). In this respect, it is reasonable to look first at the foundational framework

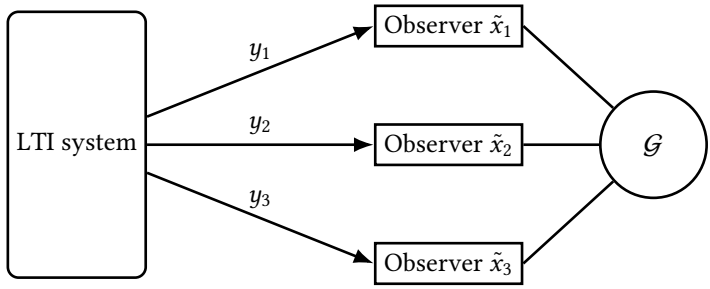


Figure 2.1: Framework of distributed state estimation consisting of a system, the observer nodes and the graph of the communication network

of *distributed state estimation for general linear systems*.

In the distributed state estimation scenario we consider linear time-invariant systems described by

$$\begin{aligned} \dot{x} = Ax, \qquad & y_1 = C_1 x \\ & \vdots \\ & y_N = C_N x, \end{aligned} \tag{2.2}$$

with state $x \in \mathbb{R}^n$ and system dynamics matrix A, but instead of a single observer, the output is partitioned and divided among several observer nodes. Each of the N nodes obtains one of the possible vector-valued partial outputs $y_i \in \mathbb{R}^{m_i}$, $i \in \{1, \dots, N\}$. The only assumption we make on the system matrices in (2.2) is that the pair $(A, \begin{bmatrix} C_1^\mathsf{T} & \cdots & C_N^\mathsf{T} \end{bmatrix}^\mathsf{T})$ is observable, meaning that the state can be reconstructed from the complete output. In relevant applications the system matrices might have some sparsity pattern (moreover controlled inputs and perturbations might have to to be accounted for); however, as a starting point we look at this basic general case.

The goal is that for every observer node the local estimate $\tilde{x}_i$ of the state of the system (2.2) converges to x with any desired rate. Generally we are interested in the situation where this is not possible with the partial output of a single node (i.e. none of the pairs (A, C_i) are observable), so the nodes have to exchange information. This is depicted in the figure by a graph $\mathcal{G}$, which describes the communication topology of the network.

The straightforward solution would be to let the observer nodes exchange their par-

tial outputs and implement each node as a Luenberger observer. However, this would require an all-to-all communication network if the system is observable from the complete output only. After a brief revision of graph theoretic terms, we will explain that it is actually enough when the network is minimally connected. In the following section we will then present designs which allow to estimate the state at each node in a distributed way.

2.2.1 Some graph theory

It is practical to describe the communication network by an unweighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of order $|\mathcal{V}| = N$. Each vertex represents an observer node and is labeled by the index from $\mathcal{V} = \{1, \dots, N\}$ which corresponds to the subscript of the partial output y_i it obtains. An edge $(j, i) \in \mathcal{E}$ is directed from j to i if the observer node corresponding to vertex i can receive information from j. The set of neighbors $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}\}$ of an observer node i consists of the vertices of the observer nodes from which it can receive information.

A graph $\mathcal{G}$ is also fully characterized by its Laplacian matrix $\mathcal{L} = \mathcal{A} - \mathcal{D}$, where the (i, j)-entry of the adjacency matrix $\mathcal{A}$ is 1 if there is a directed edge from j to i, otherwise it is zero, and $\mathcal{D} = \operatorname{diag}(|\mathcal{N}_1|, \dots, |\mathcal{N}_N|)$ is the in-degree matrix. A graph is strongly connected if there exists a path (a sequence of edges, such that the starting vertex of an edge is the end vertex of the preceding one) from each node to every other node. A strongly connected component is a subgraph formed by a set of vertices and the edges connecting them, such that no additionally vertex of $\mathcal{G}$ can be added to the set without violating strong connectedness.

We present a lemma which was exploited in the context of distributed state estimation by Han et al. (2019).

Lemma 2.1 (Yu et al. (2010))**.** *For a strongly connected graph with Laplacian matrix $\mathcal{L}$, there exists a vector r with strictly positive entries such that $\mathcal{L}^\mathsf{T} r = 0$. Furthermore,*

$$\hat{\mathcal{L}} \coloneqq \mathcal{L}^\mathsf{T} \operatorname{diag}(r) + \operatorname{diag}(r)\mathcal{L}$$

is the Laplacian matrix of an undirected weighted graph.

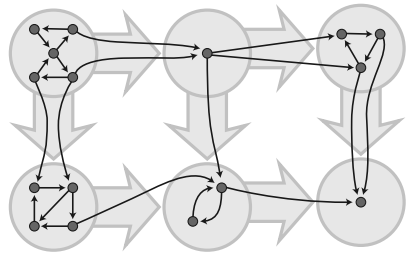


Figure 2.2: A directed graph with a single source component and its condensation in light gray (Wikipedia)

For any directed graph an acyclic graph can be derived, by collapsing the set vertices of a strongly connected component into a single representative (its condensation), see Figure 2.2. Strongly connected components are called source components in Park and Martins (2017) if there is no edge from any vertex outside of the graph to a vertex of the component (in analogy that in the condensation they will be source nodes, i.e. nodes with no incoming edges).

A useful permutation of the numeration of the vertices of a directed graph with $\bar{k}$ strongly connected components of which k are source components is such that the resulting Laplcian matrix has the form

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ 0 & \cdots & \mathcal{L}_k & 0 \\ \times & \dots & \times & \mathcal{M} \end{bmatrix} \tag{2.3}$$

where $\mathcal{L}_1, \dots, \mathcal{L}_k$ are the Laplacian matrices of the source components, with a block-triangular matrix

$$\mathcal{M} = \begin{bmatrix} \mathcal{L}_{k+1} + D_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \times & \cdots & \mathcal{L}_{\bar{k}} + D_{\bar{k}-k} \end{bmatrix}$$

where $\mathcal{L}_{k+1}, \dots, \mathcal{L}_{\bar{k}}$ are Laplacian matrices of strongly connected graphs and $D_1, \dots, D_{\bar{k}-k}$ are non-negative diagonal matrices with at least one non-zero entry (stemming from an incoming edge).

2.2.2 A necessary and sufficient condition

To summarize, the underlying constituents of the distributed estimation problem are a dynamical system (modelling the plant), a directed simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ (describing the observer node network) and their interplay by having each vertex i of the graph assigned to a partial output y_i of the system (the locally available information).

This brings us to the question: Given a system model (2.2), what are the minimum requirements on the topology of the communication network such that the state can be reconstructed at every node? The question leads us to the following extension of the notion of observability to the distributed case by considering also the graph (cf. Kar and Moura, 2011; Doostmohammadian and Khan, 2014).

Definition 2.2. *System* (2.2) *is called distributively observable with respect to the graph* $\mathcal{G}$ *if for every source component of* $\mathcal{G}$ *the pair* $(A, [C_i^\mathsf{T}]_{i\in\mathcal{V}_k}^\mathsf{T})$*, where* $\mathcal{V}_k$ *is the vertex set of the source component, is observable.*

It is clear that this notion of observability is a necessary condition to the existence of a solution to the distributed state estimation problem. By finding a feasible design for the observer nodes in the next section which relies solely on the condition posed in the definition, we can show that this is also a sufficient condition.

2.3 Distributed observer design

The challenge of the distributed state estimation problem is two-fold: local observability and limited communication. The former as each observer node usually has only access to a part of the output from which the system is not completely observable; the latter since only neighboring nodes can directly exchange information (whereas in applications the graph of the observer network is usually not complete).

To solve it, Olfati-Saber (2007) proposed to use consensus seeking between neighboring observer nodes. The observer algorithm for each node then takes the form of a Luenberger observer with a diffusive coupling term over the set of neighbours $\mathcal{N}_i$

of an observer node

$$\dot{\tilde{x}}_i = A\tilde{x}_i - L_i(C_i\tilde{x}_i - y_i) - \sum_{j\in\mathcal{N}_i} H_{i,j}\left(\tilde{x}_i - \tilde{x}_j\right) \tag{2.4}$$

where $\tilde{x}_i$ is the local estimate of state x, computed at the $i$the observer node. We will call (2.4) for all $i \in \{1,\dots,N\}$ as a whole a distributed (Luenberger) observer. For each node we can introduce the local error $e_i = \tilde{x}_i - x$ which is dynamically coupled with the errors of the neighboring nodes

$$\dot{e}_i = Ae_i - L_iC_ie_i - \sum_{j\in\mathcal{N}_i} H_{i,j}\left(e_i - e_j\right).$$

In order to solve the distributed state estimation problem, it is necessary to find the output-feedback gains L_i and coupling gains $H_{i,j}$ such that the error of the observer network $E^\mathsf{T} = \begin{bmatrix} e_1^\mathsf{T} & \cdots & e_N^\mathsf{T} \end{bmatrix}$ converges to zero. Equivalently, we look at the stability of the error dynamics

$$\dot{E} = (I_N \otimes A)E - \text{blkdiag}(L_1C_1,\dots,L_NC_N)E - \mathcal{H}E, \tag{2.5}$$

with the algebraically constrained matrix

$$\mathcal{H} = \begin{bmatrix} \sum_{j\in\mathcal{N}_1} H_{1,j} & -H_{1,2} & \dots & -H_{1,N} \\ & & \ddots & \\ -H_{N,1} & \dots & -H_{N,N-1} & \sum_{j\in\mathcal{N}_N} H_{N,j} \end{bmatrix}$$

where $H_{i,j} = 0$ if $j \notin \mathcal{N}_i$.

Three basic questions arise when proposing (2.4): 1. Under what conditions can stability of (2.5) be achieved, given the algebraically constrained (i.e. the entries are subject to equalities) gain matrices? 2. How to design the gains L_i and $H_{i,j}$ of the output feedback and coupling, respectively? 3. What are the benefits of such a design?

2.3.1 Consensus-based design

The initial idea of Kim et al. (2016b) is that the consensus dynamics, induced by the last term in the right-hand side of (2.4), should only act on the unobservable subspace of the node, since it the subspace that cannot be reconstructed from the partial output. Conversely, the dynamics of the complementary subspace can be assigned with the output-feedback gain. This brings us to the following choice of gains

$$L_i = T_{i\mathrm{o}} L_{i\mathrm{o}}, \qquad H_{i,j} = T_{i\mathrm{u}} G_{i,j} T_{i\mathrm{u}}^{\mathsf{T}} \tag{2.6}$$

where $T_i = [T_{i\mathrm{o}} T_{i\mathrm{u}}]$ is the transformation matrix described in section 2.1.2 for the observability decomposition of the pair (A, C_i).

With this choice, the observer algorithm (2.4) can be expressed in local coordinates $\tilde{x}_{i\mathrm{o}}$ and $\tilde{x}_{i\mathrm{u}}$ for the observable part

$$\dot{\tilde{x}}_{i\mathrm{o}} = A_{i\mathrm{o}} \tilde{x}_{i\mathrm{o}} - L_{i\mathrm{o}} (C_{i\mathrm{o}} \tilde{x}_{i\mathrm{o}} - y_i) \tag{2.7}$$

and unobservable part

$$\dot{\tilde{x}}_{i\mathrm{u}} = A_{i\mathrm{r}} \tilde{x}_{i\mathrm{o}} + A_{i\mathrm{u}} \tilde{x}_{i\mathrm{u}} - \sum_{j \in \mathcal{N}_i} G_{i,j} T_{i\mathrm{u}}{}^{\mathsf{T}} (T_{i\mathrm{o}} \tilde{x}_{i\mathrm{o}} + T_{i\mathrm{u}} \tilde{x}_{i\mathrm{u}} - T_{j\mathrm{o}} \tilde{x}_{j\mathrm{o}} - T_{j\mathrm{u}} \tilde{x}_{j\mathrm{u}}), \tag{2.8}$$

respectively, where $\tilde{x}_i = T_{i\mathrm{o}} \tilde{x}_{i\mathrm{o}} + T_{i\mathrm{u}} \tilde{x}_{i\mathrm{u}}$.

In the following we will analyze the resulting error dynamics to show that under the distributed observability condition, there exists gains $G_{i,j}$ and $L_{i\mathrm{o}}$ such that the system is asymptotically stable with any prescribed decay rate. Namely, we will choose the consensus gain $G_{i,j} = \gamma I$ for some sufficiently large $\gamma > 0$.

Looking at the error of observer network (2.2), (2.4) with (2.6), $G_{i,j} = \gamma I$ in local coordinates with a rearranged order

$$E = \begin{bmatrix} \mathrm{blkdiag}(T_{\mathrm{o}1}, \ldots, T_{\mathrm{o}N}) & \mathrm{blkdiag}(T_{\mathrm{u}1}, \ldots, T_{\mathrm{u}N}) \end{bmatrix} \begin{bmatrix} E_{\mathrm{o}} \\ E_{\mathrm{u}} \end{bmatrix}$$

(note that for the most general case some matrices have to be interpreted as empty

n-by-0 matrices), we obtain the error dynamics in block-triangular form

$$\begin{bmatrix} \dot{E}_{\mathrm{o}} \\ \dot{E}_{\mathrm{u}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{\mathbf{o}} - \boldsymbol{L}_{\mathbf{o}}\boldsymbol{C}_{\mathbf{o}} & 0 \\ \boldsymbol{A}_{\mathbf{r}} + \gamma \boldsymbol{T}_{\mathbf{u}}^{\mathsf{T}}(\mathcal{L} \otimes I)\boldsymbol{T}_{\mathbf{o}} & \boldsymbol{A}_{\mathbf{u}} - \gamma \boldsymbol{T}_{\mathbf{u}}^{\mathsf{T}}(\mathcal{L} \otimes I)\boldsymbol{T}_{\mathbf{u}} \end{bmatrix} \begin{bmatrix} E_{\mathrm{o}} \\ E_{\mathrm{u}} \end{bmatrix} \tag{2.9}$$

where bold typeset is used to denote block-diagonal matrix concatenations over the sequence of observer nodes (e.g. $\boldsymbol{A}_{\mathbf{o}} := \mathrm{blkdiag}(A_{1\mathrm{o}}, \ldots, A_{N\mathrm{o}})$).

Due to the block-triangular structure, the error of the observable parts E_{o} and unobservable parts E_{u} of the network can be treated separately such their internal dynamics is exponentially stable. It is evident that the block-diagonal matrix $\boldsymbol{A}_{\mathbf{o}} - \boldsymbol{L}_{\mathbf{o}}\boldsymbol{C}_{\mathbf{o}}$ can be made stable with an appropriate choice of the observer gains $L_{i\mathrm{o}}$ since each pair $(A_{i\mathrm{o}}, C_{i\mathrm{o}})$ is observable. For the unobservable part it would be sufficient if $-\boldsymbol{T}_{\mathbf{u}}^{\mathsf{T}}(\mathcal{L} \otimes I)\boldsymbol{T}_{\mathbf{u}}$ is stable, since then there exists a sufficiently large $\gamma > 0$ to compensate the possibly unstable matrix $\boldsymbol{A}_{\mathbf{u}}$.

The compressed Laplacian matrix

In the following we will show that under the necessary condition of distributed observability, the matrix $-\boldsymbol{T}_{\mathbf{u}}^{\mathsf{T}}(\mathcal{L} \otimes I)\boldsymbol{T}_{\mathbf{u}}$ is indeed Hurwitz (thus it is also a sufficient condition). We will take multiple steps, starting from a connected undirected graph, for which we know that its Laplacian matrix has a unique zero eigenvalue with corresponding eigenvector 1_N and all the other eigenvalues are larger.

Lemma 2.2. *If system* (2.2) *is observable and the graph of the observer network is undirected and connected, then matrix* $\boldsymbol{T}_{\mathbf{u}}^{\mathsf{T}}(\mathcal{L} \otimes I)\boldsymbol{T}_{\mathbf{u}}$ *in* (2.9) *is positive definite.*

Proof. Clearly, the matrix is positive semi-definite. Assume that there exists a vector $v^{\mathsf{T}} = \begin{bmatrix} v_1^{\mathsf{T}} & \ldots & v_N^{\mathsf{T}} \end{bmatrix} \neq 0$, $v_i \in \mathbb{R}^n$ for all $i \in \{1, \ldots, N\}$, with $v^{\mathsf{T}}\boldsymbol{T}_{\mathbf{u}}^{\mathsf{T}}(\mathcal{L} \otimes I)\boldsymbol{T}_{\mathbf{u}}v = 0$. Then $\boldsymbol{T}_{\mathbf{u}}v$ must be in $\ker(\mathcal{L} \otimes I) = \{1_N \otimes w | w \in \mathbb{R}^n\}$. Due to $\mathrm{Im}(T_{i\mathrm{u}}) \cap \mathrm{Im}(T_{2\mathrm{u}}) \cdots \cap \mathrm{Im}(T_{N\mathrm{u}}) = \{0\}$ (i.e. observability of the system from the complete output) there does not exist $w \neq 0$ and v for which $T_{i\mathrm{u}}v_i = w$ for all $i \in \{1, \ldots, N\}$. Therefore, $\boldsymbol{T}_{\mathbf{u}}v \notin \ker(\mathcal{L} \otimes I)$ for $v \neq 0$, which implies $\boldsymbol{T}_{\mathbf{u}}^{\mathsf{T}}(\mathcal{L} \otimes I)\boldsymbol{T}_{\mathbf{u}} > 0$. ∎

We extend the result to directed graphs which are strongly connected using Lemma 2.1,

but also introduce diagonal stability, since it will be important later in section 2.3.4 (as well as for the proof regarding distributed finite-time observers in Chapter 3).

Definition 2.3. *A matrix A is called diagonally stable if there exists a positive definite diagonal matrix D such that $A^\mathsf{T}D + DA$ is negative definite.*

The following result can also be inferred from the proof of a proposition presented by Wang et al. (2019a).

Lemma 2.3. *If system* (2.2) *is observable and the graph of the observer network is strongly connected, then the matrix $-\boldsymbol{T}_\mathbf{u}^\mathsf{T}(\mathcal{L}^\mathsf{T} \otimes I)\boldsymbol{T}_\mathbf{u}$ in* (2.9) *is diagonally stable.*

Proof. Due to Lemma 2.1 there exists a null vector r of $\mathcal{L}^\mathsf{T}$ with strictly positive elements such that $\hat{\mathcal{L}} = \mathcal{L}^\mathsf{T}\operatorname{diag}(r) + \operatorname{diag}(r)\mathcal{L}$ is the Laplacian matrix of a connected undirected weighted graph. Therefore, by Lemma 2.2 we have $\boldsymbol{T}_\mathbf{u}^\mathsf{T}(\hat{\mathcal{L}} \otimes I)\boldsymbol{T}_\mathbf{u} \succ 0$, or

$$\boldsymbol{T}_\mathbf{u}^\mathsf{T}(\mathcal{L}^\mathsf{T} \otimes I)(\operatorname{diag}(r) \otimes I)\boldsymbol{T}_\mathbf{u} + \boldsymbol{T}_\mathbf{u}^\mathsf{T}(\operatorname{diag}(r) \otimes I)(\mathcal{L} \otimes I)\boldsymbol{T}_\mathbf{u} \succ 0.$$

From

$$(\operatorname{diag}(r) \otimes I)\boldsymbol{T}_\mathbf{u} = \boldsymbol{T}_\mathbf{u} \operatorname{blkdiag}(r_1 I_{n_{1\mathrm{u}}}, \ldots, r_N I_{n_{N\mathrm{u}}}) \tag{2.10}$$

with $n_{i\mathrm{u}}$ being the dimension of the unobservable subspace of node i, we can conclude that there exits a positive definite diagonal matrix $D = \operatorname{blkdiag}(r_1 I_{n_{1\mathrm{u}}}, \ldots, r_N I_{n_{N\mathrm{u}}})$. ∎

Turning to a general directed graph we will apply the reordering (2.3) and see that indeed the observability attribute of the components which are not sources does not play a role for the (diagonal) stability of the matrix $-\boldsymbol{T}_\mathbf{u}^\mathsf{T}(\mathcal{L} \otimes I)\boldsymbol{T}_\mathbf{u}$.

Proposition 2.1. *If system* (2.2) *is distributively observable, then the matrix $-\boldsymbol{T}_\mathbf{u}^\mathsf{T}(\mathcal{L} \otimes I)\boldsymbol{T}_\mathbf{u}$ in* (2.9) *is diagonally stable.*

Proof. Looking at an ordering of the nodes such that the Laplacian matrix $\mathcal{L}$ of the graph of the network is block-triangular as in (2.3), we can treat every diagonal block in $-\boldsymbol{T}_\mathbf{u}^\mathsf{T}(\mathcal{L}\otimes I)\boldsymbol{T}_\mathbf{u}$ separetly. The diagonal stability of the diagonal blocks corresponding to the k source components follows from Lemma 2.3, since the system is observable from the nodes belonging to a source component. For each of the remaining diagonal

blocks corresponding to the $\bar{k} - k$ connected components which are not sources we first examine a single diagonal block of $\mathcal{M}$. For example applying Lemma 2.1 with some r to the $(k+1)$th strongly connected component we note that

$$\hat{\mathcal{L}}_{k+1} + 2\,\mathrm{diag}(r)D_1 \succ 0,$$

since both terms in the sum are positive semi-definite, but $\ker(\hat{\mathcal{L}}_{k+1}) \cap \ker(D_1) = \{0\}$. Using (2.10) reveals then that the corresponding diagonal block in $-\boldsymbol{T}_{\mathbf{u}}^{\mathsf{T}}(\mathcal{L} \otimes I)\boldsymbol{T}_{\mathbf{u}}$ is diagonally stable. ■

We end this section with a theorem concluding the above discussion. A proof of the design was also derived by Kim et al. (2019) (where directly the fact was used that $\mathcal{M}$ in (2.3) is a nonsingular M-matrix, since its diagonal blocks are irreducibly diagonally dominant (Horn and Johnson, 2013)). They establish a lower bound for γ to guarantee a certain decay rate of the error which depends on the graph topology and all the local unobservable subspaces (of a strongly connected component). Moreover, a decentralized design scheme is obtained (without guaranteeing a decay rate) by choosing an adaptive $G_{i,j}(t) = \gamma_i(t)$. Essentially, the adaption corresponds to a proportional-integral feedback of the disagreement on the state estimate between the neighboring nodes and it might be a way to offer flexibility with respect to a sudden change of nodes in the distributed observer network.

Theorem 2.1. *Consider a system* (2.2) *which is distributively observable with respect to the graph of the communication network of the distributed observer* (2.4)*, where using local observability decompositions the gains are of the form* (2.6)*. By selecting the* L_{io} *(e.g. with pole placement) and* $G_{i,j} = \gamma > 0$ *sufficiently high, the observer nodes can be designed such that their local state estimate converges towards the state of the system with any desired rate.*

2.3.2 Pole placement

The idea of using the observability decomposition can be taken a step further, as was done by del Nozal et al. (2019). The key is to look at the p-hop neighborhood of a vertex, which is the set consisting not only of the direct neighbors, but also the vertices which are connected by a path of length less or equal then p (the direct

neighbors are the 1-hop neighbors). In addition to dividing the state space for each node into the observable and unobservable subspace, the unobservable subspace is further recursively divided into mutually orthogonal complements for each p-hop neighborhood. The pth subspace is the subspace which is in the observable subspace if the partial outputs of the p-hop neighbours are included. Such a decomposition is called the multi-hop subspace decomposition.

To be formal, let for every $i \in \{1, \dots, N\}$ and $\rho \in \{1, \dots, \ell_i\}$

$$C_{i,p} = [C_j^\mathsf{T}]_{j \in \mathcal{N}_{i,p}}^\mathsf{T}$$

where $\mathcal{N}_{i,p}$ is the set of vertices in a subgraph of $\mathcal{G}$ centered around i of radius p, $\mathcal{N}_{i,0} = \{i\}$ and ℓ_i is the smallest number such that $\mathcal{O}(A, C_{\ell_i,i})$ has full rank. Such an ℓ_i exists for every i if the system is distributively observable with respect to $\mathcal{G}$. The coordinate transformation $T_i = \begin{bmatrix} T_{i,0} & \dots & T_{i,\ell_i} \end{bmatrix}$ is then set up of mutually orthogonal columns such that the columns of $T_{i,0} = T_{i\mathrm{o}}$ are a basis of the observable subspace of node i and

$$\mathrm{Im}(T_{i,p}) = \ker \left(\mathcal{O}(A, C_{i,p})\right)^\perp \cap \ker \left(\mathcal{O}(A, C_{i,p-1}\right)$$

where $\perp$ means orthogonal complement.

The gains in (2.4) are chosen as

$$L_i = T_{i,0} L_{i\mathrm{o}}, \qquad H_{i,j} = \sum_{p=1}^{\ell_i} T_{i,p} G_{i,j,p} T_{j,p-1}^\mathsf{T}$$

which is for the output feedback the same as in the consensus-based design, but in the diffusive coupling term there is a gain matrix $G_{i,j,p}$ for each p-hop observable subspace and there is first a projection to the senders own $(p-1)$-hop observable subspace.

By definition the spaces $\mathrm{Im}(T_{i,p})$ are A-invariant subspaces so that the dynamics ma-

trix in transformed coordinates is block-triangular

$$T_i^{\mathsf{T}} A T_i = \begin{bmatrix} T_{i,0}^{\mathsf{T}} A T_{i,0} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \times & \cdots & T_{i,\ell_i}^{\mathsf{T}} A T_{i,\ell_i} \end{bmatrix}.$$

The transformation

$$E = \begin{bmatrix} \operatorname{blkdiag}(T_{1,0}, \ldots, T_{1,\ell_i}) & \cdots & \operatorname{blkdiag}(T_{N,0}, \ldots, T_{N,\ell_N}) \end{bmatrix} \begin{bmatrix} E_0 \\ \vdots \\ E_{\max_i \ell_i} \end{bmatrix},$$

makes the dynamics of the error of the network (2.5) block-triangular with diagonal entries $A_{io} - L_{io}C_{io}$ and $T_{i,p}^{\mathsf{T}} A T_{i,p} - \sum_{j \in \mathcal{N}_i} G_{i,j,p} T_{j,p-1}^{\mathsf{T}} T_{i,p}$ for $p > 0$. It is apparent that it is possible to assign the spectrum of the error dynamics of the observer network with appropriate entries of $[G_{i,j,p}]_{j \in \mathcal{N}_i}$, since the rank of $[T_{j,p-1}]_{j \in \mathcal{N}_i}^{\mathsf{T}} T_{i,p}$ is equal to the dimension of $T_{i\rho}^{\mathsf{T}} T_{i\rho}$ (hence the pair $(T_{i,p}^{\mathsf{T}} A T_{i,p}, [T_{j,p-1}]_{j \in \mathcal{N}_i}^{\mathsf{T}} T_{i,p})$ is trivially observable).

2.3.3 Controller based design

Park and Martins (2012) recognized that the design of the observers gains in (2.5) can also be cast into a particular cooperative stabilization problem with static control of a network of N systems with identical dynamics matrix. Inspired by this, Wang and Morse (2018) selected the observer nodes of the form (2.4), except the dynamics of one arbitrary observer node (in every source component), say the ith as

$$\dot{\tilde{x}}_i = A\tilde{x}_i - (L_i + \bar{D})(C_i\tilde{x}_i - y_i) - \sum_{j \in \mathcal{N}_i} H_{i,j} \left(\tilde{x}_i - \tilde{x}_j\right) + \bar{C}z$$

where $z \in \mathbb{R}^{nN}$ is an addtional state with dynamics and output feedback

$$\dot{z} = \bar{A}z - \bar{B}(C_i\tilde{x}_i - y_i). \tag{2.11}$$

It is proven that if system (2.2) is observable and the graph strongly connected, then for almost any choice of the output-feedback and coupling gains of the observer nodes (2.4) the resulting error system of the observer network is controllable with respect to the input matrix $\begin{bmatrix} 0 & \cdots & \underbrace{I}_{i\text{th}} & \cdots & 0 \end{bmatrix}^{\mathsf{T}}$ and observable from the partial output $\begin{bmatrix} 0 & \cdots & C_i & \cdots & 0 \end{bmatrix} E = C_i e_i$.

This means that if the observer state of one node is augmented by $z \in \mathbb{R}^{nN}$, then (2.11) can be implemented as an Luenberger observer to estimate E and following the separation principle the poles of the closed-loop can be freely assigned with $\bar{B}$ and $\bar{C}$. Furthermore, it was established that the controllability index is (whenever it is non-zero) always N. Therefore, it is pointed out that the spectrum can (in theory) be selected with $\bar{A}, \bar{B}, \bar{C}$ and $\bar{D}$ by implementing (2.11) as a dynamic controller with order $N - 1$.

2.3.4 Design using linear matrix inequality

Ugrinovskii (2011) proposed a linear matrix inequality (LMI) for the design using a vector Lyapunov function approach. However, in the paper it is not clarified under which conditions a solution to the LMI problem exists. It presumes that there exists a block-diagonal Lyapunov-matrix for the error dynamics of the network. We will show that under the distributed observability condition this is the case and derive a straightforward method to design the gains with an LMI (formulated later in section 2.5). This LMI approach is also at the basis for designing distributed observers for systems with delay in the information exchange in Chapter 4.

The stability of the error dynamics (2.5) can be analyzed with a quadratic Lyapunov function $V(E) = E^{\mathsf{T}} P E$ leading to the LMI

$$(I_N \otimes A)^{\mathsf{T}} P + P(I_N \otimes A) - \boldsymbol{C}^{\mathsf{T}} \boldsymbol{L}^{\mathsf{T}} P - P \boldsymbol{L} \boldsymbol{C} - \mathcal{H}^{\mathsf{T}} P - P \mathcal{H} \prec -Q,$$

with positive definite matrices P and Q. To design the gains L_i and $H_{i,j}$ we prescribe a block-diagonal structure to the Lyapunov-matrix $P = \operatorname{blkdiag}(P_1, \ldots, P_N)$. The substitution $Z_{i,j} = P_i H_{i,j}$ and $Y_i = P_i C_i$ then leads to an LMI for the design of the gains while preserving the structural and algebraic constraint on the matrices $\boldsymbol{L}$ and

$\mathcal{H}$, respectively.

Such a block-diagonal stability property is dependent on the coordinate system (Carlson et al., 1992). We show that indeed there exist gains such that (2.5) is block-diagonally stable in its original coordinates. For the proof we use the diagonal stability property established in Proposition 2.1 (the multi-hop subspace decomposition of section 2.3.2 could also be used instead).

Theorem 2.2. *If system* (2.2) *is distributively observable, then there exist output-feedback and coupling gains such that the error dynamics* (2.5) *of the observer network* (2.4) *admits a block-diagonal Lyapunov-matrix in its original coordinates.*

Proof. Consider the gains (2.6) and the corresponding error dynamics in transformed coordinates (2.9). For the internal dynamics of E_o and E_u there exist Lyapunov-matrices
$\mathrm{blkdiag}(P_{1\mathrm{o}}, \dots, P_{N\mathrm{o}}) \succ 0$ and $\mathrm{blkdiag}(P_{1\mathrm{u}}, \dots, P_{N\mathrm{u}}) \succ 0$, respectively (the former is clear, the later is due to Proposition 2.1). Consequenly, system (2.9) has a Lyapunov matrix of the form $\left[\begin{smallmatrix} \boldsymbol{P}_\mathrm{o} & 0 \\ 0 & \gamma \boldsymbol{P}_\mathrm{u} \end{smallmatrix}\right]$ with some $\gamma > 0$. Returning to original coordinates, the Lyapunov-matrix is transformed to

$$
\begin{aligned}
P &= \begin{bmatrix} \boldsymbol{T}_\mathrm{o} & \boldsymbol{T}_\mathrm{u} \end{bmatrix} \begin{bmatrix} \boldsymbol{P}_\mathrm{o} & 0 \\ 0 & \gamma \boldsymbol{P}_\mathrm{u} \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_\mathrm{o}^\mathsf{T} \\ \boldsymbol{T}_\mathrm{u}^\mathsf{T} \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{blkdiag}(T_{1\mathrm{o}} P_{1\mathrm{o}} T_{1\mathrm{o}}^\mathsf{T}, \dots, T_{N\mathrm{o}} P_{N\mathrm{o}} T_{N\mathrm{o}}^\mathsf{T}) & 0 \\ 0 & \mathrm{blkdiag}(T_{1\mathrm{u}} P_{1\mathrm{u}} T_{1\mathrm{u}}^\mathsf{T}, \dots, T_{N\mathrm{u}} P_{N\mathrm{u}} T_{N\mathrm{u}}^\mathsf{T}) \end{bmatrix}.
\end{aligned}
$$

■

2.4 Discrete-time setting

The discrete-time setting, where a linear time-invariant system is described by

$$
\begin{aligned}
x_{k+1} = A x_k, \qquad & y_{1,k} = C_1 x_k \\
& \vdots \\
& y_{N,k} = C_N x_k
\end{aligned}
\tag{2.12}
$$

has received more attention in the literature, especially to find a Kalman-filter like optimal solution to the distributed state estimation problem. It comes naturally from the fact that controllers are digital in applications. Moreover, the delays caused by communicating over a network are explicitly included, since in one time step the information can reach only a neighboring node.

To illustrate this, we contrast the exchange of state estimates with the exchange of partial outputs in the situation where the observer nodes are connected in a directed cycle. For simplification, we examine the special case of a system dynamics matrix in observer form

$$A = \begin{bmatrix} \underbrace{a_1}_{a} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2} & 0 & \cdots & 1 & 0 \\ a_{n-1} & 0 & \cdots & 0 & 1 \\ a_n & 0 & \cdots & 0 & 0 \end{bmatrix}$$

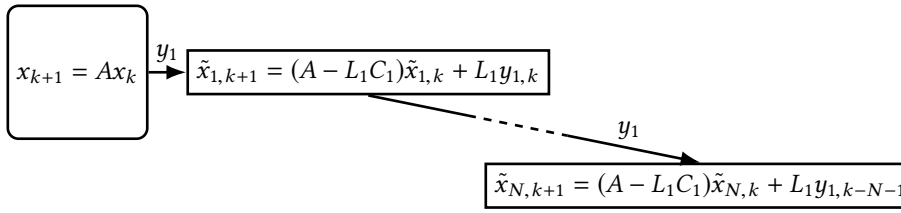
with $a \neq 0$ and that the first observer node can estimate the state independently

$$C_1 = \begin{bmatrix} 1 & \dots & 0 \end{bmatrix},$$

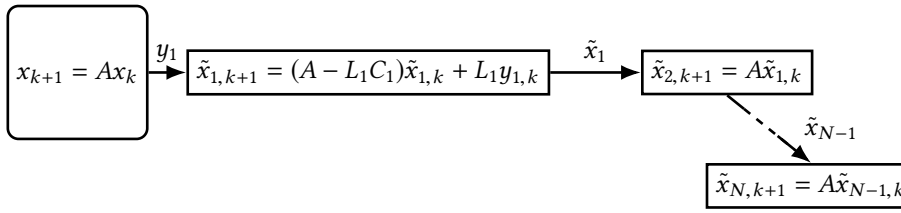
while the other nodes cannot access the system at all ($C_i = 0$ for $i > 1$), thus fully rely on communication. For the first observer we can select a dead-beat observer with output-feedback gain $L_1 = a$, such that it will reconstruct the state of system (2.12) exactly after n steps at most. For the other nodes the outcome will depend on what information it receives, see Figure 2.3.

If the output is forwarded, it will take $N-1$ steps to pass it from the first node to the farthest node and the reconstructed state will therefore be also delayed by the same amount of time steps. However, if instead the local state estimates are forwarded and the first node has reconstructed the state, it takes additional $N-1$ steps for the nodes to synchronize end there will be no delay in the estimate.

In the continuous-time setting, where communication is instantaneous if delays are not considered, there is no such difference in output or state coupling. In Chapter 4 we will argue that in the context of distributed state estimation it is nonetheless more suitable to keep a continuous-time framework while modeling the (real-world)



(a) Static output coupling



(b) Dynamic state coupling

Figure 2.3: Illustration of an observer network connected in a chain with different kind of information exchange. In case (a), unlike in (b), the estimate at the last node will have a steady delay.

effects of the network. The simple example with a dead-beat observer does not have a continuous-time analogue, instead we will show the difference in static and dynamic coupling by comparing the minimal transmission rate to guarantee a certain performance in terms of convergence rate.

Returning to the general case, the diffusively coupled observer nodes in discrete-time are selected as

$$\tilde{x}_{i,k+1} = A\tilde{x}_{i,k} - L_i(C_i\tilde{x}_i - y_{i,k}) - \sum_{j\in\mathcal{N}_i} H_{i,j}\left(\tilde{x}_{i,k} - \tilde{x}_{j,k}\right). \tag{2.13}$$

2.4.1 Information diffusion

It is immediately clear that the pole-placement method of section 2.3.2 is also directly applicable to the discrete-time setting. By placing all the eigenvalues at zero a distributed dead-beat observer is obtained. The error dynamics is then a nilpotent matrix with a maximal index $n + N - 1 \leq nN$ occurring in the worst-case described above. In general cases, the distributed dead-beat observer will be faster then n by

taking advantage of the multiple outputs of the system (provided that the network is not needlessly large). Note however, that in a situation where the system is completely observable from a partial output the distributed dead-beat observer designed using the multi-hop subspace decomposition will not make use of this advantage. Due to the particular choice of the coupling gains no incoming communication will take place. This highlights that a more general choice of the coupling gains and a better design criteria are necessary.

Mitra and Sundaram (2018) propose a single global decomposition where the same issue arises. For each source component of the graph of the observer network pairwise essentially disjoint subspaces (intersection is only $\{0\}$) are assigned to the observer nodes. Using an arbitrary ordering of the nodes and starting from the last, its subspace is the unobservable subspace which arises if its partial output is removed from the output (this can be the zero space). A next node's subspace is such that the direct sum with the subspaces of the previous nodes is the enlarged unobservable subspace formed by removing the partial output of the node as well (if orthogonality is imposed the decomposition equals the multi-hop subspace decomposition of a single node on a graph where each p-hop neighborhood increases by one vertex).

Let $T = \begin{bmatrix} T_1 & \dots & T_N \end{bmatrix}$, where the columns of $T_1 \in \mathbb{R}^{n \times n_1}, \dots, T_N \in \mathbb{R}^{n \times n_N}$ form basis vectors of the respective subspaces described above, then $T^{-1}AT = A_1 + A_2$, where A_1 is block-diagonal with diagonal block dimensions $n_1 \times n_1, \dots, n_N \times n_N$ and A_2 is stricly lower block-triangular. The observers gains in (2.13) using a global decomposition are then

$$L_i = T \begin{bmatrix} 0 \\ \vdots \\ \bar{L}_i \\ \vdots \\ 0 \end{bmatrix}, \qquad H_{i,j} = TA_1 \operatorname{blkdiag}(w_{i,j,1} I_{n_1}, \dots, w_{i,j,N} I_{n_N}) T^{-1}.$$

To choose the weights $w_{i,j,k}$ we consider for each subspace k, spanning-trees (an acyclic subgraph such that there is a path to every other vertex) from the vertex of the designated node for this subspace. A weight $w_{i,j,k}$ is non-zero only if there is a path from the node for the subspace k to j in such a spanning-tree and moreover

$\sum_{j\in\mathcal{V}_i} w_{i,j,k} = 1$. It can be shown that in this case the eigenvalues of the resulting error dynamics can be positioned with the feedback gains $\bar{L}_i$ into the inner unit circle, and therefore the distributed observer (2.13) converges towards the state of system (2.12). In extension, a distributed dead-beat observer can be obtained in which case the estimates reach exactly the state in at most $n+2N(N-1)$ steps (Mitra et al., 2019).

Some tentative observations

Recall the presence of matrix A_1 in the coupling gain. For the continuous-time version of this design the matrix is replaced with the identity matrix. The difference can be explained when comparing the synchronization of identical linear systems via (state) diffusive coupling (Scardovi and Sepulchre, 2009). Looking at the conditions for synchronization in discrete-time explains also the normalization of the sum of the weights to 1 (diffusive coupling equals then to a convex combination, as in average consensus algorithms).

In the continuous-time algorithm the weighting in the transformed coordinates can be omitted (i.e. $w_{i,j,k} = 1$ for all k) if a sufficiently high scalar coupling gain is selected instead. However, compared to the local observability decomposed design (section 2.3.1), the gains $\bar{L}_i$ of the output feedback of the observable space cannot be tuned independently in this case [2]. The situation is different in the discrete-time setting, if instead of the ordered weights (in the transformed space) a weighted average of the estimates of the neighbours is taken in original coordinates. It is not possible to obtain faster synchronization with a high gain (the sum of the weights has to be smaller then 1), but instead more steps of averaging are necessary.

This leads to designs with multiple consensus iterations in between the time steps of the system. Açıkmeşe et al. (2014) show for a connected graph, that if the output-feedback gains L_i are selected such that the eigenvalues of $A + \frac{1}{N}\sum_{i=1}^{N} L_i C_i$ are in the inner unit circle, then for a uniform selection of consensus weights, the numbers of consensus iterations for stability of the error dynamics is finite. Following Rego et al. (2016), the same gains can be used for strongly connected graphs by considering non-uniform scalar weights (which are independent of the system matrices

[2] In numerical tests an inverse relationship has been observed between how high the synchronization gain is and how "fast" the output feedback has to be.

provided that the consensus matrix is doubly stochastic and irreducible with positive diagonal entries). In the next section, we look more in detail at a method with multiple iterations which employs matrix-valued weights, since it can be regarded as the discrete-time equivalent to section 2.3.1.

Finally, an interessting design for discrete-time systems without operations between time steps and constant weights was presented by Rego et al. (2017) relating to stationary Kalman-filtering. Like other distributed Kalman filters, the tuning or optimality of the gains are not obvious and it does not rely on the diffusive coupling structure (2.13). In fact, stable error dynamics can even be obtained if solely one node uses its own estimate in its recurrence relation (the consensus matrix has to be row stochastic and primitive).

2.4.2 Consensus-based design

The design in section 2.3.1 can be regarded as a high-gain approach, in the sense that sufficiently high gain matrices $G_{i,j}$ are used to force the errors of the estimate of the unobservable parts into consensus (Wang et al., 2019a). In the discrete-time setting faster convergence is not achieved by high-gain, since a stability matrix in discrete-time is a Schur matrix not a Hurtwitz matrix. Instead, the discrete-time counterpart has to use multiple inner iterations for the consensus between time steps and a higher number of inner iterations corresponds to an increased continuous-time consensus gain.

The design with a multiple consensus iteration approach using local observability decompositions was presented by Wang et al. (2019b). The observer nodes for the system (2.12) are selected as

$$\tilde{x}_{i,k+1} = A\bar{x}_{i,k,q} - T_{io}L_{io}\left(C_i\tilde{x}_{i,k} - y_{i,k}\right) \tag{2.14}$$

where the averaged estimate $\bar{x}_{i,k,q}$ is obtained from an inner iteration with q steps of the form

$$\begin{aligned} \bar{x}_{i,k,0} &= \tilde{x}_{i,k} \\ \bar{x}_{i,k,l+1} &= (I - P_i)\bar{x}_{i,k,l} + \frac{1}{|\mathcal{N}_i|}P_i \sum_{j\in\mathcal{N}_i} \bar{x}_{j,k,l} \end{aligned} \tag{2.15}$$

with the orthogonal projection matrix $P_i = T_{i\mathrm{u}} T_{i\mathrm{u}}^{\mathsf{T}}$ onto the unobservable subspace $\mathrm{Im}(T_{i\mathrm{u}})$ of the node i. The iteration (2.15) is similar to a method for solving distributively algebraic equations (Mou et al., 2015). It is interessting to note that (2.15) is a convex combination and for $q = 1$ the observer is in the form of (2.4) with

$$L_i = T_{i\mathrm{o}} L_{i\mathrm{o}} \qquad H_{i,j} = T_{i\mathrm{u}} \frac{A_{i\mathrm{u}}}{|\mathcal{N}_i|} T_{i\mathrm{u}}^{\mathsf{T}},$$

but it will be apparent that generally more inner iterations are required.

Expressing the average after q steps (with $P_i^2 = P_i$ for a projection matrix)

$$\bar{x}_{i,k,q} = (I-P_i)^q \tilde{x}_{i,k} + \frac{1}{|\mathcal{N}_i|} \sum_{l=0}^{q-1} (I-P_i)^{q-l-1} P_i \sum_{j \in \mathcal{N}_i} \bar{x}_{j,k,l} = (I-P_i)\tilde{x}_{i,k} + \frac{1}{|\mathcal{N}_i|} P_i \sum_{j \in \mathcal{N}_i} \bar{x}_{j,k,q-1}$$

the observer dynamics (2.14) can be decomposed in local coordinates into

$$\tilde{x}_{i\mathrm{o},k+1} = A_{\mathrm{o}} \tilde{x}_{i\mathrm{o},k} - L_{i\mathrm{o}} \left(C_{i\mathrm{o}} \tilde{x}_{i\mathrm{o},k} - y_{i,k} \right)$$

for the observable part and

$$\tilde{x}_{i\mathrm{u},k+1} = A_{\mathrm{r}} \tilde{x}_{i\mathrm{o},k} + A_{\mathrm{u}} T_{i\mathrm{u}}^{\mathsf{T}} \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \bar{x}_{j,k,q-1}$$

for the unobservable part, where $\tilde{x}_{i,k} = T_{i\mathrm{o}} \tilde{x}_{i\mathrm{o},k} + T_{i\mathrm{u}} \tilde{x}_{i\mathrm{u},k}$.

Analysis of the error dynamics

Using $(I - \boldsymbol{P}(I - (\mathcal{F} \otimes I))(1_N \otimes x_k) = 1_N \otimes x_k$ and $(I - \boldsymbol{P}(I - (\mathcal{F} \otimes I))^q = I - \boldsymbol{P} - (\boldsymbol{P}(\mathcal{F} \otimes I))^q$ (or more directly that $x_k = (I - P_i)x_k - P_i x_k$) the error of the inner loop is described by

$$\bar{E}_{k,q} = \begin{bmatrix} \bar{x}_{1,k,q} - x_k \\ \vdots \\ \bar{x}_{N,k,q} - x_k \end{bmatrix} = \boldsymbol{T}_{\mathbf{o}} E_{\mathrm{o},k} - (\boldsymbol{P}(\mathcal{F} \otimes I))^q \boldsymbol{T}_{\mathbf{o}} E_{\mathrm{o},k} - (\boldsymbol{P}(\mathcal{F} \otimes I))^q \boldsymbol{T}_{\mathbf{u}} E_{\mathrm{u},k}$$

where the flocking matrix $\mathcal{F} = \mathcal{D}^{-1}\mathcal{A}$ of a graph appears (here without self-loops). The error dynamics in transformed coordinates can then be compactly written as

$$\begin{bmatrix} E_{\mathrm{o},k+1} \\ E_{\mathrm{u},k+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_\mathbf{o} - \boldsymbol{L}_\mathbf{o}\boldsymbol{C}_\mathbf{o} & 0 \\ \boldsymbol{A}_\mathbf{r} - \boldsymbol{A}_\mathbf{u}(\boldsymbol{T}_\mathbf{u}^\mathsf{T}(\mathcal{F}\otimes I)\boldsymbol{T}_\mathbf{u})^{q-1}\boldsymbol{T}_\mathbf{u}^\mathsf{T}(F\otimes I)\boldsymbol{T}_\mathbf{o} & -\boldsymbol{A}_\mathbf{u}(\boldsymbol{T}_\mathbf{u}^\mathsf{T}(\mathcal{F}\otimes I)\boldsymbol{T}_\mathbf{u})^q \end{bmatrix} \begin{bmatrix} E_{\mathrm{o},k} \\ E_{\mathrm{u},k} \end{bmatrix}. \tag{2.16}$$

Similarly to the continuous-time setting we can find $L_{i\mathrm{o}}$ such that the eigenvalues of $\boldsymbol{A}_\mathbf{o} - \boldsymbol{L}_\mathbf{o}\boldsymbol{C}_\mathbf{o}$ are in the inner unit circle. The next lemma summarizing the results of Wang et al. (2019b) is an analog to Lemma 2.3 for a flocking matrix.

Lemma 2.4. *If system* (2.12) *is observable and the graph of the observer network strongly connected, then there exists an induced matrix norm such that* $\|(\boldsymbol{T}_\mathbf{u}^\mathsf{T}(\mathcal{F}\otimes I)\boldsymbol{T}_\mathbf{u})\| < 1$ *in* (2.16).

As a consequence it is possible to select the number of inner iterations q such that the spectral radius $\rho\left(\boldsymbol{A}_\mathbf{u}(\boldsymbol{T}_\mathbf{u}^\mathsf{T}(\mathcal{F}\otimes I)\boldsymbol{T}_\mathbf{u})^q\right) \leq \|\boldsymbol{A}_\mathbf{u}\|\|(\boldsymbol{T}_\mathbf{u}^\mathsf{T}(\mathcal{F}\otimes I)\boldsymbol{T}_\mathbf{u})\|^q < 1$. The local state estimates will therefore converge towards the state of the system (2.12) and to increase the convergence rate more inner iterations are necessary.

2.5 Exchange of artificial outputs

A shortcoming of the diffusive coupling term in (2.4) is that for large-scale systems it might not be viable to transmit the complete state estimate. However, in case of the consensus based design presented in section 2.3.1, if a node j knows the value of $T_{i\mathrm{u}}$ of its neighbor i, it can just transmit the projection ${T_{i\mathrm{u}}}^\mathsf{T}\tilde{x}_j$ instead of $\tilde{x}_j$ to reduce the size of the vector. Note that in Proposition 2.1 we established that the compressed Laplacian matrix is Hurwitz so that a high consensus-gain can be applied to make the error dynamics stable, but in theory matrix $\boldsymbol{A}_\mathbf{u}$ can be made Hurwitz with a low-rank correction matrix instead.

This motives us to consider a design of the coupling gain

$$H_{i,j} = G_{i,j}F_{i,j}$$

such that $H_{i,j}$ is low-rank and the neighboring node j only sends $F_{i,j}\tilde{x}_j$. Following

section 2.3.4, for given values of $F_{i,j}$ for the connected nodes, the design of the gains $G_{i,j}$ can be stated in terms of an LMI feasibility problem.

Theorem 2.3. *Let there exist positive definite matrices $P_1, \ldots, P_N$, matrices $Z_{1,j}$ for all $j \in \mathcal{N}_1, \ldots, Z_{N,j}$ for all $j \in \mathcal{N}_N$ and $Y_1, \ldots, Y_N$ that satisfy the LMI $\Phi \succ 0$ with diagonal blocks*

$$\Phi_{ii} = A^\mathsf{T} P_i + P_i A + C_i^\mathsf{T} Y_i^\mathsf{T} + Y_i C_i + 2\gamma P_i$$

and off-diagonal blocks

$$\Phi_{ij} = \Phi_{ij}^\mathsf{T} = Z_{i,j} F_{i,j} + F_{i,j}^\mathsf{T} Z_{i,j}^\mathsf{T},$$

then the distributed observer (2.4) *with gains $L_i = P_i^{-1} Y_i$ and $H_{i,j} = P_i^{-1} Z_{i,j} F_{i,j}$ converges to the state of system* (2.2) *with decay rate γ.*

From Theorem 2.2 we know that for $F_{i,j} = I, j \in \mathcal{N}_i$ the LMI is feasible under distributed observability, as well as for $F_{i,j} = T_{i\mathrm{u}}^\mathsf{T}$. An heuristic approach to find an $F_{i,j}$ with even lower dimension would be to select its rows as a subset of the rows of $T_{i\mathrm{u}}^\mathsf{T}$. We will use this approach in Chapter 4 to design a distributed observer on an academic example. [3]

If the $F_{i,j}$ are treated as variables, then $\Phi \succ 0$ is not an LMI but a bilinear matrix inequality (BMI). Generally, BMIs are difficult to solve. However, what are feasible apriori choices of $F_{i,j}$ is an open problem (for example the conditions given by Ugrinovskii (2013) are neither necessary nor sufficient). Choosing the $F_{i,j}$ will also inevitably lead to questions of optimality, for example on the selection of the topology of the graph, that is, when to choose non-zero $F_{i,j}$ under some criteria.

[3] A MATLAB script for this and other design methods mentioned in this chapter is available at `http://twr.cs.kuleuven.be/research/software/delay-control/distributed_observers.m`

Chapter 3

Distributed Finite-Time Observers

Besides successfully estimating the state, depending on the application context, observers can be selected with various other criteria in mind. The distributed estimation problem by its very novelty is especially susceptible to non-conventional approaches. One of such is the design of observers with non-asymptotic time characteristics which have previously been investigated for the centralized case by Levant (2003), Perruquetti et al. (2008), Angulo et al. (2013), and Lopez-Ramirez et al. (2018), to name a few. Such a design can be useful for networked systems, where time constraints have to be met due to time-dependent interconnections.

Finite-time observers have been derived with the help of homogeneity (Bhat and Bernstein, 2005). Homogeneous systems can exhibit either exponential, finite-time or fixed-time convergence depending whether the degree is zero, negative or positive, respectively. This can be contrasted to linear systems, which can attain only exponential convergence of the estimation error. The design of homogeneity based finite-time observers in the centralized case relies on the fact that the dynamics of the system can be eliminated with a preliminary linear feedback transformation. In the distributed estimation problem the system is not observable from the partial output of an observer node alone; therefore, a design for distributed finite-time observers does not follow immediately.

The refined selection of coupling gains using the multi-hop subspace decomposition derived by del Nozal et al. (2019) allows to assign the poles of the error dynamics. It might open up the possibility to design (next to distributed dead-beat observers for discrete-time systems) distributed finite-time observers for continuous-time systems

with the generalized homogeneity concept used for multi-output systems (see Zimenko et al., 2020a and 2020b). In this case the structural and algebraic constraints on the gains have to be considered. By comparison, the approach derived in the present thesis using just standard observability decompositions offers a design which is rather simple.

First, we recapitulate some of the definitions and seminal results on (weighted) homogeneous systems. Then a summary of the centralized finite-time observer design is given for the case of a single output system. In extension to previous works we arrive to a bound on the degree of homogeneity which is obtained by solving a standard Lyapunov equation. This design will be useful in the distributed case to estimate locally the observable part of the state at each node in finite-time. As an intermediate step we show a design which is applicable to the special case of a system consisting of disconnected subsystems. Finally, turning to general linear systems, we design a distributed finite-time observer by combining the consensus-based linear design of the previous chapter with homogeneity concepts. The proofs are collected in the end of the chapter for the sake of clarity.

3.1 Preliminaries

Consider a nonlinear system

$$\dot{x}(t) = f\left(x(t), d(t)\right), \ t \geq 0, \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $d(t) \in \mathbb{R}^m$ is the input, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ ensures forward existence and uniqueness of the system's solutions (at least locally) for $d \in \mathcal{L}_\infty$ and $f(0,0) = 0$. For an initial condition $x_0 \in \mathbb{R}^n$ and an input d, the corresponding solution is called $X(t, x_0, d)$, for all $t \geq 0$ for which the solution exists.

Stability properties

Following Roxin (1966), Khalil (2002), and Polyakov (2012), let Ω be an open neighborhood of the origin in $\mathbb{R}^n$.

Definition 3.1. *At the steady state $x = 0$ the system* (3.1) *with $d = 0$ is said to be*

- stable *on Ω if for any $x_0 \in \Omega$, $X(t, x_0, 0)$ is defined for all $t \geq 0$, and for any $\epsilon > 0$ there is $\delta > 0$ such that for any $x_0 \in \Omega$, if $\|x_0\| \leq \delta$ then $\|X(t, x_0, 0)\| \leq \epsilon$, for all $t \geq 0$;*
- asymptotically stable *on Ω if it is stable and for any $\kappa > 0$ and $\epsilon > 0$ there exists $T(\kappa, \epsilon) \geq 0$ such that for any $x_0 \in \Omega$, if $\|x_0\| \leq \kappa$ then $\|X(t, x_0, 0)\| \leq \epsilon$ for all $t \geq T(\kappa, \epsilon)$;*
- finite-time stable *on Ω if it is stable and* finite-time converging from Ω, *i.e. for any $x_0 \in \Omega$ there exists $0 \leq T < +\infty$ such that $X(t, x_0, 0) = 0$ for all $t \geq T$. The function $T_0(x_0) = \inf\{T \geq 0 : X(t, x_0, 0) = 0,\ \forall t \geq T\}$ is called the* settling time *of the system* (3.1)*;*
- fixed-time stable *on Ω if it is finite-time stable and $\sup_{x_0 \in \Omega} T_0(x_0) < +\infty$.*

The set Ω is called the domain of stability/attraction.

If $\Omega = \mathbb{R}^n$, then the corresponding properties are called *global* stability / asymptotic stability / finite-time / fixed-time stability of (3.1) at $x = 0$. Similarly, these stability notions can be defined with respect to a compact invariant set, by replacing the distance to the origin in Definition 3.1 with the distance to the invariant set. In the case of an input $d \neq 0$ we are interested in the following stability property (see Dashkovskiy et al. (2011):

Definition 3.2. *System* (3.1) *is called input-to-state stable (ISS), if there are some functions $\beta \in \mathcal{KL}, g \in \mathcal{K}$ such that for any input $d \in \mathcal{L}_\infty$ and any $x_0 \in \mathbb{R}^n$*

$$\|X(t, x_0, d)\| \leq \beta(\|x_0\|, t) + g(\|d\|_{[0,\,t)})$$

for all $t \geq 0$.

Weighted homogeneity

Following Zubov (1958) and Bacciotti and Rosier (2001), for strictly positive numbers $r_1, \ldots, r_n$ called weights and $\lambda > 0$, define:

- the *vector of weights* $\mathbf{r} = \begin{bmatrix} r_1 & \cdots & r_n \end{bmatrix}^\mathsf{T}$, $r_{\max} = \max_{1 \le i \le n} r_i$ and $r_{\min} = \min_{1 \le i \le n} r_i$;
- the *dilation matrix* function $\Lambda_{\mathbf{r}}(\lambda) = \operatorname{diag}\left(\begin{bmatrix} \lambda^{r_1} & \dots & \lambda^{r_n} \end{bmatrix}^\mathsf{T} \right)$. Note that for all $x \in \mathbb{R}^n$ and for all $\lambda > 0$ we have $\Lambda_{\mathbf{r}}(\lambda)x = \begin{bmatrix} \lambda^{r_1} x_1 & \cdots & \lambda^{r_n} x_n \end{bmatrix}^\mathsf{T}$;
- the $\mathbf{r}$*–homogeneous norm* $\|x\|_r = \left(\sum_{i=1}^n |x_i|^{\frac{\rho}{r_i}} \right)^{\frac{1}{\rho}}$ for any $x \in \mathbb{R}^n$ and $\rho \ge r_{\max}$. This is not a norm in the standard sense, since the triangle inequality is not satisfied for $\|\cdot\|_r$, however there exist $\underline{\sigma}, \overline{\sigma} \in \mathcal{K}_\infty$ such that

$$\underline{\sigma}(\|x\|_r) \le \|x\| \le \overline{\sigma}(\|x\|_r) \quad \text{for all } x \in \mathbb{R}^n;$$

- for $\rho \ge 0$, the *sphere* and the *ball in the homogeneous norm* $S_r(\rho) = \{x \in \mathbb{R}^n : \|x\|_r = \rho\}$ and $B_r(\rho) = \{x \in \mathbb{R}^n : \|x\|_r \le \rho\}$.

Definition 3.3. *A function $g : \mathbb{R}^n \to \mathbb{R}$ is* $\mathbf{r}$–homogeneous with degree $\mu \in \mathbb{R}$ *if for all $x \in \mathbb{R}^n$ and for all $\lambda > 0$ we have $\lambda^{-\mu} g(\Lambda_{\mathbf{r}}(\lambda)x) = g(x)$. A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is* $\mathbf{r}$–homogeneous with degree $\nu \in \mathbb{R}$, *with $\nu \ge -r_{\min}$ if for all $x \in \mathbb{R}^n$ and for all $\lambda > 0$ we have $\lambda^{-\nu} \Lambda_{\mathbf{r}}^{-1}(\lambda) f(\Lambda_{\mathbf{r}}(\lambda)x) = f(x)$ (this means that the ith component of f is a $\mathbf{r}$-homogeneous function of degree $r_i + \nu$).*

System (3.1) *with $d = 0$ is* $\mathbf{r}$*–homogeneous of degree ν if the vector field f is* $\mathbf{r}$*–homogeneous of the degree ν.*

Theorem 3.1 (Zubov, 1958; Rosier, 1992; Bhat and Bernstein, 2000)**.** *For the system* (3.1) *with $d = 0$ and* $\mathbf{r}$*–homogeneous and continuous function f the following properties are equivalent:*

- *the system* (3.1) *is locally asymptotically stable;*
- *there exists a continuously differentiable* $\mathbf{r}$*–homogeneous Lyapunov function V :*

$\mathbb{R}^n \to \mathbb{R}_+$ *such that*

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|),$$
$$\frac{\partial V}{\partial x}(x) f(x) \le -\alpha(\|x\|),$$
$$\lambda^{-\mu} V(\Lambda_{\mathbf{r}}(\lambda)x) = V(x),\ \mu > r_{\max},$$

for all $x \in \mathbb{R}^n$ *and for all* $\lambda > 0$*, for some* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and* $\alpha \in \mathcal{K}$*;*

- *there is a compact strictly positively invariant set* S *containing the origin (i.e.* $X(t, x_0, 0) \in S \setminus \partial S$ *for all* $t > 0$ *and all* $x_0 \in \partial S$*).*

Theorem 3.2 (Nakamura et al., 2002)**.** *If* (3.1) *with* $d = 0$ *is* $\mathbf{r}$*–homogeneous of degree* ν *and asymptotically stable at the origin, then it is*

- *globally finite-time stable at the origin if* $\nu < 0$*;*
- *globally exponentially stable at the origin if* $\nu = 0$*;*
- *globally fixed-time stable with respect to a ball, if* $\nu > 0$*.*

3.2 Centralized design

Consider a linear time-variant system

$$\dot{x} = Ax, \quad y = c^\mathsf{T} x, \tag{3.2}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}$ is the output of the system, $A \in \mathbb{R}^{n \times n}$ is the dynamics matrix and $c \in \mathbb{R}^n$ is the output vector such that the system is observable from y (i.e. the observability matrix $O(A, c^\mathsf{T})$ is invertible).

A way to construct a finite-time observer for a linear system with scalar output (3.2) is by adding a nonlinear output feedback term to a Luenberger observer

$$\dot{\tilde{x}} = A\tilde{x} - k(c^\mathsf{T}\tilde{x} - y) - T\,\mathrm{diag}(\ell)\lceil 1_n (c^\mathsf{T}\tilde{x} - y)\rfloor^g \tag{3.3}$$

where $1_n \in \mathbb{R}^n$ is the vector with all entries equal to 1 and the operation

$$\lceil a \rfloor^b := \begin{bmatrix} \operatorname{sign}(a_1)|a_1|^{b_1} & \cdots & \operatorname{sign}(a_n)|a_n|^{b_n} \end{bmatrix}^\mathsf{T}$$

with vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ denotes the sign preserving element-wise exponentiation. The gain of the linear term $k = A^n O(A, c^\mathsf{T})^{-1} \begin{bmatrix} 0 & \cdots & 1 \end{bmatrix}^\mathsf{T}$ is obtained through Ackermann's formula such that all the eigenvalues of $A - kc^\mathsf{T}$ are zero. The matrix $T = O(A - kc^\mathsf{T}, c^\mathsf{T})$ is a coordinate transformation such that $T(A - kc^\mathsf{T})T^{-1}$ is a chain of integrators and $c^\mathsf{T}T^{-1} = \begin{bmatrix} 1 & \cdots & 0 \end{bmatrix}$.

In the following we will discuss the selection of the gains $\ell \in \mathbb{R}^n_+$ and $g \in \mathbb{R}^n_+$ such that for any initial conditions of (3.2) and (3.3) the state estimate $\tilde{x}$ reaches x in a finite-time. To this end, we investigate the dynamics of the error $e = T(\tilde{x} - x)$ in transformed coordinates

$$\dot{e} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} e - \operatorname{diag}(\ell) \lceil 1_n e_1 \rfloor^g . \tag{3.4}$$

It can be verified that for a proper selection of the entries of g the right-hand side of (3.4) is homogeneous with any desired degree:

Lemma 3.1. *The error dynamics* (3.4) *is weighted homogeneous if*

$$g = \begin{bmatrix} 1 + \alpha & \ldots & 1 + n\alpha \end{bmatrix}^\mathsf{T}$$

and the sign of the degree of homogeneity is equal to the sign of $\alpha \in \mathbb{R}$.

To establish stability of (3.4), we first observe that with the entries of $\ell = \begin{bmatrix} \ell_1 & \cdots & \ell_n \end{bmatrix}^\mathsf{T}$ selected as the coefficients of a monic Hurwitz polynomial (i.e. the real-parts of the

roots are all negative), the dynamics of the system for $\alpha = 0$

$$\dot{e}|_{\alpha=0} = \Omega(\ell)e, \quad \Omega(\ell) \coloneqq \begin{bmatrix} -\ell_1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\ell_{n-2} & 0 & \cdots & 1 & 0 \\ -\ell_{n-1} & 0 & \cdots & 0 & 1 \\ -\ell_n & 0 & \cdots & 0 & 0 \end{bmatrix}$$

are globally asymptotically stable at the origin. In particular for any $Q \succ 0$ there exists $P \succ 0$ such that

$$\Omega(\ell)^\mathsf{T} P + P\Omega(\ell) + Q = 0. \tag{3.5}$$

To design the finite-time observer it remains to obtain a lower bound on $\alpha < 0$ which guarantees that that asymptotic stability of (3.4) at the origin is preserved. Since it is additionally homogeneous with negative degree (Lemma 3.1), it is globally finite-time stable according to Theorem 3.2.

Theorem 3.3. *System* (3.4) *is globally finite-time stable at the origin, if ℓ is the coefficient vector of a monic Hurwitz polynomial of degree $n+1$ and $g = \begin{bmatrix} 1+\alpha & \dots & 1+n\alpha \end{bmatrix}^\mathsf{T}$, with*

$$-\frac{\eta}{n(\sqrt{n}+\eta)} < \alpha < 0, \quad \eta = \frac{\mathrm{e}}{2}\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}\frac{1}{\|P\,\mathrm{diag}(\ell)\|}$$

where $P \succeq I$ and $Q \succ 0$ satisfying the Lyapunov equation (3.5).

3.3 Distributed finite-time observer for integrator chains

In this section we look for a distributed finite-time observer applied to a system of a specific form, that allows to use the same homogeneity concept as in the previous section. In this scenario the state of a set of agents with integrator dynamics is estimated by a strongly connected network of N observer nodes. The problem has similarities to the finite-time leader-following and consensus problems studied in literature (e.g. Du et al., 2017).

For simplicity, we assume that each agent is measured by exactly one observer node.

The system is then described by

$$\dot{x} = \begin{bmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_N \end{bmatrix} x, \quad y_i = c_i^{\mathsf{T}} x \tag{3.6}$$

where

$$A_i = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad c_i^{\mathsf{T}} = \begin{bmatrix} 0 & \dots & \underbrace{1}_{p_i\text{th}} & \dots & 0 \end{bmatrix},$$

with $A_i \in \mathbb{R}^{n_i \times n_i}$, $c_i \in \mathbb{R}^n$, $n = \sum_{i=1}^N n_i$ and $p_i = 1 + \sum_{k=1}^{i-1} n_k$ for every $i \in \{1, \dots, N\}$.

To obtain a distributed finite-time observer, we combine the principles of section 3.2 with the ideas of Chapter 2. This leads to an observer node with nonlinear output feedback complemented by a nonlinear consensus term over the neighboring nodes

$$\dot{\tilde{x}}_i = A\tilde{x}_i - \operatorname{diag}(\ell_i) \left\lceil 1_n (c_i^{\mathsf{T}} \tilde{x}_i - y_i) \right\rfloor^g - \gamma \sum_{j \in \mathcal{N}_i} \lceil \tilde{x}_i - \tilde{x}_j \rfloor^h \tag{3.7}$$

with gains $\ell_i \in \mathbb{R}_+^n$, $\gamma > 0$ and exponents $g \in \mathbb{R}_+^n$ and $h \in \mathbb{R}_+^n$.

Introducing $e_i = \tilde{x}_i - x$, the local errors are governed by

$$\dot{e}_i = Ae_i - \operatorname{diag}(\ell_i) \lceil 1_n e_i \rfloor^g - \gamma \sum_{j \in \mathcal{N}_i} \lceil e_i - e_j \rfloor^h.$$

We are now interested in the finite-time stability of the error of the observer network $E^{\mathsf{T}} = \begin{bmatrix} e_1^{\mathsf{T}} & \cdots & e_N^{\mathsf{T}} \end{bmatrix}$. Again, we have to first ensure stability in the linear case where $g = 1_n$ and $h = 1_n$. The error dynamics can be expressed then as

$$\dot{E} = (I_N \otimes A - \operatorname{blkdiag}(\ell_1 c_1^{\mathsf{T}}, \dots, \ell_N c_N^{\mathsf{T}}) - \gamma \mathcal{L} \otimes I_n)E,$$

where $\mathcal{L}$ is the Laplacian matrix of the graph of the network.

Assumption 1. *The gains ℓ_i and γ are such that there exists a solution $P \succ 0$ and*

$Q \succ 0$ *for the Lyapunov equation*

$$(I \otimes A - \operatorname{blkdiag}(\ell_1 c_1^\mathsf{T}, \dots, \ell_N c_N^\mathsf{T}) - \gamma \mathcal{L} \otimes I)^\mathsf{T} P + P(I \otimes A - \operatorname{blkdiag}(\ell_1 c_1^\mathsf{T}, \dots, \ell_N c_N^\mathsf{T}) - \gamma \mathcal{L} \otimes I) = -Q. \quad (3.8)$$

Having established stability in the linear case (i.e. for zero degree of homogeneity), we have to find a choice of the exponents g and h such that the error dynamics is homogeneous with negative degree.

Lemma 3.2. *The dynamics of the error* $E^\mathsf{T} = \begin{bmatrix} e_1^\mathsf{T} & \cdots & e_N^\mathsf{T} \end{bmatrix}$ *is weighted homogeneous with negative degree if*

$$g_i = \left[\begin{bmatrix} \frac{\alpha_i}{\alpha_1} + \alpha_i & \frac{\alpha_i}{\alpha_1} + 2\alpha_i & \cdots & \frac{\alpha_i}{\alpha_1} + n_1\alpha_i \end{bmatrix} \cdots \begin{bmatrix} \frac{\alpha_i}{\alpha_N} + \alpha_i & \frac{\alpha_i}{\alpha_N} + 2\alpha_i & \cdots & \frac{\alpha_i}{\alpha_N} + n_N\alpha_i \end{bmatrix}\right]^\mathsf{T}$$
$$h = \left[\begin{bmatrix} 1 + \alpha_1 & \frac{1+2\alpha_1}{1+\alpha_1} & \cdots & \frac{1+n_1\alpha_1}{1+(n_1-1)\alpha_1} \end{bmatrix} \cdots \begin{bmatrix} 1 + \alpha_N & \frac{1+2\alpha_N}{1+\alpha_N} & \cdots & \frac{1+n_N\alpha_N}{1+(n_N-1)\alpha_N} \end{bmatrix}\right]^\mathsf{T}$$

where $\alpha_1 < 0, \dots, \alpha_N < 0$.

To obtain an estimate on the bounds of the exponents such that the error dynamics will be homogeneous with a negative degree but retains asymptotic stability, we select $\alpha_i = \alpha$ (Previously, Silm et al. (2019a) selected $\alpha_i = \frac{\alpha}{1+(p_i-1)\alpha}$. Considering that for the estimate the smallest entry of g is used, the present choice is more favorable). This gives the following result:

Theorem 3.4. *Consider a system* (3.6) *and a strongly connected network of observer nodes* (3.7) *with* $|\mathcal{E}|$ *edges. The error dynamics is globally finite-time stable provided that Assumption 1 is fulfilled and*

$$g = \left[\begin{bmatrix} 1 + \alpha & \cdots & 1 + n_1\alpha \end{bmatrix} \cdots \begin{bmatrix} 1 + \alpha & \cdots & 1 + n_N\alpha \end{bmatrix}\right]^\mathsf{T}$$
$$h = \left[\begin{bmatrix} 1 + \alpha & \cdots & \frac{1+n_1\alpha}{1+(n_1-1)\alpha} \end{bmatrix} \cdots \begin{bmatrix} 1 + \alpha & \cdots & \frac{1+n_N\alpha}{1+(n_N-1)\alpha} \end{bmatrix}\right]^\mathsf{T},$$

with

$$-\frac{(2\bar{n}-1)z + a + b - \sqrt{(z+a+b)^2 - 4bz}}{2\left((a+\bar{n}z)(\bar{n}-1) + b\bar{n}\right)} < \alpha < 0 \quad (3.9)$$

where

$$\bar{n} = \max_i n_i$$
$$a = \|P \operatorname{blkdiag}(\operatorname{diag}(\ell_1), \dots, \operatorname{diag}(\ell_N))\| \sqrt{Nn}\, \bar{n}$$
$$b = 2\gamma \|P \operatorname{blkdiag}(1_{|\mathcal{N}_1|}, \dots, 1_{|\mathcal{N}_N|})^{\mathsf{T}}\| \ln(2) \sqrt{|\mathcal{E}|n}, \quad z = \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$$

and $P \geq I$ and $Q > 0$ satisfy the Lyapunov equation (3.8).

3.4 Distributed finite-time observer for linear systems

We now turn to the distributed state estimation of a general linear system which multiple outputs

$$\dot{x} = Ax, \qquad y_1 = C_1 x, \quad \dots \quad y_N = C_N x. \tag{3.10}$$

As in Chapter 2, we consider the observability decomposition $x_i = \begin{bmatrix} T_{i\mathrm{o}} & T_{i\mathrm{u}} \end{bmatrix} \begin{bmatrix} x_{i\mathrm{o}} \\ x_{i\mathrm{u}} \end{bmatrix}$, where the columns of $T_{i\mathrm{u}}$ are $T_{i\mathrm{o}}$ are orthogonal bases for the locally unobservable subspace and its orthogonal complement, respectively.

A finite-time observer for the estimate $\tilde{x}_{i\mathrm{o}}$ of the observable part can be designed independently of $\tilde{x}_{i\mathrm{u}}$ and in practice any centralized method of finite-time estimation could be employed. In the case that each partial output y_i is scalar, the application of the results in section 3.2 is straightforward. This finite-time observer structure relies on a preliminary feedback which eliminates the dynamics of the system from the local error. The same approach for the whole state estimate is therefore not directly viable, since for none of the observer nodes the pair (A, C_i) is observable.

Instead, we exploit the fact the consensus-based diffusive coupling term as introduced in Chapter 2 ensures stability with arbitrary fast but asymptotic rates. As we

will see, adding a second consensus term embedded in a fractional power

$$\dot{\tilde{x}}_{i\mathrm{u}} = A_{i\mathrm{r}}\tilde{x}_{i\mathrm{o}} + A_{i\mathrm{u}}\tilde{x}_{i\mathrm{u}} - \gamma \sum_{j\in\mathcal{N}_i}\left(\tilde{x}_{i\mathrm{u}} - T_{i\mathrm{u}}{}^{\mathsf{T}}\tilde{x}_j\right) - \left[\sum_{j\in\mathcal{N}_i}\left(\tilde{x}_{i\mathrm{u}} - T_{i\mathrm{u}}{}^{\mathsf{T}}\tilde{x}_j\right)\right]^{1_{n_{i\mathrm{u}}}\beta} \tag{3.11}$$

with $0 < \beta < 1$, will lead to a finite-time observer for the unobservable part on the grounds of homogeneity arguments.

The intuitive idea is that, while the linear term drives the observer nodes into consensus and thus asymptotically to the correct estimate (if each one estimates its observable part correctly), the increasing gain of the nonlinear term ensures that this happens in a finite time. We will show the efficacy of this approach by analyzing the resulting error dynamics for the unobservable part

$$\dot{E}_{\mathrm{u}} = \left(\boldsymbol{A}_{\mathbf{u}} - \gamma \boldsymbol{T}_{\mathbf{u}}^{\mathsf{T}}(\mathcal{L}\otimes I)\boldsymbol{T}_{\mathbf{u}}\right)E_{\mathrm{u}} + \left(\boldsymbol{A}_{\mathbf{r}} + \gamma \boldsymbol{T}_{\mathbf{u}}^{\mathsf{T}}\mathcal{A}\boldsymbol{T}_{\mathbf{o}}\right)E_{\mathrm{o}} - \lceil \boldsymbol{T}_{\mathbf{u}}^{\mathsf{T}}(\mathcal{L}\otimes I)\boldsymbol{T}_{\mathbf{u}}E_{\mathrm{u}} - \boldsymbol{T}_{\mathbf{u}}^{\mathsf{T}}\mathcal{A}\boldsymbol{T}_{\mathbf{o}}E_{\mathrm{o}}\rfloor^{1_{n_{i\mathrm{u}}}\beta}. \tag{3.12}$$

The right-hand side of (3.12) for the case when $E_{\mathrm{o}} = 0$ (all the nodes have estimated their observable subspace correctly) can be written as a sum of vector-fields $\dot{E}_{\mathrm{u}}\big|_{E_{\mathrm{o}}=0} = f_1(E_{\mathrm{u}}) + f_2(E_{\mathrm{u}})$, where

$$f_1(E_{\mathrm{u}}) = \left(\boldsymbol{A}_{\mathrm{u}} - \gamma \boldsymbol{T}_{\mathrm{u}}^{\mathsf{T}}(\mathcal{L}\otimes I)\boldsymbol{T}_{\mathrm{u}}\right)E_{\mathrm{u}}$$
$$f_2(E_{\mathrm{u}}) = -\lceil \boldsymbol{T}_{\mathrm{u}}^{\mathsf{T}}(\mathcal{L}\otimes I)\boldsymbol{T}_{\mathrm{u}}E_{\mathrm{u}}\rfloor^{1_{n_{i\mathrm{u}}}\beta}$$

are homogeneous with degree 0 and $\beta-1$ with respect to the dilation matrix $\Lambda_{\mathbf{r}}(\lambda) = I$, respectively. We can now exploit the fact that as in Lyapunov's indirect method, local stability at the origin can be inferred from the vector field with the lowest degree (Hermes, 1990).

With the choice of $0 < \beta < 1$ we have that $f_2(E_{\mathrm{u}})$ has lower degree than $f_1(E_{\mathrm{u}})$, thus it dominates at the origin. Moreover, the degree is negative, which following Theorem 3.2 means that $\dot{E}_{\mathrm{u}} = f_2(E_{\mathrm{u}})$ is finite-time stable if it is asymptotically stable at the origin. We can conclude that in this case if (3.12) is globally asymptotically stable, it is also globally finite-time stable. Andrieu et al. (2008, Corollary 2.24) derive a detailed proof of this assertion using the notion of homogeneous approximation (for a sum of homogeneous vector-fields the one with the lowest degree ν is the

homogeneous approximation at the origin, since $\lim_{\lambda\to 0} \lambda^{-\nu-1}(f_1(\lambda x) + f_2(\lambda x)) = f_2(x)$).

So far we looked at the case where the observable part E_o of the observer network has attained zero. This will happen eventually in a finite time, if for each node the observer for the observable subspaces has been designed for example according to section 3.2. However, until then it has to be ensured that the transient E_o does not destabilize (3.12). For this we can consider E_o as an external input (since its dynamics is independent of E_u) and verify that (3.12) is ISS.

To establish ISS, which conveniently also directly implies global asymptotic stability for the case when $E_\mathrm{o} = 0$, we will take advantage of two specific properties of (3.12): The diagonal stability of the compressed Laplacian matrix (Proposition 2.1) and that the nonlinearity f_2 is confined into the 1st and 3rd quadrant. Both of these properties together will allow us to use the following lemma adapted from Efimov and Aleksandrov (2019) to prove our main result afterwards.

Lemma 3.3. *Consider the system*

$$\dot{x} = M_1 x + M_2 f(x) + d \tag{3.13}$$

where $f(x) = \begin{bmatrix} f_1(x_1) & \cdots & f_n(x_n) \end{bmatrix}^\mathsf{T} \in \mathbb{R}^n$, f_i *are strictly increasing and satisfy the quadrant condition* $x_i f_i(x_i) \geq 0$, $\lim_{t\to\pm\infty} f_i(t) = \pm\infty$ *and* $d \in \mathcal{L}_\infty^n$. *Let there exist* $P \succ 0$, $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \succ 0$ *and* $\Upsilon = \operatorname{diag}(\upsilon_1, \ldots, \upsilon_n) \succ 0$ *such that*

$$\Phi = \begin{bmatrix} M_1^\mathsf{T} P + P M_1 & M_1^\mathsf{T}\Lambda + P M_2 + \Upsilon \\ * & M_2^\mathsf{T}\Lambda + \Lambda M_2 \end{bmatrix} \preccurlyeq 0. \tag{3.14}$$

Then system (3.13) *is ISS.*

To bring (3.12) into the form of (3.13) we use the affine coordinate transformation

$$x = \boldsymbol{T}_\mathrm{u}^\mathsf{T}(\mathcal{L} \otimes I)\boldsymbol{T}_\mathrm{u} E_\mathrm{u} - \boldsymbol{T}_\mathrm{u}^\mathsf{T}\mathcal{A}\boldsymbol{T}_\mathrm{o} E_\mathrm{o}, \tag{3.15}$$

whose regularity is guaranteed if (3.10) is distributively observable (see Definition 2.2) with respect to the graph of the communication network.

Theorem 3.5. *Consider a system* (3.10) *which is distributively observable with respect*

to the graph of the communication network of the distributed observer consisting of (3.3) *and* (3.11) *for each locally observable and unobservable subspace, respectively. Let the observers for the locally observable subspace be designed according to Theorem 3.3 and* $0 < \beta < 1$. *Then there exists a sufficiently large* $\gamma > 0$ *such that the LMI* (3.14) *with*

$$M_1 = \boldsymbol{T}_{\mathrm{u}}^{\mathsf{T}}(\mathcal{L} \otimes I)\boldsymbol{T}_{\mathrm{u}}\boldsymbol{A}_{\mathrm{u}}(\boldsymbol{T}_{\mathrm{u}}^{\mathsf{T}}(\mathcal{L} \otimes I)\boldsymbol{T}_{\mathrm{u}})^{-1} - \gamma \boldsymbol{T}_{\mathrm{u}}^{\mathsf{T}}(\mathcal{L} \otimes I)\boldsymbol{T}_{\mathrm{u}}$$
$$M_2 = -\boldsymbol{T}_{\mathrm{u}}^{\mathsf{T}}(\mathcal{L} \otimes I)\boldsymbol{T}_{\mathrm{u}}$$

is feasible, guaranteeing that the local state estimates reach the state of the system in finite-time for any initial condition.

3.4.1 Example

The system's dynamics matrix is

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -8 & 1 & 1 & -1 & -2 & 0 \\ 4 & -0.5 & 0.5 & 0 & 0 & -4 \end{bmatrix}$$

with output matrices

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$
$$C_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$
$$C_4 = \begin{bmatrix} 2 & 0 & 5 & 0 & 0 & 0 \end{bmatrix}.$$

The observer network is given as in Figure 3.1.

The observer gains for the observable subspace were selected such that the eigenvalues of the error systems are all -3 for $\alpha_i = 0$ and then $\alpha_i = -0.1$ was chosen. The consensus gains were selected to be $\gamma = 10$ and the exponent $\beta = 0.7$. To check if γ is high enough any standard LMI solver can be used to check the feasibility of (3.14).

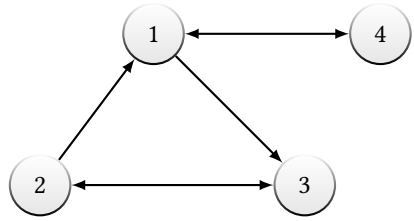


Figure 3.1: Communication graph of the distributed finite-time observer

Figure 3.2 shows the simulation result in semi-logarithmic scaling to high-light the finite-time convergence. The result of a second simulation is shown in linear scaling in, where band-limited white noise of power 10^{-2} was added to each output. It demonstrates the robustness of the distributed finite-time observer with respect to measurement noise. Moreover, the effect of measurement noise is more smooth in the unobservable part since it is filtered by the dynamics of the observable part of the network.

3.5 Proofs

Proof of Lemma 3.1. It has to be verified that the right-hand side of $\dot{e} = f(e)$ in (3.4) fulfills the restrictions imposed in Definition 3.3 for a dilation $\Lambda_{\mathbf{r}}(\lambda)$ with some weights $\mathbf{r} = \begin{bmatrix} r_1 & \cdots & r_n \end{bmatrix}^{\mathsf{T}}$ and homogeneity degree ν. Comparing

$$f_s\left(\Lambda_{\mathbf{r}}(\lambda)e\right) = \begin{cases} \lambda^{r_{s+1}} e_{s+1} + \lambda^{r_1 g_s} \ell_s \lceil e_1 \rfloor^{g_s} & \text{if } s \in \{1, \dots n-1\} \\ \lambda^{r_1 g_s} \ell_s \lceil e_1 \rfloor^{g_s} & \text{if } s = n. \end{cases}$$

with

$$\lambda^{r_s+\nu} f_s\left(e\right) = \begin{cases} \lambda^{r_s+\nu} e_{s+1} + \lambda^{r_s+\nu} \ell_s \lceil e_1 \rfloor^{g_s} & \text{if } s \in \{1, \dots n-1\} \\ \lambda^{r_s+\nu} \ell_s \lceil e_1 \rfloor^{g_s} & \text{if } s = n. \end{cases}$$

shows that for any selection of weights which satisfy $r_{s+1} = r_s + \nu = r_1 + s\nu$, the system is homogenous if

$$g_s = \frac{r_s + \nu}{r_1} = 1 + s\alpha,$$

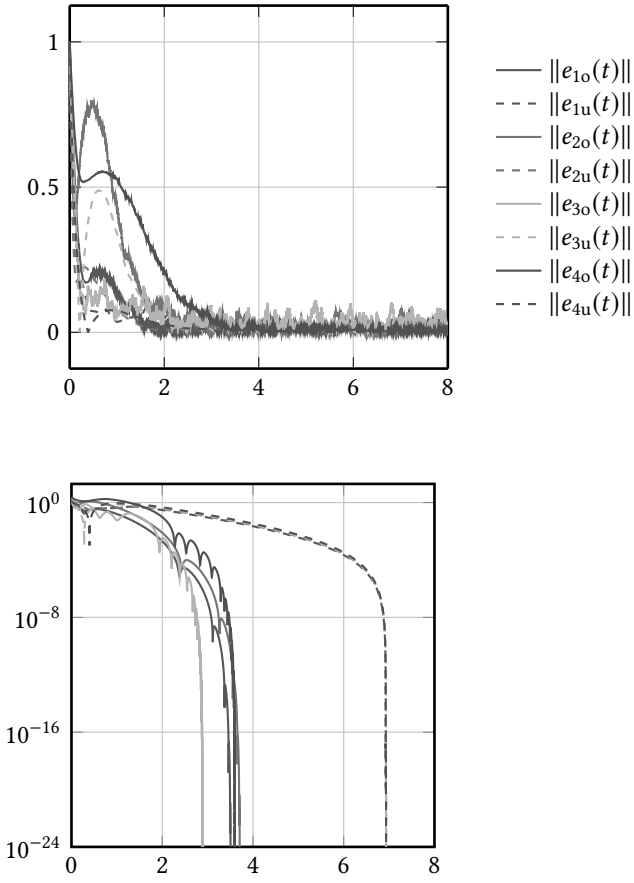


Figure 3.2: Local errors of the observer network in transformed coordinates with (top) and without measurement noise (bottom)

where $\alpha = \nu/r_1$. In particular, the sign of the degree of homogeneity is determined by α, because $r_1 > 0$. ■

Proof of Theorem 3.3. We show that (3.4) has a strictly positively invariant set according to Theorem 3.1. To this end we rewrite the dynamics by adding and substracing $\ell \begin{bmatrix} 1 & \dots & 0 \end{bmatrix}$ as follows:

$$\dot{e} = \Omega(\ell)e - \operatorname{diag}(\ell)\left(\lceil e_1 \rfloor^g - 1_n e_1\right).$$

Taking the Lyapunov function $V(e) = e^{\mathrm{T}}Pe$ we can estimate with the Euclidean norm $\|\cdot\|$ its time-derivatives along the solutions

$$\dot{V} = -e^{\mathrm{T}}Qe + 2e^{\mathrm{T}}P\operatorname{diag}(\ell)\left(\lceil 1e_1 \rfloor^g - 1e_1\right) \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}V + 2\|e\|\|P\operatorname{diag}(\ell)\|\|\lceil 1e_1 \rfloor^g - 1e_1\|$$

where Q satisfies (3.5). Let us consider the set $\partial S = \{e \in \mathbb{R}^n : V(e) = 1\}$ and select Q such that $P \geq I$, which ensures $\|e\|^2 \leq V(e)/\lambda_{\min}(P) \leq 1$ for all $e \in \partial S$, then

$$\dot{V} \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} + 2\|P\operatorname{diag}(\ell)\|\sqrt{\sum_{s=1}^{n}(|e_1|^{g_s} - |e_1|)^2}, \quad e \in \partial S. \tag{3.16}$$

For $\alpha = 0$ (i.e. $g_s = 1$) the above expression is reduced to

$$\dot{V} \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}, \quad e \in \partial S$$

and by continuity $\dot{V} < 0$ is satisfied for α close to zero. By obtaining a limit of the additional term in (3.16) with respect to $-1 \leq \alpha < 0$, we can find a lower bound on α, which guarantees that $\dot{V}$ is negative definite on ∂S.

We calculate an upper bound for the $\left||e_1|^{g_s} - |e_1|\right|$ taking in mind that $|e_1| \leq 1$ for $e \in \partial S$. The idea is to consider the function $\kappa(x) = |x^{\xi} - x|$ and its maximum for $x \in [0, 1]$ (an example of κ is given in Figure 3.3 for $\xi = 0.8$). Inspired by the graph of κ, we apply the mean value theorem

$$\tilde{\kappa}(a) - \tilde{\kappa}(b) = g'(\theta)(a - b), \quad \theta \in [a, b]$$

to the function $\tilde{\kappa} : \xi \mapsto x^{\xi}$, considering x as a parameter and ξ as the argument. For

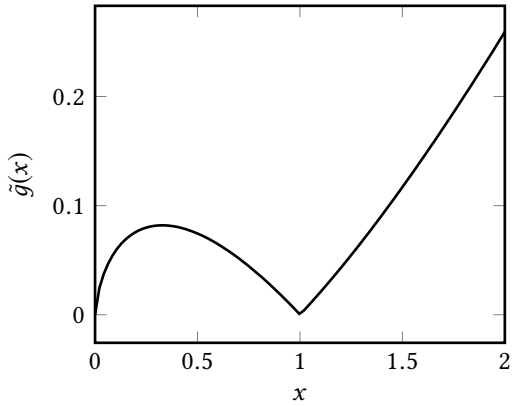


Figure 3.3: Example of $\kappa(x)$ with $\xi = 0.8$. To estimate the bound, the maximum for $x \in [0, 1]$ is relevant.

$\xi \in [0, 1)$, we obtain

$$\tilde{\kappa}(\xi) - \tilde{\kappa}(1) = \chi(x, \theta)(\xi - 1)$$

with $\chi(x, \theta) = x^\theta \ln(x)$ for some $\theta \in [\xi, 1]$. For any such fixed θ, $\chi(0, \theta) = \chi(1, \theta) = 0$ and $\chi(x, \theta) \leq 0$ for any $x \in [0, 1]$. The minimal value of χ with respect to $x \in [0, 1]$ is therefore reached at $x_{\text{opt}} = \mathrm{e}^{-1/\theta}$ and $\chi(x_{\text{opt}}, \theta) = -\mathrm{e}^{-1}\theta^{-1}$.

Thus, we can use the bound

$$|e_1|^{g_s} - |e_1| \leq \mathrm{e}^{-1} \frac{1 - g_s}{g_s} = \mathrm{e}^{-1} \frac{-s\alpha}{1 + s\alpha}.$$

Additionally, using $\left|\frac{s\alpha}{1+s\alpha}\right| \leq \left|\frac{n\alpha}{1+n\alpha}\right|$ for $s \in \{1, \ldots, n\}$ in (3.16) leads to

$$\dot{V} \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} + \frac{2}{\mathrm{e}} \|P \operatorname{diag}(\ell)\| \sqrt{n} \frac{n|\alpha|}{1 - n|\alpha|}.$$

Now a bound for α can be selected as

$$|\alpha| < \frac{\eta}{n(\sqrt{n} + \eta)}, \quad \eta = \frac{\mathrm{e}}{2} \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} \frac{1}{\|P \operatorname{diag}(\ell)\|},$$

which ensures that $\dot{V}(e) < 0$ for $e \in \partial S$ and that the set $S = \{e \in \mathbb{R}^n | V(e) \leq 1\}$ is strictly positively invariant for (3.4).

Hence, according to Theorem 3.1 the origin of (3.4) is asymptotically stable and since it is $\mathbf{r}$-homogeneous with negative degree for $\alpha < 0$, it is globally finite-time stable by Theorem 3.2. ∎

Proof of Lemma 3.2. Let us first check the relation for homogeneity for some general g_i and β_i. Clearly $r_1 = r_{n+1} = \cdots = r_{(N-1)n+1}, \ldots, r_n = r_{2n} = \cdots = r_{nN}$. Then we note from

$$\lambda^{r_s+\nu}\left(\mu_s e_{i,s+1} + \ell_{i,s}\lceil e_{i,n_i}\rfloor^{g_{i,s}} + \gamma\sum_{j\in\mathcal{N}_i}\lceil e_{i,s}-e_{j,s}\rfloor^{h_s}\right)$$
$$= \mu_s\lambda^{r_{s+1}}e_{i,s+1} + \lambda^{r_{p_i}g_{i,s}}\ell_{i,s}\lceil e_{i,n_i}\rfloor^{g_{i,s}} + \lambda^{r_s h_s}\gamma\sum_{j\in\mathcal{N}_i}\lceil e_{i,s}-e_{j,s}\rfloor^{h_s} \quad (3.17)$$

where $\mu_s = 0$ if $s \in \{p_1, \ldots, p_N\}$ and $\mu_s = 1$ otherwise. We obtain the relations

$$\begin{aligned} r_{s+1} &= r_s + \nu \quad \text{for } s \notin \{p_1, \ldots, p_N\} \\ g_{i,s} &= \frac{r_s+\nu}{r_{p_i}} \\ h_s &= \frac{r_s+\nu}{r_s} \end{aligned}$$

and substitute the unspecified $\alpha_i = \frac{\nu}{r_{p_i}}$. ∎

Proof of Theorem 3.4. The error dynamics can be written as

$$\dot{E} = (I_N\otimes A)E - \operatorname{diag}\left(\begin{bmatrix}\ell_1^\mathsf{T} & \cdots & \ell_N^\mathsf{T}\end{bmatrix}^\mathsf{T}\right)\left\lceil\begin{bmatrix}e_{1,p_1} & \ldots & e_{N,p_N}\end{bmatrix}\otimes 1_n\right\rfloor^{1_N\otimes g}$$
$$-\gamma\operatorname{blkdiag}(1_{|\mathcal{N}_1|}^\mathsf{T}\otimes I_n, \ldots, 1_{|\mathcal{N}_N|}^\mathsf{T}\otimes I_n)\left\lceil(\mathcal{I}\otimes I_n)^\mathsf{T}E\right\rfloor^{1_{|\mathcal{E}|}\otimes h}.$$

where $\mathcal{I} = \left[\left[\begin{bmatrix}0\cdots\underbrace{1}_{i\text{th}}\cdots\underbrace{-1}_{j\text{th}}\cdots 0\end{bmatrix}^\mathsf{T}\right]_{j\in\mathcal{N}_i}\right]_{i\in\mathcal{V}}$ is the incidence matrix.

Taking the Lyapunov function $V(E) = E^\mathsf{T}PE$, we repeat the steps as in the proof of

Theorem 3.3, and for $E \in \partial S = \{E \in \mathbb{R}^{Nn} : V(E) = 1\}$ we obtain

$$\dot{V} \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} + 2\Bigg(\|P \operatorname{blkdiag}(\operatorname{diag}(\ell_1), \dots, \operatorname{diag}(\ell_N))\| \sqrt{\sum_{i=1}^{N} \sum_{s=1}^{n} \left(|e_{i,p_i}|^{g_s} - |e_{i,p_i}| \right)^2} \\ + \gamma \|P \operatorname{blkdiag}(1_{|\mathcal{N}_i|}^{\mathsf{T}} \otimes I_n, \dots, 1_{|\mathcal{N}_N|}^{\mathsf{T}} \otimes I_n)\| \sqrt{\sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \sum_{s=1}^{n} \left(|e_{i,s} - e_{j,s}|^{h_s} - |e_{i,s} - e_{j,s}| \right)^2} \Bigg). \tag{3.18}$$

The first square root can be bound similarly to the proof of Theorem 3.3,

$$\sqrt{\sum_{i=1}^{N} \sum_{s=1}^{n} \left(|e_{i,p_i}|^{g_s} - |e_{i,p_i}| \right)^2} \leq \frac{\sqrt{Nn}}{\mathrm{e}} \frac{\bar{n}|\alpha|}{1 - \bar{n}|\alpha|}; \tag{3.19}$$

however, for the second square root we have to take into account that if $\|E\| \leq 1$, then $|e_{i,s} - e_{j,s}| \leq 2$. Therefore we consider for $\xi \in [0, 1)$ the function $\kappa(x) = |x^{\xi} - x|$ for $x \in [0, 2]$. It is easy to check that $\kappa(0) = \kappa(1) = 0$, then $\kappa(x) = x^{\xi} - x$ for $x \in [0, 1]$ and the maximal value of κ on this interval is reached for $\tilde{x}_{\text{opt}} = \xi^{\frac{1}{1-\xi}}$ with $\sup_{0 \leq x \leq 1} \kappa(x) = \xi^{\frac{\xi}{1-\xi}} - \xi^{\frac{1}{1-\xi}}$. For $x \in [1, 2]$, a simple analysis of $\frac{\mathrm{d}\kappa(x)}{\mathrm{d}x}$ shows that κ is a strictly increasing function, then $\sup_{1 \leq x \leq 2} \kappa(x) = 2 - 2^{\xi}$. For $\xi \in [0, 1)$, the second derivatives with respect to ξ of $\xi^{\frac{\xi}{1-\xi}} - \xi^{\frac{1}{1-\xi}}$ and $2 - 2^{\xi}$ are

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \xi^{\frac{1}{1-\xi}} (\xi^{-1} - 1) = -\frac{\xi^{-1/(\xi-1)-2} \left((1-\xi)(\xi-1)^2 + \xi(\ln(\xi) - (\xi-1))^2 \right)}{(\xi-1)^3} > 0,$$

which means it is convex, and $\dfrac{\mathrm{d}^2}{\mathrm{d}\xi^2}(2 - 2^{\xi}) = -2^{\xi} \ln^2(2) < 0$, which means it is concave, respectively. Since both functions meet at $\xi = 0$ and $\xi = 1$, we obtain $\xi^{\frac{\xi}{1-\xi}} - \xi^{\frac{1}{1-\xi}} \leq 2 - 2^{\xi}$ for $\xi \in [0, 1)$, see Figure 3.4. Consequently, returning to the second square root in (3.18)

$$\left| |e_{j,s} - e_{i,s}|^{h_s} - |e_{j,s} - e_{i,s}| \right| \leq 2 - 2^{h_s} \leq 2^{\theta} \ln(2)(1 - h_s) \leq 2 \ln(2)(1 - h_s) \\ \leq 2 \ln(2) \frac{-\alpha}{1 + (\bar{n} - 1)\alpha}$$

for all $s \in \{1, \dots, n\}$, where the mean value theorem with $\theta \in [h_s, 1]$ and $h_s \leq$

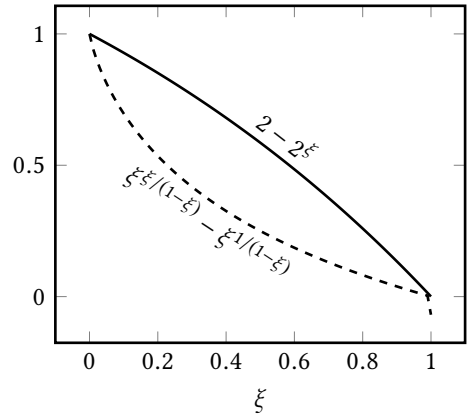


Figure 3.4: Comparison of the maximum value of $\kappa(x)$ on $[0,1]$ (dotted) and $[2,1]$ (solid) with respect to $\xi \in [0,1]$

$\frac{1+\bar{n}\alpha}{1+(\bar{n}-1)\alpha}$ has been used, and in total we get

$$\sqrt{\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sum_{s=1}^{n}\left(|e_{i,s}-e_{j,s}|^{h_s}-|e_{i,s}-e_{j,s}|\right)^2} \leq 2\ln(2)\sqrt{|\mathcal{E}|n}\frac{|\alpha|}{1-(\bar{n}-1)|\alpha|}. \quad (3.20)$$

Inserting (3.19), (3.20) into (3.18) gives with $\dot{V} < 0$ a quadratic equation in $|\alpha|$, whose solution is provided in the formulation (3.9) of Theorem 3.4. ■

Proof of Lemma 3.3. A sufficient condition for a system to be ISS is that there exist a positive definite and radially unbounded ISS Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ for which $\dot{V} \leq -\gamma(\|x\|) + \sigma(\|d\|)$ holds for all x and d with some $\gamma \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ (Dashkovskiy et al., 2011). Consider a Lyapunov function $V(x) = V_1(x) + V_2(x)$ where $V_1(x) = x^\mathsf{T} P x$ and

$$V_2(x) = 2\sum_{i=1}^{n}\lambda_i \int_0^{x_i} f_i(s)\,\mathrm{d}s,$$

which is positive definite and radially unbounded due to the conditions of the lemma.

Taking the derivative

$$\dot{V}_1 = x^\mathsf{T}(M_1^\mathsf{T}P + PM_1)x + 2f(x)^\mathsf{T}M_2^\mathsf{T}Px + 2d^\mathsf{T}Px$$
$$\dot{V}_2 = f(x)^\mathsf{T}(M_2^\mathsf{T}\Lambda + \Lambda M_2)f(x) + 2x^\mathsf{T}M_1^\mathsf{T}\Lambda f(x) + 2d^\mathsf{T}\Lambda f(x)$$

and adding and subtracting $2x^\mathsf{T}\Upsilon f(x) + d^\mathsf{T}\Gamma d$ with some positive definite matrix Γ leads to

$$\dot{V} = \begin{bmatrix} x \\ f(x) \\ d \end{bmatrix}^\mathsf{T} \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{Φ}} & P \\ & & \Lambda \\ \hline P & \Lambda & -\Gamma \end{array}\right) \begin{bmatrix} x \\ f(x) \\ d \end{bmatrix} - 2x^\mathsf{T}\Upsilon f(x) + d^\mathsf{T}\Gamma d.$$

If $\Phi \preccurlyeq 0$, then there exists a $\Gamma \succ 0$ such that $\dot{V} \leq -2x^\mathsf{T}\Upsilon f(x) + d^\mathsf{T}\Gamma d$, where the first term after the inequality is negative definite and radially unbounded due to the imposed restrictions of the lemma, which implies that the system is ISS. ∎

Proof of Theorem 3.5. According to Theorem 3.3 the error E_o of the observable part will be zero in finite-time. It remains to show that the error dynamics of the unobservable part (3.12) is ISS and that its homogeneous approximation at the origin for $E_\mathrm{o} = 0$ is finite-time stable. To this end we use the coordinate transformation (3.15) to arrive to at system of the form (3.13) where in addition

$$d = \boldsymbol{T}_\mathrm{u}^\mathsf{T}(\mathcal{L}\otimes I)\boldsymbol{T}_\mathrm{u}\left(\boldsymbol{A}_\mathrm{u} - \alpha\boldsymbol{T}_\mathrm{u}^\mathsf{T}(\mathcal{L}\otimes I)\boldsymbol{T}_\mathrm{u}\right)\boldsymbol{T}_\mathrm{u}^\mathsf{T}\mathcal{A}\boldsymbol{T}_\mathrm{o}E_\mathrm{o} + \boldsymbol{T}_\mathrm{u}^\mathsf{T}(\mathcal{L}\otimes I)\boldsymbol{T}_\mathrm{u}\left(\boldsymbol{A}_\mathrm{r} + \alpha\boldsymbol{T}_\mathrm{u}^\mathsf{T}\mathcal{A}\boldsymbol{T}_\mathrm{o}\right)E_\mathrm{o} - \boldsymbol{T}_\mathrm{u}^\mathsf{T}\mathcal{A}\boldsymbol{T}_\mathrm{o}\dot{E}_\mathrm{o}$$

is composed by all terms which are dependent only on the error of the observable part.

The nonlinear functions are all $f_i(t) = \operatorname{sign}(t)|t|^\beta$ and therefore fulfill the conditions of Lemma 3.3 for $\beta > 0$. Due to diagonal stability (Proposition 2.1) there exists $Q \succ 0$ and a diagonal matrix $\Lambda \succ 0$ such that $M_2^\mathsf{T}\Lambda + \Lambda M_2 = -Q$. It follows that $V_2(x)$ is a Lyapunov-function of the homogeneous approximation $\dot{x} = -M_2\lceil x\rfloor^{1_{n_\mathrm{iu}}\beta}$ with degree $\beta - 1$ and is therefore finite-time stable for $0 < \beta < 1$.

To prove the ISS of the approximated system we show that for sufficiently high $\gamma > 0$ the LMI (3.14) is fulfilled. We take $P = \gamma\Lambda$ and note that $M_1 = S + \gamma M_2$, where

$S = \boldsymbol{T}_{\mathbf{u}}^{\mathsf{T}}(\mathcal{L} \otimes I)\boldsymbol{T}_{\mathbf{u}}\boldsymbol{A}_{\mathbf{u}}(\boldsymbol{T}_{\mathbf{u}}^{\mathsf{T}}(\mathcal{L} \otimes I)\boldsymbol{T}_{\mathbf{u}})^{-1}$. The LMI is equivalent to

$$\begin{bmatrix} \frac{\sqrt{\gamma}}{\gamma} I & 0 \\ 0 & \sqrt{\gamma} I \end{bmatrix} \begin{bmatrix} \gamma(S + \gamma M_2)^{\mathsf{T}}\Lambda + \gamma\Lambda(S + \gamma M_2) & (S + \gamma M_2)^{\mathsf{T}}\Lambda + \gamma\Lambda M_2 + \Upsilon \\ * & M_2^{\mathsf{T}}\Lambda + \Lambda M_2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{\gamma}}{\gamma} I & 0 \\ 0 & \sqrt{\gamma} I \end{bmatrix} \preccurlyeq 0$$

and after some rearrangement

$$\begin{bmatrix} S + S^{\mathsf{T}} & S^{\mathsf{T}} \\ S & 0 \end{bmatrix} - \begin{bmatrix} \Upsilon & 0 \\ 0 & \Upsilon \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes (\gamma Q - \Upsilon) \preccurlyeq 0,$$

can be shown to be fulfilled if $\left[\begin{smallmatrix} \Upsilon & 0 \\ 0 & \Upsilon \end{smallmatrix}\right] \succeq \left[\begin{smallmatrix} S+S^{\mathsf{T}} & S^{\mathsf{T}} \\ S & 0 \end{smallmatrix}\right]$ and $\gamma Q \succcurlyeq \Upsilon$. Such a diagonal matrix Υ always exists if γ can be arbitrary large. ■

Chapter 4

Distributed Observers with Delays

Communication is neither instantaneous nor totally reliable and disregarding this can have severe effects on the performance. In the preceding chapters, the continuous-time observer nodes were designed without considering any delays, but when applied in a control system, the effects of the network do not only hamper the performance but they can actually destabilize it (Zhang et al., 2001). To make such a design work closely to the specifications, requires an amount of bandwidth and quality of service which is currently not foreseen in wireless senor technology.

One way to go about it is to adopt discrete-time modelling and use buffered communication to accommodate the network effects (Hespanha et al., 2007). However, such an approach will introduce conservatism and not offer the flexibility expected in a distributed estimation scenario. Instead, in this chapter we will keep a continuous-time model while still taking the digital nature of communication into account. This approach is especially well suited for networked sensors. It is possible to equip the observer nodes with enough processing power and measurements to keep up with the dynamics of the plant, while the communication capacities are more limited as argued before by Dörfler et al. (2013).

The main attributes which characterize packet-switched communication include delays, variable rates, packet-dropouts and scheduling (Richard and Divoux, 2007). Together with noise and quanitzation they pose a trade-off in the selection of a communication channel. Various approaches exists to model their effects when analyzing a networked control system with continuous-time dynamics (Hetel et al., 2017). The main ones are the hybrid systems framework and time-varying delay approach of

Fridman et al. (2004) and Kruszewski et al. (2012). In this work we adopt the latter one to guarantee stability for the whole spectrum of effects and together with the Lyapunov-Krasovskii functional approach (Richard, 2003) it leads to effective design methods.

The literature is very rich in what concerns the design of observers in networked control systems. As an early example, Dačić and Nešić (2008) have considered the performance of observer-protocols for a fully connected network where each node sends it output to every other node. For the distributed estimation problem a first work is by Millán et al. (2012) in discrete-time where delays and dropouts were included. Ugrinovskii and Fridman (2014) designed continuous-time distributed observers under a scheduling protocol. More recently, Li et al. (2018) investigate the case of asynchronous communication with a hybrid systems approach. Basu and Yoon (2019) follow the Lyapunov-Krasovskii functional approach for the constant-delay case.

The objective of this chapter is twofold. First, we provide a method for designing distributed observers which will function as specified in presence of a digital communication channel with synchronized clocks and time stamps. The performance is derived using the Lyapunov-Krasovskii functional method for systems with time-varying delays which leads to design based on a linear matrix inequality (LMI). The performance of the distributed observer is guaranteed under the obtained bounds for the time-varying delay.

A second goal is to make an assessment of the claim that the distributed observer concept can outperform centralized ones when delays are present. This goes further then the fact that local communication is less demanding then interacting over the whole network. When exchanging estimates of states or of artificial outputs instead of the actual partial outputs, the influence of the communication network is shifted from the output feedback to the diffusive coupling term. The gain in flexibility allows to find designs which can tolerate higher delays. Admittedly, this is a preliminary step, as no general statement can be done based on a single academic example.

We start off this chapter by describing the differences in what we will call the centralized and the distributed case and how they are differently affected by the communication network. Then we will elaborate how time-varying delays serve to take

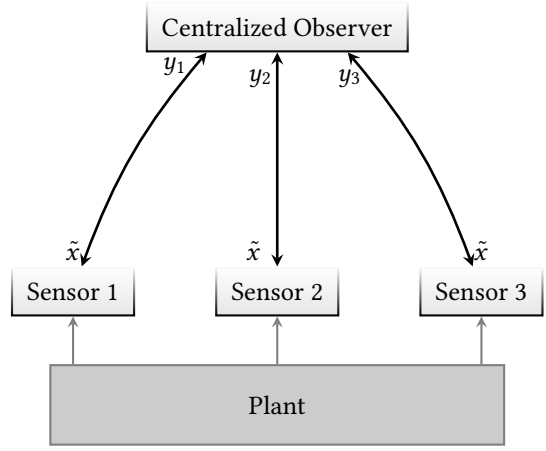
into account the network effects and present the Lyapunov-Krasovskii method to analyze the convergence. Next we formulate the error dynamics for both cases to which we formulate LMI conditions for the design of the gains which ensure exponential convergence with a given rate, up to a guaranteed bound on the delay. The design serves as a method to make a comparison of the centralized observer and distributed observer approach on an example. We will discuss the result and make a brief extension, by allowing the observer nodes to exchange only a projection of the complete state estimate. A final section considers the adaption of the vector Lyapunov approach to the time-varying delay case. Proofs can be found in the end of the chapter.

4.1 Centralized and distributed estimation structures

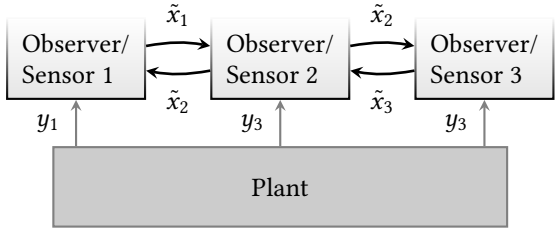
In this section, we examine a centralized observer and a distributed observer for the estimation of a large-scale continuous-time plant measured by a (wireless) sensor network. The underlying goal is to have an estimate of the full state, which is further processed at sensor level by a decentralized control system for example.

In the centralized case, all the measurements from the sensors are collected at a central entity to obtain the state estimate which is sent back to the sensors (Figure 4.1a). In the distributed case, each sensor carries an observer node with its own state estimate; however, to reconstruct the state all measurements are needed, so the nodes have to communicate with each other. Contrarily to the straightforward approach of exchanging measurements (which is essentially the same as a centralized structure), the observer nodes communicate only locally (subject to a connected graph) by exchanging their state estimate with neighbouring nodes (Figure 4.1a).

A first comparison of both concepts shows as an advantage of the distributed case that the communication is only local, while the transmission of the measurements to a remote centralized observer and the return of the state estimate is possibly impacted by large delays. Of course, there are other factors to consider, for example that the centralized observer suffers form a bottleneck in the communication (so a



(a) Centralized observer structure



(b) Distributed observer structure

Figure 4.1: Comparison of the observer structures. Collecting all measurements y_i at a centralized entity and returning the state estimate $\tilde{x}$ is possibly impacted by large delays. Moreover, for an increasing number of sensors there is a potential bottleneck at the observer. In the distributed structure the delays are most likely smaller and the bottleneck is avoided. However, exchanging full state-estimates $\tilde{x}_i$ requires more bandwidth.



Figure 4.2: Model for the transmission of a continuous signal y over a digital communication channel. A signal sample $y(s_k)$ at time instant s_k arrives after a delay $\eta_k = t_k - s_k$. Passed the holder, it is as if the signal was delayed by a time-varying delay τ.

scheduling approach might be necessary), while the distributed case requires computational capacities at every sensor and exchanging the full state-estimates takes up more bandwidth.

We assume that in both cases the estimation algorithms are implemented on fast processors (compared to the plant's dynamics) and therefore the dynamics of the observers can be described as continuous-time systems. At the same time, communication resources in sensor networks are typically limited, such that significant delays and data rate constraints are inevitable. We pose ourselves now the question, weather the relative performance of the two structures may additionally differ due to what is communicated.

To answer this question, we will compare the designs of a centralized observer and of a distributed observer on an example. To this end, we describe in the next section how the effects of transmitting the signal from the source to the destination over a digital communication channel (between the sensor and centralized observer in the centralized case; between two observer nodes in the distributed case) can be modeled using a time-varying delay.

4.2 Time-varying delay approach

We consider an abstraction of transmitting a continuous signal y from a sender to a receiver over a digital communication channel, see Figure 4.2. It consists of a sampler, communication network and holder. With this description, we focus on the effects on sampling and delay, but do not include for example channel coding. As the result the signal at the receiver is shifted by a time-varying delay τ.

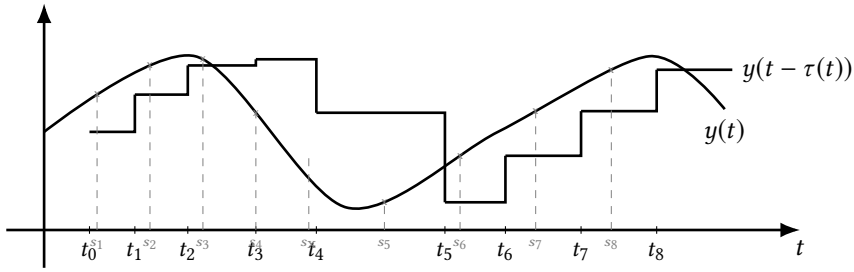


Figure 4.3: Comparison of the senders output $y(t)$ and the sampled and delayed input $y(t-\tau(t))$ at the receiver.

The sampling instances s_k are $s_0 = 0$ and

$$s_{k+1} = s_k + h_k,$$

$k \in \mathbb{Z}^{\leq 0}$, where $h_k > 0$ is the transmission interval. Its nominal value is selected according to the available channel capacity, but it can also vary, for example, causes by sampling jitter or an event-triggered scheme. Failed transmissions due to dropouts can also be accounted for.

After the data has traversed through the communication network, it arrives at the holder at updating time instant t_k. Due to the communication network they do not coincide with the sampling instances, but suffer from a network induced delay

$$\eta_k = t_k - s_k.$$

The delay of the communication network can have various sources, for example it can be caused by the routing or processing of the communication protocol.

During the updating instances, the holder keeps its output constant at the received value (Figure 4.3). The resulting piecewise constant signal can be expressed with a sawtooth-like shaped delay τ as shown in Figure 4.4 where in between updating instances

$$\tau(t) = t - t_k + \eta_k \quad \text{for } t_k \leq t < t_{k+1} \tag{4.1}$$

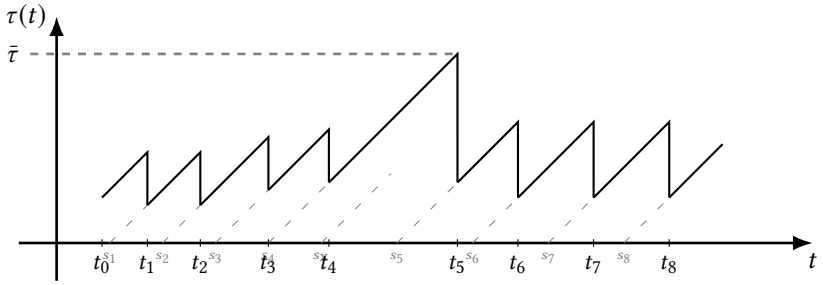


Figure 4.4: Graph of a time-varying delay induced by the communication channel. It illustrates a scenario where the delay in the communication channel increases (e.g. due to congestion in the communication network) until a package dropout incurs to the message with $y(s_x)$. In response the sampling interval is increased.

with $\dot{\tau} = 1$ for $t \neq t_k$. Its peak value right before an update can be expressed as

$$t_{k+1} - t_k + \eta_k = t_{k+1} - s_k = \eta_{k+1} + h_k.$$

The maximum of the time-varying delay over all transmission instances is denoted as

$$\bar{\tau} = \max_k(\eta_{k+1} + h_k).$$

4.2.1 Performance criteria for estimation

Clearly the aberration shown in Figure 4.3 caused by the communication channel in the form of a time-varying delay (even when not so extreme) can seriously hamper the estimation performance of an observer. That means there is a maximum allowable transmission interval (MATI) and a maximum allowable delay (MAD) to satisfy some performance criteria (Heemels et al., 2010). Our goal in the next sections is to estimate their sum for a given converge rate of the observer and to compare them in the centralized and the distributed case.

To employ the time-delay approach, we consider time-delay systems of the form

$$\dot{x}(t) = f(x_t) \tag{4.2}$$

where $x(t) \in \mathbb{R}^n$ is the state, $x_t \in C_{[-\bar{\tau},0]}$ is the state trajectory from $t-\bar{\tau}$ until t (i.e. $x_t(\theta) = x(t+\theta)$ with $\theta \in [-\bar{\tau}, 0]$), $C_{[-\bar{\tau},0]}$ is the space of continuous functions from the interval $[-\bar{\tau}, 0]$ to $\mathbb{R}^n$ and $\bar{\tau}$ is the maximum delay. The functional $f : C_{[-\bar{\tau},0]} \mapsto \mathbb{R}^n$ is assumed to be locally Lipschitz continuous, $f(0) = 0$, and we let $W_{[-\bar{\tau},0]}$ be the subspace of absolutely continuous functions with square integrable derivative equipped with the norm $\|\phi\|_W = \max_{t\in[t-\bar{\tau},t]} \|\phi(t)\| + \sqrt{\int_{t-\bar{\tau}}^{t} \|\dot{\phi}(s)\|^2 \mathrm{d}s}$.

Definition 4.1 (Fridman, 2014)**.** *System* (4.2) *is said to be globally exponentially stable with a decay rate $\alpha > 0$, if for any initial condition $x_0 \in W_{[-\bar{\tau},0]}$ there exits a constant k such that*

$$\|x(t)\| \leq k\mathrm{e}^{-\alpha t}\|x_0\|_W$$

for all $t \geq 0$.

With the above definition a sufficient condition for exponential stability using a Lyapunov-Krasovskii functional follows from the comparison lemma (Khalil, 2002).

Theorem 4.1. *System* (4.2) *is globally exponentially stable with decay rate α, if there exists a continuously differentiable functional $V : W_{[-\bar{\tau},0]} \mapsto \mathbb{R}_+$ such that*

$$k_1\|\phi(0)\|^2 \leq V(\phi) \leq k_2\|\phi\|_W^2$$

for some positive k_1, k_2 and

$$\dot{V} := \limsup_{h\to 0^+} \frac{1}{h}\left(V(x_h) - V(\phi)\right) \leq -2\alpha V$$

for all initial conditions $x_0 = \phi \in W_{[-\bar{\tau},0]}$.

4.3 Observer design

We consider a linear time-invariant plant with multiple outputs

$$\dot{x}(t) = Ax(t), \qquad y_1(t) = C_1 x(t), \quad \dots \quad y_N(t) = C_N x(t) \tag{4.3}$$

where each output $y_i(t) \in \mathbb{R}^{m_i}$ corresponds to one of the N sensors nodes. We assume that the sensors, as well as the sensors with the centralized observer, respectively, have synchronized clocks and know the delay (e.g. using a network time protocol and including the sampling instances in the transmission).

We are interested in finding a design of a centralized and a distributed Luenberger observer as introduced in Chapter 2 while taking into account the digital communication network. Henceforth, we derive new methods to obtain the gains such that a preselected convergence rate of the observers is preserved in face of a time-varying delay.

4.3.1 Centralized observer

For simplicity, the sampling instances and the delays from the sensor nodes to the centralized observer are taken to be identical for all sensors. It allows to combine the individual measurement outputs of the N sensors to

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = Cx$$

where $C^\mathsf{T} = \begin{bmatrix} C_1^\mathsf{T} & \cdots & C_N^\mathsf{T} \end{bmatrix}$. Following section 4.2, the observer receives the samples $y(s_k)$ at time-instances $t_k = s_k + \eta_k$ with delays $\eta_k > 0$. The observer is then implemented as a continuous-time Luenberger observer with gain $L \in \mathbb{R}^n$ to obtain the estimate $\tilde{x}$ of the plant's state

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) - L\left(C\tilde{x}(t_k - \eta_k) - y(t_k - \eta_k)\right) \text{ for } t_k \le t < t_{k+1}. \tag{4.4}$$

The output feedback is kept constant with corresponding estimate $\tilde{x}(s_k)$ in between the receptions.

Introducing the error $e = \tilde{x} - x$ leads to dynamics

$$\dot{e}(t) = Ae(t) - LCe(t_k - \eta_k) \quad \text{for } t_k \le t < t_{k+1}.$$

As described in section 4.2, it can be rewritten using (4.1) as a time-delay system

$$\dot{e}(t) = Ae(t) - LCe(t - \tau(t)) \tag{4.5}$$

with a piecewise linear time-varying delay τ where $\tau(t_k) = \eta_k$ and $\dot{\tau}(t) = 1$ for $t \neq t_k$.

Now, we wish to find an observer gain L which makes the error system (4.5) exponentially stable with a preselected decay rate up to the highest possible value of the bound $\bar{\tau}$ on $\tau(t)$. By applying the Lyapunov-Krasovskii approach for linear systems with time-varying delays, we can then derive an LMI based design.

Proposition 4.1. *Given $\bar{\tau} > 0$, $\alpha > 0$ and a tuning parameter ε, if there exist positive definite matrices P, S, R, a nonsingular matrix P_2 as well as matrices Y and S_{12} that satisfy the LMI*

$$\begin{bmatrix} \Phi_{11} & P - P_2^{\mathsf{T}} + \varepsilon A^{\mathsf{T}} P_2 & \mathrm{e}^{-2\alpha\bar{\tau}} S_{12} & -YC + \mathrm{e}^{-2\alpha\bar{\tau}}(R - S_{12}) \\ * & -\varepsilon(P_2 + P_2^{\mathsf{T}}) + \bar{\tau}^2 R & 0 & -\varepsilon YC \\ * & * & -\mathrm{e}^{-2\alpha\bar{\tau}}(S + R) & \mathrm{e}^{-2\alpha\bar{\tau}}(R - S_{12}) \\ * & * & * & \mathrm{e}^{-2\alpha\bar{\tau}}(-2R + S_{12} + S_{12}^{\mathsf{T}}) \end{bmatrix} \le 0 \tag{4.6a}$$

$$\begin{bmatrix} R & S_{12} \\ * & R \end{bmatrix} \ge 0 \tag{4.6b}$$

where $\Phi_{11} = A^{\mathsf{T}} P_2 + P_2^{\mathsf{T}} A + 2\alpha P + S - \mathrm{e}^{-2\alpha\bar{\tau}} R$, then the observer error dynamics (4.5) *for system* (4.3) *with gain $L = (P_2^{\mathsf{T}})^{-1} Y$ is guaranteed to be globally exponentially stable with decay rate α for any time-varying delay τ subject to $0 \le \tau(t) \le \bar{\tau}$.*

This proposition is a combination of results from Fridman (2014), for completeness we repeat the proof at the end of the chapter.

Since the observer design relies on sufficient conditions, the actual convergence rate

might be preserved also for time-varying delays which exceed the guaranteed bound (i.e. even when the LMI with $Y = P_2 L$ is not feasible). To obtain a less conservative estimate on the bound of the time-varying delay a different analysis method can be used a posteriori. Moreover, the obtained L is suboptimal in the sense that there might be a gain which allows to preserve the decay rate for even higher delays. This is the gain we are actually interested in for the comparison of the centralized and the distributed case; however, finding it is not within reach of current design methods in general.

4.3.2 Distributed observer

For the distributed observer structure in Figure 4.1b, we consider neither sampling nor delays in the measurement inputs. For one thing, this can be assumed since each observer node is implemented on the corresponding sensor and hence no transmission network is involved. Even if sensor delays arise in some applications, for example due to the processing of sensory data, the distributed and the centralized structure are equally affected. Therefore, for the purpose of the comparison they can be disregarded. In contrast, since the sensors are distributed over a large-scale plant, the communication lag between the sensors can be significant and, in addition, constrained in the transmission rate.

Again, for the sake of the comparison we assume that all observer nodes receive the information at the same time instance t_k and suffer from the same delay η_k. An observer node i is then selected as a Luenberger observer with an additional diffusively coupled term over the neighboring nodes $\mathcal{N}_i$ as explained in Chapter 2

$$\dot{\tilde{x}}_i(t) = A\tilde{x}_i(t) - L_i\left(C_i\tilde{x}_i(t) - y_i(t)\right) - \sum_{j\in\mathcal{N}_i} H_{i,j}\left(\tilde{x}_i(t_k - \eta_k) - \tilde{x}_j(t_k - \eta_k)\right)$$

for $t_k \leq t < t_{k+1}$. The state estimate $\tilde{x}_i$ of the node in the diffusive coupling term is shifted to be in synchrony with the received delayed estimates.

Introducing the error of each observer node $e_i = \tilde{x}_i - x$, their dynamics can be written

with a time-varying delay

$$\dot{e}_i(t) = (A - L_i C_i)e_i(t) - \sum_{j \in \mathcal{N}_i} H_{i,j} \left(e_i(t - \tau(t)) - e_j(t - \tau(t)) \right)$$

and for the error of the observer network $E^{\mathsf{T}} = \begin{bmatrix} e_1^{\mathsf{T}} & \cdots & e_N^{\mathsf{T}} \end{bmatrix}$, we obtain the dynamics with a point-wise delay of the form

$$\dot{E}(t) = (I \otimes A - \operatorname{blkdiag}(L_1 C_1, \ldots, L_N C_N))\, E(t) - \mathcal{H} E(t - \tau(t)) \tag{4.7}$$

with

$$\mathcal{H} = \begin{bmatrix} \sum_{j \in \mathcal{N}_1} H_{1,j} & -H_{1,2} & \cdots & -H_{1,N} \\ & & \ddots & \\ -H_{N,1} & \cdots & -H_{N,N-1} & \sum_{j \in \mathcal{N}_N} H_{N,j} \end{bmatrix}$$

where $H_{i,j} = 0$ if $j \notin \mathcal{N}_i$. As in the section before we formulate an LMI for the design of the output-feedback and coupling gains.

Proposition 4.2. *Given $\bar{\tau} > 0$, $\alpha > 0$ and a tuning parameter $\epsilon = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_N) \otimes I$, let there exist positive definite matrices P, S, R, nonsingular matrices $P_{2,1}, \ldots, P_{2,N}$, matrices $Z_{1,j}$ for all $j \in \mathcal{N}_1, \ldots, Z_{N,j}$ for all $j \in \mathcal{N}_N$, $Y_1, \ldots, Y_N$ and S_{12} that satisfy the LMI*

$$\begin{bmatrix} \Phi_{11} & P - \operatorname{blkdiag}(P_{2,i}^{\mathsf{T}})_{i \in \mathcal{V}} + \epsilon \operatorname{blkdiag}(A^{\mathsf{T}} P_{2,i} - C_i^{\mathsf{T}} Y_i^{\mathsf{T}})_{i \in \mathcal{V}} & \mathrm{e}^{-2\alpha\bar{\tau}} S_{12} & -\mathcal{Z} + \mathrm{e}^{-2\alpha\bar{\tau}}(R - S_{12}) \\ * & -\epsilon \operatorname{blkdiag}(P_{2,i} + P_{2,i}^{\mathsf{T}})_{i \in \mathcal{V}} + \bar{\tau}^2 R & 0 & -\epsilon\mathcal{Z} \\ * & * & -\mathrm{e}^{-2\alpha\bar{\tau}}(S + R) & \mathrm{e}^{-2\alpha\bar{\tau}}(R - S_{12}) \\ * & * & * & \mathrm{e}^{-2\alpha\bar{\tau}}(-2R + S_{12} + S_{12}^{\mathsf{T}}) \end{bmatrix} \preceq 0 \tag{4.8a}$$

$$\begin{bmatrix} R & S_{12} \\ * & R \end{bmatrix} \succeq 0 \tag{4.8b}$$

where $\Phi_{11} = \operatorname{blkdiag}(A^{\mathsf{T}} P_{2,i} + P_{2,i}^{\mathsf{T}} A - C_i^{\mathsf{T}} Y_i^{\mathsf{T}} - Y_i C_i)_{i \in \mathcal{V}} + 2\alpha P + S - \mathrm{e}^{-2\alpha\bar{\tau}} R$ and

$$\mathcal{Z} = \begin{bmatrix} \sum_{j \in \mathcal{N}_1} Z_{1,j} & -Z_{1,2} & \cdots & -Z_{1,N} \\ & & \ddots & \\ -Z_{N,1} & \cdots & -Z_{N,N-1} & \sum_{j \in \mathcal{N}_N} Z_{N,j} \end{bmatrix},$$

then the error of the distributed observer network (4.7) *for the system* (4.3) *with gains*

$L_i = (P_{2,i}^{\mathsf{T}})^{-1} Y_i$ and $H_{i,j} = (P_{2,i}^{\mathsf{T}})^{-1} Z_{i,j}$ is guaranteed to be globally exponentially stable with decay rate α for any time-varying delay τ subject to $0 \leq \tau(t) \leq \bar{\tau}$.

The LMI (4.8) is derived the same way as the centralized counterpart with difference that, to take into account the structural constraint on blkdiag$(L_1 \ldots, L_N)$ and the algebraic constraint on $\mathcal{H}$, the auxiliary matrix P_2 is taken block-diagonal.

As a side note, without compensating for the delay in the output feedback the centralized observer would be practically stable (in the sense of a bounded error) for any delay. Nonetheless, when the observer is used for control purposes there would be a feedback which could indeed cause instability. In the distributed case there is already a feedback between the observer nodes, therefore in any case it is reasonable to compensate the delays if they are known.

4.4 Example

To verify the presented methods, the design of a centralized observer and a distributed observer is carried out on a numerical example and tested in a simulation with a sawtooth shaped delay (i.e. $\eta_k = 0$ for all k). Moreover, the two designs are compared in three different ways. The feasibility of the LMIs used for the design gives a sufficient bound, which might be more conservative than the bounds obtained by the LMIs for analysis (i.e. without variable substitution). Lastly, the bounds obtained with the simulations are compared. This is done for the same convergence rate to ensure that the designs are comparable.

The example for system (4.3) was adapted from Han et al. (2019)

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -8 & 1 & 1 & -1 & -2 & 0 \\ 4 & -0.5 & 0.5 & 0 & 0 & -4 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix}$$

$$C_4 = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 & 0 \end{bmatrix}$$

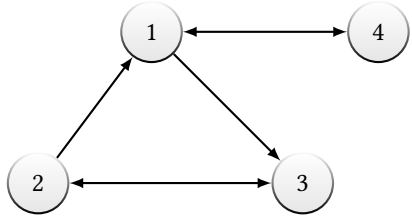


Figure 4.5: Communication graph of the distributed observer

with a slight change to ensure that the system dynamics is unstable and no single pair (A, C_i) is therefore detectable. The communication graph for the distributed observer is shown in Figure 4.5.

Centralized observer

The observer gain L was designed according to Proposition 4.1 with YALMIP (Lofberg, 2004) as solver for the LMI. The convergence rate was first fixed to $\alpha = 1$, then $\bar{\tau}$ was maximized while consecutively adapting ε. The maximal possible $\bar{\tau}$ was found to be 0.204 with $\varepsilon = 0.5$. Then the LMI for analysis (see (4.11) below) was found feasible with $\bar{\tau} = 0.204$ for $\alpha = 1$ and with $\bar{\tau} = 0.287$ for $\alpha = 0$. The design was verified in a simulation where the measurement input was sampled with a rate of $\bar{\tau}^{-1}$, showing that the desired rate of convergence is attained (Figure 4.6).

Next the sampling rate was gradually decreased, showing that from $\bar{\tau} = 0.430$ the convergence rate is less than 1 (Figure 4.7). Lastly, from $\bar{\tau} = 0.494$ on, the error will start to diverge; the observer becomes unstable.

Distributed observer

For the distributed observer, Proposition 4.2 is applied. The LMI was found feasible with $\bar{\tau} = 0.257$ for $\alpha = 1$ with $\varepsilon_1 = 0.499$ and $\varepsilon_i = 0.5$ for $i \in \{2, \ldots, 4\}$. With the designed $L_i, H_{i,j}$ the analysis LMI (i.e. without imposing structure on P_2) was found feasible with $\bar{\tau} = 0.257$ for $\alpha = 1$ ($\bar{\tau} = 0.394$ for $\alpha = 0$).

In the simulation with the same sawtooth shaped time-varying delay for all commu-

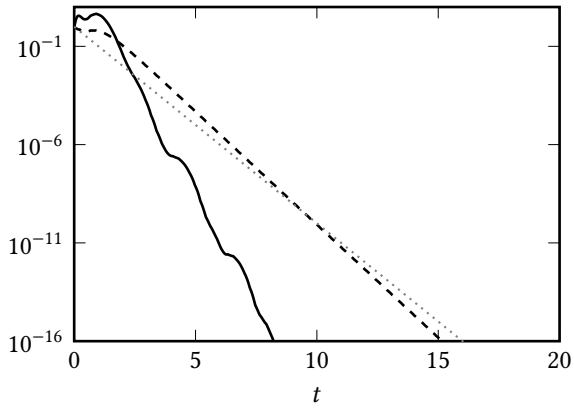


Figure 4.6: Error norm of the distributed (solid) and the centralized (dashed) observer in logarithmic scaling for $\bar{\tau} = 0.25$

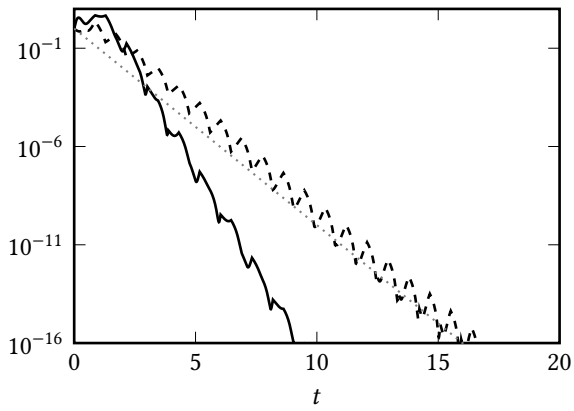


Figure 4.7: Error norm of the distributed (solid) and the centralized (dashed) observer in logarithmic scaling for $\bar{\tau} = 0.430$

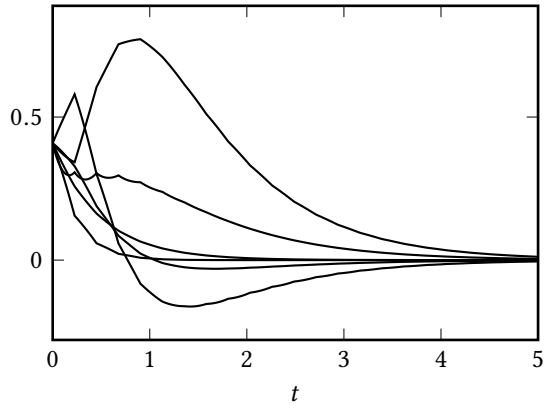


Figure 4.8: Error of the centralized observer for $\bar{\tau} = 0.25$

nications (i.e. modeling sampled communication with equal rate), the convergence rate is breached with $\bar{\tau} = 0.577$ and the estimates diverge for $\bar{\tau} = 0.69$. Table 4.1 summarizes the comparison of the centralized and the distributed observer. Figure 4.8 and Figure 4.9 show the coordinates of the error in linear scaling, illustrating the non-smoothness due to the sampled communication.

	centralized	distributed
for convergence rate 1		
design	0.204	0.257
analysis	0.204	0.257
simulation	0.430	0.577
without prescribed rate (i.e. $\alpha = 0$)		
analysis	0.287	0.394
simulation	0.494	0.690

Table 4.1: Summary of maximal obtained $\bar{\tau}$ and simulation result with sawtooth shaped delay for the centralized and the distributed observer. It shows that for this example the distributed observer design outperforms the centralized one.

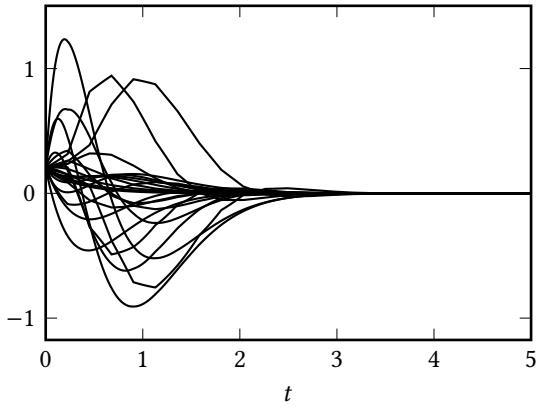


Figure 4.9: Error of each node of the distributed observer for $\bar{\tau} = 0.25$

4.4.1 Discussion of the results

The comparison shows that for the selected example the sum of MATI and MAD is higher for the distributed observer design (the results based on the feasibility of the LMIs and the simulations are consistent). Additionally, the centralized observer is less robust than the distributed one, as it can be seen from the decline in convergence rate for decreasing communication rate. Of course we have to be aware that there exists gains for which these values are higher. Nevertheless, since the same method was used for the design and there is no reason for the centralized case to be more conservative, there is little doubt that the result will be the same for the optimal gains too.

Notice the considerable gap in the result of the simulation and the bound obtained with the LMI. The fact that the LMI is based on a sufficient condition is an important reason for it, but also that the bound has to hold for fast-varying delays of any shape. A remedy for this is to use an LMI which takes the sawtooth shape into account or generally is more tailored for analysis than design (here the analysis LMI was not less conservative). Silm et al. (2018) assumed constant delays and carried out the design as well as the analysis for a bound of the achievable delay margin directly on the approximate eigenvalues of the linear operator of the error dynamics.

Even though this is only a single example, it still suggests that in the distributed case

it is indeed beneficial for the observer nodes to be diffusively coupled. A plain explanation of this is that there are more degrees of freedom in the coupling gains then in the output feedback gain (due to the fact that the outputs are lower-dimensional maps of the state). However the transmission of full state-estimates between the nodes requires more bandwidth and might not be feasible for a high-dimensional plant.

4.4.2 Transmission of reduced data

Already in the delay-free case in Chapter 2 it was shown that it is not necessary for the observer nodes to actually exchange the full state estimate. Instead, it suffices to send the projection to the unobservable subspace of dimension $n_{j\mathrm{u}}$ of a neighboring node j. However, this means that there are less degrees of freedom to design the coupling gains. To get an idea how it might impact the achievable convergence rate, we carry out a design also for this case.

As in the previous chapters the observer design can be separated for the observable and unobservable subspace of an observer node (see section 2.3.1) by selecting

$$L_i = T_{i\mathrm{o}} L_{i\mathrm{o}}, \qquad H_{i,j} = T_{i\mathrm{u}} \bar{G}_{i,j} T_{i\mathrm{u}}^{\mathsf{T}}$$

such that only $T_{i\mathrm{u}}^{\mathsf{T}} \tilde{x}_j$, a vector of smaller dimension, needs to be sent from node j to node i. In particular the observable subspace can be estimated solely from the measurements, which we consider delay-free. Therefore, if the observers for the observable subspace are tuned to be sufficiently fast, we can ignore their contribution to the error dynamics for the unobservable part

$$\dot{E}_{\mathrm{u}}(t)|_{E_{\mathrm{o}}=0} = \operatorname{blkdiag}(A_{1\mathrm{u}}, \dots, A_{N\mathrm{u}}) E_u(t) - \mathcal{G} \operatorname{blkdiag}(T_{1\mathrm{u}}, \dots, T_{N\mathrm{u}}) E_{\mathrm{u}}(t - \tau(t))$$

where

$$\mathcal{G} = \begin{bmatrix} \sum_{j \in \mathcal{N}_1} G_{1,j} T_{1\mathrm{u}}^{\mathsf{T}} & -G_{1,2} T_{1\mathrm{u}}^{\mathsf{T}} & \dots & -G_{1,N} T_{1\mathrm{u}}^{\mathsf{T}} \\ & & \ddots & \\ -G_{N,1} T_{N\mathrm{u}}^{\mathsf{T}} & \dots & -G_{N,N-1} T_{N\mathrm{u}}^{\mathsf{T}} & \sum_{j \in \mathcal{N}_N} G_{N,j} T_{N\mathrm{u}}^{\mathsf{T}} \end{bmatrix}.$$

Deriving an LMI similarly to (4.2) and following the design procedure we get the same bound as for the considered numerical example. This demonstrates that the even when full state-estimates are exchanged, they do not contribute to the observable subspace of a node in the LMI based design.

Moreover, we can consider the case where we do not even transmit the projection onto the whole subspace, but only to a subspace of the unobservable subspace. The observer nodes are then designed as

$$\dot{\tilde{x}}_i(t) = A\tilde{x}_i(t) - T_{io}L_{io}\left(C_i\tilde{x}_i(t) - y_i(t)\right) - T_{iu}\sum_{j\in\mathcal{N}_i} G_{i,j}F_i\left(\tilde{x}_i(t_k-\eta_k) - \tilde{x}_j(t_k-\eta_k)\right)$$

with $G_{i,j} \in \mathbb{R}^{n_{iu}\times n_{if}}$ where $F_i \in \mathbb{R}^{n_{if}\times n}$ is obtained by selecting a subset of the rows of T_{iu}^{T}, therefore a node receives only vectors of dimension $n_{if} < n_{iu}$. For the considered example, we can select $n_{1f} = n_{1u}-2 = 2, n_{2f} = n_{2u} = 1, n_{3f} = n_{3u}-2 = 3, n_{4f} = n_{4u} = 1$ and still find a design without decreasing $\bar{\tau}$. In a situation where all observer nodes exchange only scalars, $n_{4f} = n_{3f} = 1$, the guaranteed bound decreased to $\bar{\tau} = 0.203$, which is comparable with the centralized case. Nonetheless, with the added freedom in designing the topology, the idea that diffusive coupling with artificial overtrumps direct exchange of outputs is further supported.

4.5 Vector Lyapunov approach

The feasibility of the Lyapunov equation with a block-diagonal Lyapunov matrix in the delay-free case as shown in section 2.3.4 implies the existence of a Lyapunov function of the form $\bar{V} = p^T v$, where $v = \begin{bmatrix} V_1 & \dots & V_N \end{bmatrix}^{\mathsf{T}}$ with $V_i = e_i^{\mathsf{T}} P_i e_i$, $P_i \succ 0$ and $p \in \mathbb{R}_+^N$. This allows to formulate an exponential stability criterion for coupled systems using individual Lyapunov functions (Lakshmikantham et al., 1991).

Considering element-wise $\dot{v} < Mv$ with a Hurwitz-Metzler matrix M, there exists a $p \in \mathbb{R}_+^N$, such that element-wise $p^{\mathsf{T}}M < -\alpha p^{\mathsf{T}}$, where α is the spectral abscissa of M. Therefore

$$\dot{\bar{V}} = p^{\mathsf{T}}\dot{v} < -\alpha p^{\mathsf{T}} v < -\alpha\bar{V},$$

which due to the comparison lemma is a sufficient condition for exponential stability

with decay rate $\alpha/2$.

Bases on this result, Ugrinovskii and Fridman (2014) derived an LMI method for the design of a distributed observer under a round-robin communication protocol using Wirtinger's inequality. In the presence of time-varying delays in the coupling of systems a similar approach can be applied by considering Halanay's inequality instead (Fridman and Blighovsky, 2012).

Lemma 4.1. *Consider a system $\dot{E} = f(E_t)$, where $E^{\mathsf{T}} = \begin{bmatrix} e_1^{\mathsf{T}} & \dots & e_N^{\mathsf{T}} \end{bmatrix}$, f is locally Lipschitz continuous and $f(0) = 0$. If for all $i \in \{1, \dots, N\}$ there exists Lyapunov-Krasovskii functionals V_i satisfying $\underline{k}_i \|e_i(t)\|^2 \leq V_i(t, e_i, \dot{e}_i) \leq \overline{k}_i \|e_{t,i}\|_W^2$ with $\underline{k}_i > 0$ and $\overline{k}_i > 0$, $m_{ii} > 0$ and $m_{ij} > 0$ with $j \in \mathcal{N}_i \subseteq \{1, \dots, N\}$ for which the inequalities*

$$\dot{V}_i \leq -m_{ii} V_i + \sum_{j \in \mathcal{N}_i} m_{ij} V_j(t - \tau(t)) \tag{4.9}$$

hold with $\delta_0 > \delta_1 > 0$ where $\delta_0 = \min_i m_{ii}$ and $\delta_1 = \max_i \sum_{j:i \in \mathcal{N}_j} m_{ji}$, then the system is exponentially stable with convergence rate bounded by the solution α of $2\alpha - \delta_0 + \delta_1 \mathrm{e}^{2\alpha\bar{\tau}} = 0$.

Taking V_i as Lyapunov-Krasovskii functionals for exponential convergence with time-varying delays as used for (4.5), the conditions of Lemma 4.1 will lead to an alternative to Proposition 4.2 for the design of the distributed observer gains in face of delayed coupling, namely using a system of coupled LMIs.

Proposition 4.3. *For all $i \in \{1, \dots, N\}$, given $\bar{\tau} > 0$, $m_{ii} > 0$, $m_{ij} > 0$ with $j \in \mathcal{N}_i$, with $\delta_0 = \min_i m_{ii} > \delta_1 = \max_i \sum_{j:i \in \mathcal{N}_j} m_{ji}$ and tuning parameters ε_i, let there exist matrices $P_i \succ 0, R_i \succ 0, S_i \succ 0$ and $P_{2,i}, Y_i, Z_{i,j}$ with $j \in \mathcal{N}_i$, $S_{12,i}$, which satisfy the system of coupled LMIs $\Phi_1 \preceq 0, \dots, \Phi_N \preceq 0$, with*

$$\Phi_i = \begin{bmatrix} \Xi_{i,11} + \Psi_{i,11} & \Xi_{i,12} + P_i & \Psi_{i,12} & \Xi_{i,13} + \Psi_{i,13} & \Xi_{i,14} \\ * & \Xi_{i,22} + \bar{\tau}^2 R_i & 0 & \Xi_{i,23} & \Xi_{i,24} \\ * & * & \Psi_{i,22} & \Psi_{i,23} & 0 \\ * & * & * & \Psi_{i,33} & 0 \\ * & * & * & * & -\operatorname{blkdiag}(m_{ij} P_j)_{j \in \mathcal{N}_i} \end{bmatrix}$$

where

$$\Xi_{i,11} = A^\mathsf{T} P_{2,i} + P_{2,i}^\mathsf{T} A - C_i^\mathsf{T} Y_i^\mathsf{T} - Y_i C_i \qquad \Xi_{i,22} = -\varepsilon_i (P_{2,i} + P_{2,i}^\mathsf{T})$$
$$\Xi_{i,12} = -P_{2,i}^\mathsf{T} + \varepsilon_i A^\mathsf{T} P_{2,i} - \varepsilon_i C_i^\mathsf{T} Y_i^\mathsf{T} \qquad \Xi_{i,23} = -\varepsilon_i \sum_{j \in \mathcal{N}_i} Z_{i,j}$$
$$\Xi_{i,13} = -\sum_{j \in \mathcal{N}_i} Z_{i,j} \qquad \Xi_{i,24} = \varepsilon_i [Z_{i,j}]_{j \in \mathcal{N}_i}$$
$$\Xi_{i,14} = [Z_{i,j}]_{j \in \mathcal{N}_i}$$

$$\Psi_i = \begin{bmatrix} \Psi_{i,11} & \Psi_{i,12} & \Psi_{i,13} \\ \Psi_{i,21} & \Psi_{i,22} & \Psi_{i,23} \\ \Psi_{i,31} & \Psi_{i,32} & \Psi_{i,33} \end{bmatrix} =$$
$$\begin{bmatrix} m_{ii} P_i + S_i - \mathrm{e}^{-2\alpha_i \bar{\tau}} R_i & \mathrm{e}^{-2\alpha_i \bar{\tau}} S_{12,i} & \mathrm{e}^{-2\alpha_i \bar{\tau}} (R_i - S_{12,i}) \\ * & -\mathrm{e}^{-2\alpha_i \bar{\tau}} (S_i + R_i) & \mathrm{e}^{-2\alpha_i \bar{\tau}} (R_i - S_{12,i}) \\ * & * & \mathrm{e}^{-2\alpha_i \bar{\tau}} (-2R_i + S_{12,i} + S_{12,i}^\mathsf{T}) \end{bmatrix}$$

and $\left[\begin{smallmatrix} R_i & S_{12,i} \\ * & R_i \end{smallmatrix}\right] \geq 0$, *then the error of the distributed observer network* (4.7) *for system* (4.3) *with* $L_i = (P_{2,i}^\mathsf{T})^{-1} Y_i$ *and* $H_{i,j} = (P_{2,i}^\mathsf{T})^{-1} Z_{i,j}$ *converges exponentially with decay rate* α *obtained as a solutions of* $2\alpha - \delta_0 + \delta_1 \mathrm{e}^{2\alpha\bar{\tau}} = 0$ *for any time-varying delay* τ *subject to* $0 \leq \tau(t) \leq \bar{\tau}$.

One possible benefit of the above stability criterion compared to (4.8) is that, since the coupling involves just the neighbours of an observer node, it be might useful for a distributed design procedure, similar to Wu et al. (2015) for the delay-free case. A distributed design procedure offers scalability by allowing to calculate the gains in parallel, but also flexibility with respect to a change of the observer nodes or transmission rates, by allowing the observer nodes to calculate the gains in an adaptive manner solely with local interactions.

4.6 Proofs

Proof of Proposition 4.1. Take as a Lyapunov-Krasovskii functional for systems with fast-varying delays and exponential convergence

$$V(t, e_t, \dot{e}_t) = e^{\mathsf{T}}(t)Pe(t) + \int_{t-\bar{\tau}}^{t} \mathrm{e}^{-2\alpha(t-s)} e^{\mathsf{T}}(s)Se(s)\mathrm{d}s + \bar{\tau} \int_{t-\bar{\tau}}^{t} \mathrm{e}^{-2\alpha(t-s)}(\bar{\tau}+s-t)\dot{e}^{\mathsf{T}}(s)R\dot{e}(s)\mathrm{d}s.$$

With

$$\lambda_{\min}(P)\|e(t)\| \leq V \leq (\lambda_{\max}(P) + \bar{\tau}\,(\lambda_{\max}(S) + \lambda_{\max}(R)))\,\|e_t\|_W$$

the first condition of Theorem 4.1 is fulfilled for all $t \geq 0$ and $e_t \in W_{[-\bar{\tau},0]}$. Taking the derivative leads to

$$\begin{aligned}\dot{V} + 2\alpha V = {} & 2\dot{e}^{\mathsf{T}}(t)Pe(t) + e^{\mathsf{T}}(t)(2\alpha P + S)e(t) - \mathrm{e}^{-2\alpha\bar{\tau}} e^{\mathsf{T}}(t-\bar{\tau})Se(t-\bar{\tau}) \\ & + \bar{\tau}^2 \dot{e}^{\mathsf{T}}(t)R\dot{e}(t) - \bar{\tau} \int_{t-\bar{\tau}}^{t} \mathrm{e}^{-2\alpha(t-s)} \dot{e}^{\mathsf{T}}(s)R\dot{e}(s)\mathrm{d}s. \quad (4.10)\end{aligned}$$

After using $\mathrm{e}^{-2\alpha(t-s)} \geq \mathrm{e}^{-2\alpha\bar{\tau}}$ for $s \in [t-\bar{\tau}, t]$ the following result is applied to the remaining integral:

Lemma. *Under assumption* (4.6b) *for some matrix* S_{12} *the following inequality holds:*

$$-\bar{\tau} \int_{t-\bar{\tau}}^{t} \dot{e}^{\mathsf{T}}(s)R\dot{e}(s)\mathrm{d}s \leq \begin{bmatrix} e(t) \\ e(t-\bar{\tau}) \\ e(t-\tau(t)) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} -R & S_{12} & R - S_{12} \\ * & -R & R - S_{12} \\ * & * & -R + S_{12} + S_{12}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} e(t) \\ e(t-\bar{\tau}) \\ e(t-\tau(t)) \end{bmatrix}.$$

Proof. Using Jensen's inequality together with the reciprocally convex approach Park et al. (2011). ∎

In line with Fridman's (2014) descriptor method the system dynamics (4.5) is incorporated by adding

$$(P_2 e(t) + P_3 \dot{e}(t))^{\mathsf{T}} (Ae(t) - LCe(t-\tau(t)) - \dot{e}(t)) = 0$$

to (4.10) where P_2 and P_3 are auxiliary matrices. Then

$$\dot{V} + 2\alpha V \leq \eta^{\mathsf{T}}(t)\Phi\eta(t),$$

with $\eta(t)^{\mathsf{T}} = \begin{bmatrix} e(t)^{\mathsf{T}} & \dot{e}(t)^{\mathsf{T}} & e(t-\bar{\tau})^{\mathsf{T}} & e(t-\tau(t))^{\mathsf{T}} \end{bmatrix}$ and

$$\Phi = \begin{bmatrix} \Phi_{11} & P - P_2^{\mathsf{T}} + A^{\mathsf{T}}P_3 & \mathrm{e}^{-2\alpha\bar{\tau}}S_{12} & -P_2^{\mathsf{T}}LC + \mathrm{e}^{-2\alpha\bar{\tau}}(R - S_{12}) \\ * & -P_3 - P_3^{\mathsf{T}} + \bar{\tau}^2 R & 0 & -P_3^{\mathsf{T}}LC \\ * & * & -\mathrm{e}^{-2\alpha\bar{\tau}}(S+R) & \mathrm{e}^{-2\alpha\bar{\tau}}(R - S_{12}) \\ * & * & * & \mathrm{e}^{-2\alpha\bar{\tau}}(-2R + S_{12} + S_{12}^{\mathsf{T}}) \end{bmatrix} \tag{4.11}$$

where $\Phi_{11} = A^{\mathsf{T}}P_2 + P_2^{\mathsf{T}}A + 2\alpha P + S - \mathrm{e}^{-2\alpha\bar{\tau}}R$. Therefore, the feasibility of the LMI $\Phi \preceq 0$ implies exponential stability with decay rate α according to Theorem 4.1. To design the gain L, we make the substitution $P_3 = \varepsilon P_2$ with some $\varepsilon > 0$ and $Y = P_2^{\mathsf{T}}L$. ∎

Proof of Lemma 4.1. The inequality (4.9) also implies

$$\dot{V}_i \leq -m_{ii}V_i + \sum_{j \in \mathcal{N}_i} m_{ij} \sup_{-\overline{\tau} \leq \theta \leq 0} V_j(t+\theta) \leq 0.$$

Taking the Lyapunov function $\bar{V} = \sum_{i=1}^{N} V_i(t, e_i, \dot{e}_i)$ gives ($\sum_{j:i\in\mathcal{N}_j}$ denotes the sum over the nodes which use the estimate of node i)

$$\dot{\bar{V}} \leq -\sum_{i=1}^{N} m_{ii}V_i + \sum_{i=1}^{N}\left(\sum_{j:i\in\mathcal{N}_j} m_{ij}\right) \sup_{-\bar{\tau}\leq\theta\leq 0} V_i(t+\theta) \leq -\delta_0\bar{V} + \delta_1 \sup_{-\bar{\tau}\leq\theta\leq 0} \bar{V}(t+\theta).$$

Therefore, according to Halanay's inequality (1966, p. 378)

$$\dot{V}(t) \leq \sup_{-\bar{\tau}\leq\theta\leq 0} \dot{V}(\theta)\mathrm{e}^{-2\alpha t}.$$

∎

Proof of Proposition 4.3. Adding to the inequality (4.9) the error dynamics of the ob-

server node i via the descriptor method

$$2\Big(P_{2,i}e_i(t) + P_{3,i}\dot{e}_i(t)\Big)^{\mathsf{T}} \times \Big((A - L_iC_i)e_i(t) - \sum_{j\in\mathcal{N}_i} H_{i,j}e_i(t-\tau(t)) + [H_{i,j}]_{j\in\mathcal{N}_i}[e_i(t-\tau(t))^{\mathsf{T}}]^{\mathsf{T}}_{j\in\mathcal{N}_i} - \dot{e}_i\Big) = 0,$$

it follows similarly to the proof of Proposition 4.1 that

$$\dot{V}_i + m_{ii}V_i - \sum_{j\in\mathcal{N}_i} m_{ij}V_j(t-\tau(t)) \le \eta_i(t)^{\mathsf{T}}\Phi_i\eta_i(t),$$

with $\eta_i(t)^{\mathsf{T}} = \Big[e_i(t)^{\mathsf{T}} \quad \dot{e}_i(t)^{\mathsf{T}} \quad e_i(t-\bar{\tau})^{\mathsf{T}} \quad e_i(t-\tau(t))^{\mathsf{T}} \quad [e_i(t-\tau(t))^{\mathsf{T}}]_{j\in\mathcal{N}_i}\Big]$ and

$$\Phi_i = \begin{bmatrix} \Xi_{i,11}+\Psi_{i,11} & \Xi_{i,12}+P_i & \Psi_{i,12} & \Xi_{i,13}+\Psi_{i,13} & \Xi_{i,14} \\ * & \Xi_{i,22}+\bar{\tau}^2R_i & 0 & \Xi_{i,23} & \Xi_{i,24} \\ * & * & \Psi_{i,22} & \Psi_{i,23} & 0 \\ * & * & * & \Psi_{i,33} & 0 \\ * & * & * & * & -\operatorname{blkdiag}(\beta_{i,j}P_j)_{j\in\mathcal{N}_i} \end{bmatrix}$$

where

$$\begin{aligned} \Xi_{i,11} &= (A-L_iC_i)^{\mathsf{T}}P_{2,i} + P_{2,i}^{\mathsf{T}}(A-L_iC_i) & \Xi_{i,22} &= -P_{3,i} - P_{3,i}^{\mathsf{T}} \\ \Xi_{i,12} &= -P_{2,i}^{\mathsf{T}} + (A-L_iC_i)^{\mathsf{T}}P_{3,i} & \Xi_{i,23} &= -P_{3,i}^{\mathsf{T}}\sum_{j\in\mathcal{N}_i} H_{i,j} \\ \Xi_{i,13} &= -P_{2,i}^{\mathsf{T}}\sum_{j\in\mathcal{N}_i} H_{i,j} & \Xi_{i,24} &= P_{3,i}^{\mathsf{T}}[H_{i,j}]_{j\in\mathcal{N}_i}, \\ \Xi_{i,14} &= P_{2,i}^{\mathsf{T}}[H_{i,j}]_{j\in\mathcal{N}_i} \end{aligned}$$

which leads with Lemma 4.1 to an LMI for the design after substituting $P_{3,i} = \varepsilon_i P_{2,i}$, $Y_i = P_{2,i}^{\mathsf{T}}L_i$ and $Z_{i,j} = P_{2,i}^{\mathsf{T}}H_{i,j}$. ■

Chapter 5

Conclusion

Taking some distance, the thesis can be summarized in yielding the design of distributed observers for three successive cases in each chapter: the case of convergence with arbitrary exponential rate, the case of convergence in a finite time and the case of convergence with limited rates due to delays. In this respect, the general view and results which were gathered in Chapter 2 for the linear time-variant systems laid the foundation. However, it has to be considered that almost all of these results were obtained in parallel or in conjunction with the thesis; in consequence thereof, the different contributions are strongly interrelated. We will discuss them in the following section and provide directions for further research.

5.1 Discussion

In bringing the various solutions for distributed state estimation together in Chapter 2, we showed how they are particular designs relying on diffusively coupled observer nodes. Notably, the notion of distributed observability with respect to a graph as introduced here provides a necessary and sufficient condition under which the designs are feasible. By pointing out the difference in the proposed discrete-time and continuous-time solutions we highlighted the relation to synchronization problems (e.g. Su et al., 2020), which can be helpful for further developments. In particular, the network of the individual error systems can be interpreted in terms of dynamic consensus of heterogeneous agents in the sense of Panteley and Loría (2017).

The discrete-time perspective provided two additional insights by turning to distributed dead-beat observers. First, depending how stringent the gains are chosen, less of the global information is utilized leading to a loss in performance. This showed that a more general choice, like with the LMI and possible in combination with a selection of article outputs, is imperative. Next, we showed how resorting to static coupling with partial outputs will incur a steady delay. These effects are readily visible for the simple example where a single node has all the information, but general systems are not exempt from them. In the continuous-time case the effects manifest themselves in higher bounds on the time-varying delay when choosing general coupling gains over the exchange of actual outputs.

Demonstrating the advantage of distributed observers in a networked setting is an important achievement. It constitutes a primary reason why distributed observers should be considered over centralized ones in the first place. Thus, if delays cannot be neglected, design methods which take them into account are of paramount importance. Nevertheless, this would be a preliminary step. Allowing for a lower transmission rate is not enough to reduce the required bandwidth, if it is gained by increasing the size of the exchanged information. An optimal design would have to solve this trade-off, in line with the continuous efforts of taking the limited information capacity of communication channels into account (see e.g. Andrievsky et al., 2010; Voortman et al., 2019).

The LMI for the design of the gains in Chapter 4 was derived using the Lyapunov-Krasovskii functional approach. To this end, various overestimations are used with the consequence that the LMI might not be feasible even if in reality there exits a gain for a higher bound on the time-varying delay. However, by convexity and showing that the error dynamics in the delay-free case is block-diagonally stable in original coordinates, we can give an important guarantee. Namely, the distributed observability condition guarantees to find at least some non-zero bound with this method. In other words, imposing the structure does not render the LMI useless for the selection of gains, even if it is conservative. Recent results by Sootla et al. (2019) show that block-stability can be further exploited in the numerical solution of LMIs, perhaps to address the demand (as voiced throughout recent literature) for a distributed design procedure.

Furthermore, a nonlinear distributed observer was presented in Chapter 3 such that

the estimates reach the state in finite-time, based on the unique properties of homogeneous systems. The design is warranted by a Lyapunov function which relies on the diagonal stability property of the compressed Laplacian matrix of the consensus-based linear design. Sufficient bounds for the gain parameters were obtained for both the centralized observer used for the observable substate and consensus-based one used for the unobservable substate. Note that assuming the partial outputs to be scalar is not restrictive, since each entry in the output vector can be treated as belonging to a virtual node of a distributed observer. This means also, that the proposed solution can be regarded as a comparatively simple (albeit high-dimensional) design of a centralized finite-time observer for multi-output systems.

5.2 Future work

So far the analysis of proposed distributed finite-time observer focused on its qualitative convergence properties, while additional advantages of the nonlinear gain, which locally corresponds to an infinite gain on the synchronization manifold, were not examined. Homogeneous systems have by themselves enhanced robustness with respect to delays (Zimenko et al., 2019) and it remains to be explored how this property can be exploited for networked control systems. Figuratively speaking, such a link would place the three main chapters in a triangle. Finite-time convergence is not only useful for state observers but also for a range of problems linked to estimation such as identification (estimate of the parameters), differentiation (estimate the derivatives), left-inversion (estimate unknown inputs) and filtering (estimate a signal hidden by noise) which ought to be distributed as well.

Admittedly, the distributed state estimation problem was treated only in theory. In particular, disturbances as well as noise were not considered and the inputs were excluded. Turning to applications they play an important role in choosing a design. In this sense, the experimental verification should be a priority and a proof of concept may also boost further developments. Ultimately, such an application would entail strict performance requirements for example due to time-varying networks and the available energy. Moreover, communication would not be considered anymore in an abstract setting, but will be given by the specifications of the used communication

protocol (e.g. Wang et al. (2017) considered FlexRay in an observer design).

In simulations, distributed observers have been designed for a range of applications (see Rinaldi et al., 2020; Zhang et al., 2019; Tian et al., 2019; Kyriacou et al., 2018). As assumptions they require that the subsystems are locally detectable with only weak interconnections. The diffusively coupled distributed observer investigated in this thesis might allow to relax these assumptions for a gain in performance and enlarge the spectrum of applications. Other ideal candidates are distributed parameters systems (Battistelli et al., 2017) as well as general decentralized control schemes (e.g. Dileep et al., 2020). The deployment of mobile sensors for environmental monitoring is another prominent example of a task requiring flexible networks and local interactions to achieve a common goal.

The above applications are technological and can provide means to continuously improve our living standards. But the ongoing quest for understanding networks and patterns in society and nature (e.g. Rogov et al., 2019) might also benefit from a general theory of distributed estimation. For example, according to Hayek (1945) each individual is an observer with limited knowledge of the economic complexity, with prices being a communicated signal.

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