Stable Lifting Construction of Non-Uniform Biorthogonal Spline Wavelets with Compact Support

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Outline

1. Motivation
   - Some examples
   - The key is stability
2. Construction of wavelets
   - Multiresolution decomposition and lifting
   - Stability over all levels
3. Examples and numerical experiments
   - Linear spline wavelets on $\Omega \subset \mathbb{R}$
   - Linear spline wavelets on $\Omega \subset \mathbb{R}^2$
   - Powell-Sabin spline wavelets on $\Omega \subset \mathbb{R}^2$
4. Customizing wavelets on uniform grids
   - Piecewise Hermite cubics in $\mathbb{R}$

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A simple example

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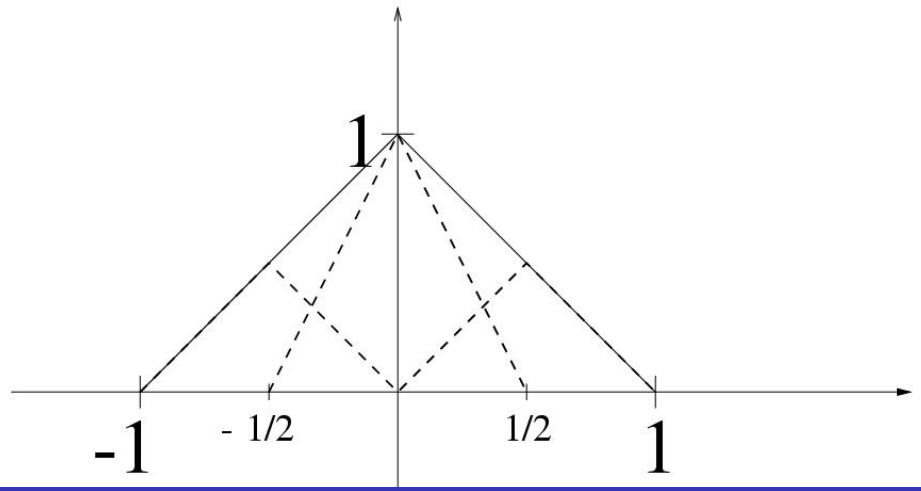
$$-u'' + u = f \text{ on } [0,1], \quad \text{with } u(0) = u(1) = 0.$$

weak formulation

$$\langle u', v' \rangle + \langle u, v \rangle = \langle f, v \rangle, \qquad v \in H_0^1([0,1]).$$

Hat function

$$\Phi(t) = \begin{cases} 1+t, & -1 \le t \le 0, \\ 1-t, & 0 \le t \le 1, \\ 0, & \text{else} \end{cases}$$



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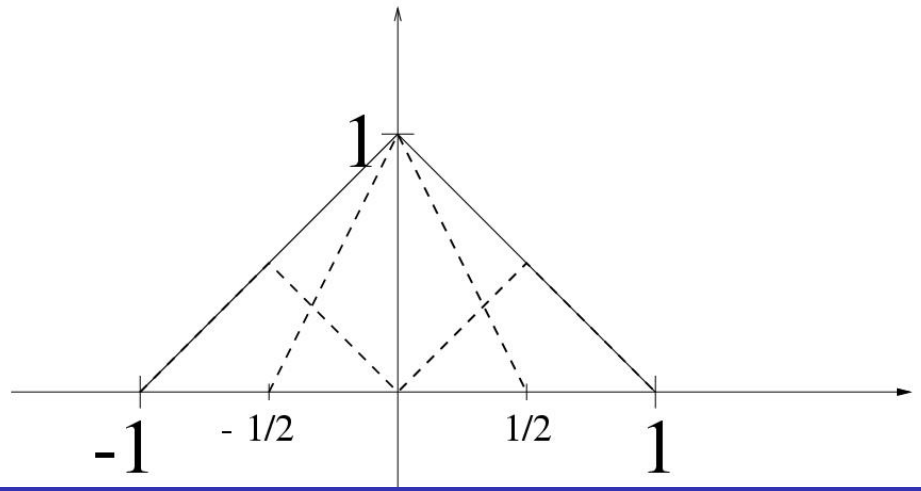

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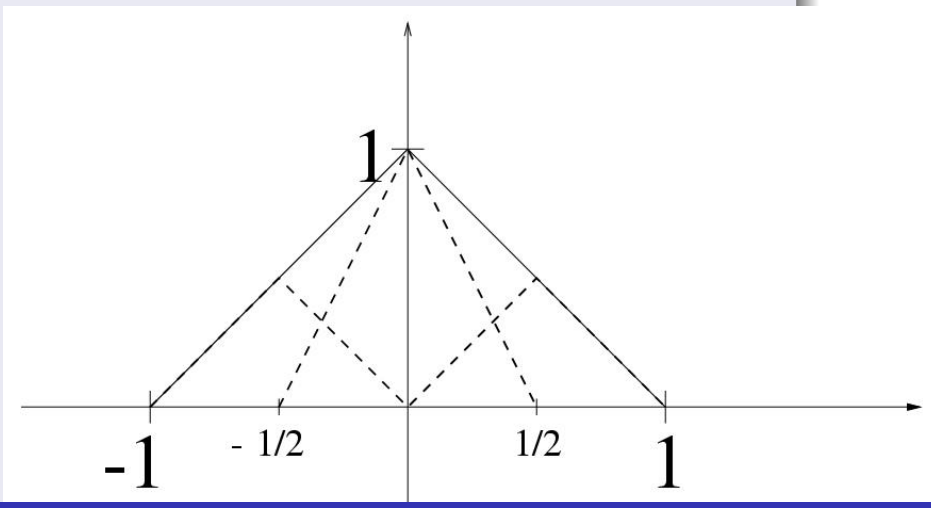

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multilevel splitting

$$\Phi(t) = \Phi(2t) + (\Phi(2t-1) + \Phi(2t+1))/2$$

$$\phi_{j,k} = \phi_{j+1,2k} + (\phi_{j+1,2k-1} + \phi_{j+1,2k+1})/2$$

hierarchical bases (HB)

$$\Psi_j := \{\psi_{j,k} := \phi_{j+1,2k+1}\}, \qquad \Psi_{-1} := \{\phi_{0,k}\}$$

system to solve: $Ac = b$

$$\left\langle \frac{d}{dx} \sum_{j,k} c_{j,k}\psi_{j,k}, \frac{d}{dx}\psi_{l,m} \right\rangle + \left\langle \sum_{j,k} c_{j,k}\psi_{j,k}, \psi_{l,m} \right\rangle = \langle f, \psi_{l,m} \rangle$$

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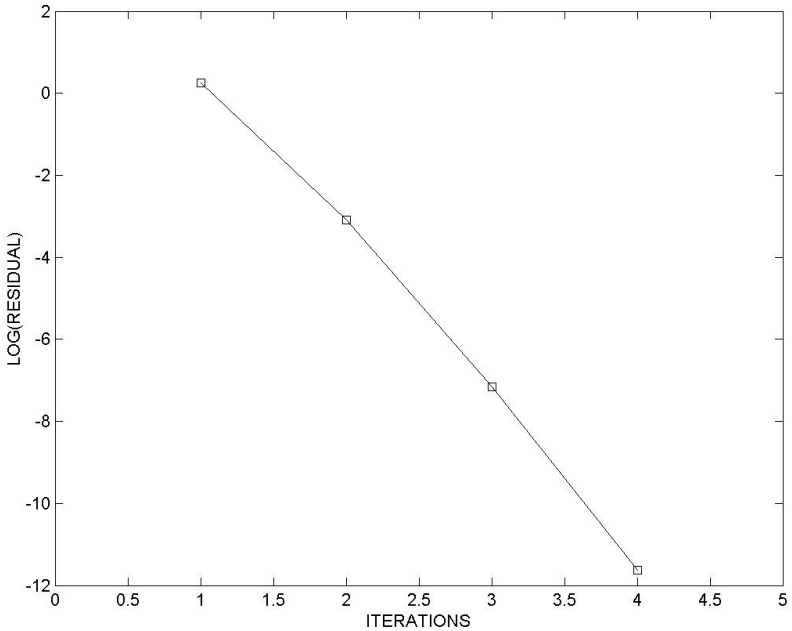

- CG iteration
- $\kappa(A) \approx 1.29$
- Good choice of basis functions

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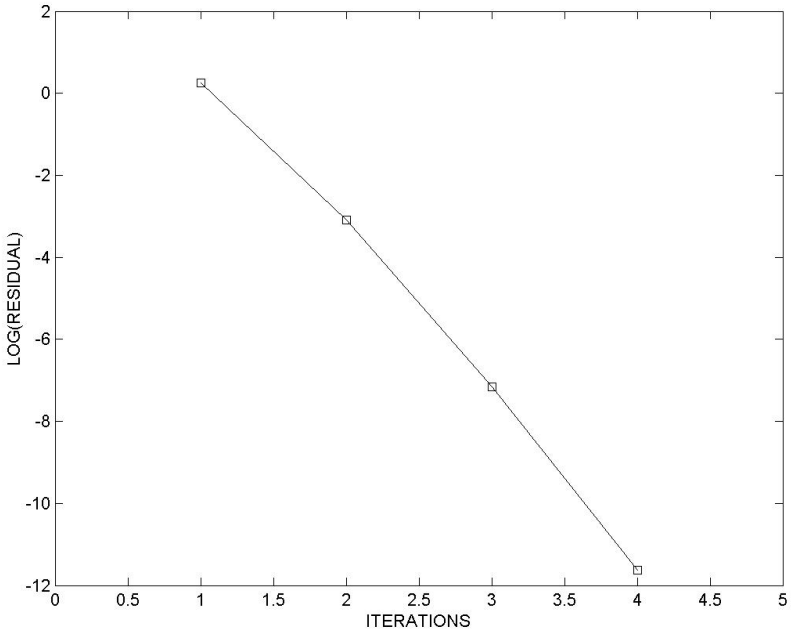

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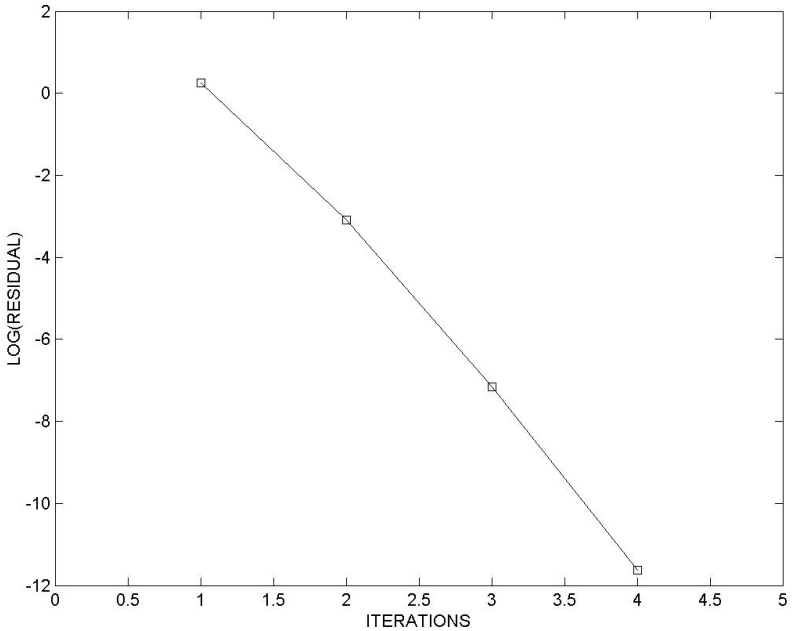

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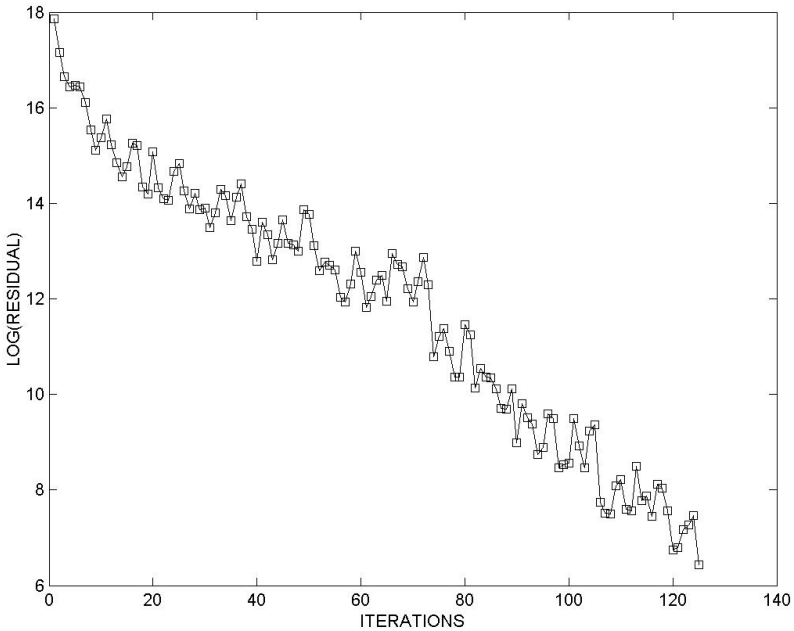

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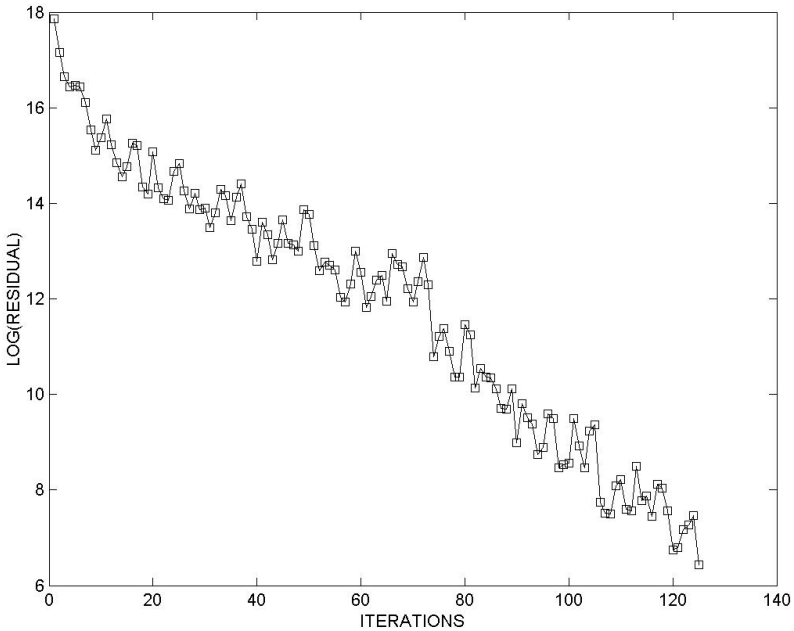

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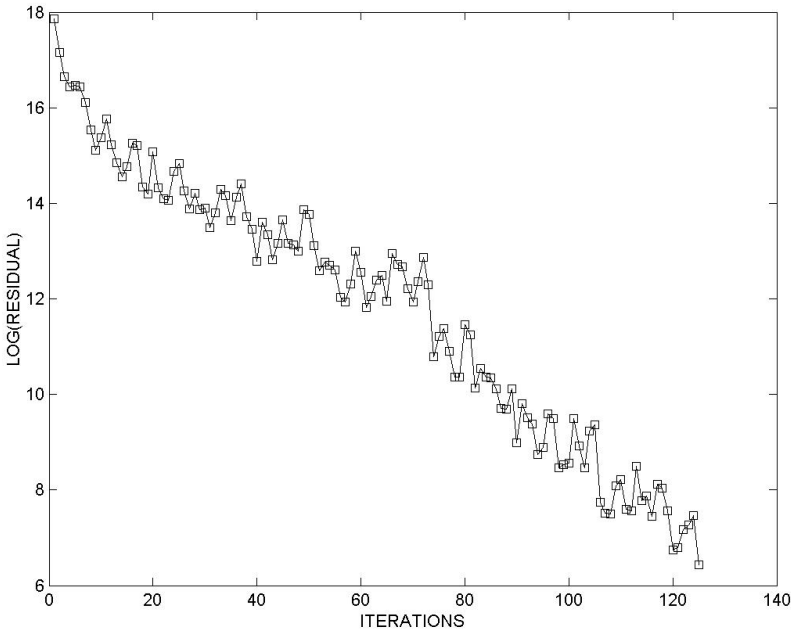

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The key is stability

- $a(u,v) = \langle f, v \rangle$ with $a(\cdot,\cdot)$ a symmetric bilinear form
- $a(v,v) \sim \|v\|_{\mathcal{H}}^2$ with $\mathcal{H}$ a Hilbert space
- Example 1: $a(u,v) = \langle u', v' \rangle + \langle u, v \rangle, \quad a(v,v) \sim \|v\|_{H^1}^2$

Stability determines condition

Suppose that

$$\gamma \sum_{j,k} |c_{j,k}|^2 \leq \left\| \sum_{j,k} c_{j,k} \psi_{j,k} \right\|_{\mathcal{H}}^2 \leq \Gamma \sum_{j,k} |c_{j,k}|^2,$$

then

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A large range of stability is preferable

$$a(u,v) = \langle u', v' \rangle + q\langle u, v \rangle$$

- for small q we have $a(v,v) \sim \|v\|_{H^1}^2$
- as q enlarges, the term $\langle u, v \rangle$ becomes much more important. Hence we practically have $a(v,v) \sim \|v\|_{L_2}^2$.

a large range of stability

We want a multilevel system $\bigcup_j \Psi_j$ that is a stable basis for H^s for a large range of smoothness exponents s.

Solution

We stabilize the HB $\Rightarrow$ Wavelets

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Multiresolution decomposition

- Multiresolution:

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset L_2(\Omega), \quad \overline{\bigcup_j V_j} = L_2(\Omega)$$

- Single scale bases: $V_j = \operatorname{span} \Phi_j, \quad \Phi_j := \{\phi_{j,k}, k \in I_j\}$

$$\|\mathbf{c}\|_{l_2(I_j)} \sim \|\Phi_j \mathbf{c}\|_{L_2}$$

- Decomposition: $V_{j+1} = V_j \oplus W_j$
- Complement bases:
  $W_j = \operatorname{span} \Psi_j, \quad \Psi_j := \{\psi_{j,\lambda}, \lambda \in J_j := I_{j+1} \setminus I_j\}$
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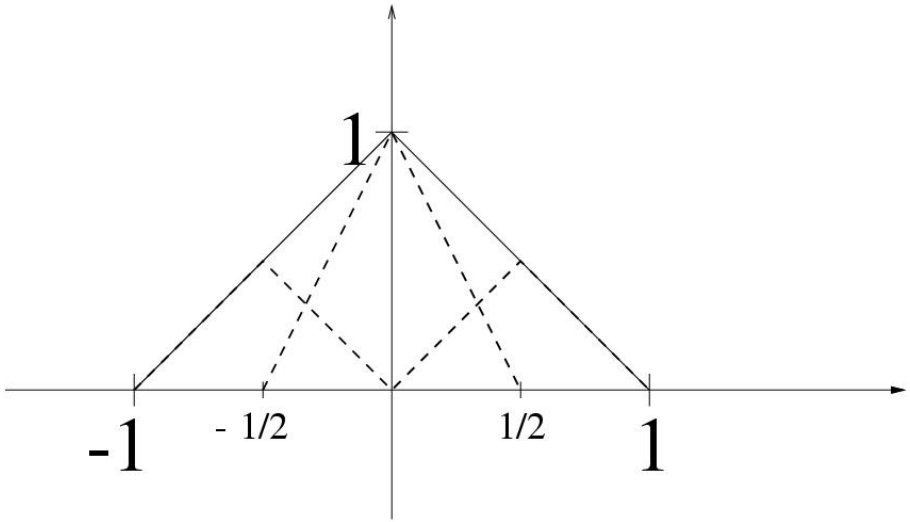
The Lifting Scheme (Sweldens)

- $\Phi_j = \mathbf{P}_j \Phi_{j+1}$
- $\Phi_{j+1} = [\mathcal{O}_{j+1} \quad \mathcal{N}_{j+1}]$
- $[\Phi_j \quad \Psi_j] = \Phi_{j+1}[\mathbf{P}_j \quad \mathbf{Q}_j]$
- HB: $\Psi_j = \mathcal{N}_{j+1}$ and $\mathbf{Q}_j = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}$



Lifting

$$\Psi_j = \mathcal{N}_{j+1} - \Phi_j \mathbf{U}_j$$

with $\mathbf{U}_j$ the update matrix. We find a relation of the form

$$[\Phi_j \quad \Psi_j] = \Phi_{j+1}[\mathbf{P}_j \quad \mathbf{Q}_j - \mathbf{P}_j \mathbf{U}_j]$$

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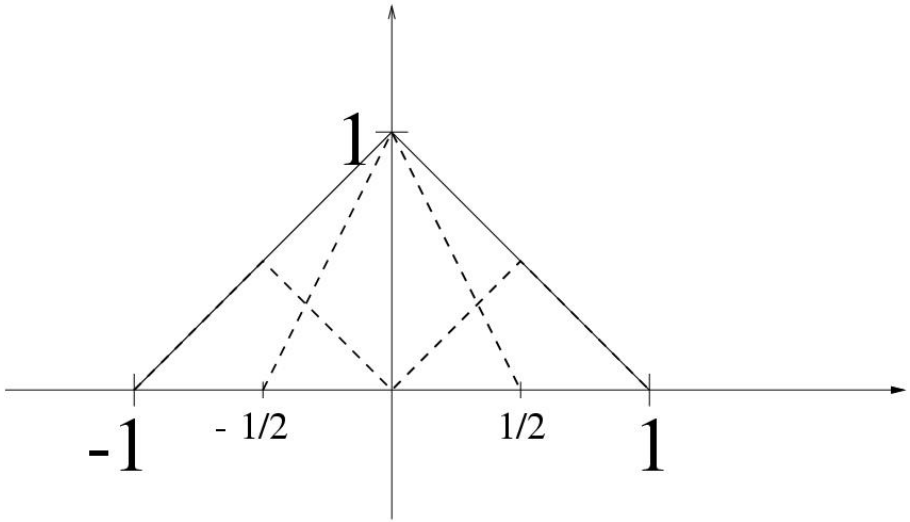

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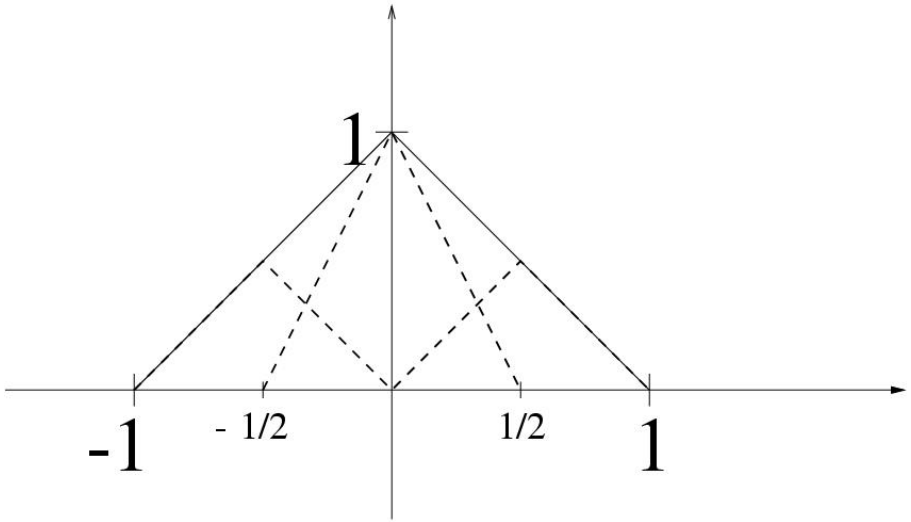

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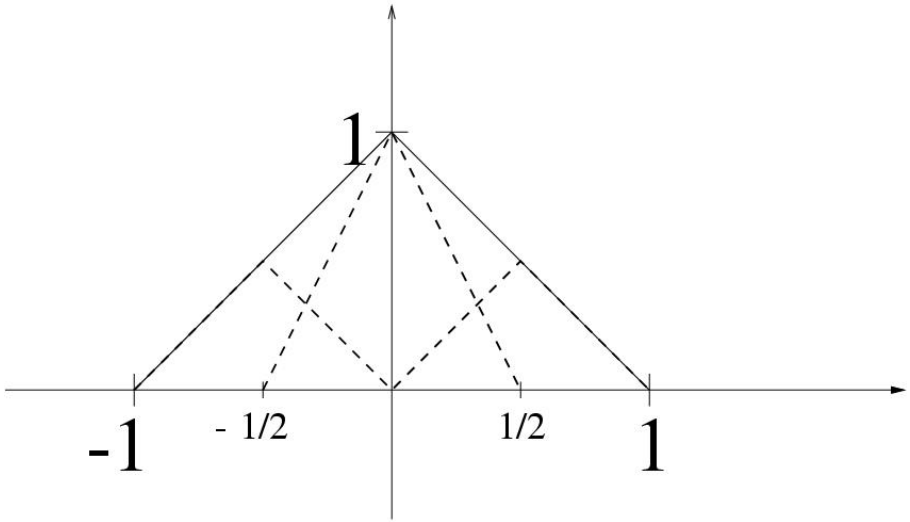

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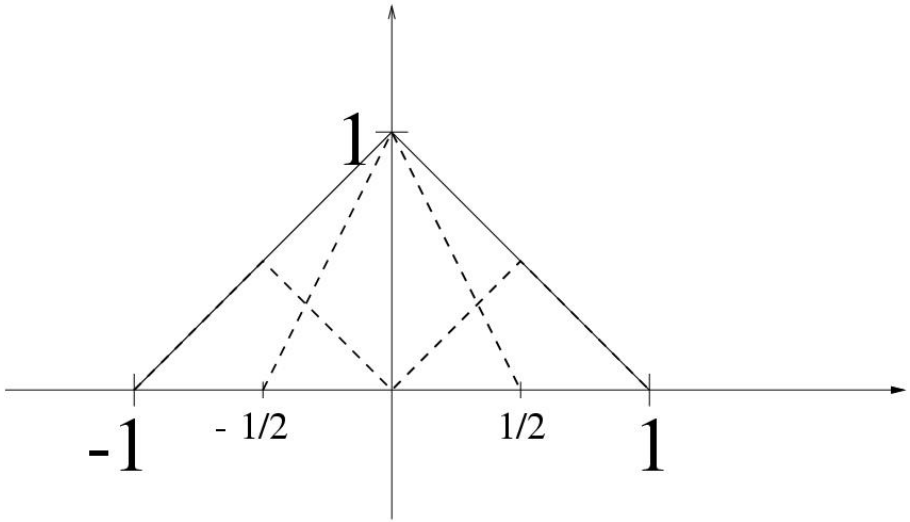

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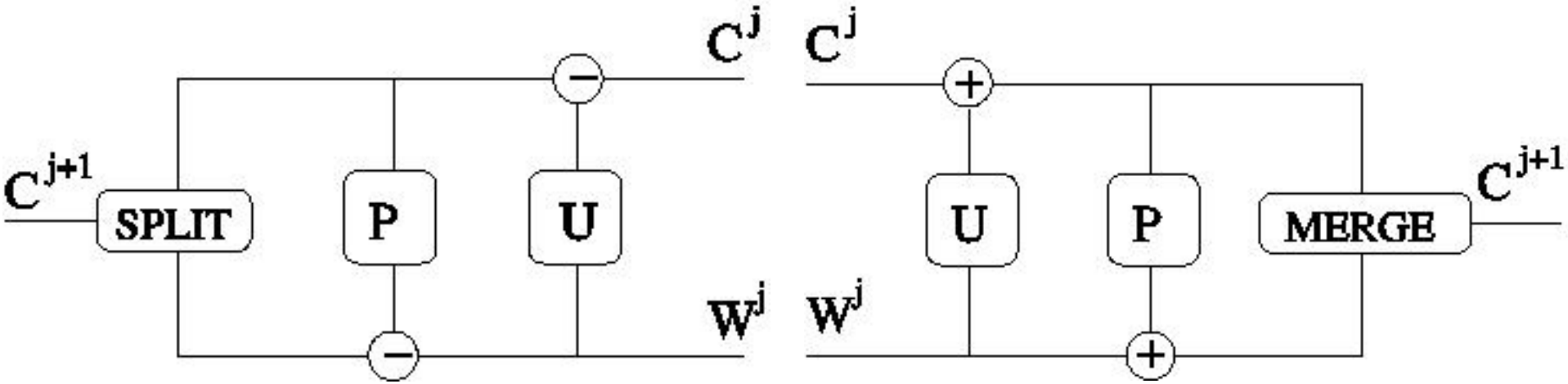
$$[\Phi_j \quad \Psi_j] = \Phi_{j+1}[\mathbf{P}_j \quad \mathbf{Q}_j - \mathbf{P}_j \mathbf{U}_j]$$

The Lifting Scheme (Sweldens)



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In order to have one level stability for the wavelets, i.e.

$$\|\mathbf{d}\|_{l_2(J_j)} \sim \|\Psi_j \mathbf{d}\|_{L_2},$$

it is sufficient to prove that

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2. Semiorthogonal lifting
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- Goal is to obtain wavelets with N vanishing moments
- Fix an update stencil: $\psi_{j,m} = \mathcal{N}_{j+1,m} - \mathbf{U}_{j,m}\Phi_{j,m}$
- Solve the $N \times N$ system (1D)

$$\mathbf{U}_{j,m} = \begin{bmatrix} \langle \Phi_{j,m}, p_0 \rangle \\ \vdots \\ \langle \Phi_{j,m}, p_{N-1} \rangle \end{bmatrix}^{-1} \begin{bmatrix} \langle \mathcal{N}_{j+1,m}, p_0 \rangle \\ \vdots \\ \langle \mathcal{N}_{j+1,m}, p_{N-1} \rangle \end{bmatrix}$$

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1. Non-singularity of system is an open question
2. Lifting coefficients can be unbounded, see J. Simoens and S. Vandewalle. Stabilized Lifting Construction On Irregular Meshes. SIAM J. Sci. Comput., 24(4):1356-1378,2002.

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- Also known as Prewavelets:

$$V_{j+1} = W_j \oplus_{\perp_{L_2}} W_{j-1} \oplus_{\perp_{L_2}} \cdots \oplus_{\perp_{L_2}} V_0$$

- Solve the system

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- Vanishing moments for free
- Stable wavelets for $H^s(\Omega)$ with $|s|$ bounded by the Sobolev regularity of the scaling functions.

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The lifting matrix in semiorthogonal lifting has exponential off-diagonal decay of a rate that is independent of the resolution level j.

Orthogonalize $\Psi_{j,m}$ to a subspace of V_j spanned by scaling functions in the neighbourhood of $\Psi_{j,m}$. Taking the update stencil sufficiently large should provide a good approximation to semiorthogonalization.

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Smoothness determines stability

Range of stability

We are interested in the range of s for which the multiscale basis $\bigcup_j \Psi_j$ forms a Riesz basis for the Sobolev space $H^s(\Omega)$.

The range of such s is determined by the Sobolev regularity of the scaling functions Φ_j and the Sobolev regularity of the dual scaling functions $\widetilde{\Phi}_j$.
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The lifting scheme automatically generates a dual multiscale basis $\widetilde{\Phi}_j \cup \widetilde{\Psi}_j$. This basis is biorthogonal to $\Phi_j \cup \Psi_j$ in the sense that

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The bases $\Phi_j \cup \Psi_j$ and $\widetilde{\Phi}_j \cup \widetilde{\Psi}_j$ are only formally biorthogonal. It is not guaranteed that the dual basis actually belongs to L_2.

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Theorem by W. Dahmen

Assume that $\Psi(:= \Phi_0 \cup \bigcup_j \Psi_j)$ and $\widetilde{\Psi}$ are dual L_2 stable multilevel bases with compactly supported basis functions. Then

$$\Psi \text{ is a Riesz basis in } H^s \quad \Leftrightarrow \quad -s_{\widetilde{\Phi}_0} < s < s_{\Phi_0},$$
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From properties of the Lifting Scheme we can find a refinement relation

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In the shift-dilation invariant case techniques exist to compute s_Φ given a refinement relation of the form

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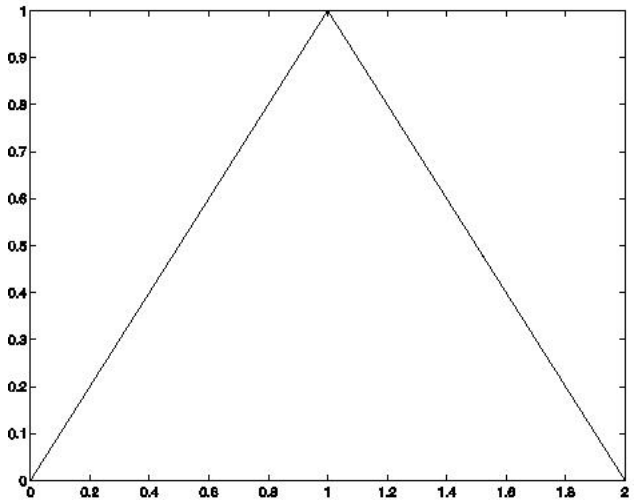
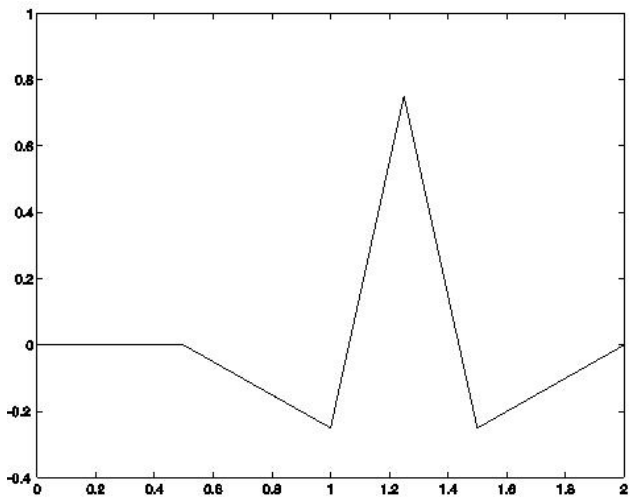
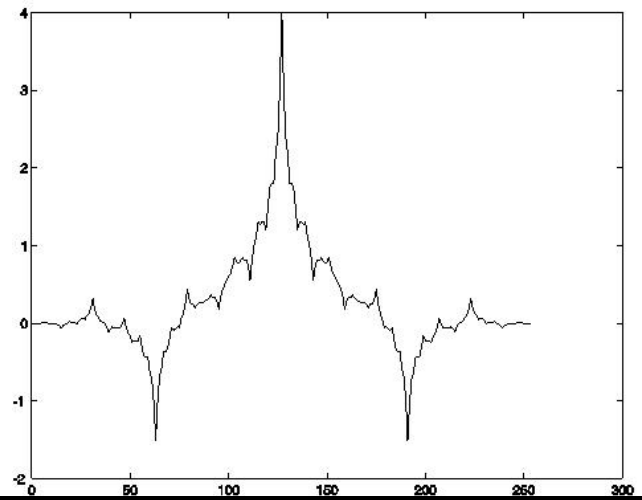
Linear spline wavelets on $\mathbb{R} \Rightarrow$ CDF(2,2)







$$s_\Phi = 1.5$$
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Riesz basis of H^s with
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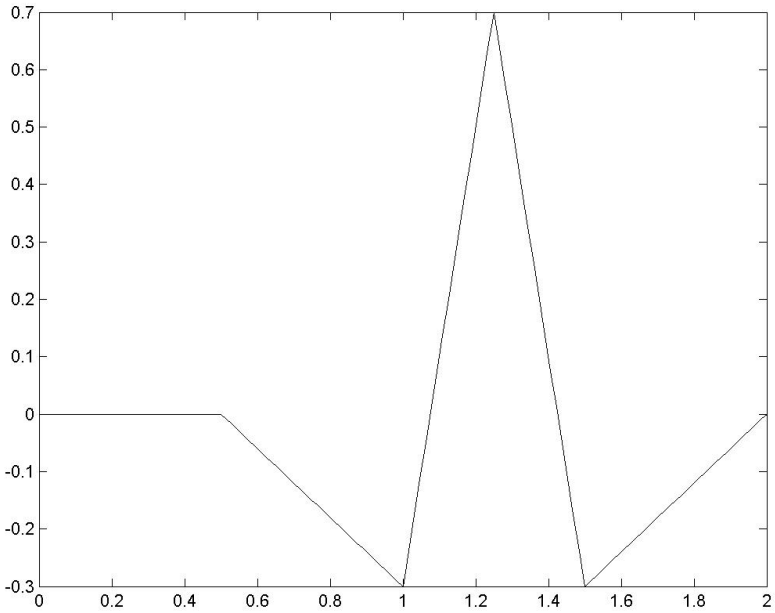

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Boundary value problem

$$-u'' + qu = f \quad \text{on} \quad \Omega := [0, 2], \qquad u(0) = u(2) = 0.$$

$q = 1$	Linear wavelets			Linear HB		
J	residual	#iter	κ	residual	#iter	κ
4	5.46e-07	11	4.2707	2.08e-07	4	1.2880
5	1.23e-06	11	4.6169	7.70e-07	3	1.2887
6	3.65e-07	12	4.8821	1.95e-07	3	1.2889
7	1.98e-07	12	5.0910	4.92e-08	3	1.2890
8	8.73e-08	12	5.2576	1.23e-08	3	1.2890
9	4.81e-08	12	5.3925	5.84e-08	2	1.2890
10	2.81e-08	12	5.5031	1.49e-08	2	1.2890
11	1.64e-08	12	5.5949	3.73e-09	2	1.2890
12	8.78e-09	12	5.6717	9.34e-10	2	1.2890

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$$-u'' + qu = f \quad \text{on} \quad \Omega := [0, 2], \qquad u(0) = u(2) = 0.$$

$q = 10^8$	Linear wavelets			Linear HB		
J	residual	#iter	κ	residual	#iter	κ
4	1.11e-06	8	6.10	5.79e-07	14	121
5	6.75e-07	7	6.64	9.50e-07	16	288
6	4.88e-07	5	7.16	6.16e-07	16	667
7	2.51e-07	4	7.58	2.45e-07	15	1.51e+03
8	1.38e-07	2	7.96	1.43e-07	10	3.38e+03
9	6.82e-08	2	8.28	7.45e-08	11	7.40e+03
10	2.29e-08	2	8.53	3.72e-08	7	1.55e+04
11	1.09e-08	2	8.68	1.86e-08	12	2.89e+04
12	4.45e-09	1	8.71	8.75e-09	32	4.31e+04

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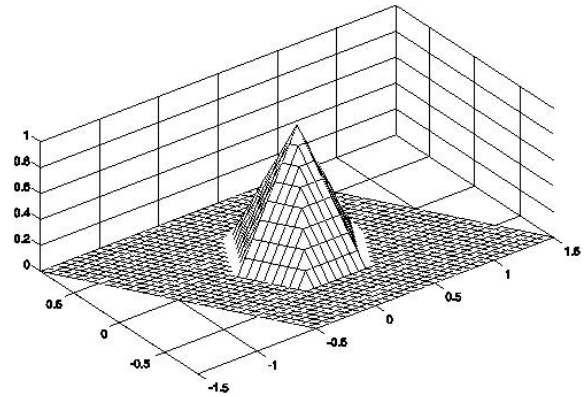

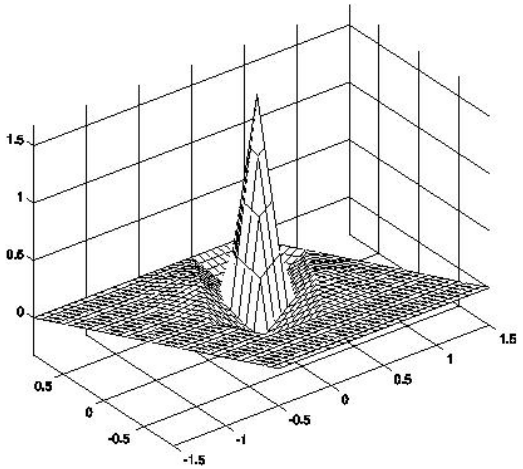

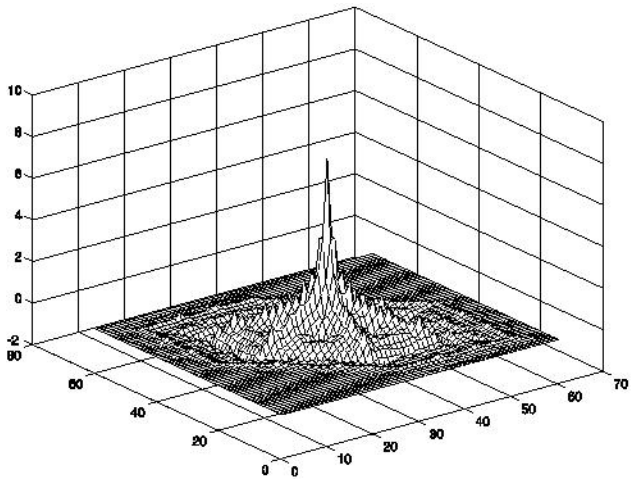

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Riesz basis of H^s with $-0.328857 < s < 1.5$

2nd order elliptic problem

$$-\Delta u + qu = f \quad \text{on} \quad \Omega := [0,1]^2, \qquad u|_{\partial\Omega} = 0.$$

$q = 1$	Linear wavelets			Linear HB		
J	residual	#iter	κ	residual	#iter	κ
2	7.86e-06	14	9.56	1.14e-05	14	10.65
3	6.23e-06	17	12.66	6.70e-06	18	19.64
4	2.45e-06	19	14.88	2.74e-06	21	32.03
5	1.30e-06	20	16.50	1.35e-06	25	47.41
6	8.03e-07	20	17.73	9.86e-07	28	65.73

2nd order elliptic problem

$$-\Delta u + qu = f \quad \text{on} \quad \Omega := [0,1]^2, \qquad u|_{\partial\Omega} = 0.$$

$q = 10^8$	Linear wavelets			Linear HB		
J	residual	#iter	κ	residual	#iter	κ
2	1.21e-05	15	14.61	1.30e-05	16	108
3	7.11e-06	18	19.39	5.86e-06	35	608
4	3.97e-06	19	23.89	3.97e-06	58	3.13e+03
5	1.97e-06	20	27.73	1.30e-06	75	1.53e+04
6	6.84e-07	21	31.03	9.01e-07	92	7.25e+04

Outline

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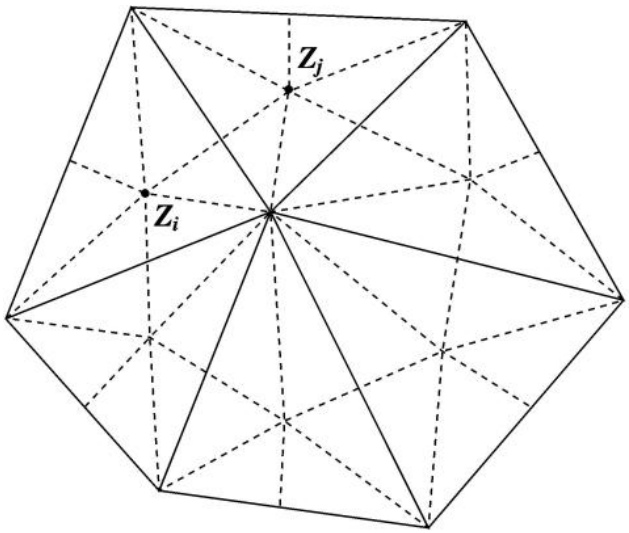
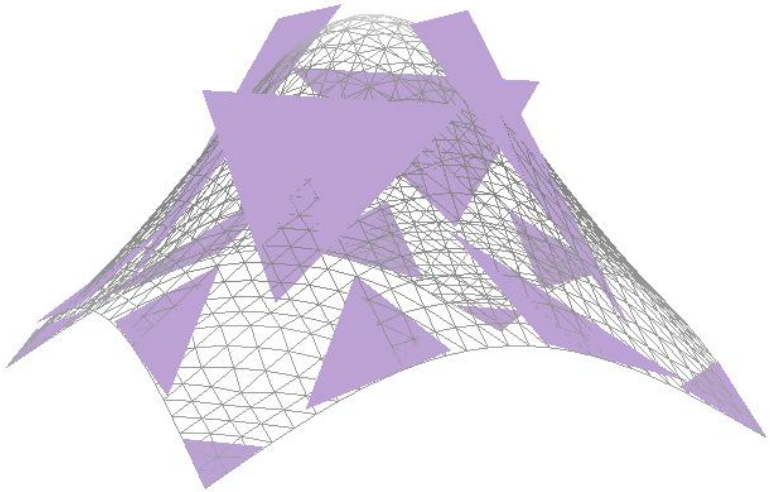
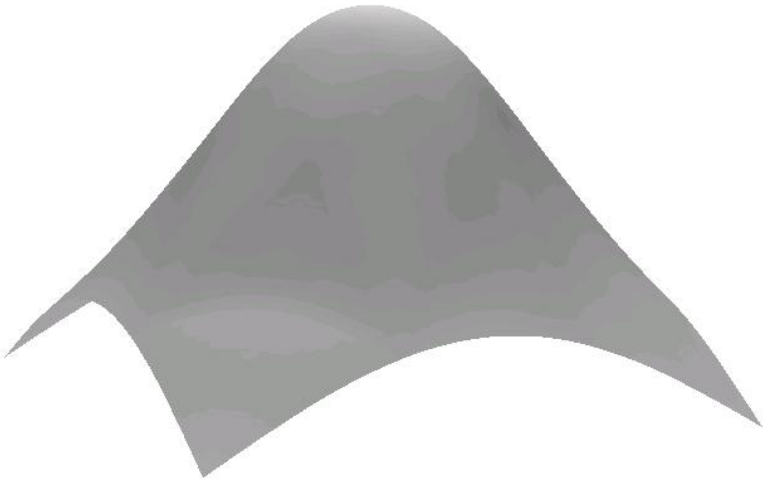
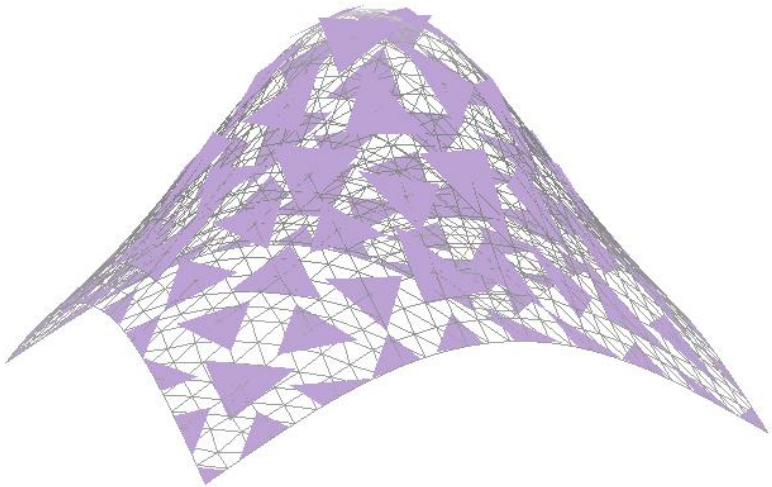
   - Some examples
   - The key is stability
2. Construction of wavelets
   - Multiresolution decomposition and lifting
   - Stability over all levels
3. Examples and numerical experiments
   - Linear spline wavelets on $\Omega \subset \mathbb{R}$
   - Linear spline wavelets on $\Omega \subset \mathbb{R}^2$
   - Powell-Sabin spline wavelets on $\Omega \subset \mathbb{R}^2$
4. Customizing wavelets on uniform grids
   - Piecewise Hermite cubics in $\mathbb{R}$

C^1 piecewise quadratic polynomials

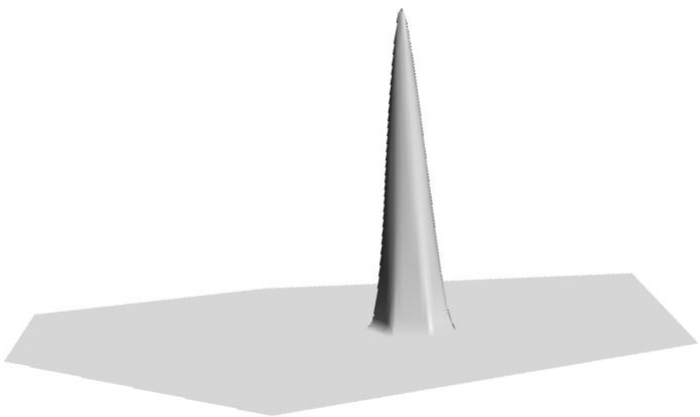


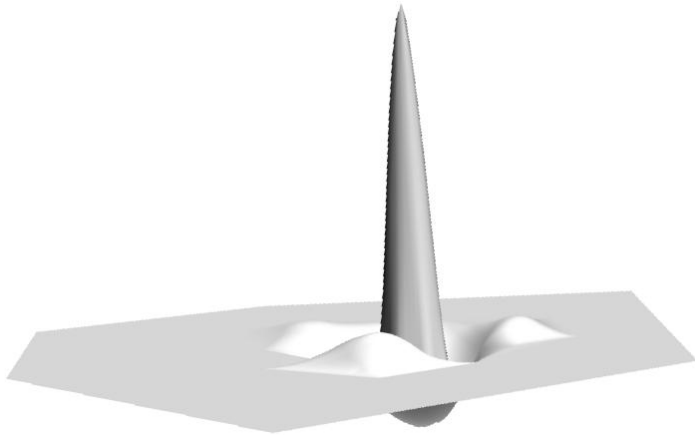




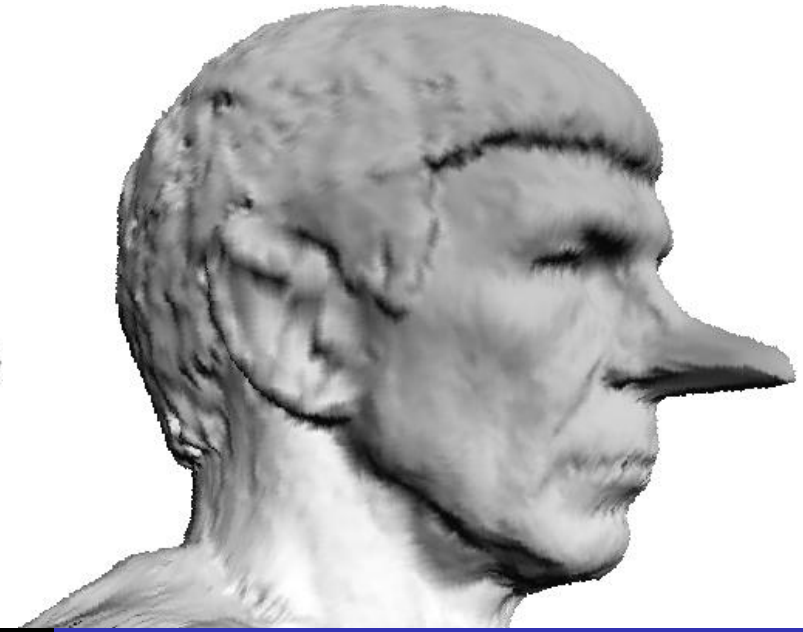


Powell-Sabin spline wavelets on $\Omega \subset \mathbb{R}^2$









Stability of Powell-Sabin spline wavelets

$$s_\Phi = 2.5, \qquad \widetilde{\Phi} \text{ not in } L_2$$

Riesz basis of H^s with $0.802774 < s < 2.5$

Suitable for the biharmonic equation

$$\Delta^2 u = f$$

	PS Wav		PS HB	
J	κ	$\frac{\kappa(J)}{\kappa(J-1)}$	κ	$\frac{\kappa(J)}{\kappa(J-1)}$
3	21.7298	/	29.1919	/
4	35.9492	1.65	60.7308	2.08
5	47.6441	1.33	104.9501	1.73

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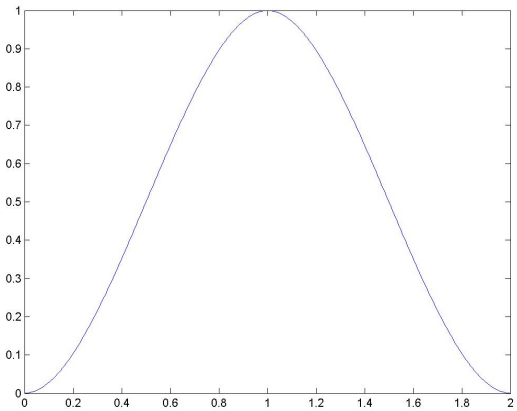
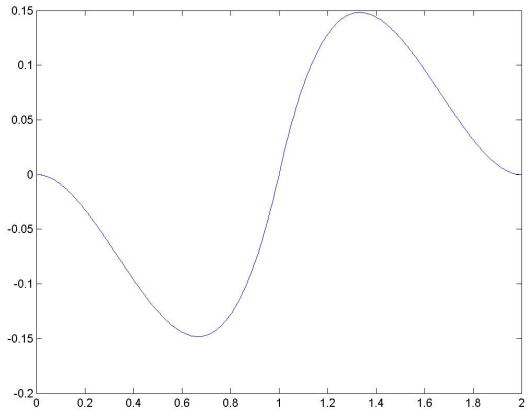
Piecewise Hermite cubics on the interval





Lifting:

$$\Psi(x) = \Phi(2x-1) - \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} \Phi(x) - \begin{bmatrix} \alpha_3 & \alpha_4 \\ \beta_3 & \beta_4 \end{bmatrix} \Phi(x-1).$$

Wish list

1. 2 vanishing moments
2. ψ_1 symmetric
3. ψ_2 anti-symmetric

We need 6 degrees of freedom to accomplish our wish list $\Rightarrow$ 2 degrees of freedom remain.

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Refinement equation of the dual generator

$$\widetilde{\Phi}(x) = \begin{bmatrix} -\frac{9}{40} - \frac{\beta_2}{5} & -\frac{7}{60} - \frac{\beta_2}{15} \\ \alpha_2 + \frac{3\beta_2}{2} & \frac{\alpha_2+\beta_2}{2} \end{bmatrix} \widetilde{\Phi}(2x+2)$$

$$+ \begin{bmatrix} \frac{1}{2} & \frac{1}{30} - \frac{4\beta_2}{15} \\ -2\alpha_2 & 2\beta_2 \end{bmatrix} \widetilde{\Phi}(2x+1) + \begin{bmatrix} \frac{29}{20} + \frac{2\beta_2}{5} & 0 \\ 0 & 4 - \alpha_2 + \beta_2 \end{bmatrix} \widetilde{\Phi}(2x)$$

$$+ \begin{bmatrix} \frac{1}{2} & -\frac{1}{30} + \frac{4\beta_2}{15} \\ 2\alpha_2 & 2\beta_2 \end{bmatrix} \widetilde{\Phi}(2x-1) + \begin{bmatrix} -\frac{9}{40} - \frac{\beta_2}{5} & \frac{7}{60} + \frac{\beta_2}{15} \\ -\alpha_2 - \frac{3\beta_2}{2} & \frac{\alpha_2+\beta_2}{2} \end{bmatrix} \widetilde{\Phi}(2x-2).$$

Optimal stability

Optimalization

Choose the remaining degrees of freedom in such a way that $s_{\widetilde{\Phi}}$ is maximal.

We find $\alpha_2 = \frac{11}{7}$ and $\beta_2 = -\frac{6}{11}$.

The resulting wavelets are stable for $H^s(\mathbb{R})$ with $-0.828823 < s < 2.5$.

This result is comparable to the Hermite spline wavelets constructed in W. Dahmen, B. Han, R.-Q. Jia, and A. Kunoth. Biorthogonal multiwavelets on the interval: Cubic Hermite splines. Constr. Approx. 16:221-259, 2000.

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Cubic Hermite spline wavelets

