Analysis and Computation of the $\mathcal{H}_2$ norm of Delay Differential Algebraic Equations

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***Abstract*—We consider a class of dynamical systems described by linear Delay Differential Algebraic Equations (DDAEs) called strangeness free, which is broader than the class commonly studied within the control theory field. Two problems arise in the study of the $\mathcal{H}_2$ norm of DDAEs: the first one is that it may be infinite even if the system is stable or has no seemingly feedthrough term; and the second one is the computation. In this paper, both problems are addressed. We provide a necessary and sufficient condition for the finiteness of the $\mathcal{H}_2$ norm, which is based on controllability and observability properties of the delay-difference part of the system, and we present a formula for computing the $\mathcal{H}_2$ norm whenever it is finite, that is obtained by means of a neutral type system whose transfer matrix is equivalent to the transfer matrix of DDAEs.**

***Index Terms*—Delay differential algebraic systems, $\mathcal{H}_2$ norm, Lyapunov matrix.**

I. Introduction

We consider systems described by Delay Differential Algebraic Equations (DDAEs) of the form

$$\begin{aligned} \frac{d}{dt}(Ex(t)) &= A_0x(t) + A_1x(t-h) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where matrix $E \in \mathbb{R}^{n\times n}$ is singular with $\operatorname{rank} E = r < n$, $A_0, A_1 \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_u}$, $C \in \mathbb{R}^{n_y\times n}$, $x(t) \in \mathbb{R}^n$ represents the state, $u(t) \in \mathbb{R}^{n_u}$ is the input and $y(t) \in \mathbb{R}^{n_y}$ is the output. DDAEs, also called singular, implicit, or descriptor systems with delay, have shown to be suitable for modeling different classes of engineering systems [1], [2] and for studying a broad class of interconnected systems with delays (see Section 2 in [3]). The next example illustrates their generality by showing that a general class of neutral type systems can be written as a system of the form (1).

Example 1. *Consider the neutral type system*

$$\begin{aligned} \frac{d}{dt}\left(\hat{x}(t) + D\hat{x}(t-h)\right) &= \hat{A}_0\hat{x}(t) + A_1\hat{x}(t-h) \\ &\quad + \hat{B}_0u(t) + \hat{B}_1u(t-h), \\ y(t) = \hat{C}_0\hat{x}(t) + \hat{C}_1\hat{x}(t-h) &+ \hat{D}_0u(t) + \hat{D}_1u(t-h). \end{aligned} \tag{2}$$

Let us introduce

$$\begin{aligned} &\xi_1(t) = \hat{x}(t) + D\hat{x}(t-h), \\ &0 = -\xi_2(t) + u(t), \\ &0 = -\xi_3(t) + \hat{C}_0\hat{x}(t) + \hat{C}_1\hat{x}(t-h) + \hat{D}_0\xi_2(t) + \hat{D}_1\xi_2(t-h), \end{aligned}$$

then, by setting $x^T(t) = \begin{pmatrix}\xi_1^T(t) & \hat{x}^T(t) & \xi_2^T(t) & \xi_3^T(t)\end{pmatrix}$ we arrive at a system of the form (1) *with matrices*

$$E = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ A_0 = \begin{pmatrix} 0 & \hat{A}_0 & \hat{B}_0 & 0 \\ -I & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & \hat{C}_0 & \hat{D}_0 & -I \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & \hat{A}_1 & \hat{B}_1 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \hat{C}_1 & \hat{D}_1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 0 \\ I \\ 0 \end{pmatrix},\ C^T = \begin{pmatrix} 0 \\ 0 \\ 0 \\ I \end{pmatrix},$$

where I and 0 denote the identity and null matrices of appropriate dimensions, respectively. Neutral type systems with multiple commensurate delays in the state, in the input and in the output can also be written as a system of the form (1) *by an appropriate choice of the vector state.*

As a price for their generality, models of the form (1) may present some peculiar properties that make their study more difficult. For instance, existence and uniqueness of solutions are in general not guaranteed [4]. Solutions might be impulsive [5] and models of form (1) may even describe systems of advanced type. Stability is not necessarily related with the location of the spectrum [6], [7]. These difficulties have been avoided by considering a particular class of systems, called regular impulse-free, in the study of the $\mathcal{H}_\infty$ norm [8], [3], and the stability [9]. Inspired by the concepts for time-variant differential algebraic equations (see [10]), a broader class of DDAEs called *strangeness free*, which also avoids the above undesired properties, is introduced in [7]. A characteristic of this class of systems is that they can be reformulated into a stepped form, where the differential-difference, difference and algebraic parts explicitly appear.

In this contribution, we address the characterization of the $\mathcal{H}_2$ norm of strangeness-free DDAEs. The relevance of the $\mathcal{H}_2$ norm in the field of systems and control is well known, as it provides a measure of robustness with respect to noise or external disturbance, see e.g., [11]. The $\mathcal{H}_2$ norm has been object of study with applications to model order reduction and control design for time-delay systems in the last decade (see [12], [13], [14], and the references therein). However, despite its importance, the $\mathcal{H}_2$ norm of DDAEs has not received the adequate attention.

In contrast to time-delay systems of retarded and neutral type, the $\mathcal{H}_2$ norm of a DDAE may be infinite even if the system is stable and has seemingly no feedthrough term (this is illustrated by Example 1, where there is a feedthrough term in the original system that only implicitly appears in the output of the transformed system). Moreover, the application of a

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Lyapunov matrix based formula is not possible as it is for time-delay systems [15], [16], since there is no such concept for DDAEs in the literature.

In this paper, two distinctive contributions are presented. We provide a necessary and sufficient condition for the finiteness of the $\mathcal{H}_2$ norm given in terms of the controllability and observability matrices of the delay-difference part of strangeness-free DDAEs, and we present an explicit formula for its computation. In order to obtain the formula for computing the $\mathcal{H}_2$ norm, we introduce a regular time-delay system of neutral type that is equivalent to the DDAE in the frequency domain. This equivalence allows us to use the results in [15] and compute the $\mathcal{H}_2$ norm of DDAEs by using the so-called delay Lyapunov matrix for neutral type systems (the reader is referred to [17] for a comprehensive study of Lyapunov matrices for time-delay systems).

Unlike [18] and [19], where the focus is on establishing a connection between system (1) and neutral systems expressed in the original state variables whose matrices are low rank updates of the original matrices, we consider a change of coordinates in the state of the strangeness-free system with the algebraic part shifted in time, which enables us to obtain a neutral type system with no derivative in the input.

The paper is organized as follows. In Section II, we introduce some basic facts on DDAEs and strangeness-free systems. The reformulation via a change of coordinates of strangeness-free systems, which is the starting point of the subsequent results, is introduced in Section III. The main result of the paper is presented in Section IV, which is split in three subsections: in the first one, we provide a necessary and sufficient condition for finiteness of the $\mathcal{H}_2$ norm; in the second one, we introduce a neutral type system associated with a strangeness-free DDAEs; in the third subsection, we recall the Lyapunov matrix based formula from [15] and relate it with the computation of the $\mathcal{H}_2$ norm. We illustrate the theoretical results by some examples in Section V and end the contribution with some final remarks in Section VI.

Throughout the paper, the symbol $AC\left([-h,0],\mathbb{R}^n\right)$ denotes the space of absolutely continuous vector functions on $[-h,0]$. The rank and kernel of a matrix A are denoted by $\operatorname{rank} A$ and $\ker A$, respectively. A diagonal matrix with elements $a_1,\dots,a_n$ is denoted by $\operatorname{diag}(a_1,\dots,a_n)$. The notation I_n and $0_{n,m}$ represent an identity matrix of dimension n and a null matrix of n rows and m columns, respectively. The subscripts are omitted if no confusion is possible. The notation $j=\overline{1,p}$ means that j takes integer values $1,\dots,p$.

II. Preliminaries

We consider initial functions $\varphi\in AC\left([-h,0],\mathbb{R}^n\right)$ and call them consistent with (1) if the corresponding associated initial value problem (1) has at least one solution [7]. A function $x(t,\varphi)$ is called a (classical) solution of system (1) if it is absolutely continuous and satisfies (1) almost everywhere on $[0,\infty)$, and $x(\theta,\varphi)=\varphi(\theta)$ for $\theta\in[-h,0]$, where φ is a consistent initial function with (1). The class of systems called strangeness free is defined as follows:

Definition 1 ([7], [18]). *System* (1) *is strangeness free if there exists a non-singular matrix*

$$Q=\begin{pmatrix}Q_1\\Q_2\\Q_3\end{pmatrix}\in\mathbb{R}^{n\times n},$$

such that

$$QE=\begin{pmatrix}Q_1E\\0\\0\end{pmatrix},\ QA_0=\begin{pmatrix}Q_1A_0\\Q_2A_0\\0\end{pmatrix},$$
$$QA_1=\begin{pmatrix}Q_1A_1\\Q_2A_1\\Q_3A_1\end{pmatrix},\ QB=\begin{pmatrix}Q_1B\\Q_2B\\0\end{pmatrix},$$

and

$$\operatorname{rank}\begin{pmatrix}Q_1E\\Q_2A_0\\Q_3A_1\end{pmatrix}=n.$$

Here, $Q_1\in\mathbb{R}^{r\times n}$, $Q_2\in\mathbb{R}^{(\hat r-r)\times n}$, $Q_3\in\mathbb{R}^{(n-\hat r)\times n}$, with $r=\operatorname{rank}E$ and $\hat r=\operatorname{rank}\begin{pmatrix}E & A_0\end{pmatrix}$.

The previous definition is equivalent to Definition 2.5 introduced in [7] up to the condition on QB, which is introduced in [18] for systems with input. Notice that Q_3B is assumed zero in order to keep the causality principle.

Remark 1. *The class of strangeness-free systems is broader than the class of systems called regular impulse-free, which has been the starting point of a number of stability and robust stability results in, e.g., [5], [8], [9], [3], [20]. Indeed, if $\hat r=n$, i.e., $Q=\begin{pmatrix}Q_1\\Q_2\end{pmatrix}$, then the definition of strangeness-free systems is equivalent to the property that the pair $\begin{pmatrix}E & A_0\end{pmatrix}$ is regular and impulse-free [19].*

If system (1) is strangeness free, then a premultiplication of the system by a matrix Q splits the system into a set of differential-difference equations, a set of delay-difference equations and an algebraic part as follows:

$$\begin{aligned}\frac{d}{dt}(Q_1Ex(t)) &= Q_1A_0x(t)+Q_1A_1x(t-h)+Q_1Bu(t),\\ 0 &= Q_2A_0x(t)+Q_2A_1x(t-h)+Q_2Bu(t),\\ 0 &= Q_3A_1x(t-h),\\ y(t) &= Cx(t).\end{aligned}\tag{3}$$

For a continuously differentiable input function the space of consistent initial functions is then given by

$$X:=\{\varphi\in AC\left([-h,0],\mathbb{R}^n\right):Q_2A_0\varphi(0)+Q_2A_1\varphi(-h)+Q_2Bu(0)=0,Q_3A_1\varphi(\theta)=0,\forall\theta\in[-h,0]\}.$$

Moreover, for every initial function belonging to X, a forward solution of the strangeness-free system is uniquely defined [6], [7] (see also [4] for a detailed study of solution properties of DDAEs). We say that system (1) is exponentially stable if there exist constants $\gamma>0$ and $\sigma>0$ such that, for $u(t)=0$ and for all $\varphi\in X$, the solution $x(t,\varphi)$ satisfies

$$\|x(t,\varphi)\|\leqslant\gamma e^{-\sigma t}\sup_{\theta\in[-h,0]}\|\varphi(\theta)\|,\ t\geqslant 0.$$

The stability of strangeness-free systems is studied in [7], where the following theorem is introduced (see also [4]).

Theorem 1. *Strangeness-free system* (1) *with* $u \equiv 0$ *is exponentially stable if and only if the supremum among the real part of the spectrum of the system is strictly less than zero, i.e.,*

$$\sup\left\{\Re(s) : \det\left(sE - A_0 - A_1 e^{-sh}\right) = 0\right\} < 0.$$

The transfer matrices of system (1) and system (3) are the same and given by

$$G(s) := C\left(sE - A_0 - A_1 e^{-sh}\right)^{-1} B =$$
$$C\left(\begin{pmatrix} Q_1 E \\ 0 \\ 0 \end{pmatrix} s - \begin{pmatrix} Q_1 A_0 \\ Q_2 A_0 \\ 0 \end{pmatrix} - \begin{pmatrix} Q_1 A_1 \\ Q_2 A_1 \\ Q_3 A_1 \end{pmatrix} e^{-sh}\right)^{-1} \begin{pmatrix} Q_1 B \\ Q_2 B \\ 0 \end{pmatrix}.$$

Shifting the time in the algebraic part of system (3) modifies the set X (see, for instance, [6]), but it does not affect its spectrum nor its transfer matrix, and the system can be written as [7]

$$\begin{aligned} \frac{d}{dt}(Q_1 E x(t)) &= Q_1 A_0 x(t) + Q_1 A_1 x(t-h) + Q_1 B u(t), \\ 0 &= Q_2 A_0 x(t) + Q_2 A_1 x(t-h) + Q_2 B u(t), \\ 0 &= Q_3 A_1 x(t), \\ y(t) &= C x(t). \end{aligned} \tag{4}$$

It should be noted that, for (4), forward solutions can be constructed using the method of steps [4], [6].

III. Reformulation of the system

In this section, we assume that system (1) is strangeness-free and rewrite it by introducing a change of coordinates in the state. Notice that an infinite number of non-singular matrices Q exist that lead to the stepped form (4). However, the dimensions of the blocks Q_1, Q_2 and Q_3 remain independent of this choice. We set the matrices (see [7])

$$\begin{aligned} Q_1 &= U_1 \in \mathbb{R}^{r\times n},\ Q_2 = \widetilde{U}_1 U_2 \in \mathbb{R}^{a\times n}, \\ Q_3 &= \widetilde{U}_2 U_2 \in \mathbb{R}^{(n-r-a)\times n}, \end{aligned} \tag{5}$$

where $a = \operatorname{rank}(U_2 A_0)$, U_i and $\widetilde{U}_i$, $i = 1, 2$, are such that the matrices

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \text{ and } \begin{pmatrix} \widetilde{U}_1 \\ \widetilde{U}_2 \end{pmatrix}$$

are the left factor of the Singular Value Decomposition (SVD) of the matrices E and $U_2 A_0$, respectively, and accordingly partitioned. If $\widetilde{U}_2 U_2 B \neq 0$ the system is not strangeness free by definition.

For our purpose, we apply a transformation to the system coordinates that allows us to make matrix E diagonal and whose nonzero elements are the singular values. More precisely, we consider a matrix Q conformed by the matrices in (5) and matrix

$$V = \begin{pmatrix} V_1 & V_2 \end{pmatrix},\ V_1 \in \mathbb{R}^{n\times r},\ V_2 \in \mathbb{R}^{n\times n-r},$$

which is the right factor of the SVD of matrix E, such that

$$QEV = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix},\ QB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},\ CV = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$$
$$\begin{pmatrix} Q_1 A_0 \\ Q_2 A_0 \\ Q_3 A_1 \end{pmatrix} V = \begin{pmatrix} A_{01} & A_{02} \\ A_{03} & A_{04} \end{pmatrix},\ \begin{pmatrix} Q_1 A_1 \\ Q_2 A_1 \\ 0 \end{pmatrix} V = \begin{pmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{pmatrix},$$

where

$$A_{k1} = U_1 A_k V_1, \qquad A_{k2} = U_1 A_k V_2,\ k = 0, 1,$$
$$A_{03} = \begin{pmatrix} \widetilde{U}_1 U_2 A_0 V_1 \\ \widetilde{U}_2 U_2 A_1 V_1 \end{pmatrix},\ A_{04} = \begin{pmatrix} \widetilde{U}_1 U_2 A_0 V_2 \\ \widetilde{U}_2 U_2 A_1 V_2 \end{pmatrix},$$
$$A_{13} = \begin{pmatrix} \widetilde{U}_1 U_2 A_1 V_1 \\ 0 \end{pmatrix},\ A_{14} = \begin{pmatrix} \widetilde{U}_1 U_2 A_1 V_2 \\ 0 \end{pmatrix},$$
$$B_1 = U_1 B,\ B_2 = \begin{pmatrix} \widetilde{U}_1 U_2 B \\ 0 \end{pmatrix},\ C_1 = CV_1,\ C_2 = CV_2,$$

and $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$, with σ_j, $j = \overline{1, r}$, the singular values of matrix E. The dimensions of the matrices are as follows: $A_{k1} \in \mathbb{R}^{r\times r}$, $A_{k2} \in \mathbb{R}^{r\times(n-r)}$, $A_{k3} \in \mathbb{R}^{(n-r)\times r}$ and $A_{k4} \in \mathbb{R}^{(n-r)\times(n-r)}$, $k = 0, 1$.

Since the system is strangeness-free, $\operatorname{rank} A_{04} = n - r$, i.e. matrix A_{04} is non-singular. Hence, we can assume without loss of generality that $A_{04} = I$, and by setting

$$x(t) = \begin{pmatrix} V_1 & V_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

system (4) is written as

$$\begin{aligned} \frac{d}{dt}(\Sigma x_1(t)) &= A_{01} x_1(t) + A_{02} x_2(t) + \\ &\quad A_{11} x_1(t-h) + A_{12} x_2(t-h) + B_1 u(t) \\ 0 &= A_{03} x_1(t) + x_2(t) + \\ &\quad A_{13} x_1(t-h) + A_{14} x_2(t-h) + B_2 u(t) \\ y(t) &= C_1 x_1(t) + C_2 x_2(t). \end{aligned} \tag{6}$$

Notice that if A_{04} is not the identity matrix, one can bring the system to the form (6) by premultiplying the delay-difference part by A_{04}^{-1}. The transfer matrix and the spectrum of strangeness-free system (1) and (6) are the same and given by

$$G(s) = \begin{pmatrix} C_1 & C_2 \end{pmatrix} H^{-1}(s) \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

and

$$\Lambda := \{s \in \mathbb{C} \,:\, \det(H(s)) = 0\}, \tag{7}$$

respectively, where

$$\begin{aligned} H(s) &= \left(\begin{array}{c|c} H_1(s) & H_2(s) \\ \hline H_3(s) & H_4(s) \end{array}\right) \\ &:= \left(\begin{array}{c|c} s\Sigma - A_{01} - A_{11} e^{-sh} & -A_{02} - A_{12} e^{-sh} \\ \hline -A_{03} - A_{13} e^{-sh} & -I - A_{14} e^{-sh} \end{array}\right). \end{aligned}$$

In contrast to [7] or [19], here we do introduce a change of coordinates in the state. It allows us to explicitly express the conditions for the finiteness of the $\mathcal{H}_2$ norm in terms of the controllability and observability matrices of the delay-difference equation of system (6).

IV. Characterization of the $\mathcal{H}_2$ norm

The $\mathcal{H}_2$ norm of system (6) and equivalently of strangeness-free system (1), provided that the system is exponentially stable, is defined as

$$\|G\|_{\mathcal{H}_2} := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}\left(G^*(i\omega)G(i\omega)\right) d\omega. \tag{8}$$

Two problems arise in the study of the $\mathcal{H}_2$ norm of system (6). The first one is that, unlike systems with non-singular matrix E, it can be infinite even if the system is stable; the second one is its computation. Example 1 illustrates the former. Indeed, system (1) with corresponding matrices E, A_0, A_1, B and C can be written back as neutral type system (2), where one observes that there exists a feedthrough term from u to y.

In this section, we provide a necessary and sufficient condition for the finiteness of the $\mathcal{H}_2$ norm of system (6), and we show that there exists a neutral type system equivalent in the frequency domain to that system. Finally, the computation of the $\mathcal{H}_2$ norm of system (6) is done by using the so-called delay Lyapunov matrix of neutral type systems [15], [17].

A. Finiteness of the $\mathcal{H}_2$ norm

Assume that s is such that

$$\det(H_4(s)) \neq 0, \ \det F(s) \neq 0, \tag{9}$$

where

$$F(s) := H_1(s) - H_2(s)H_4^{-1}(s)H_3(s). \tag{10}$$

Applying the formula for the inversion of a two-by-two block matrix to $H(s)$ then yields

$$H(s)^{-1} = \begin{pmatrix} F^{-1}(s) & 0 \\ 0 & H_4^{-1}(s) \end{pmatrix} \times \begin{pmatrix} I & -H_2(s)H_4^{-1}(s) \\ -H_3(s)F^{-1}(s) & I + H_3(s)F^{-1}(s)H_2(s)H_4^{-1}(s) \end{pmatrix}.$$

Inspired by [3], this allows us to decompose transfer function G as

$$G(s) = G_b(s) - G_a(s) \tag{11}$$

where

$$G_a(s) := -C_2H_4^{-1}(s)B_2 = C_2(I + A_{14}e^{-sh})^{-1}B_2$$

and

$$G_b(s) := \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} F^{-1}(s) & 0 \\ 0 & H_4^{-1}(s) \end{pmatrix} \times \begin{pmatrix} I & -H_2(s)H_4^{-1}(s) \\ -H_3(s)F^{-1}(s) & H_3(s)F^{-1}(s)H_2(s)H_4^{-1}(s) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

We can now state the following lemma.

Lemma 1. *Let system* (6) *be exponentially stable. Then the $\mathcal{H}_2$ norm of transfer matrix G_b is finite.*

Proof. The finite singularities (including removable singularities) of G_b correspond to points where (9) is not satisfied. Due to the exponential stability assumption (which also implies that eigenvalues of A_{14} are inside the unit circle [9]), such points must be confined to the open left half plane.

Functions H_2, H_3 and H_4^{-1}, which appear in the definition of G_b, can be uniformly bounded in the closed right half plane. Furthermore, we can write

$$F^{-1}(s) = \frac{1}{s} \times \left(\Sigma - \frac{1}{s}\left(A_{01} + A_{11}e^{-sh} + H_2(s)H_4^{-1}(s)H_3(s)\right)\right)^{-1}.$$

From the invertibility of Σ we conclude that $\lim_{\omega\to\infty} ||G_b(i\omega)||_2 = 0$ and that the assertion holds. □

We introduce the criterion for the finiteness of the $\mathcal{H}_2$ norm of system (6) in the next theorem.

Theorem 2. *Let system* (6) *be exponentially stable. The $\mathcal{H}_2$ norm of system* (6) *is finite if and only if*

$$C_2A_{14}^jB_2 = 0, \ j = \overline{0, n-r-1}.$$

Proof. By the exponential stability of (6), decomposition (11) holds for all s in the closed right half plane. From this property and the finiteness of $\|G_b\|_{\mathcal{H}_2}$ we have

$$\|G_a\|_{\mathcal{H}_2} - \|G_b\|_{\mathcal{H}_2} \leqslant \|G\|_{\mathcal{H}_2} \leqslant \|G_a\|_{\mathcal{H}_2} + \|G_b\|_{\mathcal{H}_2}.$$

Combining these inequalities with the property that function $\omega \mapsto G_a(i\omega)$ is periodic, we conclude that $\|G\|_{\mathcal{H}_2}$ is finite *if and only if* $G_a(i\omega)$ is zero for all $\omega \in \mathbb{R}$. In what follows we prove that this is the case *if and only if* the condition of the theorem holds.

Let us introduce the function

$$\widetilde{G}_a(z) := C_2z(zI + A_{14})^{-1}B_2,$$

and notice that the equality $G_a(i\omega) = 0$ for all $\omega \in \mathbb{R}$ is equivalent to

$$\widetilde{G}_a(z) = 0, \text{ for all } z \in \mathbb{S}, \tag{12}$$

where $\mathbb{S} := \{z \in \mathbb{C} : |z| = 1\}$. Since $\widetilde{G}_a(z)$ is a meromorphic function, and the eigenvalues of the matrix A_{14} are inside the unit circle [9], we can take a convergent Taylor expansion within an open disk $\mathbb{S}_0$ centered at $z_0 \in \mathbb{S}$ and extended out to the nearest eigenvalue λ_j of the matrix A_{14}. As $\widetilde{G}_a^{(l)}(z_0) = 0$, $z_0 \in \mathbb{S}$, for $l = 0, 1, \ldots$, we have that $\widetilde{G}_a(z) = 0$ for all $z \in \mathbb{S}_0$. By successively applying the same argument for different values of z within convergent sets, the whole complex plane is covered, and (12) is equivalent to

$$\widetilde{G}_a(z) = 0, \text{ for all } z \in \mathbb{C}\backslash\Delta, \tag{13}$$

where Δ is the spectrum of A_{14} and the elements of Δ correspond to removable singularities. Then, by applying the inverse $\mathcal{Z}$-transform we get

$$\mathcal{Z}^{-1}\{z(zI + A_{14})^{-1}\}(k) = (-1)^kA_{14}^k,$$

where $k = 0, 1, \ldots$. It follows from the previous equality and the Cayley-Hamilton theorem that equation (13) holds if and only if

$$C_2A_{14}^jB_2 = 0, \ j = \overline{0, n-r-1}.$$

The equivalence between $G_a(i\omega) = 0$ for all $\omega \in \mathbb{R}$ and (13) concludes the proof. □

A sufficient condition for the finiteness of (8) is directly deduced from the above theorem for the particular cases in which $B_2 = 0$ and $C_2 = 0$. Condition $C_2 = CV_2 = 0$ is equivalent to $\operatorname{Ker} E \subseteq \operatorname{Ker} C$, which is the classical assumption made in early works for computing the $\mathcal{H}_2$ norm of delay-free differential algebraic equations (see [21]).

Remark 2. *Function G_a, which describes the behavior of G at high frequencies, can be interpreted as transfer function of delay-difference equation*

$$\xi(t) = -A_{14}\xi(t-h) + B_2 u(t), \;\; y(t) = C_2\xi(t).$$

Consistently with Theorem 2, its impulse response $g(t)$ is

$$g(t) = \sum_{j=0}^{\infty} (-1)^j C_2 A_{14}^j B_2 \; \delta(t - jh),$$

with δ the Dirac delta function. The coefficients of the shifted delta functions in the expression for g in turn be interpreted as Markov parameters of discrete-time system $\eta(k+1) = -A_{14}\eta(k) + B_2 u(k)$, $y(k) = C_2 x(k)$.

The condition of Theorem 2 can be written as

$$C_2\mathcal{C}_o(A_{14}, B_2) = 0 \text{ or } \mathcal{O}_b(A_{14}, C_2)B_2 = 0,$$

where

$$\mathcal{C}_o(A_{14}, B_2) = \begin{pmatrix} B_2 & A_{14}B_2 & \dots & A_{14}^{n-r-1}B_2 \end{pmatrix}$$

and

$$\mathcal{O}_b(A_{14}, C_2) = \begin{pmatrix} C_2^T & (C_2A_{14})^T & \dots & (C_2A_{14}^{n-r-1})^T \end{pmatrix}^T.$$

Notice that under the condition of Theorem 2 matrices $\mathcal{C}_o$ and $\mathcal{O}_b$ have no full rank, which means that in order to have a finite $\mathcal{H}_2$ norm, the intersection of the controllable and observable subspaces of the delay-difference part of system (6) must be null. Based on these observations, we introduce the next corollary of Theorem 2, which is helpful in the next subsection.

Corollary 1. *Let system* (6) *be exponentially stable, and $B_2 \neq 0$ and $C_2 \neq 0$. A matrix $T_c \in \mathbb{R}^{(n-r)\times(n-r)}$ such that*

$$T_c^{-1}A_{14}T_c = \begin{pmatrix} A_u & 0 \\ A_{cu} & A_c \end{pmatrix}, \; T_c^{-1}B_2 = \begin{pmatrix} 0 \\ B_c \end{pmatrix}, \quad C_2T_c = \begin{pmatrix} C_u & 0 \end{pmatrix} \tag{14}$$

exists if and only if the $\mathcal{H}_2$ norm of system (6) *is finite. Here, $B_c \in \mathbb{R}^{r_2 \times n_u}$ and $C_u \in \mathbb{R}^{n_y \times r_1}$, where $r_1 + r_2 = n - r$.*

Proof. Suppose first that the $\mathcal{H}_2$ norm is finite. As $C_2 \neq 0$, matrix $\mathcal{C}_o$ is rank-deficient, hence, there exists a matrix T_c that brings the matrices A_{14} and B_2 to the corresponding controllable canonical form with $C_2T_c = \begin{pmatrix} C_u & C_c \end{pmatrix}$. Since $B_2 \neq 0$, then $C_2B_2 = 0$ implies that $C_c = 0$.

Suppose now that there exists a matrix T_c that satisfies (14). We have

$$A_{14} = T_c \begin{pmatrix} A_u & 0 \\ A_{cu} & A_c \end{pmatrix} T_c^{-1}, \; B_2 = T_c \begin{pmatrix} 0 \\ B_c \end{pmatrix},$$

$C_2 = \begin{pmatrix} C_u & 0 \end{pmatrix} T_c^{-1}$, therefore $C_2A_{14}^jB_2 = 0$, $j = \overline{0, n-r-1}$. It follows from Theorem 2 that the $\mathcal{H}_2$ norm of the system is finite. □

If $B_2 = 0$ or $C_2 = 0$, we do not require any particular transformation as pointed out in the next subsection.

The finiteness of the $\mathcal{H}_2$ norm of a general class of neutral type systems can be deduced from Theorem 2. Consider the system from Example 1 presented in the introduction. Premultiplication of the matrices A_0, A_1 and B by the matrix

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & \hat{C}_0 & -\hat{D}_0 & -I \end{pmatrix}$$

leads to a system of the form (6) with

$$A_{14} = \begin{pmatrix} D & 0 & 0 \\ 0 & 0 & 0 \\ \hat{C}_0D - \hat{C}_1 & -\hat{D}_1 & 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 \\ -I \\ -\hat{D}_0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & I \end{pmatrix}.$$

The controllability matrix is given by the block matrix

$$\mathcal{C}_o(A_{14}, B_2) = \begin{pmatrix} 0 & 0 & 0 \\ -I & 0 & 0 \\ -\hat{D}_0 & \hat{D}_1 & 0 \end{pmatrix},$$

hence $C_2\mathcal{C}_o(A_{14}, B_2) = \begin{pmatrix} -\hat{D}_0 & \hat{D}_1 & 0 \end{pmatrix}$ and from Theorem 2 follows that the $\mathcal{H}_2$ norm of the neutral type system of Example 1 is finite if and only if $\hat{D}_0 = 0$ and $\hat{D}_1 = 0$, which is a well-known result (see, for instance, [11]).

B. The associated neutral type system

We next show the equivalence between the transfer matrix and spectrum of a neutral type system and system (6), which is key in order to compute the $\mathcal{H}_2$ norm in the next subsection.

Let us assume that system (6) has a finite $\mathcal{H}_2$ norm. It implies that if $B_2 \neq 0$ and $C_2 \neq 0$ a matrix $T = T_c$ and numbers r_1 and r_2 exist under the conditions of Corollary 1, which allows us to introduce the system

$$\frac{d}{dt}\left(\mathcal{E}z(t) + \mathcal{D}z(t-h)\right) = \mathcal{A}_0 z(t) + \mathcal{A}_1 z(t-h) + \mathcal{B}u(t), \quad y(t) = \mathcal{C}z(t), \tag{15}$$

where $z^T(t) = \begin{pmatrix} x_1^T(t) & \xi^T(t) \end{pmatrix}$, $\xi(t) = Tx_2(t)$,

$$\mathcal{E} = \begin{pmatrix} \Sigma & A_{02}TP_2 \\ P_1T^{-1}A_{03} & -I \end{pmatrix},$$

$$\mathcal{D} = \begin{pmatrix} 0 & A_{12}TP_2 \\ P_1T^{-1}A_{13} & P_1T^{-1}A_{14}T + T^{-1}A_{14}TP_2 \end{pmatrix},$$

$$\mathcal{A}_0 = \begin{pmatrix} A_{01} & A_{02}T \\ T^{-1}A_{03} & I \end{pmatrix}, \; \mathcal{A}_1 = \begin{pmatrix} A_{11} & A_{12}T \\ T^{-1}A_{13} & T^{-1}A_{14}T \end{pmatrix},$$

$$\mathcal{B}^T = \begin{pmatrix} B_1^T & B_2^TT^{-T} \end{pmatrix}, \; \mathcal{C} = \begin{pmatrix} C_1 & C_2T \end{pmatrix},$$

and

$$P_1 = -\begin{pmatrix} I_{r_1} & 0 \\ 0 & 0_{r_2,r_2} \end{pmatrix} \text{ and } P_2 = -\begin{pmatrix} 0_{r_1,r_1} & 0 \\ 0 & I_{r_2} \end{pmatrix}.$$

In case $B_2 = 0$ or $C_2 = 0$, we consider T as an arbitrary non-singular matrix. If $B_2 = 0$, we set $r_2 = 0$ and $P_2 = 0$, and if $C_2 = 0$, we set $r_1 = 0$ and $P_1 = 0$.

The regularity of system (15), i.e. the non-singularity of the matrix $\mathcal{E}$ follows from the fact that $\det(\mathcal{E}) \neq 0$. Indeed, since $P_2 P_1 = 0$,

$$\det \mathcal{E} = (-1)^r \det\left(\Sigma + A_{02} T P_2 P_1 T^{-1} A_{03}\right) = \\ = (-1)^r \det \Sigma \neq 0.$$

The transfer matrix and spectrum of system (15) are

$$\mathcal{G}(s) := \mathcal{C}\mathcal{H}^{-1}(s)\mathcal{B},$$

and

$$\widetilde{\Lambda} := \{ s \in \mathbb{C} : \det \mathcal{H}(s) = 0 \}, \tag{16}$$

respectively, where

$$\mathcal{H}(s) = s(\mathcal{E} + e^{-sh}\mathcal{D}) - \mathcal{A}_0 - \mathcal{A}_1 e^{-sh}.$$

We next show that the spectrum of system (15) is given in terms of the spectrum of system (6), and that their transfer matrices are the same, whenever the $\mathcal{H}_2$ norm is finite.

Lemma 2. *Let system* (6) *be exponentially stable and have a finite $\mathcal{H}_2$ norm. The following assertions hold:*

1) *The spectrum of system* (15) *satisfies* $\widetilde{\Lambda} = \Lambda \cup \{-1\}$, *where Λ and $\widetilde{\Lambda}$ are defined in* (7) *and* (16)*, respectively.*
2) *The equality $\mathcal{G}(s) = G(s)$, $s \in \mathbb{C} \setminus \widetilde{\Lambda}$, is satisfied.*

Proof. Item 1: Let us introduce the matrices

$$\widetilde{P}_1 = \begin{pmatrix} 0_{r,r} & 0 \\ 0 & P_1 \end{pmatrix}, \ \widetilde{P}_2 = \begin{pmatrix} 0_{r,r} & 0 \\ 0 & P_2 \end{pmatrix}, \ \widetilde{T} = \begin{pmatrix} I_r & 0 \\ 0 & T \end{pmatrix}.$$

Notice that the following holds:

$$\widetilde{P}_1 \widetilde{T}^{-1} H(s) \widetilde{T} = -\begin{pmatrix} 0 & 0 \\ P_1 T^{-1} A_{03} & P_1 \end{pmatrix} \\ - \begin{pmatrix} 0 & 0 \\ P_1 T^{-1} A_{13} & P_1 T^{-1} A_{14} T \end{pmatrix} e^{-sh},$$

$$\widetilde{T}^{-1} H(s) \widetilde{T} \widetilde{P}_2 = -\begin{pmatrix} 0 & A_{02} T P_2 \\ 0 & P_2 \end{pmatrix} - \begin{pmatrix} 0 & A_{12} T P_2 \\ 0 & T^{-1} A_{14} T P_2 \end{pmatrix} e^{-sh},$$

and by Corollary 1

$$\widetilde{P}_1 \widetilde{T}^{-1} H(s) \widetilde{T} \widetilde{P}_2 = 0.$$

From the previous equalities, we have that

$$s \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} - s\widetilde{P}_1 \widetilde{T}^{-1} H(s) \widetilde{T} - s\widetilde{T}^{-1} H(s) \widetilde{T} \widetilde{P}_2 = s\mathcal{E} + s\mathcal{D} e^{-sh}$$

and in turn that

$$\mathcal{H}(s) = \left(I - s\widetilde{P}_1\right) \widetilde{T}^{-1} H(s) \widetilde{T} \left(I - s\widetilde{P}_2\right). \tag{17}$$

Hence,

$$\det \mathcal{H}(s) = (s+1)^{r_1 + r_2} \det H(s),$$

and the assertion immediately follows.

Item 2: From equation (17) we get

$$\mathcal{G}(s) = \mathcal{C}(I - s\widetilde{P}_2)^{-1} \widetilde{T}^{-1} H^{-1}(s) \widetilde{T} (I - s\widetilde{P}_1)^{-1} \mathcal{B}, \ s \in \mathbb{C} \setminus \widetilde{\Lambda}.$$

Notice that

$$\mathcal{C}(I - s\widetilde{P}_2)^{-1} = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \widetilde{T},$$

and

$$(I - s\widetilde{P}_1)^{-1} \mathcal{B} = \widetilde{T}^{-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

Using these expressions, we arrive at

$$\mathcal{G}(s) = \begin{pmatrix} C_1 & C_2 \end{pmatrix} H^{-1}(s) \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = G(s), \ s \in \mathbb{C} \setminus \widetilde{\Lambda}.$$

□

The construction of neutral type system (15) is not trivial, therefore, the ideas behind it are explained step by step. We depart from a strangeness-free system written in the form (6) with finite $\mathcal{H}_2$ norm. For the sake of a better understanding, we illustrate each step by the system

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \frac{d}{dt} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \left(\begin{array}{c|cc} -10 & 1 & 3 \\ \hline 5 & 1 & 0 \\ 2 & 0 & 1 \end{array}\right) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \\ \left(\begin{array}{c|cc} 1 & 2 & 3 \\ \hline -2 & -2 & 0 \\ -3 & 3 & 1 \end{array}\right) \begin{pmatrix} x_1(t-h) \\ x_2(t-h) \end{pmatrix} + \left(\begin{array}{c} 0 \\ \hline 0 \\ 2 \end{array}\right) u(t), \\ y(t) = \left(\begin{array}{c|cc} 2 & 3 & 0 \end{array}\right) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \tag{18}$$

where it is observed that the block matrices corresponding to the delay-difference part are already in the form (14).

1. We set $x_2(t) = T\xi(t)$ and apply the operator $I - P_1 \frac{d}{dt}$ to the delay-difference part. A neutral type system of the form (15) is obtained with matrices

$$\mathcal{E} = \begin{pmatrix} \Sigma & 0 \\ P_1 T^{-1} A_{03} & P_1 \end{pmatrix}, \ \mathcal{D} = \begin{pmatrix} 0 & 0 \\ P_1 T^{-1} A_{13} & P_1 T^{-1} A_{14} T \end{pmatrix},$$

and the same $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{B}$ and $\mathcal{C}$. Notice that as

$$T^{-1} B_2 = \begin{pmatrix} 0 \\ B_c \end{pmatrix},$$

the input is not affected by the derivative term of the operator. Let us illustrate the step with system (18). Since the block matrices corresponding to the delay-difference part of system (18) are already in canonical form, we can take $T = I$, i.e., $x_2(t) = \xi(t)$. Then, applying the operator $I_2 + \operatorname{diag}\left(0, \frac{d}{dt}\right)$ to the part corresponding to $\xi(t)$, we obtain the neutral system

$$\frac{d}{dt}\left(\begin{pmatrix} 1 & 0 & 0 \\ -5 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} z(t-h) \right) = \\ \begin{pmatrix} -10 & 1 & 3 \\ 5 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} z(t) + \begin{pmatrix} 1 & 2 & 3 \\ -2 & -2 & 0 \\ -3 & 3 & 1 \end{pmatrix} z(t-h) + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} u(t), \\ y(t) = \begin{pmatrix} 2 & 3 & 0 \end{pmatrix} z(t), \tag{19}$$

where $z^T(t) = \begin{pmatrix} x_1(t) & \xi^T(t) \end{pmatrix}$. As pointed out, the input is not affected by the derivative.

2. We consider now the dual form of the neutral type system obtained in the previous step, i.e. a system with matrices

$$\left(\mathcal{E}^T, \mathcal{D}^T, \mathcal{A}_0^T, \mathcal{A}_1^T, \mathcal{C}^T, \mathcal{B}^T\right),$$

and apply the operator $I - P_2 \dfrac{d}{dt}$ to the part corresponding to $\xi(t)$. We arrive at a neutral type system of the form (15) with matrices

$$\mathcal{E} = \begin{pmatrix} \Sigma & A_{03}^T T^{-T} P_1 \\ P_2 T^T A_{02}^T & -I \end{pmatrix},$$
$$\mathcal{D} = \begin{pmatrix} 0 & A_{13}^T T^{-T} P_1 \\ P_2 T^T A_{12}^T & T^T A_{14}^T T^{-T} P_1 + P_2 T^T A_{14}^T T^{-T} \end{pmatrix},$$

and the same $\mathcal{A}_0^T$, $\mathcal{A}_1^T$, $\mathcal{C}^T$ and $\mathcal{B}^T$. As in the previous step, the input is not affected by the derivative term of the operator since $C_2 T = \begin{pmatrix} C_u & 0 \end{pmatrix}$. The dual form of system (19) is obtained by transposing the matrices and swapping the input and output matrices. It is given by

$$\begin{aligned} &\frac{d}{dt}\left(\begin{pmatrix} 1 & -5 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} z(t-h)\right) = \\ &\begin{pmatrix} -10 & 5 & 2 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} z(t) + \begin{pmatrix} 1 & -2 & -3 \\ 2 & -2 & 3 \\ 3 & 0 & 1 \end{pmatrix} z(t-h) + \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} u(t), \\ &y(t) = \begin{pmatrix} 0 & 0 & 2 \end{pmatrix} z(t). \end{aligned} \tag{20}$$

Applying the operator $I_2 + \operatorname{diag}\left(0, \frac{d}{dt}\right)$ to the part corresponding to $\xi(t)$ of (20) does not affect the input and we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\begin{pmatrix} 1 & -5 & 0 \\ 0 & -1 & 0 \\ -3 & 0 & -1 \end{pmatrix} z(t) + \begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ -3 & 0 & -1 \end{pmatrix} z(t-h)\right) = \\ &\begin{pmatrix} -10 & 5 & 2 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} z(t) + \begin{pmatrix} 1 & -2 & -3 \\ 2 & -2 & 3 \\ 3 & 0 & 1 \end{pmatrix} z(t-h) + \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} u(t), \\ &y(t) = \begin{pmatrix} 0 & 0 & 2 \end{pmatrix} z(t). \end{aligned} \tag{21}$$

3. Finally, taking the dual form of the system obtained in the previous step, i.e. transposing the matrices and swapping the input and output matrices, we arrive at the neutral type system (15).

C. Computation of the $\mathcal{H}_2$ norm

Analogously to delay free systems, the $\mathcal{H}_2$ norm of neutral type systems is computed by the so-called delay Lyapunov matrix in [15]. This result is recalled in the next theorem. For the basic properties of the delay Lyapunov matrix of neutral type system the reader is referred to [17, Chapter 6].

Theorem 3 ([15]). *Let system* (15) *be exponentially stable. Then, the $\mathcal{H}_2$ norm satisfies*

$$\|\mathcal{G}\|_{\mathcal{H}_2} = \sqrt{\operatorname{Tr}\left(\mathcal{B}^T U(0) \mathcal{B}\right)},$$

where matrix $U(t)$, $t \in [-h, h]$, is the delay Lyapunov matrix of system (15)*, associated with matrix $\mathcal{C}^T\mathcal{C}$.*

Based on Theorem 3 and Lemma 2, we arrive at the following result.

Theorem 4. *Let system* (6) *be exponentially stable. If the $\mathcal{H}_2$ norm of system* (6) *is finite then,*

$$\|G\|_{\mathcal{H}_2} = \sqrt{\operatorname{Tr}\left(\mathcal{B}^T U(0) \mathcal{B}\right)}, \tag{22}$$

where $U(t)$, $t \in [-h, h]$, is the delay Lyapunov matrix associated with matrix $\mathcal{C}^T\mathcal{C}$ of neutral system (15)*, constructed from system* (6)*.*

Proof. By Lemma 2, we have that system (15) is exponentially stable and $G(s) = \mathcal{G}(s)$, $s \in \mathbb{C} \setminus \widetilde{\Lambda}$, hence, from Theorem 3

$$\|G\|_{\mathcal{H}_2} = \|\mathcal{G}\|_{\mathcal{H}_2} = \sqrt{\operatorname{Tr}\left(\mathcal{B}^T U(0) \mathcal{B}\right)}.$$

□

V. Examples

We present two examples. In the first one, we further elaborate on Example 1 from the introduction. The second example is taken from [18] and illustrates the complete numerical procedure departing from system (1). The Lyapunov matrix is computed via the semianalytic method [17, Section 6.6].

Example 2. *We consider the neutral type system introduced in Example 1 with the following matrices proposed in [16]:*

$$D = \begin{pmatrix} -0.5 & -1 \\ 0 & -0.5 \end{pmatrix}, \hat{A}_0 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \hat{A}_1 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix},$$

$\hat{B}_0 = I$, $\hat{B}_1 = I$, $\hat{C}_0 = I$, $\hat{C}_1 = 0$. As pointed out in Subsection IV-A, in order to have a finite $\mathcal{H}_2$ norm $\hat{D}_0$ and $\hat{D}_1$ must be zero. With the previous matrices, we construct a system of the form (15)*, apply Theorem 4 and obtain*

$$\|G\|_{\mathcal{H}_2} = 2.5425.$$

Example 3 ([18]). *We consider system* (1) *with matrices*

$$E = \begin{pmatrix} 2 & 4 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; A_0 = \begin{pmatrix} 0 & 0 & 2 \\ -3 & 3 & 5 \\ -3 & 3 & 6 \end{pmatrix},$$
$$A_1 = \begin{pmatrix} 0 & -2 & -2 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \; C = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}^T.$$

The matrix E is singular with $\operatorname{rank} E = 1$. We consider

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \left(\begin{array}{ccc} -0.8944 & 0.4472 & 0 \\ \hline 0 & 0 & 1 \\ 0.4472 & 0.8944 & 0 \end{array}\right),$$
$$\begin{pmatrix} \widetilde{U}_1 \\ \widetilde{U}_2 \end{pmatrix} = \left(\begin{array}{cc} -0.7454 & -0.6667 \\ \hline -0.6667 & 0.7454 \end{array}\right),$$
$$V = \begin{pmatrix} V_1 & V_2 \end{pmatrix} = \left(\begin{array}{c|cc} -0.4472 & 0 & -0.8944 \\ -0.8944 & 0 & 0.4472 \\ 0 & 1 & 0 \end{array}\right).$$

Then, the transformation matrix Q is given by

$$Q = \begin{pmatrix} -0.8944 & 0.4472 & 0 \\ -0.2981 & -0.5963 & -0.7454 \\ 0.3333 & 0.6667 & -0.6667 \end{pmatrix},$$

and system (4) *takes the form*

$$\frac{d}{dt}\begin{pmatrix}-2.2361 & -4.4721 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}x(t) =$$

$$= \begin{pmatrix}-1.3416 & 1.3416 & 0.4472\\ 4.0249 & -4.0249 & -8.0498\\ 0.6668 & 0 & -0.6667\end{pmatrix}x(t)+$$

$$\begin{pmatrix}0.4472 & 2.6833 & 1.7889\\ -0.5963 & -1.3416 & 0.5963\\ 0 & 0 & 0\end{pmatrix}x(t-h)+\begin{pmatrix}-2.2361\\ 0\\ 0\end{pmatrix}u(t),$$

$$y(t) = \begin{pmatrix}2 & 5 & 1\end{pmatrix}x(t).$$

Applying the change of coordinates with the matrix V *first, and premultiplying the delay-difference part by the matrix*

$$\begin{pmatrix}-8.0498 & -5.4\\ -0.6667 & -0.5963\end{pmatrix}^{-1}$$

then, we arrive at a system of the form (6) *with* $\Sigma = 5$,

$$A_{01} = -0.6,\ A_{02} = \begin{pmatrix}0.4472 & 1.8\end{pmatrix},\ A_{03} = \begin{pmatrix}-2.2361\\ 3\end{pmatrix},$$

$$\begin{pmatrix}A_{11} & A_{12}\\ A_{13} & A_{14}\end{pmatrix} = \left(\begin{array}{c|cc}-2.6 & 1.7889 & 0.8\\ \hline -0.7288 & -0.2963 & 0.0331\\ 0.8148 & 0.3313 & -0.0370\end{array}\right),$$

$$\begin{pmatrix}B_1\\ B_2\end{pmatrix} = \left(\begin{array}{c}-2.2361\\ \hline 0\\ 0\end{array}\right),\ \begin{pmatrix}C_1\\ C_2^T\end{pmatrix}^T = \left(\begin{array}{c}-5.3666\\ \hline 1\\ 0.4472\end{array}\right).$$

Since $B_2 = 0$, *the system has a finite* $\mathcal{H}_2$ *norm and can be transformed into a neutral type system of the form* (15) *by setting* $T = I_2$, $P_2 = 0$ *and* $P_1 = -I_2$. *The computed matrices of system* (15) *are*

$$\mathcal{E} = \begin{pmatrix}5 & 0 & 0\\ 2.2361 & -1 & 0\\ -3 & 0 & -1\end{pmatrix}, \mathcal{A}_0 = \begin{pmatrix}-0.6 & 0.4472 & 1.8\\ -2.2361 & 1 & 0\\ 3 & 0 & 1\end{pmatrix}$$

$$\mathcal{D} = \begin{pmatrix}0 & 0 & 0\\ 0.7288 & 0.2963 & -0.0331\\ -0.8148 & -0.3313 & 0.0370\end{pmatrix},$$

$$\mathcal{A}_1 = \begin{pmatrix}-2.6 & 1.7889 & 0.8\\ -0.7288 & -0.2963 & 0.0331\\ 0.8148 & 0.3313 & -0.0370\end{pmatrix},$$

$\mathcal{B}^T = \begin{pmatrix}B_1^T & B_2^T\end{pmatrix}$ *and* $\mathcal{C} = \begin{pmatrix}C_1 & C_2\end{pmatrix}$. *Finally, from Theorem 4, we obtain*

$$\|G\|_{\mathcal{H}_2} \approx 1.3631.$$

VI. Conclusions

The $\mathcal{H}_2$ norm of strangeness-free DDAEs has been studied. We provide a necessary and sufficient condition for its finiteness and present a Lyapunov matrix based formula in order to compute it. The finiteness criterion depends on the controllability and observability matrices of the delay-difference part, and the computational method relies on the equivalence in the frequency domain of DDAEs and a differential-difference equation of neutral type.

The presented results may be considered the starting point of a number of possible applications in $\mathcal{H}_2$ controller design and model order reduction of DDAEs (see, [13], [14], and the references therein).

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