The Structure of the Minimal Padé Approximation Table

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Abstract

The problem of minimal Padé approximation was conceived by A. Bultheel and M. Van Barel in their study of the Euclidean algorithm in the matrix case. In the scalar case, the minimal Padé approximants are generically equal to the classical Padé approximants, but a minimal Padé approximant is not characterized by a numerator and a denominator degree, but by two other parameters. A table with respect to these parameters is considered here. We characterize the structure of this minimal Padé approximation table by establishing connections with the classical Padé table. A property of square block structure of the table is formulated.

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The problem of minimal Padé approximation was conceived by A. Bultheel and M. Van Barel in their study of the Euclidean algorithm in the matrix case. In the scalar case, the minimal Padé approximants are generically equal to the classical Padé approximants, but a minimal Padé approximant is not characterized by a numerator and a denominator degree, but by two other parameters. A table with respect to these parameters is considered here. We characterize the structure of this minimal Padé approximation table by establishing connections with the classical Padé table. A property of square block structure of the table is formulated.

Based on the problem of Minimal Partial Realization, A. Bultheel and M. Van Barel have introduced the minimal Padé approximation problem for a given power series:

$$F(z) = \sum_{i=0}^{\infty} F_i z^i. \tag{1}$$

The problem is characterized by two parameters ω and δ. They refer to the problem as the MPA(δ) problem. It is defined as follows:

Definition 1 (Problem MPA(δ)) *Given a power series $F(z)$, a nonnegative integer ω (i.e., $\omega \in \mathbf{Z}_+$) and an integer $\delta \in [-\omega, \omega]$ (i.e.,$\delta \in [-\omega, \omega] \cap \mathbf{Z}$), find nontrivial polynomials $M(z)$ and $N(z)$ such that*

$$F(z)M(z) - N(z) = O(z^{\omega+1}), \tag{2}$$

$$\nu \leq \mu - \delta, \tag{3}$$

$$\mu \quad \textit{as low as possible}, \tag{4}$$

$$\nu \quad \textit{as low as possible}, \tag{5}$$

where $\nu = \partial N(z)$ and $\mu = \partial M(z)$.

The solution $(N(z), M(z))$ of the problem is denoted by $[\omega, \delta]$.

It should be pointed out that condition (4) and condition (5) are not at the same level. We want μ to be as low as possible first and then among all possible solutions with minimal μ, we choose one for which ν is as low as possible.

From [1] we know that, the solution of the MPA(δ) problem always exists uniquely and its solution is a Padé approximant in the sense of the classical definition of Padé approximant. However, not every Padé approximant is a solution of the MPA(δ) problem. This means that, if we arrange the solutions of the MPA(δ) problem as a two dimensional table in terms of the parameters ω and δ, then this table is not just a copy of the Padé table.

To make this clear, we first recall the following definition for the classical Padé approximation (see [1]):

Definition 2 (Problem PA) *Given a power series $F(z)$ and nonnegative integers m and n, find nontrivial polynomials $M(z)$ and $N(z)$ such that*

$$F(z)M(z) - N(z) = O(z^{m+n+1}),$$

$$\nu \leq m, \quad \mu \leq n,$$

$$\mu \quad \textit{as low as possible},$$

where $\nu = \partial N(z)$ and $\mu = \partial M(z)$.

Let the solution of the latter problem be a pair of polynomials (N, M) which we denote as (m/n). Then the solution of this problem always exists and is unique up to a constant factor. Suppose $F_0 \neq 0$. Then it is well known that the Padé table has a square block structure. Such a block has the following properties:

(i) The elements at the triangular part of the block above and on its anti-diagonal are equal to the left-top corner element.

(ii) The elements in the block on an anti-diagonal below the main anti-diagonal are equal to z times the elements at the previous line.

(iii) Any element outside the block is different from any element inside the block.

In this report, we attempt to characterize the structure of the MPA(δ) table by establishing some connections between the two tables. To begin with, we give some lemmas.

Lemma 1 *The MPA(δ) problem is always solvable.*

Proof. Since $(N(z), M(z)) = (0, z^{\omega+k})(k \geq 1)$ always satisfies (2) and (3), the set

$$S(\omega, \delta) = \{(N, M) : F(z)M(z) - N(z) = O(z^{\omega+1}), \quad \partial N \leq \partial M - \delta, \quad M \neq 0\}$$

is not empty. From this set we can choose the one which satisfies conditions (4) and (5). □

Let (m/n) denote the Padé approximant in the classical sense, we shall refer to the following table as the extended Padé table :

Extended Padé Table

$$\begin{array}{cccc} (0,1) & (0,z) & (0,z^2) & \cdots \\ (0/0) & (0/1) & (0/2) & \cdots \\ (1/0) & (1/1) & (1/2) & \cdots \\ (2/0) & (2/1) & (2/2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

The first row of this table is an artificial one. Without it the table is the ordinary Padé table.

Lemma 2 *The solution of the MPA(δ) problem is equal to one of the elements in the extended PA table.*

Proof. Let $[\omega, \delta] = (N, M)$. We shall show at first that

$$\partial N + \partial M \leq \omega. \tag{6}$$

Suppose on the contrary, $\partial N + \partial M > \omega$. Then $N \neq 0$ ($\partial 0 = -\infty$). Since $(0, z^{\omega+1}) \in S(\omega, \delta)$, we find $\partial M \leq \omega$. Consider a $(\omega - \partial M, \partial M)$ type Padé approximant (N_1, M_1). Because $\partial N_1 \leq \omega - \partial M < \partial N$ and $\partial M_1 \leq \partial M$, it follows that (N_1, M_1) can not be a solution $[\omega, \delta]$ of the MPA(δ) problem due to the minimality of ∂N. Hence

$$\partial N_1 > \partial M_1 - \delta. \tag{7}$$

Since

$$\partial N_1 + \partial M \leq \omega - \partial M + \partial M = \omega, \tag{8}$$

and

$$\begin{aligned} \partial N + \partial M_1 &\leq (\partial M - \delta) + (\partial N_1 + \delta - 1) \\ &= \partial M + \partial N_1 - 1 \leq \omega - 1, \end{aligned} \tag{9}$$

we have by condition (2) that $N_1 M = N M_1$. Hence

$$\partial N_1 - \partial M_1 = \partial N - \partial M.$$

It follows from (7) and (2) that $-\delta < -\delta$, a contradiction. Then (6) is proved.

From (6) we know that $[\omega, \delta] = (N, M)$ is the Padé approximant $(\partial N/\partial M)$ in the Padé table if $N \neq 0$. If $N = 0$, $[\omega, \delta] = (N, M)$ is in the extended Padé table. From the uniqueness of PA, we know that $[\omega, \delta]$ is unique. □

Lemma 3 *Let $(N_i, M_i), i = 1, 2$ be two elements in the PA table and both of them have order at least ω. If*

$$\partial N_1 + \partial M_2 \leq \omega, \tag{10}$$

and

$$\partial M_1 + \partial N_2 \leq \omega, \tag{11}$$

then (N_1, M_1) and (N_2, M_2) are in a same block of the Padé table.

Proof. From

$$FM_i - N_i = O(z^{\omega+1}), \quad i = 1, 2$$

(10) and (11) we have that $M_1 N_2 = N_2 M_1$. Thus $N_1/M_1 = N_2/M_2$ and therefore the lemma is true. □

Lemma 4 *Assume $\omega \in \mathbf{Z}_+$ and $\delta \in [-\omega, \omega] \cap \mathbf{Z}$ such that $\omega + \delta$ is an odd integer. Then*

$$[\omega, \delta] = [\omega, \delta - 1], \quad \textit{or} \quad [\omega, \delta] = [\omega, \delta + 1].$$

Proof. Let

$$[\omega, \delta - 1] = (N_0, M_0), \quad [\omega, \delta] = (N, M), \quad [\omega, \delta + 1] = (N_1, M_1),$$

and

$$\nu = \frac{\omega - \delta - 1}{2}, \quad \mu = \frac{\omega + \delta + 1}{2}.$$

1. If $\partial N \leq \partial M - (\delta + 1)$, then by

$$S(\omega, \delta + 1) \subset S(\omega, \delta)$$

it follows that (N, M) satisfies the same conditions as (N_1, M_1). So $[\omega, \delta] = [\omega, \delta + 1]$.
2. Now we assume that

$$\partial N = \partial M - \delta.$$

From the proof of Lemma 2, it follows that $\partial N + \partial M \leq \omega$. Thus

$$2\partial M \leq \omega + \delta.$$

It follows from the oddness of $\omega + \delta$ that

$$\partial M \leq \mu - 1, \quad \partial N \leq \nu$$

and

$$\partial N + \partial M \leq \omega - 1.$$

This implies that there is a block, which has size bigger than 1, in the Padé table such that (N, M) is its left-top corner element and $(\nu/\mu) = (\nu + 1/\mu - 1) = (N, M)$.

Now we consider $[\omega, \delta - 1]$. We shall prove that $(N, M) \neq (N_0, M_0)$ leads to a contradiction. Since $\partial N = \partial M - \delta$, it follows that if $(N, M) \neq (N_0, M_0)$, we shall have

$$\partial N_0 = \partial M_0 - (\delta - 1).$$

Since $\partial N_0 + \partial M_0 \leq \omega$, we get

$$\partial M_0 \leq \omega - \partial N_0 = \omega - \partial M_0 + (\delta - 1)$$

$$2\partial M_0 \leq \omega + \delta - 1$$

$$\partial M_0 \leq \frac{\omega + \delta - 1}{2} = \mu - 1$$

$$\partial N_0 \leq \frac{\omega + \delta - 1}{2} - (\delta - 1) = \nu + 1$$

This implies $(N_0, M_0) = (\nu + 1/\mu - 1) = (N, M)$, which is a contradiction. □

From the discussion above, one has that for each elements $[\omega, \delta]$ with $\omega + \delta$ odd, there is always another element which coincides with $[\omega, \delta]$. So we only consider the elements $[\omega, \delta]$ with $\omega + \delta$ even. We arrange all $[\omega, \delta]$ with $\omega + \delta$ being even integers into a table and refer to it as the MPA(δ) table.

Extended MPA(δ) Table

$[-1, 1]$	$[0, 2]$	$[1, 3]$	$[2, 4]$	$[3, 5]$	$\cdots$
$[0, 0]$	$[1, 1]$	$[2, 2]$	$[3, 3]$	$[4, 4]$	$\cdots$
$[1, -1]$	$[2, 0]$	$[3, 1]$	$[4, 2]$	$[5, 3]$	$\cdots$
$[2, -2]$	$[3, -1]$	$[4, 0]$	$[5, 1]$	$[6, 2]$	$\cdots$
$[3, -3]$	$[4, -2]$	$[5, -1]$	$[6, 0]$	$[7, 1]$	$\cdots$
$[4, -4]$	$[5, -3]$	$[6, -2]$	$[7, -1]$	$[8, 0]$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Remarks :

1. The first line in the table is artificial, $[i, i+2] = (-1/i+1) = (0, z^{i+1})$. Without the first line, the table will be called MPA(δ) table.

2. The position (ω, δ) in the MPA(δ) table is the same as the position $(\nu, \mu) = (\frac{\omega-\delta}{2}, \frac{\omega+\delta}{2})$ in the PA table.

3. If the PA table is normal, then

$$[\omega, \delta] = (\tfrac{\omega-\delta}{2}/\tfrac{\omega+\delta}{2}), \tag{12}$$

or

$$(\nu/\mu) = [\nu+\mu, \mu-\nu]. \tag{13}$$

4. Along the line, which is parallel to the diagonal, δ is a constant. So we call such a line a δ–line. Similarly, a line orthogonal to the diagonal will be called an ω–line.

Now we can establish a relation between the two Padé tables. As we shall see, the MPA(δ) table has a square block structure, similar to the PA table. The block in the MPA(δ) table is denoted by (See Fig. 1) $S_\delta(\omega, \delta, l)$, where (ω, δ) stands for the position of the left–top element of the block and l stands for the size of the block.

$$\begin{array}{llcr} [\omega,\delta] & [\omega+1,\delta+1] & \cdots & [\omega+l-1,\delta+l-1] \\ [\omega+1,\delta-1] & [\omega+2,\delta] & \cdots & [\omega+l,\delta+l-2] \\ \cdots & \cdots & \cdots & \cdots \\ [\omega+l-1,\delta-l+1] & [\omega+l,\delta-l+2] & \cdots & [\omega+2(l-1),\delta] \end{array}$$

Fig. 1. $S_\delta(\omega, \delta, l)$

The block $S_\delta(\omega, \delta, l)$ is defined as follows:

Definition 3 (The block $S_\delta(\omega, \delta, l)$) *It contains l^2 elements from the MPA(δ) table viz. $[r, \epsilon]$ with $r+\epsilon$ even, $\omega-\delta \le r-\epsilon \le \omega-\delta+2(l-1)$, and $\omega+\delta \le r+\epsilon \le \omega+\delta+2(l-1)$ and which are such that*

(i) $[\omega, \delta] \neq [\omega, \delta+2]$ if $\delta < \omega$ and $[\omega, \delta] \neq [\omega, \delta-2]$ if $\delta > -\omega$.

(ii) If $[r, \epsilon] \in S_\delta(\omega, \delta, l)$ and $\epsilon \le \delta$ then $[r, \epsilon] = [r, \epsilon-2] = \cdots = [r, \epsilon']$ where $[r, \epsilon']$ is the leftmost element on ω-line r inside the block.
I.e. all the elements on and below the main δ-line of a block and on the same ω-line are all equal to the bordering element in south-west direction which is considered to be just inside the block. We call the latter the SW bordering elements.

(iii) *If* $[r,\epsilon] \in S_\delta(\omega,\delta,l)$ *and* $\epsilon > \delta$ *then* $[r,\epsilon] = [r,\epsilon+2] = \cdots = [r,\epsilon'']$ *where* $[r,\epsilon'']$ *is the leftmost element on* ω*-line* r *just outside (to the right or above) the block in the extended table.*
I.e. all the elements above the main δ*-line of a block and on the same* ω*-line are all equal to the bordering element in north-east direction which is considered to be just outside the block. We call the latter the NE bordering elements.*

Let $S(\nu,\mu,m)$ be the block in the PA table with (ν,μ) being the position of its left-top element. Let m be its size. Then we have

Theorem 1 *For each block* $S(\nu,\mu,m)$ *(including* 1×1 *blocks) in the PA table, there is one and only one block* $S_\delta(\nu+\mu,\mu-\nu,m)$ *in the (extended) MPA(*δ*) table, such that*

(i) $S(\nu,\mu,m)$ *and* $S_\delta(\nu+\mu,\mu-\nu,m)$ *occupy the same position in the (extended) PA table and (extended) MPA(*δ*) table;*

(ii) *The (NE or SW) bordering elements of* $S_\delta(\nu+\mu,\mu-\nu,m)$ *are exactly the same as the bordering elements of* $S(\nu,\mu,m)$ *in the same positions,*

Notes :

1. From the definition of the block $S_\delta(\nu+\mu,\mu-\nu,m)$, it follows that the elements inside the block are equal to some bordering element. From the theorem it follows that the bordering elements are equal to bordering elements of $S(\mu,\nu,m)$. Therefore, all the elements in the extended MPA(δ) table can easily be produced from the extended PA table.

2. Since the size and position of the blocks in the extended MPA(δ) table are the same as in the extended PA table, the blocks in the MPA(δ) table will form a complete tiling of the table just as do the blocks of the PA table.

Before we prove the theorem, we give an example. Let

$F(z) = 1 + z^4 + z^5 + z^9 + \cdots$.

Then the extended PA table for this F is as follows:

Table 1.

$(0,1)$	$(0,z)$	$(0,z^2)$	$(0,z^3)$	$(0,z^4)$	$(0,z^5)$	$(0,z^6)$	$(0,z^7)$	$\cdots$
$(0/0)$	$(0/0)$	$(0/0)$	$(0/0)$	$(0/4)$	$(0/5)$	$(0/5)$	$(0/5)$	$\cdots$
$(0/0)$	$(0/0)$	$(0/0)$	$z(0/0)$	$(1/4)$	$(0/5)$	$(0/5)$	$z(0/5)$	$\cdots$
$(0/0)$	$(0/0)$	$z(0/0)$	$z^2(0/0)$	$(2/4)$	$(0/5)$	$z(0/5)$	$z^2(0/5)$	$\cdots$
$(0/0)$	$z(0/0)$	$z^2(0/0)$	$z^3(0/0)$	$(3/4)$				$\cdots$
$(4/0)$	$(4/1)$	$(4/2)$	$(4/3)$	$(4/4)$				$\cdots$
$(5/0)$	$(5/0)$	$(5/0)$	$(5/0)$					$\cdots$
$(5/0)$	$(5/0)$	$(5/0)$	$z(5/0)$					$\cdots$
$(5/0)$	$(5/0)$	$z(5/0)$	$z^2(5/0)$					$\cdots$
$(5/0)$	$z(5/0)$	$z^2(5/0)$	$z^3(5/0)$					$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Where

$(0/0) = (1, 1)$, $(0/4) = (1, 1 - z^4)$,
$(0/5) = (1, 1 - z^4 - z^5)$, $(1/4) = (1 - z, 1 - z - z^4)$,
$(2/4) = (1 - z + z^2, 1 - z + z^2 - z^4)$, $(3/4) = (1 - z + z^2 - z^3, 1 - z + z^2 - z^3 - z^4)$,
$(4/0) = (1 + z^4, 1)$, $(4/1) = (1 - z + z^4, 1 - z)$,
$(4/2) = (1 - z + z^2 + z^4, 1 - z + z^2)$, $(4/3) = (1 - z + z^2 - z^3 + z^4, 1 - z + z^2 - z^3)$,
$(5/0) = (1 + z^4 + z^5, 1)$, $(4/4) = (1 - z + z^2 - z^3 + 2z^4, 1 - z + z^2 - z^3 + z^4)$.

The block structure of the extended MPA(δ) table is like given in table 2

Table 2.

$(-1/0)$	$(-1/1)$	$(-1/2)$	$(-1/3)$	$(-1/4)$	$(-1/5)$	$(-1/6)$	$(-1/7)$	$(-1/8)$	$\cdots$
$(0/0)$	$(-1/2)$	$(-1/3)$	$(-1/4)$	$(0/4)$	$(0/5)$	$(-1/7)$	$(-1/8)$	$(0/8)$	$\cdots$
$(0/0)$	$(0/0)$	$(-1/4)$	$(0/4)$	$(1/4)$	$(0/5)$	$(0/5)$	$(0/8)$	$(1/8)$	$\cdots$
$(0/0)$	$(0/0)$	$z(0/0)$	$(1/4)$	$(2/4)$	$(0/5)$	$z(0/5)$	$z^2(0/5)$	$(2/8)$	$\cdots$
$(0/0)$	$z(0/0)$	$z^2(0/0)$	$z^3(0/0)$	$(3/4)$	$(3/5)$	$(3/6)$	$(3/7)$	$(3/8)$	$\cdots$
$(4/0)$	$(4/1)$	$(4/2)$	$(4/3)$	$(4/4)$					$\cdots$
$(5/0)$	$(4/2)$	$(4/3)$	$(4/4)$	$(5/4)$					$\cdots$
$(5/0)$	$(5/0)$	$(4/4)$	$(5/4)$	$(6/4)$					$\cdots$
$(5/0)$	$(5/0)$	$z(5/0)$	$(6/4)$	$(7/4)$					$\cdots$
$(5/0)$	$z(5/0)$	$z^2(5/0)$	$z^3(5/0)$	$(8/4)$					$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

From Table 1 we see that there are three blocks $S(0,0,4)$, $S(0,5,3)$ and $S(5,0,4)$ in the PA table with block size bigger than 1. Corresponding to these blocks, there are blocks $S_\delta(0,0,4)$, $S_\delta(5,5,3)$ and $S_\delta(5,-5,4)$ in Table 2 with block size bigger than 1.

Now we prove Theorem 1.

Suppose there is a block $S(\nu,\mu,m)$ with (N_0,M_0) as its left-top corner element in the PA table. We now want to prove that for each ω satisfying $\mu+\nu \le \omega \le \mu+\nu+2(m-1)$ and $\omega+\delta$ even, the element $[\omega,\delta]$ is given by

$$[\omega,\delta] = \begin{cases} (\omega-\mu/\mu), & \omega-\mu-\nu+1 \le m, \\ (\nu+m-1/\omega-\nu-m+1), & \text{otherwise}, \end{cases} \tag{14}$$

for

$$\begin{cases} 2\mu-\omega \le \delta \le \mu-\nu, & \omega-\mu-\nu+1 \le m, \\ \omega-2m-2\nu+2 \le \delta \le \mu-\nu, & \text{otherwise}, \end{cases} \tag{15}$$

and

$$[\omega,\delta] = \begin{cases} (\nu-1/\omega-\nu+1), & \omega-\mu-\nu+1 \le m, \\ (\omega-\mu-m/\mu+m), & \text{otherwise}, \end{cases} \tag{16}$$

for

$$\begin{cases} \mu-\nu+1 \le \delta \le \omega-2\nu, & \omega-\mu-\nu+1 \le m, \\ \mu-\nu+1 \le \delta \le 2\mu+2m-\omega-2, & \text{otherwise}. \end{cases}$$

Since the main diagonal of the block $S_\delta(\nu+\mu,\mu-\nu,m)$ is the δ–line for $\delta=\mu-\nu$, $[\omega,\delta]$ in (14) is on or below the main diagonal, while in (16) it is above the main diagonal. It can be seen that the elements in the first line of (14) are the left bordering elements of $S(\nu,\mu,m)$, those in the second line of (14) are at the bottom boundary. The first line of (16) gives top boundary elements of $S(\nu,\mu,m)$. The second line gives the elements of the right boundary both outside the block.

a. Suppose $\delta \leq \mu - \nu$.

This means that this δ–line is not above the main diagonal of the block $S(\nu, \mu, m)$. Therefore $(\nu/\mu) = (N_0, M_0)$ satisfies the degree condition (3). Hence

$$(N_1, M_1) = \begin{cases} (\omega - \mu/\mu), & \omega - \mu - \nu + 1 \leq m, \\ (\nu + m - 1/\omega - \nu - m + 1), & \text{otherwise}, \end{cases} \tag{17}$$

satisfies (3), because it is equal to (ν/μ) up to a possible factor z^k in numerator and denominator. Order condition (2) is obviously valid. So we need only check that conditions (4) and (5) are satisfied. We note first that the elements in (17) have full degree in denominator. If (4) is not satisfied by these elements, then there is another one $(N, M) = [\omega, \delta]$ satisfying (2)–(5) and

$$\partial M < \begin{cases} \mu, & \omega - \mu - \nu + 1 \leq m, \\ \omega - \nu - m + 1, & \text{otherwise}, \end{cases}$$

i.e., (N, M) is outside the block $S(\nu, \mu, m)$. Using (15) and $\partial N \leq \partial M - \delta$, we get :

$$\begin{aligned} \partial N + \partial M_1 &\leq \partial M - \delta + \partial M_1 \\ &\leq \begin{cases} 2\mu - \delta - 1, & \omega - \mu - \nu + 1 \leq m, \\ 2(\omega - \nu - m) - \delta + 1, & \text{otherwise} \end{cases} \\ &\leq \begin{cases} 2\mu - 1 - 2\mu + \omega, & \omega - \mu - \nu + 1 \leq m, \\ 2(\omega - \nu - m) - \omega + 2m + 2\nu - 1, & \text{otherwise}. \end{cases} \end{aligned}$$

Hence $\partial N + \partial M_1 \leq \omega - 1$. On the other hand,

$$\begin{aligned} \partial N_1 + \partial M &\leq \begin{cases} \nu \\ \nu + \omega - (\nu + \mu - 1) - m \end{cases} + \begin{cases} \mu - 1 \\ \omega - \nu - m \end{cases} \\ &\leq \begin{cases} \nu + \mu - 1 \\ 2\omega - (\nu + \mu - 1) - 2m \end{cases} \\ &\leq \begin{cases} \nu + \mu - 1, & \omega - \mu - \nu + 1 \leq m, \\ \omega, & \text{otherwise}. \end{cases} \end{aligned}$$

Thus $\partial N_1 + \partial M \leq \omega$. It follows from Lemma 3 that (N, M) and (N_1, M_1) are in the same block. This contradict the definition of $S(\nu, \mu, m)$. Therefore (17) satisfies (4). Similarly, (5) is also satisfied.

b. Assume $\delta > \mu - \nu$.

We note first that the elements in (16) have full degree in denominator if $\omega \geq \mu + \nu$ and $\omega \leq \mu + \nu + 2m - 2$. Secondly, we mention that the elements in the block $S(\nu, \mu, m)$ can not be selected as $[\omega, \delta]$, since they don't satisfy the degree condition (3).

Now we show that elements in (16) satisfy (4). Suppose this is not true. Then let $[\omega, \delta] = (N, M)$ be such that

$$\partial M < \begin{cases} \omega - \nu + 1, & \omega - \mu - \nu + 1 \leq m, \\ \mu + m, & \text{otherwise.} \end{cases}$$

Then we can prove in a similar manner as in case (a) that the element (N, M) and the element

$$\begin{cases} (\nu/\omega - \nu), & \omega - \mu - \nu + 1 \leq m, \\ (\omega - \mu - m + 1/\mu + m - 1), & \text{otherwise.} \end{cases}$$

are in the same block. But the element above is in the block $S(\nu, \mu, m)$. Hence (N, M) belongs to this block, and this is a contradiction. Thus (4) is satisfied by (16). Similarly, (5) is also satisfied.

c. Since any two blocks in the Padé table are not overlapping and all the blocks fill up the complete table, it follows that condition (i) of definition 3 is satisfied, but also that l is as large as possible i.e., that the bordering elements of the PA block are also the bordering elements of the MPA(δ) block.

Now the proof of the theorem is completed. □

In the definition of the MPA(δ) table, we do not include the elements $[\omega, \delta]$, for $\omega + \delta$ odd. Therefore, there arises a natural question: How to reproduce these elements from the MPA(δ) table? From Lemma 4 we know that $[\omega, \delta]$ is the same as its left neighbour or right neighbour. Therefore, it is possible to obtain these elements from the MPA(δ) table. But what we want now is to determine the one which gives the correct choice. The next theorem answers this question. At first, we note that the elements $[\omega, \delta]$ for $\omega + \delta$ odd and $< 2\omega$, are surrounded by four elements in the MPA(δ) table in the following pattern.

$$\begin{array}{ccc} [\omega - 1, \delta] & & [\omega, \delta + 1] \\ & [\omega, \delta] & \\ [\omega, \delta - 1] & & [\omega + 1, \delta] \end{array}$$

Therefore the elements $[\omega, \delta]$ for $\omega + \delta$ odd, can be embedded into the MPA(δ) table at the positions that are the intersection points of δ–lines and ω–lines. It will turn out that all the "odd" entries (see Fig. 2) are equal to their north-eastern neighbours, except those odd entries that are precisely on the main diagonal of a block. From now on, we shall mean that in the MPA(δ) table the $[\omega, \delta]$ elements for $\omega + \delta$ odd, are thus embedded. In

Fig. 2 the Es stand for original ("even") elements, and Os are embedded ("odd") ones.

$$\begin{array}{ccccccc}
\mathrm{E} & & \mathrm{E} & & \mathrm{E} & \cdots & \mathrm{E} \\
& \mathrm{O} & & \mathrm{O} & & \cdots & \\
\mathrm{E} & & \mathrm{E} & & \mathrm{E} & \cdots & \mathrm{E} \\
& \mathrm{O} & & \mathrm{O} & & \cdots & \\
\mathrm{E} & & \mathrm{E} & & \mathrm{E} & \cdots & \mathrm{E} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathrm{E} & & \mathrm{E} & & \mathrm{E} & \cdots & \mathrm{E}
\end{array}$$

Fig. 2.

A reformulation of Lemma 4 is as follows:

Theorem 2 *Each element* $[\omega, \delta]$ *for* $\omega + \delta$ *odd and* $-\omega + 1 \leq \delta \leq \omega - 1$*, is equal to its north-eastern (NE) neighbour* $[\omega, \delta + 1]$*, except when* $[\omega, \delta - 1]$ *and* $[\omega, \delta + 1]$ *are in the same block and if* $[\omega, \delta]$ *is on the main diagonal of that block, then* $[\omega, \delta]$ *is equal to its south-western (SW) neighbour* $[\omega, \delta - 1]$*.*

Proof of Theorem 2. This is what was proved in Lemma 4. Indeed, $[\omega, \delta]$ is always equal to its NE partner, except when $[\omega, \delta] = N/M$ with $\partial N = \partial M - \delta$ and N/M the left-top element of a block. This means that $[\omega, \delta]$ is on the main diagonal of this block. □

Thus if we want to describe a block structure for the complete MPA(δ) table, i.e., with even and odd entries in it, we can do this as follows : Construct the table (with blocks) for the even entries as described in Theorem 1. According to Theorem 2, duplicate each of these E-entries as an O-entry in the SW direction (see Fig. 2), except for those O-entries that then lie on the main diagonal of a block, they should be equal to their SW neighbour. Note that in this way, new O-entries are generated, which should be denoted by $[\omega, -\omega - 1]$, i.e., an extra artificial column is added to the left which borders the (already extended) MPA(δ) table, just as we introduced the artificial top row before. The earlier $m \times m$ block of E-entries is then extended with m^2 O-entries which are (with the exception of the diagonal elements) duplications in SW direction. Thus in this complete table of E- and O-entries, each $m \times m$ block contains m^2 E-entries and m^2 O-entries.

References

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