Stability Analysis of Linear Time-Varying Time-Delay Systems by Non-quadratic Lyapunov Functions with Indefinite Derivatives*

Tianrui Zhao† Bin Zhou†‡ Wim Michiels§

Abstract

This paper presents a stability analysis of linear time-varying (LTV) time-delay systems by using non-quadratic Lyapunov functions and functionals. Two types of sufficient conditions are proposed for testing several different kinds of stability, such as asymptotic stability, exponential stability and uniformly exponential stability, for LTV systems without delay. Then, by constructing suitable non-quadratic Lyapunov functions (functionals) that are respectively the time-varying weighted L_1 and L_∞ norms of the state variables and by using properties of the uniformly asymptotically stable (UAS) function and the recently established improved Razumikhin and Krasovskii stability theorems, both delay-dependent and delay-independent conditions are proposed for testing uniformly exponential stability of a class of LTV time-delay systems. The time derivatives of the non-quadratic Lyapunov functions (functionals) along the solutions are allowed to be indefinite, namely, to take both negative and positive values. Numerical examples show that the non-quadratic Lyapunov functions (functionals) based methods are more efficient than the existing ones that are based on quadratic functions (functionals).

Keywords: Linear time-varying systems; Time-delay systems; Stability analysis; Non-quadratic Lyapunov functions; Indefinite time-derivatives.

1 Introduction

The stability analysis and stabilization of linear time-varying (LTV) systems and LTV time-delay systems have received much attention in the control community and a large number of results have been reported in the literature (see [2, 9, 15, 22, 30, 32] and the reference therein). Compared to the stability analysis of linear time-invariant (LTI) systems, the stability analysis of LTV systems is much more complicated for several reasons. Firstly, stability concepts for LTV systems are much more involved than those for LTI systems, namely, there are both uniform and non-uniform stability notions (except for the periodic case [22]). Secondly, there is no obvious relationship between the stability of an LTV system and eigenvalues of its system matrix [22, 24]. Finally, the stability analysis of LTV systems relies heavily on their state transition matrices, which, however, are generally not easy to compute. Therefore, the stability analysis problem for LTV systems is challenging.

Lyapunov's second method is one of the most important approaches for the stability analysis of control systems, especially for LTV systems [3]. By this method, the asymptotic stability is guaranteed if the time-derivative of the Lyapunov function (a positive definite function) along the solutions is negative definite [22]. When the time-derivative of the Lyapunov function is negative semi-definite, stability rather than asymptotic stability follows [1]. Lyapunov's second method in the stability theory has been extended to time-delay systems by two main approaches, the Krasovskii approach and the Razumikhin approach [4, 9, 15, 23]. The Krasovskii approach relies on a positive definite functional (referred to as Krasovskii functional) that decreases in the whole state space [9]. This method is commonly used in the literature,

*This work was supported in part by the National Natural Science Foundation of China under grant number 61773140.

†T. Zhao and B. Zhou are with the Center for Control Theory and Guidance Technology, Harbin Institute of Technology, Harbin, 150001, China.

‡Corresponding author. Email: binzhoulee@163.com, binzhou@hit.edu.cn.

§W. Michiels is with the Department of Computer Science, KU Leuven, Celestijnenlaan 200A, 3001 Leuven, Belgium. Email: wim.michiels@cs.kuleuven.be.

for example, Lyapunov-Krasovskii based analysis for establishing stability and performance guarantees for nonlinear systems in the presence of nonconstant delays was provided in [4]. The Razumikhin approach relies on a positive definite function (referred to as Razumikhin function) that decreases under the so-called Razumikhin condition [9]. As noticed in [29], the Razumikhin approach may be more easy to use in the stability analysis for systems with time-varying delays.

Lyapunov's second method has been recognized as a very powerful tool in stability theory. However, applying Lyapunov's second method on time-varying systems is quite challenging. Firstly, Lyapunov's second method generally requires that the time-derivative of the Lyapunov function is negative definite along the solutions of the system, which, turns out to be very restrictive. Therefore, there are many references devoted to allowing time-derivatives of Lyapunov functions to take both negative and positive values (see, for example, [17, 18, 20, 25]). Very recently, the second author established a systematic approach in [30, 31, 32] to allow indefinite time-derivatives of Lyapunov functions and functionals by using the concept of scalar stable functions. This approach relaxes several different restrictions in previous studies and has been successfully used in [19, 26] and [33] to study stability of stochastic time-varying systems. Secondly, in most cases the Lyapunov function is chosen as a quadratic function of the state variables. Although it leads to necessary and sufficient conditions for linear systems, it may be difficult to be constructed in the LTV case as one needs to solve some time-varying Lyapunov differential equation whose closed form solution is not available in general [3, 21, 22]. As we shall see, for time-varying systems it is beneficial to consider non-quadratic Lyapunov functions rather than the conventional quadratic ones.

Considerable efforts have been devoted to the stability analysis of dynamical systems by using non-quadratic functions in the literature (see, for instance [5, 6, 8, 28]). Particularly, non-quadratic Lyapunov functions have been frequently utilized to deal with the stability analysis of delayed neural networks. For example, the polystability of a class of neural networks with constant delay was discussed in [28] by constructing a non-quadratic Lyapunov functional, global exponential stability criteria were established in [6] for a class of delayed cellular neural networks by using suitable non-quadratic functions, and the stability analysis of recurrent neural networks with time delays by using non-quadratic Lyapunov functionals was presented in [5]. The non-quadratic Lyapunov function based approach was also used to study LTV time-delay systems. For example, the stability and stabilization of LTV time-delay impulsive systems with unbounded time-varying delays by using a non-quadratic Lyapunov-like function, namely the L_1 norm of the state variables, were studied in [12]. These results obtained by non-quadratic Lyapunov functions possess many adjustable parameters and can be easily checked in practice.

Motivated by the second author's previous study in [30, 31, 32, 33] and the non-quadratic Lyapunov functions in [5, 7, 8, 13], this paper will establish some Lyapunov stability criteria for LTV time-delay systems by using suitable non-quadratic time-varying Lyapunov functions (functionals) with indefinite time-derivatives. Two types of non-quadratic Lyapunov functions, namely, the time-varying weighted L_1 and L_∞ norms of state variables, are considered in this paper. With the help of scalar uniformly asymptotically stable function [30] and the improved Krasovskii and Razumikhin approaches [32], some sufficient stability conditions are proposed for LTV time-delay systems, by constructing some suitable non-quadratic Lyapunov functions (functionals). Numerical examples show that the proposed methods can lead to much less conservative stability conditions than the existing ones.

The remainder of this paper is organized as follows. Various stability concepts for LTV and LTV time-delay systems and some important notions associated with scalar stable functions are recalled in Section 2. In Section 3, several sufficient conditions are firstly presented for testing stability of LTV delay-free systems by using non-quadratic time-varying Lyapunov functions in Subsection 3.1. The uniformly exponential stability for LTV time-delay systems is then studied in Subsection 3.2 and Subsection 3.3. Two examples borrowed from the literature are given in Section 4 to show the effectiveness of the obtained results. The paper is finally concluded in Section 5.

Notation: Throughout this paper, if not specified, we will use fairly standard notation. Denote $J = [t^\#, \infty)$ with $t^\#$ being some finite constant, $J_\tau = [t^\# - \tau, \infty)$, and $\mathbf{R}^n_{0,+} = (0, \infty)^n$. We use $\mathbf{C}(J, \mathbf{\Omega})$ and $\mathbf{PC}(J, \mathbf{\Omega})$ to denote respectively the space of $\mathbf{\Omega}$-valued continuous functions and piecewise continuous functions defined on J. Given any constants a and b with $b > a$, we let $\mathbf{C}([a, b], \mathbf{\Omega})$ denote the set of all continuous $\mathbf{\Omega}$-valued functions defined on the interval $[a, b]$. The acronym WGDR refers to "with guaranteed decay rate" and $\operatorname{sgn}(f)$ denotes the sign of f. For two integers p and q with $p \leq q$, denote $\mathbf{I}[p, q] = \{p, p+1, \ldots, q\}$. Let $\mathbf{1}_n$ be the n-dimensional column vector $[1, \ldots, 1]^{\mathrm{T}}$. We use $\|\cdot\|$ to denote the usual Euclidean norm and $\|f\|_{[t-\tau,t]}$ to denote the supremum of $\|f(s)\|$ over the interval $s \in [t-\tau, t]$. For two column vectors $\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_n]^{\mathrm{T}}$ and $\beta = [\beta_1, \beta_2, \cdots, \alpha_n]^{\mathrm{T}}$, the symbol $\alpha \preceq \beta$ means $\alpha_i \leq \beta_i, i \in \mathbf{I}[1, n]$.

2 Problem Formulation and Preliminaries

In this section, we briefly introduce some basic definitions and results for general linear time-varying (LTV) systems. More details about the LTV systems theory can be found in [30, 32] and the references therein.

2.1 System Description and Problem Formulation

We consider the following LTV system

$$\dot{x}(t) = A(t)\, x(t), \ \forall t \in J, \tag{1}$$

where $x \in \mathbf{R}^n$, and $A(t) = [a_{ij}(t)] \in \mathbf{PC}(J, \mathbf{R}^{n\times n})$ is some known function. System (1) is a special case of the following LTV time-delay system

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, x(t - h(t)), \forall t \in J, \tag{2}$$

where $A(t), B(t) = [b_{ij}(t)] \in \mathbf{PC}(J, \mathbf{R}^{n\times n})$, $h(t) \in \mathbf{PC}(J, [0, \tau])$, with $\tau > 0$, denotes the delay in the system, and the initial condition is $x_{t_0} = \phi \in \mathbf{C}([t_0 - \tau, t_0], \mathbf{\Omega})$.

Very recently, the stability analysis of the LTV system (1) was studied in [30] by establishing a Lyapunov differential inequality based approach, which provides necessary and sufficient conditions for several different stability concepts. The main feature of the approach in [30] is that the time-derivative of some Lyapunov functions are allowed to be indefinite. The approach in [30] was extended in [32] to LTV time-delay systems (2), to nonlinear systems [31], and to stochastic systems [33]. In this paper, we continue to study the stability analysis of systems (1) and (2) by further exploring the idea of allowing indefinite time-derivatives of Lyapunov functions (functionals) as in [30, 32] and [33]. Differently from [30, 32] where the Lyapunov functions (functionals) are quadratic, in this paper, motivated by [5, 7, 8] and [13], we will establish a non-quadratic Lyapunov function (functional) based approach. As in [30, 32], we will provide conditions under which time-derivatives of Lyapunov functions (functionals) are allowed to take both negative and positive values. Two different kinds of non-quadratic Lyapunov functions will be utilized, which correspond to the time-varying weighted L_1-norm and L_∞-norm of the state $x(t)$. Our improved stability theorems will be applied on LTV time-delay systems to obtain much less conservative stability conditions than those obtained in [30] and [32].

For future use, we introduce the following lemma, which characterizes several stability notions for system (1) (their definitions can be found in [30]), in terms of the state transition matrix.

Lemma 1 *Denote the state transition matrix for system (1) as $\Phi_A(t, t_0)$, $\forall t \geq t_0 \in J$. Then system (1) is*

1. *stable if and only if, for any $t_0 \in J$, there exists a $k(t_0) > 0$ such that the inequality*

$$\|\Phi_A(t, t_0)\| \leq k(t_0), \ \forall t, t_0 \in J, t \geq t_0; \tag{3}$$

holds (Theorem 5.1 in [10]);

2. *uniformly stable if and only if $k(t_0)$ in (3) is independent of t_0 (Theorem 6.4 in [22]);*

3. *asymptotically stable (AS) if and only if (3) is satisfied and in addition the condition*

$$\lim_{t\to\infty} \|\Phi_A(t, t_0)\| = 0$$

holds (Theorem 5.2 in [10]);

4. *exponentially stable (ES) [WGDR $\varepsilon > 0$] if and only if, for any $t_0 \in J$, there exists a scalar $k(t_0) > 0$ such that condition*
$$\left\|\Phi_A\left(t, t_0\right)\right\| \leq k\left(t_0\right) \mathrm{e}^{-\varepsilon\left(t-t_0\right)}, \ \forall t, t_0 \in J, t \geq t_0 \tag{4}$$
is satisfied (Lemma 1 in [30]);

5. *uniformly exponentially stable (UES) [WGDR $\varepsilon > 0$] if and only if $k(t_0)$ in (4) is independent of t_0 (Theorem 6.7 in [22]);*

6. *uniformly asymptotically stable (UAS) if and only if it is UES (Theorem 6.13 in [22]).*

Finally, we recall the definition of UES for a time-delay system. The LTV time-delay system (2) is said to be UES if there exist two positive constants α and β such that
$$\|x(t)\| \leq \beta \mathrm{e}^{-\alpha\left(t-t_0\right)}\|x\|_{\left[t_0-\tau, t_0\right]}, t, t_0 \in J, t \geq t_0.$$

2.2 Scalar Stable Functions

In this subsection, we recall results from [30] regarding a so-called scalar stable function. Consider the following scalar LTV system
$$\dot{y}\left(t\right)=\mu\left(t\right) y\left(t\right), \ \forall t \in J, \tag{5}$$
where $y \in \mathbf{R}$, and $\mu(t) \in \mathbf{PC}(J, \mathbf{R})$.

Definition 1 *[30] The function $\mu(t) \in \mathbf{PC}(J, \mathbf{R})$ is said to be an AS function if system (5) is AS, is said to be an ES function if system (5) is ES, and is said to be a UAS function if system (5) is UAS.*

By noting that the unique solution $y(t)$ to the LTV system (5) is
$$y\left(t\right)=\exp \left(\int_{t_0}^t \mu\left(s\right) \mathrm{d} s\right) y\left(t_0\right), t, t_0 \in J, t \geq t_0,$$
the following result can be obtained.

Lemma 2 *[30] A function $\mu(t) \in \mathbf{PC}(J, \mathbf{R})$ is*

1. *AS if and only if*
$$\lim _{t \rightarrow \infty} \int_{t_0}^t \mu\left(s\right) \mathrm{d} s=-\infty, t_0 \in J.$$
2. *ES if and only if there exist a constant $\varepsilon > 0$ and a strictly positive function $\delta(t_0) \in [0, \infty)$ such that*
$$\int_{t_0}^t \mu\left(s\right) \mathrm{d} s \leq-\varepsilon\left(t-t_0\right)+\delta\left(t_0\right), t, t_0 \in J, t \geq t_0. \tag{6}$$
3. *UAS if and only if $\delta(t_0)$ in (6) is independent of t_0.*

We mention that, if we let $\varepsilon = 0$ in Items 2-3 of the above lemma, then the resulting conditions imply that the scalar LTV system (5) is stable and uniformly stable, respectively. We next recall the following notions introduced in [32].

Definition 2 *[32] Let $\mu(t)$ be a UAS. The set*
$$\Omega_\mu=\left\{T>0: \sup _{t \in J}\left\{\int_t^{t+T} \mu\left(s\right) \mathrm{d} s\right\}<0\right\},$$
is said to be the uniform convergence set (UCS) of $\mu(t)$. For any given $T \geq 0$, the overshoot $\varphi_\mu(T)$ of $\mu(t)$ is defined as
$$\varphi_\mu\left(T\right)=\sup _{t \in J}\left\{\max _{\theta \in[0, T]}\left\{\int_t^{t+\theta} \mu\left(s\right) \mathrm{d} s\right\}\right\}.$$

Properties of the UCS and the overshoot of $\mu(t)$ are given in the following lemma.

Lemma 3 *[33] Let $\mu(t) \in \mathbf{PC}(J, \mathbf{R})$ be a UAS function, Ω_μ be the corresponding UCS and $\varphi_\mu(T)$ be the overshoot for a given $T \geq 0$. Then: 1). There exists a number T_0 such that $[T_0, \infty) \subset \Omega_\mu$. 2). $\varphi_\mu(T)$ is a nondecreasing function of T. 3). If $T \in \Omega_\mu$, $\varphi_\mu(T)$ is independent of T.*

According to Item 3 of Lemma 3, hereafter we can use φ_μ to denote the overshoot $\varphi_\mu(T), \forall T \in \Omega_\mu$ of $\mu(t)$ for simplicity.

3 Stability Analysis of LTV Time-Delay Systems

3.1 Stability Analysis of LTV Systems

To study the stability of the LTV time-delay system, in this subsection, we will provide a simple criterion for testing the stability of system (1) by using non-quadratic time-varying Lyapunov functions. To this end, we denote

$$\lfloor A(t) \rfloor = \begin{bmatrix} a_{11}(t) & |a_{12}(t)| & \cdots & |a_{1n}(t)| \\ |a_{21}(t)| & a_{22}(t) & \cdots & |a_{2n}(t)| \\ \vdots & \vdots & \ddots & \vdots \\ |a_{n1}(t)| & |a_{n2}(t)| & \cdots & a_{nn}(t) \end{bmatrix}.$$

The following lemma presents various stability conditions, inferred from the use of non-quadratic time-varying Lyapunov functions of the form

$$V_1(t, x(t)) = \sum_{i=1}^{n} c_i(t) |x_i(t)|, \ \forall t \in J, \tag{7}$$

and

$$V_\infty(t, x(t)) = \max_{i \in \mathbf{I}[1,n]} \left\{ \frac{|x_i(t)|}{c_i(t)} \right\}, \ \forall t \in J, \tag{8}$$

where $c_i : J \to \mathbf{R}_{0,+}$ for $i \in \mathbf{I}[1, n]$ are differentiable functions. The functions $V_1(t, x(t))$ and $V_\infty(t, x(t))$ have been respectively motivated by several references, for example, [5, 7, 8] and [13], where c_i are constants. It is readily shown that

$$c_{\min}(t) \|x(t)\| \leq V_1(t, x(t)) \leq \sqrt{n} c_{\max}(t) \|x(t)\|, \tag{9}$$

$$\frac{1}{\sqrt{n} c_{\max}(t)} \|x(t)\| \leq V_\infty(t, x(t)) \leq \frac{1}{c_{\min}(t)} \|x(t)\|, \tag{10}$$

where $c_{\max}(t) = \max_{i \in \mathbf{I}[1,n]} \{c_i(t)\}$ and $c_{\min}(t) = \min_{i \in \mathbf{I}[1,n]} \{c_i(t)\}$. In addition, consider the following two inequalities:

$$\dot{c}(t) + \lfloor A^{\mathrm{T}}(t) \rfloor c(t) \preceq \mu(t) c(t), \tag{11}$$

$$-\dot{c}(t) + \lfloor A(t) \rfloor c(t) \preceq \mu(t) c(t). \tag{12}$$

where $c : J \to \mathbf{R}_{0,+}^n$ is a vector-valued differentiable function and $\mu(t) \in \mathbf{PC}(J, \mathbf{R})$.

Condition (11) or (12) imposes that the diagonal system associated with system (1), namely, $\dot{x}(t) = \mathrm{diag}\{a_{11}(t), a_{22}(t), \cdots, a_{nn}(t)\} x(t)$, is stable with sufficient margin to dominate the weighted coupling terms $a_{ij}(t), i \neq j$, in the sense that the time-derivative of certain Lyapunov function can still be bounded by a scalar stable function in the presence of these coupling terms. This motivates us to present the following result.

Lemma 4 *System (1) is*

1. *stable if $\mu(t)$ is stable and, either (11) is satisfied with*

$$c\left(t\right) \succeq \mathbf{1}_n, \forall t \in J, \tag{13}$$

or (12) is satisfied with

$$c\left(t\right) \preceq \mathbf{1}_n, \forall t \in J. \tag{14}$$

2. *US if $\mu(t)$ is US and there exists a $k_1 > 1$ such that either (11) or (12) is satisfied with*

$$\mathbf{1}_n \preceq c\left(t\right) \preceq k_1 \mathbf{1}_n, \ \forall t \in J. \tag{15}$$

3. *AS if $\mu(t)$ is AS and, either (11) is satisfied with (13) or (12) is satisfied with (14).*
4. *ES if $\mu(t)$ is ES and, either (11) is satisfied with (13) or (12) is satisfied with (14).*
5. *UES if $\mu(t)$ is UAS and, either (11) or (12) is satisfied with (15).*

Proof. In a first stage we derive bounds on the transition matrix induced by (11) and (12). In a second stage we prove the assertions of the lemma in a point-by-point fashion, by invoking Lemma 1.

We first consider the case where (11) is satisfied. We let $c_i(t)$ denote the i-th component of $c(t)$ and consider the corresponding Lyapunov function (7). Define a two-variable function $\sigma(z_1, z_2) : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ as follows [13]:

$$\sigma(z_1, z_2) = \begin{cases} \mathrm{sgn}(z_1), & z_1 \neq 0, \\ \mathrm{sgn}(z_2), & z_1 = 0, z_2 \neq 0, \\ 0, & z_1 = 0, z_2 = 0. \end{cases}$$

It follows that $\left|\sigma\left(z_1, z_2\right)\right| \leq 1$, $z_1 \sigma\left(z_1, z_2\right) = \left|z_1\right|$ and $\mathbf{D}^+ \left|z_1\right| = \dot{z}_1 \sigma\left(z_1, \dot{z}_1\right)$ where z_1 is some differentiable function [13]. Similarly to the development in [7] and [13], the Dini derivative of $V_1(t, x(t))$ along the trajectories of system (1) can be evaluated as

$$\begin{aligned}
\mathbf{D}^+ V_1(t, x\left(t\right)) &= \sum_{i=1}^{n} c_i\left(t\right) \dot{x}_i\left(t\right) \sigma(x_i\left(t\right), \dot{x}_i\left(t\right)) + \sum_{i=1}^{n} \dot{c}_i\left(t\right) \left|x_i\left(t\right)\right| \\
&\leq \sum_{i=1}^{n} \left(\dot{c}_i\left(t\right) + c_i\left(t\right) a_{ii}(t)\right) \left|x_i(t)\right| + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \left|c_i\left(t\right) a_{ij}(t)\right| \left|x_j(t)\right| \\
&\leq \sum_{i=1}^{n} \left(\mu(t) c_i(t) - \sum_{j=1, j\neq i}^{n} \left|c_j\left(t\right) a_{ji}(t)\right|\right) \left|x_i(t)\right| + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \left|c_i\left(t\right) a_{ij}(t)\right| \left|x_j(t)\right| \\
&= \mu(t) V_1(t, x\left(t\right)),
\end{aligned} \tag{16}$$

where we have used (11). By the Gronwall inequality [16], we get

$$V_1(t, x\left(t\right)) \leq V_1(t_0, x\left(t_0\right)) \exp\left(\int_{t_0}^{t} \mu(s) \mathrm{d}s\right), \ \forall t \geq t_0 \in J. \tag{17}$$

Hence it follows from (9) and (17) that

$$\left\|x\left(t\right)\right\| \leq \sqrt{n} \frac{c_{\max}\left(t_0\right)}{c_{\min}\left(t\right)} \exp\left(\int_{t_0}^{t} \mu(s) \mathrm{d}s\right) \left\|x\left(t_0\right)\right\|.$$

By the arbitrariness of t_0 and $x(t_0)$, we obtain from the above inequality that (see, for example, [7])

$$\left\|\Phi_A\left(t, t_0\right)\right\| \leq n\sqrt{n} \frac{c_{\max}\left(t_0\right)}{c_{\min}\left(t\right)} \exp\left(\int_{t_0}^{t} \mu(s) \mathrm{d}s\right). \tag{18}$$

We next consider the case that (12) is satisfied and take (8) as the Lyapunov function. For any $t \in J$, there exists an l such that $V_\infty(t, x(t)) = \frac{|x_l(t)|}{c_l(t)}$ [13]. Then, similarly to the development in [13], we have

$$\mathbf{D}^+ V_\infty(t, x\left(t\right)) = -\dot{c}_l\left(t\right) \frac{\left|x_l\left(t\right)\right|}{c_l^2\left(t\right)} + \frac{\sigma(x_l\left(t\right), \dot{x}_l\left(t\right))}{c_l\left(t\right)} \dot{x}_l\left(t\right)$$

$$\leq \left(-\frac{\dot{c}_l(t)}{c_l^2(t)} + \frac{a_{ll}(t)}{c_l(t)}\right)|x_l(t)| + \frac{1}{c_l(t)} \sum_{j=1, j\neq l}^{n} c_j(t)\,|a_{lj}(t)|\,\frac{|x_j(t)|}{c_j(t)}$$

$$\leq \left(-\frac{\dot{c}_l(t)}{c_l^2(t)} + \frac{a_{ll}(t)}{c_l(t)}\right)|x_l(t)| + \frac{1}{c_l(t)} \sum_{j=1, j\neq l}^{n} c_j(t)\,|a_{lj}(t)|\,\frac{|x_l(t)|}{c_l(t)}$$

$$\leq \mu(t) V_\infty(t, x(t)),$$

from which and the Gronwall inequality [16] it follows that

$$V_\infty(t, x(t)) \leq V_\infty(t_0, x(t_0)) \exp\left(\int_{t_0}^{t} \mu(s)\mathrm{d}s\right), \ \forall t \geq t_0 \in J. \tag{19}$$

Hence it follows from (10) and (19) that

$$\|x(t)\| \leq \sqrt{n}\frac{1/c_{\min}(t_0)}{1/c_{\max}(t)} \exp\left(\int_{t_0}^{t} \mu(s)\mathrm{d}s\right) \|x(t_0)\|.$$

By the arbitrariness of t_0 and $x(t_0)$, we have

$$\|\Phi_A(t, t_0)\| \leq n\sqrt{n}\frac{c_{\max}(t)}{c_{\min}(t_0)} \exp\left(\int_{t_0}^{t} \mu(s)\mathrm{d}s\right). \tag{20}$$

Proof of Item 1. Since the stable function $\mu(t)$ satisfies $\exp(\int_{t_0}^{t} \mu(s)\mathrm{d}s) \leq d(t_0) < \infty$, by (13) and (18), we know that $c_{\min}(t) \geq 1 > 0$ and

$$\|\Phi_A(t, t_0)\| \leq n\sqrt{n} c_{\max}(t_0)\, d(t_0),$$

which implies by Lemma 1 that system (1) is stable. Moreover, if $c_{\max}(t) \leq 1$ in (14) and $\exp(\int_{t_0}^{t} \mu(s)\mathrm{d}s) \leq d(t_0) < \infty$ are satisfied, it follows from (20) that

$$\|\Phi_A(t, t_0)\| \leq n\sqrt{n}\frac{1}{c_{\min}(t_0)} d(t_0),$$

which implies that system (1) is stable.

Proof of Item 2. Compared with Item 1, we further have either $c_{\max}(t_0) \leq k_1$ or $1 \leq c(t) \leq k_1$, and $d(t_0)$ is independent of t_0. Thus we have $\|\Phi_A(t, t_0)\| \leq n\sqrt{n} k_1 d$, which implies that system (1) is US.

Proof of Item 3. By using Item 1 and $\lim_{t\to\infty} \int_{t_0}^{t} \mu(s)\mathrm{d}s = -\infty$, we have $\lim_{t\to\infty} \|\Phi_A(t, t_0)\| = 0$, which implies that system (1) is AS.

Proof of Item 4. This is similar to the proof of Item 1. By assumption we know that $\int_{t_0}^{t} \mu(s)\mathrm{d}s \leq -\alpha(t - t_0) + \beta(t_0)$. Hence we have either

$$\|\Phi_A(t, t_0)\| \leq n\sqrt{n} c_{\max}(t_0)\, \mathrm{e}^{\beta(t_0)} \mathrm{e}^{-\alpha(t-t_0)}$$

or

$$\|\Phi_A(t, t_0)\| \leq n\sqrt{n}\frac{1}{c_{\min}(t_0)} \mathrm{e}^{\beta(t_0)} \mathrm{e}^{-\alpha(t-t_0)},$$

which implies that system (1) is ES.

Proof of Item 5. By combining the proofs for Items 2 and 4 together, we have

$$\|\Phi_A(t, t_0)\| \leq n\sqrt{n} k_1 \mathrm{e}^{\beta} \mathrm{e}^{-\alpha(t-t_0)},$$

which implies that system (1) is UES. The proof is finished. ■

It is worth of noting that $\mu(t)$ is not required to be negative for all time. As a result, the non-quadratic Lyapunov functions $V(t, x(t))$ defined in (7) and (8) may be increasing during the interval within which $\mu(t)$ is positive [17, 20, 30]. As a didactic example, system (1) with

$$A(t) = \begin{bmatrix} -1 + 2\cos(t) & p \\ p & -3 \end{bmatrix}$$

and parameter $p \in [0, 1)$ is UES by Lemma 4, since both (11) and (12) are satisfied for $\mu(t) = -1 + p + 2\cos t$ and $c(t) = \mathbf{1}_2, \ \forall t \in J$.

3.2 The Krasovskii Approach to LTV Time-Delay Systems

In this and the next subsections, by constructing non-quadratic Lyapunov functionals and using our recently established improved time-varying Krasovskii and Razumikhin stability theorems [32], we investigate the uniformly exponential stability of the LTV time-delay system (2). For sake of conciseness, we only consider uniformly exponential stability, while the other stability notions can be studied in a similar way.

We first consider the following LTV system with a constant state delay

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,x(t-h)\,,\ t \in J, \tag{21}$$

which is a special case of system (2), where $A(t), B(t) \in \mathbf{PC}(J, \mathbf{R}^{n\times n})$, $B(t)$ is bounded, and $h > 0$ is a known constant. This system has been studied in [32] by using quadratic Lyapunov functionals. We assume that system (21) in the absence of the delayed term $B(t)x(t-h)$ is UES and conditions (11) and (15) in Lemma 4 are satisfied for some UAS function $\mu(t) \in \mathbf{PC}(J, \mathbf{R})$.

Theorem 1 *Let conditions (11) and (15) in Lemma 4 be satisfied, where $A(t)$ is defined in system (21) and $\mu(t)$ is UAS. Assume that there exist a constant $\mu_0 = \inf_{t\in J}\mu(t)$ and a constant $\rho_1 \in (0,1)$ such that*

$$\eta_1 \triangleq \frac{\rho_1 - 1}{h} - \mu_0 \geq 0,$$

and there exists a UAS function $\zeta_1(t)$ such that

$$\mu(t) + \eta_1 + \frac{1}{\rho_1}\sum_{i=1}^{n}\left|\frac{c_i(t+h)\,b_{ij}(t+h)}{c_j(t)}\right| \leq \zeta_1(t)\,,\ \forall j \in \mathbf{I}[1,n]\,,\ \forall t \in J, \tag{22}$$

is satisfied. Then system (21) is UES.

Proof. We choose the following non-quadratic Lyapunov-Krasovskii functional candidate

$$V_1(t, x_t) = \sum_{i=1}^{n} c_i(t)\,|x_i(t)| + \int_{t-h}^{t} f_0(t,\theta)\sum_{i=1}^{n}\sum_{j=1}^{n}|c_i(\theta+h)\,b_{ij}(\theta+h)x_j(\theta)|\,\mathrm{d}\theta, \tag{23}$$

where $x_t = x(t+s), s\in[-h,0]$, $f_0(t,\theta) = \frac{1}{\rho_1} + \frac{\theta-t}{h}(\frac{1}{\rho_1}-1)$ and $c_i(t)$ satisfies conditions (15) in Lemma 4. Note that $f_0(t,\theta)$ is non-negative as long as $\theta \in [t-h, t]$. The Dini derivative of $V_1(t,x_t)$ can be evaluated as

$$\begin{aligned}
\mathbf{D}^+V_1(t,x_t) =& \sum_{i=1}^{n} c_i(t)\,\dot{x}_i(t)\,\sigma(x_i(t)\,, \dot{x}_i(t)) + \sum_{i=1}^{n}\dot{c}_i(t)\,|x_i(t)| \\
&+ \frac{1}{\rho_1}\sum_{i=1}^{n}\sum_{j=1}^{n}|c_i(t+h)\,b_{ij}(t+h)x_j(t)| - \sum_{i=1}^{n}\sum_{j=1}^{n}|c_i(t)\,b_{ij}(t)x_j(t-h)| + f_1(t) \\
=& \sum_{i=1}^{n} c_i(t)\,a_{ii}(t)x_i(t)\sigma(x_i(t)\,,\dot{x}_i(t)) + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} c_i(t)\,a_{ij}(t)x_j(t)\sigma(x_i(t)\,,\dot{x}_i(t)) \\
&+ \sum_{i=1}^{n}\dot{c}_i(t)\,|x_i(t)| + \sum_{i=1}^{n} c_i(t)\sum_{j=1}^{n} b_{ij}(t)x_j(t-h)\sigma(x_i(t)\,,\dot{x}_i(t)) \\
&+ \frac{1}{\rho_1}\sum_{i=1}^{n}\sum_{j=1}^{n}|c_i(t+h)\,b_{ij}(t+h)x_j(t)| - \sum_{i=1}^{n}\sum_{j=1}^{n}|c_i(t)\,b_{ij}(t)x_j(t-h)| + f_1(t) \\
\leq& (\mu(t)+\eta_1)\sum_{i=1}^{n} c_i(t)\,|x_i(t)| + \frac{1}{\rho_1}\sum_{j=1}^{n} c_j(t)\,|x_j(t)|\,g_j(t) + f_1(t) \\
=& \sum_{i=1}^{n} c_i(t)\,|x_i(t)|\left(\mu(t)+\eta_1+\frac{1}{\rho_1}g_i(t)\right) + f_1(t)
\end{aligned}$$

$$\leq \zeta_1(t)\sum_{i=1}^{n} c_i(t)|x_i(t)| + f_1(t)$$
$$= \zeta_1(t)V_1(t,x_t) - \int_{t-h}^{t} q_1(t,\theta)\sum_{i=1}^{n}\sum_{j=1}^{n}|c_i(\theta+h)b_{ij}(\theta+h)x_j(\theta)|\,\mathrm{d}\theta,$$

where we have used (11) and (22), and denoted

$$f_1(t) = \int_{t-h}^{t}\left(-\frac{1}{h}\left(\frac{1}{\rho_1}-1\right)\right)\sum_{i=1}^{n}\sum_{j=1}^{n}|c_i(\theta+h)b_{ij}(\theta+h)x_j(\theta)|\,\mathrm{d}\theta,$$
$$g_j(t) = \sum_{i=1}^{n}\left|\frac{c_i(t+h)b_{ij}(t+h)}{c_j(t)}\right|,$$
$$q_1(t,\theta) = \frac{1}{h}\left(\frac{1}{\rho_1}-1\right) + \zeta_1(t)\left(\frac{1}{\rho_1}+\frac{\theta-t}{h}\left(\frac{1}{\rho_1}-1\right)\right).$$

For any $t \in J$, and $\theta \in [t-h,t]$, the function $q_1(t,\theta)$ satisfies

$$\begin{aligned} q_1(t,\theta) &= \frac{1}{h}\left(\frac{1}{\rho_1}-1\right) + \zeta_1(t)\left(\frac{1}{\rho_1}+\frac{\theta-t}{h}\left(\frac{1}{\rho_1}-1\right)\right) \\ &\geq \frac{1}{h}\left(\frac{1}{\rho_1}-1\right) + (\mu_0+\eta_1)\frac{1}{\rho_1} \\ &= 0,\ \theta \in [t-h,t]. \end{aligned}$$

Then we can get

$$\mathbf{D}^+V_1(t,x_t) \leq \zeta_1(t)V_1(t,x_t). \tag{24}$$

As $\zeta_1(t)$ is a UAS function, there exist two constants $\varepsilon > 0$ and $\delta \geq 0$ such that $\int_{t_0}^{t}\zeta_1(s)\mathrm{d}s \leq -\varepsilon(t-t_0)+\delta, \forall t, t_0 \in J, t \geq t_0$. Then it follows from (24) and the Gronwall inequality [16] that

$$\|x(t)\|^2 = \sum_{i=1}^{n}|x_i(t)|^2 \leq \left(\sum_{i=1}^{n}|x_i(t)|\right)^2 \leq V_1^2(t,x_t) \leq V_1^2(t_0,x_{t_0})\exp\left(\int_{t_0}^{t}2\zeta_1(s)\mathrm{d}s\right). \tag{25}$$

Let $k_2 = \sup_{t\in J}\{|b_{ij}(t)|\}, \forall i,j \in \mathbf{I}[1,n]$. By using (15) and (23), we obtain, for all $t \in J$,

$$V_1(t,x_t) \leq k_1\sqrt{n}\|x(t)\| + \frac{n\sqrt{n}k_1k_2}{\rho_1}\int_{t-h}^{t}\|x(\theta)\|\,\mathrm{d}\theta. \tag{26}$$

Combining (25) and (26) yields

$$\begin{aligned} \|x(t)\| &\leq \sqrt{n}\left(k_1\|x(t_0)\| + \frac{nk_1k_2}{\rho_1}\int_{t_0-h}^{t_0}\|x(\theta)\|\,\mathrm{d}\theta\right)\exp\left(\int_{t_0}^{t}\zeta_1(s)\mathrm{d}s\right) \\ &\leq 2\sqrt{n}\max\left\{k_1, \frac{nhk_1k_2}{\rho_1}\right\}\|x\|_{[t_0-h,t_0]}\exp\left(\int_{t_0}^{t}\zeta_1(s)\mathrm{d}s\right), \end{aligned}$$

which implies that system (21) is UES. ■

In case that h in system (21) is time-varying, then $\mathbf{D}^+V_2$ involves $\dot{h}$. This means that additional conditions must be imposed on h to obtain stability conditions. So, for the sake of simplicity, we assume here that h is a constant.

3.3 The Razumikhin Approach to LTV Time-Delay Systems

In this subsection, we investigate the exponential stability of system (2), in which $\tau \geq 0$ may be unknown. Assume also that system (2) in the absence of the delayed term $B(t)x(t-h(t))$ is UES and conditions of Lemma 4 are satisfied for some UAS function $\mu(t) \in \mathbf{PC}(J,\mathbf{R})$.

Theorem 2 *System (2) with time-varying delay is UES if one of the following conditions holds:*

1. *Conditions (11) and (15) in Lemma 4 are satisfied, where $A(t)$ is defined in system (2) and $\mu(t)$ is UAS. In addition, there exist a scalar UAS function $\zeta_{21}(t)$ and a number $q_{21} > \exp(\varphi_{\zeta_{21}})$, with $\varphi_{\zeta_{21}}$ the overshoot of ζ_{21}, such that*

$$\mu(t) + q_{21} \sum_{i=1}^{n} |c_i(t)\, b_{ij}(t)| \le \zeta_{21}(t)\,,\ \forall j \in \mathbf{I}[1,n]\,,\ \forall t \in J. \tag{27}$$

2. *Conditions (12) and (15) in Lemma 4 are satisfied, where $A(t)$ is defined in system (2) and $\mu(t)$ is UAS. In addition, there exist a scalar UAS function $\zeta_{22}(t)$ and a number $q_{22} > \exp(\varphi_{\zeta_{22}})$, with $\varphi_{\zeta_{22}}$ the overshoot of ζ_{22}, such that*

$$\mu(t) + q_{22} k_1 \sum_{i=1}^{n} \left| \frac{b_{ji}(t)}{c_j(t)} \right| \le \zeta_{22}(t)\,,\ \forall j \in \mathbf{I}[1,n]\,,\ \forall t \in J. \tag{28}$$

Proof. We only prove Item 2 since Item 1 can be proven in a similar way. Consider the nonnegative non-quadratic function $V_\infty(t, x(t))$ as in (8). Clearly, for any $t \in J$, there exists an l such that $V_\infty(t, x(t)) = \frac{|x_l(t)|}{c_l(t)}$ [13]. Then the Dini derivative of $V_\infty(t, x(t))$ can be evaluated as (see, for example, [13])

$$\begin{aligned}
\mathbf{D}^+ V_\infty(t, x(t)) =& -\dot{c}_l(t) \frac{|x_l(t)|}{c_l^2(t)} + \frac{\sigma(x_l(t), \dot{x}_l(t))}{c_l(t)} \dot{x}_l(t) \\
=& -\dot{c}_l(t) \frac{|x_l(t)|}{c_l^2(t)} + \frac{\sigma(x_l(t), \dot{x}_l(t))}{c_l(t)} (a_{ll}(t) x_l(t) \\
& + \sum_{j=1, j\neq l}^{n} a_{lj}(t) x_j(t) + \sum_{j=1}^{n} b_{lj}(t) x_j(t - h(t))) \\
\le& \mu(t) V_\infty(t, x(t)) + \frac{1}{c_l(t)} \sum_{j=1}^{n} |b_{lj}(t) x_j(t - h(t))| \\
\le& \mu(t) V_\infty(t, x(t)) + \frac{|x_j(t - h(t))|}{c_j(t - h(t))} \gamma_l(t)\,,
\end{aligned}$$

where we have used (12) and denoted (by noting (15))

$$\gamma_l(t) \triangleq \sum_{j=1}^{n} \left| \frac{b_{lj}(t) c_j(t - h(t))}{c_l(t)} \right| \le k_1 \sum_{j=1}^{n} \left| \frac{b_{lj}(t)}{c_l(t)} \right|.$$

Under the condition that $V_\infty(t+s, x(t+s)) \le q_{22} V_\infty(t, x(t))$, $\forall s \in [-\tau, 0]$, and (28) we obtain

$$\begin{aligned}
\mathbf{D}^+ V_\infty(t, x(t)) &\le \mu(t) V_\infty(t, x(t)) + \max_{j \in \mathbf{I}[1,n]} \left\{ \frac{|x_j(t - h(t))|}{c_j(t - h(t))} \right\} \gamma_l(t) \\
&= \mu(t) V_\infty(t, x(t)) + V_\infty(t - h(t), x(t - h(t))) \gamma_l(t) \\
&\le (\mu(t) + q_{22} \gamma_l(t)) V_\infty(t, x(t)) \\
&\le \left(\mu(t) + q_{22} k_1 \sum_{j=1}^{n} \left| \frac{b_{lj}(t)}{c_l(t)} \right| \right) V_\infty(t, x(t)) \\
&\le \zeta_{22}(t) V_\infty(t, x(t)).
\end{aligned}$$

Hence it follows from Theorem 1 in [32] that system (2) is UES. The proof is finished. ■

The conditions in Theorem 2 are delay-independent. To establish delay-dependent stability conditions, we need to rewrite system (2) as

$$\dot{x}(t) = (A(t) + B(t))\, x(t) - B(t)\,(x(t) - x(t - h(t)))$$

$$= (A(t) + B(t))\,x(t) - B(t)\int_{t-h(t)}^{t} \dot{x}(s)\,\mathrm{d}s$$

$$= (A(t) + B(t))\,x(t) - B(t)\int_{t-h(t)}^{t} (A(s)\,x(s) + B(s)\,x(s - h(s)))\,\mathrm{d}s, \tag{29}$$

which is a special case of the following LTV system with distributed state delay

$$\dot{x}(t) = C(t)x(t) + \int_{t-h_1(t)}^{t} (D(t,s)\,x(s) + E(t,s)\,x(s - h_2(s)))\,\mathrm{d}s,\ t \in J, \tag{30}$$

obtained with the choice $h_1(t) = h_2(t) = h(t)$ and

$$C(t) = A(t) + B(t), D(t,s) = -B(t)A(s), E(t,s) = -B(t)B(s). \tag{31}$$

In system (30) $h_1 \in \mathbf{PC}(J, [0, \tau_1])$, and $h_2 \in \mathbf{PC}(J, [0, \tau_2])$ are scalar piecewise continuous functions, $C(t) = [c_{ij}(t)] \in \mathbf{PC}(J, \mathbf{R}^{n\times n})$, $D(t,s) = [d_{ij}(t)] \in \mathbf{PC}(J \times J_{\tau_1}, \mathbf{R}^{n\times n})$, $E(t,s) = [e_{ij}(t)] \in \mathbf{PC}(J \times J_{\tau_2}, \mathbf{R}^{n\times n})$, where τ_1 and τ_2 are some constants that may be unknown, $\tau = \tau_1 + \tau_2$ is the delay of the system. Thus we can consider system (30) instead. We assume that the delay free part of system (30), namely, the LTV system

$$\dot{x}(t) = C(t)\,x(t),\ t \in J,$$

is UES. We are now ready to state the main result of Section 3.3.

Theorem 3 *Let conditions (12) and (15) in Lemma 4 be satisfied, where $\mu(t)$ is UAS and $A(t)$ is replaced by $C(t)$. Assume that there exist a scalar UAS function $\zeta_3(t)$ and a constant $q_3 > \exp(\varphi_{\zeta_3})$ such that, for all $j \in \mathbf{I}[1,n]$, $s \in [t - h_1(t), t]$ and $t \in J$,*

$$\mu(t) + \frac{q_3 k_1 h_1(t)}{c_j(t)} \sum_{i=1}^{n} (|d_{ji}(t,s)| + |e_{ji}(t,s)|) \le \zeta_3(t), \tag{32}$$

are satisfied. Then system (30) is UES.

Proof. Consider the non-quadratic function $V_\infty(t, x(t))$ defined in (8). Clearly, for any $t \in J$, there exists an l such that $V_\infty(t, x(t)) = \frac{|x_l(t)|}{c_l(t)}$ [13]. The Dini derivative of $V_\infty(t, x(t))$ can then be evaluated as

$$\begin{aligned}
\mathbf{D}^+ V_\infty(t, x(t)) =& -\dot{c}_l(t)\frac{|x_l(t)|}{c_l^2(t)} + \frac{\sigma(x_l(t), \dot{x}_l(t))}{c_l(t)}\dot{x}_l(t) \\
=& -\dot{c}_l(t)\frac{|x_l(t)|}{c_l^2(t)} + \frac{\sigma(x_l(t), \dot{x}_l(t))}{c_l(t)}\Bigg(c_{ll}(t)\,x_l(t) + \sum_{j=1, j\neq l}^{n} c_{lj}(t)x_j(t) \\
&+ \int_{t-h_1(t)}^{t}\sum_{j=1}^{n}(d_{lj}(t,s)x_j(s) + e_{lj}(t,s)x_j(s - h_2(s)))\,\mathrm{d}s\Bigg) \\
\le& \mu(t)V_\infty(t, x(t)) + \frac{1}{c_l(t)}\int_{t-h_1(t)}^{t}\sum_{j=1}^{n}(|d_{lj}(t,s)\,x_j(s)| + |e_{lj}(t,s)\,x_j(s - h_2(s))|)\,\mathrm{d}s,
\end{aligned}$$

where we have used (12). Under the condition that $V_\infty(t + s, x(t + s)) \le q_3 V_\infty(t, x(t))$, $\forall s \in [-\tau, 0]$, we obtain

$$\begin{aligned}
\mathbf{D}^+ V_\infty(t, x(t)) \le& \mu(t)V_\infty(t, x(t)) + \int_{t-h_1(t)}^{t} \gamma_l(t,s)\,q_3 V_\infty(t, x(t))\,\mathrm{d}s \\
\le& \mu(t)V_\infty(t, x(t)) + \frac{\zeta_3(t) - \mu(t)}{h_1(t)}\int_{t-h_1(t)}^{t} V_\infty(t, x(t))\,\mathrm{d}s \\
\le& \zeta_3(t)\,V_\infty(t, x(t)),
\end{aligned}$$

where we have used (32) and

$$\gamma_l(t,s) \triangleq \sum_{j=1}^{n}\left|\frac{d_{lj}(t,s)\,c_j(s)}{c_l(t)}\right| + \sum_{j=1}^{n}\left|\frac{e_{lj}(t,s)\,c_j(s-h_2(s))}{c_l(t)}\right|$$
$$\le k_1\left(\sum_{j=1}^{n}\left|\frac{d_{lj}(t,s)}{c_l(t)}\right| + \sum_{j=1}^{n}\left|\frac{e_{lj}(t,s)}{c_l(t)}\right|\right),$$

in which we have noticed (15). Since all conditions of Theorem 1 in [32] are satisfied, we conclude that system (30) is UES. This completes the proof. ■

Clearly, setting $C(t), D(t,s)$, and $E(t,s)$ in Theorem 3 as (31) provides delay-dependent stability conditions for system (29) or system (2). The details are omitted for brevity. We finally mention that, although the non-quadratic function $V_1(x_t)$ defined in (7) can also be used to analyze the stability of system (30), the corresponding results will be much more complicated than $V_\infty(x_t)$. Indeed, when $V_1(x_t)$ is used, there will be $2n$ conditions for ensuring stability, where the first n inequalities are related to the elements in matrix D, and the other n conditions are related to the elements in matrix E. While there are only n conditions when $V_\infty(x_t)$ is used. Therefore, here we will not present the results associated with $V_1(x_t)$.

4 Some Illustrative Examples

4.1 An LTV System Without Delay

We consider the following planar LTV system (see p. 252 in [27] and Example 8 in [30]),

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{1+t} & -10 \end{bmatrix} x(t) \triangleq A(t)x(t),\ t \in J = [0,\infty). \tag{33}$$

This system was proven to be asymptotically stable in [30] and, moreover, it was shown in this reference that its state transition matrix satisfies

$$\|\Phi_A(t,t_0)\| \le \frac{2\left(10t_0^2+11t_0+9\right)}{\sqrt{(1+t_0)(79t_0+7)}}\sqrt[12]{\frac{t_0+1}{t+1}}, \forall t \ge t_0 \in J. \tag{34}$$

In the following, we use the proposed approach to improve this result.

We first take the following transformation

$$y(t) = Tx(t),\ T = \begin{bmatrix} 1 & \frac{1}{10} \\ 0 & 1 \end{bmatrix}, \tag{35}$$

by which the LTV system (33) is transformed into

$$\dot{y}(t) = \begin{bmatrix} -\frac{1}{10(1+t)} & \frac{1}{100(1+t)} \\ -\frac{1}{1+t} & -10+\frac{1}{10(1+t)} \end{bmatrix} y(t) \triangleq \overline{A}(t)y(t),\ t \in J, \tag{36}$$

from which it follows that

$$\lfloor \overline{A}(t) \rfloor = \begin{bmatrix} -\frac{1}{10(1+t)} & \frac{1}{100(1+t)} \\ \frac{1}{1+t} & -10+\frac{1}{10(1+t)} \end{bmatrix}. \tag{37}$$

The aim of taking this transformation is to make the $(1,1)$ element in $\overline{A}(t)$ be negative. We then take the following vector

$$c(t) = \begin{bmatrix} 110+\frac{71}{555}\frac{1}{1+t} \\ 1+\frac{71}{550}\frac{1}{1+t} \end{bmatrix},\ t \in J, \tag{38}$$

from which we can choose

$$c_{\max}(t) = 110+\frac{71}{555}\frac{1}{1+t},\ c_{\min}(t) = 1+\frac{71}{550}\frac{1}{1+t},\ t \in J. \tag{39}$$

Consider the following function $\mu(t) : J \to \mathbf{R}$

$$\mu(t) = -\frac{1}{11}\frac{1}{1+t}. \tag{40}$$

Then, by some straightforward computation, we obtain

$$\dot{c}(t) + \left\lfloor \overline{A}^{\mathrm{T}}(t) \right\rceil c(t) - \mu(t)c(t) = \begin{bmatrix} 0 \\ -\frac{346409}{3357750}\frac{1}{(1+t)^2} - 10 \end{bmatrix} \preceq 0, \tag{41}$$

namely, inequality (11) is satisfied associated with $(c(t), \overline{A}(t))$. We then get from (18), (39), and (40) that

$$\begin{aligned} \|\Phi_{\overline{A}}(t,t_0)\| &\leq 2\sqrt{2}\frac{c_{\max}(t_0)}{c_{\min}(t)} \exp\left(\int_{t_0}^{t} \mu(s)\mathrm{d}s\right) \\ &= 2\sqrt{2}\frac{110 + \frac{71}{555}\frac{1}{1+t_0}}{1 + \frac{71}{550}\frac{1}{1+t}} \exp\left(-\int_{t_0}^{t} \frac{1}{11}\frac{1}{1+s}\mathrm{d}s\right) \\ &\leq 2\sqrt{2}\left(110 + \frac{71}{555}\frac{1}{1+t_0}\right) \sqrt[11]{\frac{1+t_0}{1+t}} \\ &\leq 222\sqrt{2}\sqrt[11]{\frac{1+t_0}{1+t}}. \end{aligned} \tag{42}$$

Since $\Phi_{\overline{A}}(t,t_0) = T\Phi_A(t,t_0)T^{-1}$, and $\|T\|\|T^{-1}\| = 1.105$, we have

$$\|\Phi_A(t,t_0)\| \leq \|T\| \left\|T^{-1}\right\| \|\Phi_{\overline{A}}(t,t_0)\| \leq 444\sqrt[11]{\frac{1+t_0}{1+t}}. \tag{43}$$

This result is significantly better than (34) since $\sqrt[11]{\frac{1+t_0}{1+t}}$ converges faster to zero than $\sqrt[12]{\frac{1+t_0}{1+t}}$ as $t \to \infty$ and the overshoot of (34), namely, $\frac{2\left(10t_0^2+11t_0+9\right)}{\sqrt{(1+t_0)(79t_0+7)}}$ is larger than 444 when $t_0 \to \infty$.

4.2 An LTV System with Delay

Consider the following scalar time-delay system studied previously in [14]

$$\dot{x}(t) = -l_1 \cos^2(t)\, x(t) + l_2 \sin(t)\, x(t-h), \tag{44}$$

where $h \geq 0$, $l_1 > 0$, and $l_2 \in \mathbf{R}$ are constants. Let $\mu(t) = -l_1\cos^2(t)$ and $c(t) = 1$, $\forall t \in J$. Obviously, $\mu(t)$ is UAS and inequalities (11) and (15) in Lemma 4 are satisfied. Let

$$\eta_1 \triangleq \frac{\rho_1 - 1}{h} - \mu_0 \geq 0, \tag{45}$$

where $\mu_0 = -l_1$. Hence, by Theorem 1, if there exist a $\rho_1 \in (\max(0, \mu_0 h + 1), 1)$ such that

$$\begin{aligned} \zeta_1(t) &= -l_1\cos^2(t) + \frac{\rho_1 - 1}{h} + l_1 + \frac{|l_2|}{\rho_1}\left|\sin(t+h)\right| \\ &= l_1\sin^2(t) + \frac{\rho_1 - 1}{h} + \frac{|l_2|}{\rho_1}\left|\sin(t+h)\right|, \end{aligned}$$

is UAS, then system (44) is UES. Notice that for any $t \geq t_0 \in J$, there exist two integers $k > k_0$ such that $t = \frac{\pi}{2}k - \delta_k$ and $t_0 = \frac{\pi}{2}k_0 + \delta_0$ where $\delta_k, \delta_0 \in [0, \frac{\pi}{2})$. Then

$$\begin{aligned} \int_{t_0}^{t} |\sin s|\,\mathrm{d}s &= \int_{\frac{\pi}{2}k_0+\delta_0}^{\frac{\pi}{2}k-\delta_k} |\sin s|\,\mathrm{d}s \\ &\leq \int_{\frac{\pi}{2}k_0}^{\frac{\pi}{2}k} |\sin s|\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=k_0}^{k-1} \int_{\frac{\pi}{2}i}^{\frac{\pi}{2}(i+1)} |\sin s| \, \mathrm{d}s \\
&= k - k_0 \\
&= \frac{2}{\pi}(t+\delta_k) - \frac{2}{\pi}(t_0-\delta_0) \\
&= \frac{2}{\pi}(t-t_0) + \frac{2}{\pi}(\delta_k+\delta_0) \\
&\leq \frac{2}{\pi}(t-t_0) + 2.
\end{aligned}
$$

Thus, for all $t \geq t_0 \in J$,

$$
\begin{aligned}
\int_{t_0}^{t} \zeta_1(s) \, \mathrm{d}s \leq & \frac{l_1}{2}\left(t - t_0 - \frac{1}{2}(\sin(2t) - \sin(2t_0))\right) \\
& + \left(\frac{\rho_1 - 1}{h}\right)(t-t_0) + \frac{|l_2|}{\rho_1}\left(\frac{2}{\pi}(t-t_0) + 2\right) \\
\leq & \left(\frac{\rho_1 - 1}{h} + \frac{l_1}{2} + \frac{2}{\pi}\frac{|l_2|}{\rho_1}\right)(t-t_0) + \frac{l_1}{2} + 2\frac{|l_2|}{\rho_1},
\end{aligned}
$$

and the conditions of Theorem 1 are satisfied if there exists a ρ_1 satisfying (45) and

$$
\frac{\rho_1 - 1}{h} + \frac{l_1}{2} + \frac{2}{\pi}\frac{|l_2|}{\rho_1} < 0,
$$

which is equivalent to

$$
2\pi\rho_1^2 - (2\pi - l_1 h\pi)\rho_1 + 4|l_2| h < 0. \tag{46}
$$

We consider three cases.

Case 1: $12|l_2| < \pi|l_1|$. In this case, we have

$$
0 \leq h < \frac{2\pi l_1 + 16|l_2| - \sqrt{64\pi l_1 |l_2| + 256|l_2|^2}}{l_1^2 \pi}.
$$

Case 2: $4|l_2| < \pi l_1 \leq 12|l_2|$. In this case, we have

$$
0 \leq h < \frac{(\pi l_1 - 4|l_2|)}{\pi l_1^2}.
$$

Case 3: $\pi l_1 \leq 4|l_2|$. In this case, we have

$$
h = 0.
$$

We next compare these results, following from Theorem 1, with the result reported in [14]. Let $l_2 = 1$. In Figure 1 we show the relationship between l_1 and the upper bounds on h guaranteeing UES, obtained by the different methods. From this figure we clearly see that qualitatively comparable results are obtained and Theorem 1 allows to improve the result in [14] when l_1 is larger than $|l_2|$.

5 Conclusion

This paper performed a stability analysis of linear time-varying (LTV) time-delay systems by using non-quadratic Lyapunov functions and functionals. The classical condition of negativity of the time-derivative of Lyapunov function(al)s was weakened in the sense that the Dini derivatives of non-quadratic Lyapunov functions and functionals are allowed to take both negative and positive values. With the help of scalar uniformly asymptotically stable (UAS) functions and the recently developed improved Razumikhin and Krasovskii stability theorems, both delay-dependent and delay-independent stability conditions were established to guarantee uniformly exponential stability of a class of LTV time-delay systems. The effectiveness of the established methods was illustrated by some numerical examples.

figure6-eps-converted-to.pdf

Figure 1: The relationships between l_1 and h with $l_2 = 1$ by different approaches

References

[1] Aeyels, D., & Peuteman, J. (1998). A new asymptotic stability criterion for nonlinear time-variant differential equations. *IEEE Transactions on Automatic Control*, 43(7), 968-971.

[2] Anderson, B. D., Ilchmann, A., & Wirth, F. R. (2013). Stabilizability of linear time-varying systems. *Systems & Control Letters*, 62(9), 747-755.

[3] Anderson, B. D. O., & Moore, J. B. (1969). New results in linear system stability. *SIAM Journal on Control*, 7(3), 398-414.

[4] Bekiaris-Liberis, N., & Krstic, M. (2013). *Nonlinear Control Under Nonconstant Delays* (Vol. 25). Siam.

[5] Cao, J., & Wang, J. (2005). Global exponential stability and periodicity of recurrent neural networks with time delays. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 52(5), 920-931.

[6] Cao, J. (2000). Periodic oscillation and exponential stability of delayed CNNs. *Physics Letters A*, 270(3), 157-163.

[7] Guo, Y. (2007). A new spectral inequality and its applications to partial stability of linear time-varying systems. *Mathematica Applicata*, 20(4):814-819.

[8] Guo, Y. (2008). New results on input-to-state convergence for recurrent neural networks with variable inputs. *Nonlinear Analysis: Real World Applications*, 9(4), 1558-1566.

[9] Hale, J. K. (1971). *Functional Differential Equations.* Springer, Berlin, Heidelberg.

[10] Harris, C. J., & Miles, J. F. (1980). *Stability of Linear Systems: Some Aspects of Kinematic Similarity.* London: Academic Press.

[11] Lewis, A. (2017). Remarks on stability of time-varying linear systems. *IEEE Transactions on Automatic Control.*

[12] Li, X. D., & Cao, J. D. (2017). An impulsive delay inequality involving unbounded time-varying delay and applications. *IEEE Transactions on Automatic Control*, 62(7), 3618-3625.

[13] Liang, X., & Wu, L. (1995). Global exponential stability of Hopfield-type neural network and its applications. *Science in China (Scientia Sinica) Series A*, 6(38), 757-768.

[14] Mazenc, F., & Malisoff, M. (2016). Stability analysis for time-varying systems with delay using linear Lyapunov functionals and a positive systems approach. *IEEE Transactions on Automatic Control*, 61(3), 771-776.

[15] Michiels, W., & Niculescu, S. I. (Eds.). (2014). *Stability, Control, and Computation for Time-Delay Systems: An Eigenvalue-based Approach*. Society for Industrial and Applied Mathematics.

[16] Morris, W.. Hirsch, & Smale, S. (1973). *Differential Equations, Dynamical Systems and Linear Algebra*. Academic Press college division.

[17] Ning, C., He, Y., Wu, M., Liu, Q., & She, J. (2012). Input-to-state stability of nonlinear systems based on an indefinite Lyapunov function. *Systems & Control Letters*, 61(12), 1254-1259.

[18] Ning, C., He, Y., Wu, M., & She, J. (2014). Improved Razumikhin-type theorem for input-to-state stability of nonlinear time-delay systems. *IEEE Transactions on Automatic Control*, 59(7), 1983-1988.

[19] Peng, S., & Deng, F. (2017). New Criteria on pth moment input-to-state stability of iImpulsive stochastic delayed differential systems. *IEEE Transactions on Automatic Control*.

[20] Peng, S., & Zhang, Y. (2010). Some new criteria on pth moment stability of stochastic functional differential equations with Markovian switching. *IEEE Transactions on Automatic Control*, 55(12), 2886-2890.

[21] Ramarajan, S. (1986). Time-varying Lyapunov functions for linear time-varying systems. *International Journal of Control*, 44(6), 1699-1702.

[22] Rugh, W. J. (1996). *Linear System Theory*. second edition. Upper Saddle River, New Jersey: Prentice Hall.

[23] Wang, Y. E., Sun, X. M., & Wu, B. (2015). Lyapunov–Krasovskii functionals for input-to-state stability of switched non-linear systems with time-varying input delay. *IET Control Theory & Applications*, 9(11), 1717-1722.

[24] Wu, M. (1974). A note on stability of linear time-varying systems. *IEEE Transactions on Automatic Control*, 19(2), 162-162.

[25] Wu, X., Zhang, W., & Tang, Y. (2013). pth Moment stability of impulsive stochastic delay differential systems with Markovian switching. *Communications in Nonlinear Science and Numerical Simulation*, 18(7), 1870-1879.

[26] Wu, X., Shi, P., Tang, Y., & Zhang, W. (2017). Input-to-state stability of nonlinear stochastic time-varying systems with impulsive effects. *International Journal of Robust and Nonlinear Control*, 27(10), 1792-1809.

[27] Zheng, D. Z. (2002). *Linear System Theory*. Press of Tsinghua University.

[28] Zhao, Z., Jian, J., & Li, L. (2014, July). Polystability in Lagrange sense for a class of delayed neural networks. *In Control Conference (CCC)*, 33rd, Chinese, pp. 6099-6102. IEEE.

[29] Zhou, B. (2014). *Truncated Predictor Feedback for Time-Delay Systems*. Springer, Berlin, Heidelberg.

[30] Zhou, B. (2016). On asymptotic stability of linear time-varying systems. *Automatica*, 68, 266-276.

[31] Zhou, B. (2017). Stability analysis of non-linear time-varying systems by Lyapunov functions with indefinite derivatives. *IET Control Theory & Applications*, 11(9), 1434-1442.

[32] Zhou, B., & Egorov, A. V. (2016). Razumikhin and Krasovskii stability theorems for time-varying time-delay systems. *Automatica*, 71, 281-291.

[33] Zhou, B., & Luo, W. (2018). Improved Razumikhin and Krasovskii stability criteria for time-varying stochastic time-delay systems. *Automatica*, 89: 382-391.