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Fair valuation of insurance liabilities: Interplay between actuarial judgement and market-consistency

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by

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Chapter 1

Introduction

1.1 Context

During the last four decades, the insurance has undergone radical changes driven by multiple and interacting forces - regulatory, economic, demographic and technological- that resulted in a complete paradigm shift.

Traditional life insurance is typically based on a diversification argument which justifies the mitigation of the risk borne by an individual by averaging out its consequences over a large pool of individuals exposed to the same risk. This valuation is performed based on historical life tables and is defined as the expectation plus an additional risk margin to cover any adverse economic-demographic development that is not diversified (e.g. changes in mortality or interest rates). Based on historical data, the actuarial valuation involves a subjective actuarial judgement on the choice of the model to be chosen, see e.g. Kaas et al. (2008) for non-life and Laurent et al. (2016) for life insurance.

In the past, actuaries mostly used deterministic models for discounting liabilities with a constant interest rate. It turned out that this deterministic approach was unable to cope with the stochastic environment of the financial market: several life insurance and pensions have shown huge deficits caused by premiums that were unsustainable in periods of financial crises and significant changes in mortality (in particular, due to a decline in interest rates coupled with an improvement of longevity).

On the regulatory side, the old standard solvency regulations

(Solvency I) were rule-based and not risk-based: the required capital was a percentage of the premiums and was not related to the underlying risk the insurers were facing. Moreover, the solvency regulations were focused on the liability side and disregarded the risk on the asset side. As a consequence, new insurance solvency regulations based on risk management (Solvency II) were introduced in the last years whose main objective is to be risk-sensitive and take market risk (major risk for life insurers) into account in capital requirements.

More specifically, modern solvency regulations for the insurance industry, such as the Swiss Solvency Test and Solvency II, require insurance undertakings to apply a *fair valuation* of their assets and liabilities. The fair value of an asset or a liability is generally understood as 'the amount for which it could be transferred (exchanged) between knowledgeable willing parties in an arm's length transaction'. A fair valuation method combines techniques from financial mathematics and actuarial science, in order to take into account and be consistent with information provided by the financial markets on the one hand and actuarial judgement based on generally available data concerning the underlying risks on the other hand. Loosely speaking, any hedgeable (part of a) claim has to be valuated at the price of its hedge. Otherwise, the value of the claim is determined by its expected present value (called the best estimate), augmented by an appropriate risk loading (called the risk margin, e.g. based on cost-of-capital arguments).[1]

Insurance liabilities are in most cases (only) partially replicable by traded assets. This may be due to the fact that the payoffs of the underlying insurance contracts are defined in terms of a combination of hedgeable and unhedgeable claims (e.g. unit-linked insurance) or due to the existence of traded insurance-linked securities of which the payoff is correlated with the payoff of the insurance liability (e.g. CAT bonds). Since not all insurance claims can be perfectly replicated by traded assets, we face the problem of valuating claims in incomplete markets (i.e. markets in which some claims are not perfectly hedgeable).

[1]Solvency II (Directive 2009/138/EC, Article 77, calculation of technical provisions): If the cash flows of the liability (or part of the cash flows) can be replicated reliably, then the value of the (part of the) cash flows is determined on the basis of the market value of these financial instruments. Otherwise, the value is equal to the sum of the best estimate and a risk margin.

Several ways of valuating unhedgeable (unreplicable) claims have been considered in the literature. Under a 'utility indifference' approach, the value of a claim is set equal to the amount which makes the agent indifferent, in terms of expected utility, between holding the claim or not. The idea for the utility indifference approach in an incomplete market setting is often attributed to Hodges and Neuberger (1989). A market-consistent insurance premium based on expected utility indifference arguments is developed in Malamud et al. (2008). A similar algorithm was proposed by Musiela and Zariphopoulou (2004) for determining indifference prices in a multiperiod binomial model. For an overview of the theory, we refer to Henderson and Hobson (2004) and Carmona (2009).

Another approach for valuating unreplicable claims starts from the observation that in an incomplete market setting no-arbitrage arguments only partially specify the pricing measure (which allows to express prices of contingent claims as discounted expectations under that measure). Therefore, one extends this partially specified measure to a 'complete' pricing measure that is used to determine the value of all contingent claims, also the ones that are not traded. The 'complete' pricing measure is chosen such that it is, in one way or another, the 'most appropriate' one. A popular choice is the minimal entropy martingale measure, see e.g. Frittelli (1995) and Frittelli (2000) in a pure financial context and Dhaene et al. (2015) in a combined financial-actuarial framework. Another possible choice is the risk-neutral Esscher measure, see Gerber and Shiu (1994). Under such a 'completing approach', the value of an unhedgeable claim can be interpreted as a reasoned estimate of what its market value would have been had it been readily traded. A formal definition of market-consistent valuation has only emerged recently, see e.g. Malamud et al. (2008), Artzner and Eisele (2010) or Pelsser and Stadje (2014).

For the determination of the solvency capital requirement, each insurance company is required to determine the fair value of its liabilities, not only today but also in future points in time. An important question in a dynamic setting is how risk valuations at different times are interrelated. In this context, time-consistency is a natural approach to glue together static valuations. It means that the same value is assigned to a position regardless of whether it is calculated over two time periods at once or in two-steps backwards in time. Time-consistent valuations have been largely studied and we refer to Acciaio and Penner (2011) for an overview.

1.2 Contribution

The main goal of this thesis is to introduce different valuation frameworks for the determination of a fair valuation for insurance liabilities. The objective of this valuation is to merge the traditional actuarial valuation based on pooling and diversification with a market-consistent approach based on hedging and replication.

We remark that the thesis is about the generic meaning of fair valuation of random payments related to liabilities in an insurance context and not about a particular technical meaning that is given to it by a particular regulation or legislation. Furthermore, we consider fair valuation in a general context without specifying the purpose it is used for. The results we present and discuss may be used not only in a reserving context (determining technical provisions) but also in a pricing context (setting premiums).

In Chapter 2, we define a *fair valuation* as a valuation which is both market-consistent and actuarial in a single period framework. This valuation is market-consistent in the sense that any hedgeable part of a claim is valuated at the price of its hedge. Moreover, the valuation is actuarial in the sense that claims with payoffs that are independent of the evolution of asset prices are valuated taking into account actuarial judgement.

We introduce and investigate 'hedge-based valuations'. Under this approach, one unbundles the unhedgeable insurance claim in a hedgeable part and a remaining part. The fair value of the claim is then set equal to the sum of the respective values of the hedgeable and the unhedgeable parts, where the hedgeable part is valuated by the financial price of its underlying hedge, while the value of the remaining part is determined via an actuarial approach. In particular, we consider the class of 'convex hedge-based valuations'. An important subclass consists of the 'mean-variance hedge-based valuations'. Further, we also investigate an adapted version of the two-step valuation approach, as introduced in Pelsser and Stadje (2014). We show that the classes of fair valuations, hedge-based valuations and two-step valuations are identical.

In Chapter 3, we extend the approach of the first chapter to a multi-period discrete time setting and focus on the 'mean-variance hedge-based valuations'. which is a two-stage valuation procedure. In a first step, a mean-variance hedge is set up for the claim, based on the available traded assets. In a second step, an actuarial valuation is applied to the remaining non-hedged part of the claim. The fair value is then defined as the sum of the price of the mean-variance hedge and the actuarial value of the residual claim. In this chapter, we will generalize the MVHB valuation approach in a dynamic investment setting and investigate properties of this valuation framework.

In Chapter 4, we incorporate time-consistency considerations in our setting and investigate the fair valuation of insurance liabilities in a dynamic multi-period setting. We define a fair dynamic valuation as a valuation which is actuarial (mark-to-model for claims independent of financial market evolutions), market-consistent (mark-to-market for hedgeable parts of claims) and time-consistent, and study their properties. In particular, we provide a complete hedging characterization for fair dynamic valuations, extending the work of the first two chapters in a dynamic setting. Moreover, we show how we can implement fair dynamic valuations through a backward iterations scheme combining risk minimization methods from mathematical finance with standard actuarial techniques based on risk measures. We remark that Pelsser and Stadje (2014) and Ghalehjooghi and Pelsser (2017) proposed time-consistent and market-consistent valuations via a so-called 'two-step market evaluation'. Compared to their papers which characterize time-consistent and market-consistent valuations in a complete financial market by operator splitting, our valuation framework is hedge-based and allows for financial market incompleteness.

In Chapter 5, we introduce two broad classes of valuations: two-step financial valuations that are market-consistent and two-step actuarial valuations that are actuarial-consistent. We provide a complete axiomatic characterization for the two-step valuations based on coherent valuations. The two-step valuations are general in the sense that they do not impose linearity constraints on the actuarial and financial valuations. Therefore, they allow to account for the diversification of actuarial risks and/or the incompleteness of the financial market (e.g. non-linear pricing with bid-ask prices).

While the two-step financial valuation is an extension of the two-step market valuation of Pelsser and Stadje (2014), the two-step actuarial valuation consists of reversing the valuation order: applying a financial valuation after conditioning on the actuarial component. We show that the two-step actuarial valuation can be decomposed into a best estimate (expected value) plus a risk margin to cover the uncertainty in the actuarial risks. The procedure will be illustrated on a portfolio of life insurance contracts with dependent financial and actuarial risks.

The various chapters in this thesis can be found in

(i) Dhaene, J., Stassen, B., Barigou, K., Linders, D., & Chen, Z. (2017). Fair valuation of insurance liabilities: merging actuarial judgement and market-consistency. *Insurance: Mathematics and Economics*, **76**, 14-27.

(ii) Barigou, K., & Dhaene, J. (2019). Fair valuation of insurance liabilities via mean-variance hedging in a multi-period setting. *Scandinavian Actuarial Journal*, **2019(2)**, 163-187.

(iii) Barigou, K., Chen, Z., & Dhaene, J. (2019). Fair dynamic valuation of insurance liabilities: Merging actuarial judgement with market-and time-consistency. *Insurance: Mathematics and Economics*, **88**, 19-29.

(iv) Barigou, K., Linders, D & Yang F. (2019). Two-step financial and actuarial valuations: Axiomatic characterization and applications. *Working paper.*

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(ii) Delong, L., Dhaene, J., & Barigou, K. (2019). Fair valuation of insurance liability cash-flow streams in continuous time: Applications. *Astin Bulletin.* 1-35. doi:10.1017/asb.2019.8

Chapter 2

Fair valuation of insurance liabilities: Merging actuarial judgement and market-consistency

This chapter is based on
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2.1 Introduction

This chapter introduces the concept of fair valuation in a single period framework. As we said in the introduction, our objective is to take market prices from the financial market on the one hand and actuarial information and judgement about non-traded risks on the other hand into account in our valuation. Since most insurance liabilities are a combination of traded financial and non-traded actuarial risks, it is important that our valuation framework considers these market-consistent and actuarial aspects. In particular, we will define a fair valuation as a valuation that is market-consistent (mark-to-market for any hedgeable part of a claim) and actuarial (mark-to-model for any claim that is independent of financial market evolutions).

Moreover, for asset and liability management purposes, the valuation of insurance liabilities should be driven by the idea of hedging and replication. In that direction, we introduce a class of hedge-based valuations, where in a first step, a 'best hedge' for the liability is set up, based on the traded assets in the market, while in a second step, the remaining part of the claim is valuated via an actuarial valuation. Finally, we also introduce a class of two-step valuations, that are closely related to the two-step valuations introduced by Pelsser and Stadje (2014) and further studied by Ghalehjooghi and Pelsser (2017). We will show that the classes of fair, hedge-based and two-step valuations are identical.

The remainder of this chapter is organized as follows. In Section 2.2, we describe the financial-actuarial world and its market of traded assets. In Section 2.3, fair valuations and the related notion of fair hedgers are introduced. Hedge-based valuations are considered in Section 2.4. An adapted version of the two-step valuations introduced by Pelsser and Stadje (2014) is considered in Section 2.5. Section 2.6 provides some concluding remarks.

2.2 The financial-actuarial world

We investigate the fair valuation of traded and non-traded payoffs in a single period financial-actuarial world. Let time 0 be 'now' and consider a set of random payoffs, which are due at time 1. These payoffs are random variables (r.v.'s) defined on a given probability space $(\Omega, \mathcal{G}, \mathbb{P})$, which is a mathematical abstraction of the combined financial-actuarial world. We call these random payoffs *(contingent) claims*. We assume that the second moments of all claims and the first moments of all products of claims that we will encounter exist and are finite under $\mathbb{P}$.

Any element of $\omega \in \Omega$ represents a possible state of the financial-actuarial world at time 1. For instance, ω could represent a set of possible outcomes for the time-1 prices of the stocks composing the Dow Jones Index and for the number of survivors at time 1 from a given closed population observed at time 0. The σ-algebra

$\mathcal{G}$ stands for the set of all events that may or may not occur in this single period world. Probabilities for these events follow from the real-world probability measure $\mathbb{P}$. We denote the set of all (contingent) claims defined on $(\Omega, \mathcal{G})$ by $\mathcal{C}$.

The financial-actuarial world $(\Omega, \mathcal{G}, \mathbb{P})$ is home to a market of $n+1$ traded assets. These assets can be bought or sold in any quantities in a deep, liquid, transparent and frictionless market (no transaction costs and other market frictions). Asset 0 is the risk-free zero coupon bond. Its current price is $y^{(0)} = 1$, while its payoff at time 1 is given by $Y^{(0)} = e^r$, where $r \geq 0$ is the (continuously compounded) deterministic interest rate r. Furthermore, there are n risky assets, denoted by $1, \ldots, n$, traded in the market. The price (or the payoff) at time 1 of each asset is a claim defined on $(\Omega, \mathcal{G})$. The current price of asset $m \in \{1, 2, \ldots, n\}$ is denoted by $y^{(m)} > 0$, whereas its non-deterministic payoff at time 1 is $Y^{(m)} \geq 0$. We introduce the notations $\mathbf{y}$ and $\mathbf{Y}$ for the vectors of the time-0 and time-1 asset prices, respectively:

$$\boldsymbol{y} = \left(y^{(0)}, y^{(1)}, \ldots, y^{(n)}\right)$$

and

$$\boldsymbol{Y} = \left(Y^{(0)}, Y^{(1)}, \ldots, Y^{(n)}\right).$$

A *trading strategy* $\boldsymbol{\theta} = \left(\theta^{(0)}, \theta^{(1)}, \ldots, \theta^{(n)}\right)$ is a real-valued vector, where the quantity $\theta^{(m)}$ stands for the number of units invested in asset m at time 0. The time-0 and time-1 values of the trading strategy $\boldsymbol{\theta}$ are given by the scalar products

$$\boldsymbol{\theta} \cdot \boldsymbol{y} = \sum_{m=0}^{n} \theta^{(m)} \ y^{(m)}$$

and

$$\boldsymbol{\theta} \cdot \boldsymbol{Y} = \sum_{m=0}^{n} \theta^{(m)} \ Y^{(m)},$$

respectively. The set of all trading strategies is denoted by Θ. The discrete, single period set-up of this chapter implies that any trading strategy is static in the sense that the hedging portfolio chosen at time 0 remains unchanged over the period $[0, 1]$.

Throughout this chapter, we assume that the $n+1$ assets are non-redundant, which means that there exists no investment strategy $\boldsymbol{\theta}$ which is different from $\mathbf{0} = (0, 0, \ldots, 0)$ such that $\boldsymbol{\theta} \cdot \boldsymbol{Y} = 0$. Hence,

$$\boldsymbol{\theta} \cdot \boldsymbol{Y} = 0 \Rightarrow \boldsymbol{\theta} = \mathbf{0}. \tag{2.1}$$

By convention, (in-)equalities between r.v.'s, such as $\boldsymbol{\theta} \cdot \boldsymbol{Y} = 0$, have to be understood in the $\mathbb{P}$-almost sure sense, unless explicitly stated otherwise.

A probability measure $\mathbb{Q}$ defined on the measurable space $(\Omega, \mathcal{G})$ is said to be an *equivalent martingale measure* (or a risk-neutral measure), further abbreviated as EMM, for the market defined above, if it fulfils the following conditions:

(1) $\mathbb{Q}$ and $\mathbb{P}$ are equivalent probability measures:

$$\mathbb{P}[A] = 0 \text{ if and only if } \mathbb{Q}[A] = 0, \quad \text{for all } A \in \mathcal{G}.$$

(2) The current price of any traded asset in the market is given by the expected value of the discounted payoff of this asset at time 1, where discounting is performed at the risk-free interest rate r and expectations are taken with respect to $\mathbb{Q}$:

$$y^{(m)} = e^{-r}\, \mathbb{E}^{\mathbb{Q}}\left[Y^{(m)}\right], \quad \text{for } m = 0, 1, ..., n.$$

Hereafter, we always assume that the market is arbitrage-free in the sense that there is no investment strategy $\boldsymbol{\theta} \in \Theta$ such that

$$\boldsymbol{\theta} \cdot \boldsymbol{y} = 0, \quad \mathbb{P}[\boldsymbol{\theta} \cdot \boldsymbol{Y} \geq 0] = 1 \quad \text{and} \quad \mathbb{P}[\boldsymbol{\theta} \cdot \boldsymbol{Y} > 0] > 0.$$

It is well-known that in our setting, the no-arbitrage condition is equivalent to the existence of a (not necessarily unique) equivalent martingale measure, whereas completeness of the arbitrage-free market is equivalent to the existence of a unique equivalent martingale measure, see e.g. Dalang et al. (1990).

Definition 1 (Hedgeable claim) A hedgeable claim S^h is an element of $\mathcal{C}$ that can be replicated by a trading strategy $\boldsymbol{\nu} = \left(\nu^{(0)}, \nu^{(1)}, \ldots, \nu^{(n)}\right) \in \Theta$:

$$S^h = \boldsymbol{\nu} \cdot \boldsymbol{Y} = \sum_{m=0}^{n} \nu^{(m)}\, Y^{(m)}. \tag{2.2}$$

We introduce the notation $\mathcal{H}$ for the set of all hedgeable claims. The time-0 price of $S^h = \boldsymbol{\nu} \cdot \boldsymbol{Y}$ is given by

$$\boldsymbol{\nu} \cdot \boldsymbol{y} = \sum_{m=0}^{n} \nu^{(m)} \ y^{(m)} = e^{-r} \ \mathbb{E}^{\mathbb{Q}} \left[S^h \right], \tag{2.3}$$

where $\mathbb{Q}$ is a generic member of the class of EMM's. The non-redundancy assumption (2.1) implies that the hedge of any hedgeable claim is uniquely determined. Hereafter, we will say that two random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ defined on $(\Omega, \mathcal{G})$ are $\mathbb{P}$-independent in case they are independent under the measure $\mathbb{P}$, and we will denote this relation by $\boldsymbol{X} \perp \boldsymbol{Y}$.

Definition 2 (Orthogonal claim) An orthogonal claim $S^{\perp}$ is an element of $\mathcal{C}$ which is $\mathbb{P}$-independent of the vector of traded claims:

$$S^{\perp} \perp \left(Y^{(1)}, Y^{(2)}, \ldots Y^{(n)} \right). \tag{2.4}$$

Hereafter, we will denote the set of all orthogonal claims by $\mathcal{O}$. The risk-free claims $a \in \mathbb{R}$ are the only claims which are both hedgeable and orthogonal. Obviously, the hedge related to the claim a due at time 1 is an investment of amount $e^{-r}a$ in zero coupon bonds.

Example 1 (Cost-of-capital principle)
Consider the liability $S^{\perp}$ related to a portfolio of one-year insurances:

$$S^{\perp} = \sum_{i=1}^{N} X_i,$$

where $N \in \mathbb{N}$ and $X_1, X_2, \ldots X_N$ are the losses of the different policies, which are assumed to be $\mathbb{P}$-independent of $\left(Y^{(1)}, Y^{(2)}, \ldots Y^{(n)} \right)$. The position of the insurer in the orthogonal liability $S^{\perp}$ cannot be hedged in the financial market. Suppose that the regulator requires the holder of this position to set up a provision $\rho \left[S^{\perp} \right]$ determined by

$$\rho \left[S^{\perp} \right] = e^{-r} \ \mathbb{E}^{\mathbb{P}} \left[S^{\perp} \right] + \ \mathrm{RM} \ \left[S^{\perp} \right], \tag{2.5}$$

and a capital buffer $e^{-r} \ \left(VaR_p \left[S^{\perp} \right] - \mathbb{E}^{\mathbb{P}} \left[S^{\perp} \right] \right)$ at time 0, for some probability level $p \in (0, 1]$. Here, $e^{-r} \ \mathbb{E}^{\mathbb{P}} \left[S^{\perp} \right]$ is the best estimate of $S^{\perp}$, where the Value at Risk of $S^{\perp}$ at a confidence level p is given by

$$VaR_p\left[S^{\perp}\right] = \inf\left\{x \mid \mathbb{P}\left[S^{\perp} \leq x\right] \geq p\right\},$$

while RM $\left[S^{\perp}\right]$ is the risk margin under the cost-of-capital approach:

$$\text{RM}\ \left[S^{\perp}\right] = e^{-r}\ i\ \left(\text{VaR}_p\left[S^{\perp}\right] - \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]\right) \tag{2.6}$$

for some cost-of-capital rate i. The risk margin RM $\left[S^{\perp}\right]$ reflects the cost related to holding the capital $e^{-r}\left(\text{VaR}_p\left[S^{\perp}\right] - \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]\right)$ to buffer the risk of $S^{\perp}$ being larger than $\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]$ at time 1.
Let us now additionally assume that under $\mathbb{P}$, the claims X_i are i.i.d. with expectation and variance given by μ and $\sigma^2 > 0$, respectively. Furthermore, let the portfolio be sufficiently large such that we can assume that

$$\mathbb{P}\left[\frac{S^{\perp} - \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]}{\sigma^{\mathbb{P}}\left[S^{\perp}\right]} \leq s\right] = \Phi\left[s\right], \qquad \text{for all } s,$$

where Φ is the cdf of a standard normal distribution. In this case, we find that $\rho\left[S^{\perp}\right]$ is given by

$$\rho\left[S^{\perp}\right] = e^{-r}\left(N\mu + i\sqrt{N}\sigma\Phi^{-1}\left[p\right]\right). \tag{2.7}$$

Due to the diversification effect, the risk margin per policy, i.e. $e^{-r}i\frac{\sigma}{\sqrt{N}}\Phi^{-1}\left[p\right]$, is a decreasing function of the portfolio size N. ◀

Many claims that insurance companies face are not perfectly hedgeable, but nevertheless not $\mathbb{P}$-independent of the payoffs of the traded assets. Such claims are neither hedgeable nor orthogonal. Instead, they belong to the class of unhedgeable and non-orthogonal claims. Hereafter, we will call the members of this class hybrid claims.

Definition 3 (Hybrid claim) A claim S is a hybrid claim in case it is neither perfectly hedgeable nor orthogonal:

$$S \in \mathcal{C} \setminus \left(\mathcal{H} \cup \mathcal{O}\right).$$

Unit-linked insurance products often have by construction a financial (hedgeable) and an actuarial (unhedgeable) part in their payoff. This means that the valuation of unit-linked insurance claims gives rise to the valuation of hybrid claims. Furthermore, the development of markets in insurance-linked securities (such as catastrophic bonds, weather derivatives, longevity bonds) creates the possibility that liabilities of insurance portfolios that are exposed to specific actuarial risks (such as those arising from natural catastrophes) become at least partially hedgeable. Hence, insurance securitization may also lead to hybrid claims in insurance portfolios.

Insurance valuation regulations are in general clear about the fair valuation of hedgeable and orthogonal claims. The former type of claims are valuated at the cost of the replicating portfolio, while the latter are valuated as the sum of their expected present value and a risk margin. However, it is usually unclear how to perform the fair valuation of hybrid claims. This thesis contributes to the development of solutions for that important issue.

2.3 Fair valuations and fair hedgers

In this section, we define different classes of valuations, which attach a value to any claim $S \in \mathcal{C}$. We also introduce different classes of hedgers, which attach a trading strategy to any claim. We show that there is a one-to-one relation between each class of valuations and its corresponding class of hedgers.

2.3.1 Fair valuations

In this subsection, we define the notion of valuation. Furthermore, we introduce the notions of market-consistent, actuarial and fair valuations, respectively.

Definition 4 (Valuation) A valuation is a mapping $\rho : \mathcal{C} \to \mathbb{R}$, attaching a real number to any claim $S \in \mathcal{C}$:

$$S \to \rho\left[S\right],$$

such that ρ is normalized:

$$\rho\left[0\right] = 0, \tag{2.8}$$

and ρ is translation invariant:

$$\rho\left[S+a\right]=\rho\left[S\right]+e^{-r}a, \qquad \text{for any } S\in\mathcal{C} \text{ and } a\in\mathbb{R}. \tag{2.9}$$

A valuation ρ attaches a real number to any claim, which we interpret as a 'value' of that claim. For any valuation ρ, we immediately find that

$$\rho\left[a\right]=e^{-r}a, \qquad \text{for any } a\in\mathbb{R}. \tag{2.10}$$

Other properties that a valuation may satisfy or not are $\mathbb{P}$-law invariance, positive homogeneity and subadditivity. A valuation ρ is said to be $\mathbb{P}$-*law invariant* if

$$\rho\left[S_1\right]=\rho\left[S_2\right] \text{ for any } S_1,S_2\in\mathcal{C} \text{ with } S_1\overset{\mathbb{P}}{=}S_2.$$

It is said to be *positive homogeneous* if

$$\rho\left[aS\right]=a\;\rho\left[S\right], \qquad \text{for any scalar } a>0 \text{ and any } S\in\mathcal{C},$$

while it is said to be *subadditive* if

$$\rho\left[S_1+S_2\right]\leq\rho\left[S_1\right]+\rho\left[S_2\right], \qquad \text{for any } S_1,S_2\in\mathcal{C}.$$

An important subclass of the class of valuations is the class of market-consistent valuations, which are defined hereafter.

Definition 5 (Market-consistent valuation)
A market-consistent valuation (MC valuation) is a valuation $\rho:\mathcal{C}\to\mathbb{R}$ such that any hedgeable part of any claim is marked-to-market:

$$\rho\left[S+\boldsymbol{\nu}\boldsymbol{\cdot}\boldsymbol{Y}\right]=\rho\left[S\right]+\boldsymbol{\nu}\boldsymbol{\cdot}\boldsymbol{y}, \qquad \text{for any } S\in\mathcal{C} \text{ and any } \boldsymbol{\nu}\cdot\mathbf{Y}\in\mathcal{H}. \tag{2.11}$$

In the literature on MC valuation, market-consistency is usually defined via condition (2.11), see e.g. Kupper et al. (2008), Malamud et al. (2008) or Artzner and Eisele (2010) and Pelsser and Stadje (2014). The mark-to-market condition (2.11) can be interpreted as an extension of the notion of translation (or cash) invariance (2.9) from scalars to hedgeable claims. The mark-to-market condition can also be stated in the following way:

$$\rho\left[S\right]=\rho\left[S-\boldsymbol{\nu}\cdot\mathbf{Y}\right]+\boldsymbol{\nu}\boldsymbol{\cdot}\boldsymbol{y}, \qquad \text{for any } S\in\mathcal{C} \text{ and } \boldsymbol{\nu}\cdot\mathbf{Y}\in\mathcal{H}. \tag{2.12}$$

In order to interpret (2.12), consider a person facing a loss S. This person could decide to transfer the whole loss S to the insurer. Alternatively, he could split his claim S into a hedgeable claim $\boldsymbol{\nu} \cdot \mathbf{Y}$, which he hedges in the financial market, while he brings the remaining part $S - \boldsymbol{\nu} \cdot \mathbf{Y}$ to the insurer. The condition (2.12) states that the claim S is equally valuated in both cases. In other words, the insurer valuates in agreement with the financial market, in the sense that he does not charge a risk margin for any hedgeable part of a claim. From (2.12), we also find that for any hedgeable claim $S^h = \boldsymbol{\nu} \cdot \mathbf{Y}$, we have that

$$\rho\left[\boldsymbol{\nu} \cdot \mathbf{Y}\right] = \boldsymbol{\nu} \bullet \boldsymbol{y}, \tag{2.13}$$

which means that the market-consistent value of a hedgeable claim is equal to the price of its underlying hedge.

Next we define actuarial valuations.

Definition 6 (Actuarial valuation) An actuarial valuation is a valuation $\rho : \mathcal{C} \to \mathbb{R}$ such that any orthogonal claim is marked-to-model:

$$\rho\left[S^{\perp}\right] = e^{-r}\ \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] + \ \mathrm{RM}\ \left[S^{\perp}\right], \qquad \text{for any } S^{\perp} \in \mathcal{O}, \tag{2.14}$$

where $\mathrm{RM} : \mathcal{O} \to \mathbb{R}$ is a mapping attaching to any orthogonal claim a real number, not depending on the current asset prices $\left(y^{(1)}, y^{(2)}, \ldots, y^{(n)}\right)$.

The mark-to-model condition (2.14) states that any orthogonal claim is valuated by the sum of its best estimate $e^{-r}\ \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]$ and a risk margin $\mathrm{RM}\ \left[S^{\perp}\right]$. In order to guarantee that an actuarial valuation is indeed a valuation, one must have that $\mathrm{RM}\ [0] = 0$ and $\mathrm{RM}\ \left[S^{\perp} + a\right] = \mathrm{RM}\ \left[S^{\perp}\right]$ for any orthogonal claim. The actuarial valuation ρ and in particular the risk margin function RM that is used in a specific situation is chosen by the actuary, the regulator or any other valuator of the claims and introduces actuarial judgement in the valuation of claims. In the traditional view on valuation in an insurance context, the existence of the financial market is ignored, except for the risk-free bank account. In such an approach, any claim S is orthogonal, and any claim is valuated via an actuarial valuation.

Our definition of an actuarial valuation is broad in the sense that the only requirement that is made concerning the risk margin is that it is not dependent on information concerning the prices of the traded assets that is available at the moment of the valuation. Alternative definitions for an actuarial valuation are possible. In a narrow setting, one could define an actuarial valuation as a valuation of the form (2.14) where RM is the risk margin function of the Cost-of-Capital principle (2.6) for a given probability level p and cost-of-capital i. In general, an actuarial valuation could be defined as a valuation satisfying a well-defined property in the set of orthogonal claims. One could consider e.g. a set of probability measures $(\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_n)$ on the measurable space $(\Omega, \mathcal{G})$, and require that for any orthogonal claim $S^\perp \in \mathcal{O}$, $\rho\left[S^\perp\right]$ only depends on the n cdf's $F_{S^\perp}^{\mathbb{P}_1}, F_{S^\perp}^{\mathbb{P}_2}, , \ldots, F_{S^\perp}^{\mathbb{P}_n}$ of $S^\perp$ under these different measures. An example is the 'worst-case' valuation for any $S^\perp \in \mathcal{O}$:

$$\rho\left[S^\perp\right] = \max\left(\rho_1\left[S^\perp\right], \rho_2\left[S^\perp\right], \ldots, \rho_n\left[S^\perp\right]\right),$$

where for each i, ρ_i is an actuarial valuation in the sense of the original definition (2.14), where the measure $\mathbb{P}$ is replaced by $\mathbb{P}_i$. It is important to notice that all the results that we will derive hereafter concerning the characterization of fair valuations remain valid under any such adapted definition of an actuarial valuation.

Current insurance solvency regulations impose mark-to-market as well as mark-to-model requirements for the valuation of assets and liabilities.[1] However, in the existing scientific literature on valuating claims in a combined financial-actuarial setting, the focus is on the mark-to-market condition as defined according to (2.11), while the mark-to-model condition, which states that non-financial components of a claim should be valued taking into account actuarial judgement, is ignored. Therefore, hereafter we introduce the class of fair valuations, which is a subset of the class of market-consistent valuations. These fair valuations are closer to

[1] In the 'Solvency II Glossary' of the 'Comité Européen des Assurances' and the 'Groupe Consultatif Actuariel Européen' of 2007, Fair Value is defined as 'the amount for which ... a liability could be settled between knowledgeable, willing parties in an arm's length transaction. This is similar to the concept of Market Value, but the Fair Value may be a mark-to-model price if no actual market price for the ... liability exists.'

the meaning of fair valuation in current insurance solvency regulations, as they satisfy a mark-to-market as well as a mark-to-model condition.

Definition 7 (Fair valuation) A fair valuation is a valuation that is both market-consistent and actuarial.

At first sight, one could wonder whether it would be more appropriate to define a fair valuation as a valuation that obeys the mark-to-market condition (2.11) for any hedgeable part of a claim, as well as the following mark-to-model condition for any orthogonal part of a claim:

$$\rho\left[S + S^{\perp}\right] = \rho\left[S\right] + \pi\left[S^{\perp}\right], \quad \text{for any } S \in \mathcal{C} \text{ and any } S^{\perp} \in \mathcal{O},$$

where π is an actuarial valuation as defined above. One can easily prove that this condition would imply that $\rho\left[S^{\perp}\right] = \pi\left[S^{\perp}\right]$ and hence,

$$\pi\left[S_1^{\perp} + S_2^{\perp}\right] = \pi\left[S_1^{\perp}\right] + \pi\left[S_2^{\perp}\right], \quad \text{for any } S_1^{\perp}, S_2^{\perp} \in \mathcal{O},$$

which would ignore the diversification benefit which is essential for valuating non-replicable insurance liabilities, see e.g. (2.7) in Example 1.

The valuation $\rho : \mathcal{C} \to \mathbb{R}$ defined by

$$\rho\left[S\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[S\right] \tag{2.15}$$

for a given EMM $\mathbb{Q}$ is an example of a valuation which is market-consistent but in general not actuarial and hence, not fair. Using the risk neutral valuation (2.15) for hybrid and orthogonal claims in insurance portfolios is in general not appropriate. Consider e.g. the orthogonal claim $S^{\perp} = \sum_{i=1}^{N} X_i$, where the claims X_i are i.i.d. and suppose that this claim is valuated by (2.15). The value per policy is then given by

$$\frac{\rho\left[S^{\perp}\right]}{N} = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[X_1\right],$$

which is independent of the size N of the portfolio and hence, ignores the diversification effect in an insurance context. This observation illustrates the fact that market-consistency is a necessary but not sufficient condition for a valuation to be appropriate in an insurance context. The insufficiency is a consequence of the ignorance of diversification concerns in a market-consistent valuation.

From the requirements (2.11) and (2.14), we find that the fair valuation for $S^{\perp} + \boldsymbol{\nu} \cdot \mathbf{Y}$ with $S^{\perp} \in \mathcal{O}$ and $\boldsymbol{\nu} \cdot \mathbf{Y} \in \mathcal{H}$ is given by

$$\rho\left[S^{\perp} + \boldsymbol{\nu} \cdot \mathbf{Y}\right] = \pi\left[S^{\perp}\right] + \boldsymbol{\nu} \cdot \mathbf{y}, \tag{2.16}$$

where π is an actuarial valuation. Hence, the fair value of the claim $S^{\perp} + S^{h}$ is given by the sum of the actuarial value of $S^{\perp}$ and the financial market price of $\boldsymbol{\nu} \cdot \mathbf{Y}$. In other words, the orthogonal part of the claim is marked-to-model, whereas the hedgeable part is marked-to-market.

Most hybrid claims observed in an insurance context are of a more complex structure than the additive structure considered in (2.16). One often encounters a multiplicative structure, where the claim S to be valuated can be expressed in the form

$$S = S^{h} \times S^{\perp}, \quad \text{with } S^{h} \in \mathcal{H} \text{ and } S^{\perp} \in \mathcal{O}. \tag{2.17}$$

Solvency regulations are in general rather vague on how to evaluate such hybrid claims. It is obvious that this claim is only partially hedgeable, and that S^{h} is hedgeable whereas $S^{\perp}$ is not. But it is not clear how to combine market prices of hedgeable claims with actuarial considerations to determine a fair value for the claim, since regulatory frameworks usually do not prescribe how to determine the hedgeable part of a non-hedgeable claim.

Example 2 (Unit-linked insurance)
Consider an insurance portfolio consisting of N insureds, with $X_i \in \mathcal{O}$ equal to 1 if insured $i = 1, 2, \ldots, N$, is alive at time 1 and equal to 0 in the other case. The orthogonal claims X_i are assumed to be i.i.d. with mean p under $\mathbb{P}$. The number of survivors at time 1 is given by

$$S^{\perp} = \sum_{i=1}^{N} X_i.$$

Each insured i has underwritten a one-year *unit-linked contract with guarantee* against the risk that asset 1 falls short of $K > 0$. The payoff of individual contract i at time 1 is given by

$$\max\left(Y^{(1)}, K\right) \times X_i.$$

Suppose that the put option with payoff $\left(K - Y^{(1)}\right)_+$ is traded at price $P\left[K\right]$. The unit-linked contracts have an unbounded upside potential and offer downward protection. The portfolio liability at time 1 is given by

$$S^h \times S^\perp = \max\left(Y^{(1)}, K\right) \times \sum_{i=1}^{N} X_i. \tag{2.18}$$

Let us now consider a valuation ρ satisfying

$$\rho\left[S^h \times S^\perp\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[S^h\right] \times \left(\mathbb{E}^{\mathbb{P}}\left[S^\perp\right] + \alpha\ \sigma^{\mathbb{P}}\left[S^\perp\right]\right),$$

with $\alpha \geq 0$. In our particular case, this expression reduces to

$$\rho\left[S^h \times S^\perp\right] = \left(y^{(1)} + P\left[K\right]\right)\left(Np + \sqrt{N}\alpha\sqrt{p(1-p)}\right).$$

It is easy to prove that in case each unit-linked contract is charged a premium equal to $\frac{\rho\left[S^h \times S^\perp\right]}{N}$ and if these premiums are fully invested in S^h, the probability that the insurer will be able to fulfil his liabilities at time 1 is given by

$$\mathbb{P}\left[\frac{S^\perp - \mathbb{E}^{\mathbb{P}}\left[S^\perp\right]}{\sigma^{\mathbb{P}}\left[S^\perp\right]} \leq \alpha\right].$$

Assuming the portfolio is sufficiently large, this probability is approximately equal to $\Phi\left[\alpha\right]$, where Φ is the cdf of a standard normal distribution. ◀

More complicated hybrid claims arise when the claim S is given by

$$S = S^h \times S', \quad \text{with } S^h \in \mathcal{H} \text{ and } S' \in \mathcal{C}, \tag{2.19}$$

where S^h and S' are not assumed to be $\mathbb{P}$-independent. Obviously, in this case the decomposition is not unique. As an example, consider the claim S defined in the previous example, where we do not assume $\mathbb{P}$-independence between $(X_1, X_2, \ldots, X_n)$ and $Y^{(1)}$.

A major simplification for valuating the claim S defined in (2.17), originating from Brennan and Schwartz (1976), see also Brennan and Schwartz (1979*a*,*b*), arises if we assume that the claim $S^\perp$ is completely diversified, in the sense that

$$S^\perp = \mathbb{E}^{\mathbb{P}}\left[S^\perp\right].$$

This assumption can be justified for very large portfolios of independent claims by the law of large numbers. Under this assumption of complete diversification, we find that $S^h \times \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]$ is a hedgeable claim, only containing financial uncertainty and hence, taking into account (2.13), we find that

$$\rho\left[S^h \times S^{\perp}\right] = \boldsymbol{\nu} \cdot \boldsymbol{y} \times \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]. \tag{2.20}$$

Taking into account that

$$\boldsymbol{\nu} \cdot \boldsymbol{y} = e^{-r}\, \mathbb{E}^{\mathbb{Q}}\left[S^h\right]$$

for any EMM $\mathbb{Q}$, we can transform the previous expression in the well-known Brennan & Schwartz-formula:

$$\rho\left[S^h \times S\right] = e^{-r}\, \mathbb{E}^{\mathbb{Q}}\left[S^h\right] \times \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]. \tag{2.21}$$

This approach based on 'complete actuarial diversification' does not answer the question of how to quantify hybrid claims of the form $S^h \times S^{\perp}$ in case the law of large numbers is not applicable for the insurance claim $S^{\perp}$. In this case, one is not able to 'average out' the insurance risk. Instead, one has to consider $S^h \times S^{\perp}$ as a claim in an incomplete market and come up with a valuation approach that reflects both financial and actuarial risk. Such valuation approaches will be considered in the following sections.

2.3.2 Fair hedgers

After having defined market-consistent, actuarial and fair valuations, we will now introduce the corresponding classes of hedgers. In particular, we will define market-consistent, actuarial and fair hedgers. We will investigate the relation between each type of valuation and its corresponding class of hedgers.

Definition 8 (Hedger) A hedger is a function $\boldsymbol{\theta} : \mathcal{C} \to \Theta$ which maps any claim S into a trading strategy $\boldsymbol{\theta}_S = \left(\theta_S^{(0)}, \theta_S^{(1)}, \ldots, \theta_S^{(n)}\right)$, such that

- $\boldsymbol{\theta}$ is normalized:

$$\boldsymbol{\theta}_0 = (0, 0, \ldots, 0).$$

• $\boldsymbol{\theta}$ is translation invariant:

$$\boldsymbol{\theta}_{S+a} = \boldsymbol{\theta}_S + \left(e^{-r}a, 0, \ldots, 0\right),$$

for any scalar $a > 0$ and any $S \in \mathcal{C}$.

The mapping $\boldsymbol{\theta} : \mathcal{C} \to \Theta$ is called a hedger, whereas for any claim S, the trading strategy $\boldsymbol{\theta}_S$ is called a hedge for S. This hedge may be a partial or a perfect hedge. The value of the hedge $\boldsymbol{\theta}_S$ of S at time 1 is given by

$$\boldsymbol{\theta}_S \cdot \boldsymbol{Y} = \sum_{m=0}^{n} \theta_S^{(m)} \; Y^{(m)}, \tag{2.22}$$

whereas its time-0 value equals

$$\boldsymbol{\theta}_S \cdot \boldsymbol{y} = \sum_{m=0}^{n} \theta_S^{(m)} \; y^{(m)} = e^{-r} \; \mathbb{E}^{\mathbb{Q}} \left[\boldsymbol{\theta}_S \cdot \boldsymbol{Y}\right], \tag{2.23}$$

where $\mathbb{Q}$ can be any element of the class of EMM's.

Definition 9 A hedger $\boldsymbol{\theta} : \mathcal{C} \to \Theta$ is said to be

• *positive homogeneous* if

$$\boldsymbol{\theta}_{a\,S} = a\,\boldsymbol{\theta}_S, \qquad \text{for any scalar } a > 0 \text{ and any } S \in \mathcal{C},$$

• *additive* if

$$\boldsymbol{\theta}_{S_1+S_2} = \boldsymbol{\theta}_{S_1} + \boldsymbol{\theta}_{S_2}, \qquad \text{for any } S_1, S_2 \in \mathcal{C}.$$

Hereafter, we introduce the subclasses of market-consistent, actuarial and fair hedgers.

Definition 10 (MC, actuarial and fair hedger)
(1) A hedger is market-consistent (MC) in case any hedgeable part $\boldsymbol{\nu} \cdot \mathbf{Y}$ of any claim is hedged by ν:

$$\boldsymbol{\theta}_{S+\boldsymbol{\nu}\cdot\mathbf{Y}} = \boldsymbol{\theta}_S + \boldsymbol{\nu}, \qquad \text{for any } S \in \mathcal{C} \text{ and any } \boldsymbol{\nu} \cdot \mathbf{Y} \in \mathcal{H}. \tag{2.24}$$

(2) A hedger is actuarial in case any orthogonal claim is hedged risk-free via an actuarial valuation ρ:

$$\boldsymbol{\theta}_{S^\perp} = \left(\rho\left[S^\perp\right], 0, \ldots, 0\right), \qquad \text{for any } S^\perp \in \mathcal{O}. \tag{2.25}$$

(3) A hedger is fair in case it is market-consistent and actuarial.

For any actuarial or fair hedger $\boldsymbol{\theta}$ with actuarial valuation ρ used to hedge claims in $\mathcal{O}$, we call ρ the underlying actuarial valuation of $\boldsymbol{\theta}$. The condition (2.24) in the definition of a market-consistent hedger can also be expressed as follows: for any hedgeable claim $\boldsymbol{\nu} \cdot \boldsymbol{Y}$ and any claim S, one has that

$$\boldsymbol{\theta}_S = \boldsymbol{\nu} + \boldsymbol{\theta}_{S-\boldsymbol{\nu}\cdot\boldsymbol{Y}}. \tag{2.26}$$

Written in this way, it is easily seen that hedging in two steps and hedging in a single step lead to the same global hedge. Indeed, first choosing a hedge $\boldsymbol{\nu}$ and then applying the hedger $\boldsymbol{\theta}$ to the remaining loss $S - \boldsymbol{\nu} \cdot \boldsymbol{Y}$ leads to the same overall hedge as immediately applying the hedger $\boldsymbol{\theta}$ on S.

The condition (2.25) in the definition of an actuarial hedger means that any orthogonal claim $S^{\perp}$ is hedged by an investment of amount $\rho\left[S^{\perp}\right]$ in zero-coupon bonds.

In the following lemma, we summarize some properties of hedgers that will be used hereafter. The proofs are straightforward and therefore omitted.

Lemma 1 Consider a claim S, an orthogonal claim $S^{\perp}$, a hedgeable claim $\boldsymbol{\nu} \cdot \mathbf{Y}$ and a scalar a.
(1) For any hedger $\boldsymbol{\theta}$, one has that

$$\boldsymbol{\theta}_a = \left(e^{-r}\ a, 0, \ldots, 0\right). \tag{2.27}$$

(2) For any market-consistent hedger $\boldsymbol{\theta}$, one has that

$$\boldsymbol{\theta}_{\boldsymbol{\nu}\cdot\mathbf{Y}} = \boldsymbol{\nu}. \tag{2.28}$$

(3) For any fair hedger $\boldsymbol{\theta}$ with underlying actuarial valuation ρ, one has that

$$\boldsymbol{\theta}_{S^{\perp}+\boldsymbol{\nu}\cdot\mathbf{Y}} = \left(\rho\left[S^{\perp}\right], 0, \ldots, 0\right) + \boldsymbol{\nu}. \tag{2.29}$$

In the proofs of a number of forthcoming theorems, we will consider a hedge $\boldsymbol{\mu}_S$ for any claim S which is defined as the sum of another hedge $\boldsymbol{\theta}_S$ of S and an actuarial hedge of the remaining risk $S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}$. Some properties of such hedgers are considered in the following lemma.

Lemma 2 Consider a hedger $\boldsymbol{\theta}$ and a valuation ρ. Define the hedger $\boldsymbol{\mu}$ by

$$\boldsymbol{\mu}_S = \boldsymbol{\theta}_S + \left(\rho\left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}\right], 0, \ldots, 0\right), \qquad \text{for any } S \in \mathcal{C}. \quad (2.30)$$

(a) If $\boldsymbol{\theta}$ is a MC hedger, then $\boldsymbol{\mu}$ is a MC hedger.

(b) If $\boldsymbol{\theta}$ is an actuarial hedger and ρ is an actuarial valuation, then $\boldsymbol{\mu}$ is an actuarial hedger with underlying actuarial valuation ρ.

(c) If $\boldsymbol{\theta}$ is a fair hedger and ρ is an actuarial valuation, then $\boldsymbol{\mu}$ is a fair hedger with underlying actuarial valuation ρ.

Proof: (a) Suppose that $\boldsymbol{\theta}$ is a MC hedger. For any claim S and any hedgeable claim $S^h = \boldsymbol{\nu} \cdot \mathbf{Y}$, we find that

$$\begin{aligned}
\boldsymbol{\mu}_{S+S^h} &= \boldsymbol{\theta}_{S+S^h} + \left(\rho\left[S + S^h - \boldsymbol{\theta}_{S+S^h} \cdot \boldsymbol{Y}\right], 0, \ldots, 0\right) \\
&= \boldsymbol{\theta}_S + \boldsymbol{\nu} + \left(\rho\left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}\right], 0, \ldots, 0\right) \\
&= \boldsymbol{\mu}_S + \boldsymbol{\nu}.
\end{aligned}$$

We can conclude that $\boldsymbol{\mu}$ is a MC hedger.

(b) Next, suppose that $\boldsymbol{\theta}$ is an actuarial hedger with underlying actuarial valuation π. Further, suppose that ρ is an actuarial valuation. For any orthogonal claim $S^{\perp}$, we have

$$\begin{aligned}
\boldsymbol{\mu}_{S^{\perp}} &= \boldsymbol{\theta}_{S^{\perp}} + \left(\rho\left[S^{\perp} - \boldsymbol{\theta}_{S^{\perp}} \cdot \boldsymbol{Y}\right], 0, \ldots, 0\right) \\
&= \left(\pi\left[S^{\perp}\right] + \rho\left[S^{\perp} - e^r\, \pi\left[S^{\perp}\right]\right], 0, \ldots, 0\right) \\
&= \left(\rho\left[S^{\perp}\right], 0, \ldots, 0\right),
\end{aligned}$$

where in the last step, we used the translation invariance of ρ. We can conclude that $\boldsymbol{\mu}$ is an actuarial hedger with underlying actuarial valuation ρ.

(c) Finally, suppose that $\boldsymbol{\theta}$ is a fair hedger with underlying actuarial valuation π, while ρ is an actuarial valuation. From (a) and (b) it follows immediately that $\boldsymbol{\mu}$ is a fair hedger with underlying actuarial valuation ρ. ■

In Section 2.4.3 , we will consider mean-variance hedging and the related mean-variance hedger which will be shown to be a fair hedger, see Corollary 2 hereafter. The mean-variance hedger is defined as follows:

$$\boldsymbol{\theta}_S^{MV} = \arg\min_{\boldsymbol{\mu} \in \Theta} \mathbb{E}^{\mathbb{P}}\left[(S - \boldsymbol{\mu} \cdot \mathbf{Y})^2\right], \qquad \text{for any } S \in \mathcal{C}. \quad (2.31)$$

In the following theorem it is shown that any MC valuation can be represented as the time-0 price of a MC hedger. Similar properties hold for actuarial and fair valuations.

Theorem 1 Consider the valuation $\rho : \mathcal{C} \to \mathbb{R}$.
(a) ρ is a MC valuation if and only if there exists a MC hedger $\boldsymbol{\theta}^m$ such that

$$\rho[S] = \boldsymbol{\theta}_S^m \cdot \boldsymbol{y}, \qquad \text{for any } S \in \mathcal{C}. \tag{2.32}$$

(b) ρ is an actuarial valuation if and only if there exists an actuarial hedger $\boldsymbol{\theta}^a$ such that

$$\rho[S] = \boldsymbol{\theta}_S^a \cdot \boldsymbol{y}, \qquad \text{for any } S \in \mathcal{C}. \tag{2.33}$$

(c) ρ is a fair valuation if and only if there exists a fair hedger $\boldsymbol{\theta}^f$ such that

$$\rho[S] = \boldsymbol{\theta}_S^f \cdot \boldsymbol{y}, \qquad \text{for any } S \in \mathcal{C}. \tag{2.34}$$

Proof: (a) Let ρ be a MC valuation. Consider a MC hedger $\boldsymbol{\theta}$, e.g. the mean-variance hedger defined in (2.31). For any claim S, we find from (2.12) that

$$\begin{aligned} \rho[S] &= \rho[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}] + \boldsymbol{\theta}_S \cdot \boldsymbol{y} \\ &= \boldsymbol{\theta}_S^m \cdot \boldsymbol{y} \end{aligned}$$

with

$$\boldsymbol{\theta}_S^m = \boldsymbol{\theta}_S + (\rho[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}], 0, \ldots, 0). \tag{2.35}$$

From Lemma 2 we know that $\boldsymbol{\theta}^m$ is a MC hedger.
(a') Suppose that the valuation ρ is defined by (2.32) for some MC hedger $\boldsymbol{\theta}^m$. For any hedgeable claim $\boldsymbol{\nu} \cdot \boldsymbol{Y}$, we find that

$$\begin{aligned} \rho[S + \boldsymbol{\nu} \cdot \boldsymbol{Y}] &= \boldsymbol{\theta}_{S+\boldsymbol{\nu}\cdot\boldsymbol{Y}}^m \cdot \boldsymbol{y} \\ &= (\boldsymbol{\theta}_S^m + \boldsymbol{\nu}) \cdot \boldsymbol{y} \\ &= \rho[S] + \boldsymbol{\nu} \cdot \boldsymbol{y}. \end{aligned}$$

We can conclude that ρ is a MC valuation.
(b) Let ρ be an actuarial valuation. Consider the hedger $\boldsymbol{\theta}^a$ with

$$\boldsymbol{\theta}_S^a = (\rho[S], 0, \ldots, 0),$$

for any claim S. Obviously, $\boldsymbol{\theta}^a$ is an actuarial hedger. Then we find that

$$\rho[S] = \boldsymbol{\theta}_S^a \cdot \boldsymbol{y}, \qquad \text{for any } S \in \mathcal{C}.$$

(b') Suppose that the valuation ρ is defined by (2.33) for some actuarial hedger $\boldsymbol{\theta}^a$ with underlying actuarial valuation π. For any orthogonal claim $S^\perp$, we have

$$\rho\left[S^\perp\right] = \boldsymbol{\theta}^a_{S^\perp} \cdot \boldsymbol{y} = \pi\left[S^\perp\right].$$

We can conclude that the valuation ρ is actuarial.

(c) Let ρ be a fair valuation. Consider a fair hedger $\boldsymbol{\theta}$, e.g. the mean-variance hedger, with underlying actuarial valuation π. From (a) we know that for any claim S, $\rho\left[S\right]$ can be expressed as

$$\rho\left[S\right] = \boldsymbol{\theta}^m_S \cdot \boldsymbol{y},$$

with the MC hedger $\boldsymbol{\theta}^m$ given by (2.35). Furthermore, for any orthogonal claim $S^\perp$, we find that

$$\begin{aligned}\boldsymbol{\theta}^m_{S^\perp} &= \boldsymbol{\theta}_{S^\perp} + \left(\rho\left[S^\perp - \boldsymbol{\theta}_{S^\perp}\cdot \boldsymbol{Y}\right], 0, \ldots, 0\right)\\ &= \left(\pi\left[S^\perp\right], 0, \ldots, 0\right) + \left(\rho\left[S^\perp - e^r\, \pi\left[S^\perp\right]\right], 0, \ldots, 0\right)\\ &= \left(\rho\left[S^\perp\right], 0, \ldots, 0\right).\end{aligned}$$

As ρ is an actuarial valuation, we can conclude that the hedger $\boldsymbol{\theta}^m$ is not only market-consistent but also actuarial and hence, a fair hedger.

(c') Suppose that the valuation ρ is defined by (2.34) for some fair hedger $\boldsymbol{\theta}^f$. From (a) and (b) we can conclude that the valuation ρ is market-consistent and actuarial, which means that it is fair. ■

From Theorem 1, we know that any fair value $\rho\left[S\right]$ can be considered as the time-0 price of a fair hedge:

$$\rho\left[S\right] = e^{-r}\, \mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\theta}^f_S \cdot \boldsymbol{Y}\right],$$

where $\mathbb{Q}$ is an EMM and $\boldsymbol{\theta}^f_S$ is a fair hedger. We remark that this result is mainly of a theoretical nature, and often not really useful in practice, as the fair hedge $\boldsymbol{\theta}^f_S$ is only implicitly specified, see (2.35). Moreover, notice that the fair hedger attached to a fair valuation is not uniquely determined.

Consider the fair valuation characterized by

$$\rho\left[S\right] = \boldsymbol{\theta}^f_S \cdot \boldsymbol{y}, \qquad \text{for any } S \in \mathcal{C}, \tag{2.36}$$

where $\boldsymbol{\theta}^f$ is a fair hedger with underlying actuarial valuation π. Furthermore, consider the claim $S^h \times S^\perp$, where $S^h = \boldsymbol{\nu} \cdot \boldsymbol{Y} \in \mathcal{H}$ and $S^\perp \in \mathcal{O}$. In case

$$\boldsymbol{\theta}^f_{S^h \times S^\perp} = \boldsymbol{\nu} \times e^r \; \pi \left[S^\perp \right] \tag{2.37}$$

holds, the fair value $\rho \left[S^h \times S^\perp \right]$ can be expressed as

$$\rho \left[S^h \times S^\perp \right] = \mathbb{E}^{\mathbb{Q}} \left[S^h \right] \times \pi \left[S^\perp \right], \tag{2.38}$$

for any EMM $\mathbb{Q}$. The condition (2.37) is always satisfied when $S^\perp = \mathbb{E}^{\mathbb{P}} \left[S^\perp \right]$. In this case, we have that $\pi \left[S^\perp \right] = e^{-r} \; \mathbb{E}^{\mathbb{P}} \left[S^\perp \right]$ and (2.38) reduces to the well-known Brennan & Schwartz-formula (2.21). In this sense, the expression (2.38) is a generalization of the Brennan & Schwartz result. As we will see in Section 2.4.3, the assumption (2.37) is satisfied and hence, the expression (2.38) holds in case $\boldsymbol{\theta}^f$ is the mean-variance hedger.

2.4 Hedge-based valuations

In this section, we present and investigate a class of fair valuations, the members of which we will call *hedge-based valuations*. We show that the classes of fair and hedge-based valuations are identical.

2.4.1 The general class of hedge-based valuations

In order to determine a hedge-based value of S, one first splits this claim into a hedgeable claim, which (partially) replicates S, and a remaining claim. The value of the claim S is then defined as the sum of the financial price of the hedgeable claim and the value of the remaining claim, determined according to a pre-specified actuarial valuation.

Definition 11 (Hedge-based valuation)
The valuation $\rho : \mathcal{C} \to \mathbb{R}$ is a hedge-based valuation (HB valuation) if for any claim S, the value $\rho[S]$ is determined by

$$\rho[S] = \boldsymbol{\theta}_S \cdot \boldsymbol{y} + \pi[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}]. \tag{2.39}$$

where $\boldsymbol{\theta}$ is a fair hedger and π is an actuarial valuation.

For any claim S, we call $\rho[S]$ a hedge-based value of S. It is easy to verify that the mapping ρ defined in (2.39) is normalized and translation invariant, and hence, a valuation as defined above.

From the definition above, we find that any HB valuation ρ reduces to an actuarial valuation for orthogonal claims:

$$\rho[S^{\perp}] = \pi[S^{\perp}], \qquad \text{for any } S^{\perp} \in \mathcal{O}.$$

Moreover, the HB value of any hedgeable claim is equal to the price of the underlying hedge:

$$\rho[S^h] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[S^h\right], \qquad \text{for any } S^h \in \mathcal{H}.$$

Sufficient conditions for positive homogeneity and subadditivity of hedge-based valuations are considered in the next theorem.

Theorem 2 For any HB valuation ρ with fair hedger $\boldsymbol{\theta}$ and actuarial valuation π, the following properties hold:
(1) If $\boldsymbol{\theta}$ and π are positive homogeneous, then ρ is positive homogeneous:

$$\rho\left[a\ S\right] = a\ \rho\left[S\right], \qquad \text{for any } a > 0 \text{ and } S \in \mathcal{C}. \tag{2.40}$$

(2) If $\boldsymbol{\theta}$ is additive and π is subadditive, then ρ is subadditive:

$$\rho\left[S_1 + S_2\right] \leq \rho\left[S_1\right] + \rho\left[S_2\right], \qquad \text{for any } S_1, S_2 \in \mathcal{C}. \tag{2.41}$$

The proof of the theorem is straightforward.

In the following theorem, it is proven that the class of hedge-based valuations is equal to the class of fair valuations.

Theorem 3 A mapping $\rho : \mathcal{C} \to \mathbb{R}$ is a HB valuation if and only if it is a fair valuation.

Proof:
(a) Consider the HB valuation ρ defined in (2.39). For any claim S, we can rewrite $\rho\left[S\right]$ as

$$\rho\left[S\right] = \boldsymbol{\mu}_S \cdot \boldsymbol{y}$$

with

$$\boldsymbol{\mu}_S = \boldsymbol{\theta}_S + \left(\pi\left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}\right], 0, \ldots, 0\right). \tag{2.42}$$

From Lemma 2, it follows that $\boldsymbol{\mu}$ is a fair hedger with underlying actuarial valuation π. Theorem 1 leads then to the conclusion that ρ is a fair valuation.
(b) Consider the fair valuation ρ. From Theorem 1, we know that there exists a fair hedger $\boldsymbol{\theta}^f$ such that $\rho[S] = \boldsymbol{\theta}^f_S \cdot \boldsymbol{y}$ for any claim S. Define the valuation ρ' by

$$\rho'\left[S\right] = \boldsymbol{\theta}^f_S \cdot \mathbf{y} + \rho[S - \boldsymbol{\theta}^f_S \cdot \mathbf{Y}]. \tag{2.43}$$

Obviously, ρ' is a HB valuation. Moreover, it is easy to verify that

$$\rho[S - \boldsymbol{\theta}^f_S \cdot \mathbf{Y}] = 0.$$

We can conclude that $\rho \equiv \rho'$, and hence, ρ is indeed a HB valuation. ■

One could define a broader class of HB valuations by requiring that the hedger $\boldsymbol{\theta}$ in (2.39) is a market-consistent hedger and π is an actuarial valuation. In this case the hedger $\boldsymbol{\mu}$ defined in (2.42) is market-consistent, but not necessarily fair, implying that such a generalized HB valuation is market-consistent but not necessarily fair.

2.4.2 Convex hedge-based valuations

We start this subsection by introducing a class of hedgers, which we baptize convex hedgers.

Definition 12 (Convex hedger) Consider a strictly convex non-negative function u with $u(0) = 0$. The convex hedger $\boldsymbol{\theta}^u$ is determined by

$$\boldsymbol{\theta}^u_S = \arg\min_{\boldsymbol{\mu}\in\Theta} \mathbb{E}^{\mathbb{P}}\left[u\left(S - \boldsymbol{\mu}\cdot\boldsymbol{Y}\right)\right], \qquad \text{for any } S \in \mathcal{C}. \tag{2.44}$$

The convex hedger $\boldsymbol{\theta}^u : \mathcal{C} \to \Theta$ attaches the hedge $\boldsymbol{\theta}^u_S$ to any claim S, such that the claim and the time-1 value of the hedge are 'close to each other' in the sense that the $\mathbb{P}$-expectation of the u-value of their difference is minimized. The choice of the convex function u determines how severe deviations are punished.

Theorem 4 The convex hedger $\boldsymbol{\theta}^u$ is a fair hedger with underlying actuarial valuation π^u satisfying

$$\pi^u\left[S^{\perp}\right] = \arg\min_{s\in\mathbb{R}} \mathbb{E}^{\mathbb{P}}\left[u\left(S^{\perp} - e^r\ s\right)\right], \qquad \text{for any } S^{\perp} \in \mathcal{O}. \tag{2.45}$$

Proof: Consider the convex hedger $\boldsymbol{\theta}^u$ defined in (2.44). We have to prove that $\boldsymbol{\theta}^u$ satisfies the conditions (2.24) and (2.25) of the definition of a fair hedger.
(a) For any hedgeable claim $S^h = \boldsymbol{\nu} \cdot \mathbf{Y}$, we have that

$$\begin{aligned}
\boldsymbol{\theta}^u_{S+\boldsymbol{\nu}\cdot\mathbf{Y}} &= \arg\min_{\boldsymbol{\mu}\in\Theta} \mathbb{E}^{\mathbb{P}}\left[u\left(S - (\boldsymbol{\mu}-\boldsymbol{\nu})\cdot\mathbf{Y}\right)\right] \\
&= \boldsymbol{\nu} + \arg\min_{\boldsymbol{\mu}'\in\Theta} \mathbb{E}^{\mathbb{P}}\left[u\left(S - \boldsymbol{\mu}'\cdot\mathbf{Y}\right)\right] \\
&= \boldsymbol{\nu} + \boldsymbol{\theta}^u_S,
\end{aligned}$$

which means that the condition (2.24) is satisfied.
(b) Consider the orthogonal claim $S^\perp \in \mathcal{O}$. Taking into account the independence of $S^\perp$ and $\mathbf{Y}$ as well as Jensen's inequality, we find for any trading strategy $\boldsymbol{\mu} \in \Theta$ that

$$\mathbb{E}^{\mathbb{P}}\left[u\left(S^\perp - \boldsymbol{\mu}\cdot\mathbf{Y}\right) \mid S^\perp\right] \geq u\left(S^\perp - \boldsymbol{\mu}\cdot\mathbb{E}^{\mathbb{P}}\left[\mathbf{Y}\right]\right).$$

Taking expectations on both sides leads to

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[u\left(S^\perp - \boldsymbol{\mu}\cdot\mathbf{Y}\right)\right] &\geq \mathbb{E}^{\mathbb{P}}\left[u\left(S^\perp - \boldsymbol{\mu}\cdot\mathbb{E}^{\mathbb{P}}\left[\mathbf{Y}\right]\right)\right] \\
&\geq \mathbb{E}^{\mathbb{P}}\left[u\left(S^\perp - e^r\ \pi^u\left[S^\perp\right]\right)\right],
\end{aligned}$$

which holds for any $\boldsymbol{\mu} \in \Theta$. Taking into account that $e^r\ \pi^u\left[S^\perp\right]$ can be rewritten as

$$e^r\ \pi^u\left[S^\perp\right] = \left(\pi^u\left[S^\perp\right], 0, \ldots, 0\right)\cdot\mathbf{Y},$$

with $\left(\pi^u\left[S^\perp\right], 0, \ldots, 0\right)$ being an element of Θ, we find that

$$\boldsymbol{\theta}^u_{S^\perp} = \left(\pi^u\left[S^\perp\right], 0, \ldots, 0\right).$$

Let us now extend the definition (2.45) of π^u to all $S \in \mathcal{C}$. It is easy to verify that π^u is an actuarial valuation. We can conclude that also the condition (2.25) is satisfied. ■

Definition 13 (Convex hedge-based valuation)
Consider a strictly convex non-negative function u with $u(0) = 0$. The valuation $\rho : \mathcal{C} \to \mathbb{R}$ defined by

$$\rho\left[S\right] = \boldsymbol{\theta}^u_S\cdot\mathbf{y} + \pi[S - \boldsymbol{\theta}^u_S\cdot\mathbf{Y}],$$

with convex hedger $\boldsymbol{\theta}^u$ and actuarial valuation π is called a convex hedge-based valuation (CHB valuation).

Corollary 1 Any CHB valuation is a fair valuation.

The proof of the corollary follows from observing that any CHB valuation is a HB valuation, implying that it is a fair valuation.

2.4.3 Mean-variance hedge-based valuations

A particular example of a convex hedge-based valuation arises when using the convex hedger with quadratic function $u(s) = s^2$. This hedger is called the mean-variance hedger.

Definition 14 (Mean-variance hedger) For any $S \in \mathcal{C}$, the mean-variance hedge $\boldsymbol{\theta}_S^{MV}$ (MV hedge) is the hedge for which the $\mathbb{P}$-expected quadratic hedging error is minimized:

$$\boldsymbol{\theta}_S^{MV} = \arg\min_{\boldsymbol{\mu}\in\Theta} \mathbb{E}^{\mathbb{P}}\left[(S - \boldsymbol{\mu}\cdot\mathbf{Y})^2\right]. \tag{2.46}$$

For an overview on the general theory of mean-variance hedging, we refer to Schweizer (2001).

Corollary 2 The mean-variance hedger $\boldsymbol{\theta}^{MV} : \mathcal{C} \to \Theta$ is a fair hedger with underlying actuarial valuation satisfying

$$\pi^{MV}\left[S^{\perp}\right] = e^{-r}\,\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right], \qquad \text{for any } S^{\perp} \in \mathcal{O}. \tag{2.47}$$

Proof: The MV hedger is a convex hedger, implying that it is a fair hedger. From (2.45) it follows that it has an actuarial valuation which satisfies (2.47). ■

In the following theorem, we present the unique solution $\boldsymbol{\theta}_S^{MV} = \left(\theta_S^{(0)}, \ldots, \theta_S^{(n)}\right)$ of the minimization problem (2.46), which is a standard result from least squares optimization. We use the notation $\mathbf{A}^{\intercal}$ for the transpose of a matrix $\mathbf{A}$.

Theorem 5 The mean-variance hedge $\boldsymbol{\theta}_S^{MV}$ of $S \in \mathcal{C}$ is uniquely determined from

$$\mathbb{E}^{\mathbb{P}}\left[\mathbf{Y}^{\intercal}\mathbf{Y}\right]\;\left(\boldsymbol{\theta}_S^{MV}\right)^{\intercal} = \mathbb{E}^{\mathbb{P}}\left[S\mathbf{Y}^{\intercal}\right]. \tag{2.48}$$

Proof: Taking partial derivatives of the objective function in (2.46) leads to (2.48). As the market of traded assets is assumed to be non-redundant, for any $\boldsymbol{\theta} \neq \mathbf{0}$, one has that

$$\boldsymbol{\theta}\;\mathbb{E}^{\mathbb{P}}\left[\mathbf{Y}^{\intercal}\mathbf{Y}\right]\;\boldsymbol{\theta}^{\intercal} = \mathbb{E}^{\mathbb{P}}\left[(\boldsymbol{\theta}_S\cdot\mathbf{Y})^2\right] > 0.$$

We can conclude that the matrix $\mathbb{E}^{\mathbb{P}}\left[\mathbf{Y}^{\intercal}\mathbf{Y}\right]$ is positive definite and hence, non-singular. This implies that the mean-variance hedge $\boldsymbol{\theta}_S^{MV}$ is uniquely determined and follows from (2.48). ■

It is a straightforward exercise to show that the system of equations (2.48) to determine $\boldsymbol{\theta}_S^{MV} = \left(\theta_S^{(0)}, \ldots, \theta_S^{(n)}\right)$ can be transformed into

$$\begin{cases} \sum_{m=1}^{n} \operatorname{cov}^{\mathbb{P}}\left[Y^{(k)}, Y^{(m)}\right] \ \theta_S^{(m)} = \operatorname{cov}^{\mathbb{P}}\left[Y^{(k)}, S\right], \quad \text{for } k = 1, ..., n \\ \theta_S^{(0)} = e^{-r}\left(\mathbb{E}^{\mathbb{P}}\left[S\right] - \sum_{m=1}^{n} \mathbb{E}^{\mathbb{P}}\left[Y^{(m)}\right] \ \theta_S^{(m)}\right) \end{cases} \tag{2.49}$$

In the following theorem, we provide some well-known properties of the MV hedger.

Theorem 6 The mean-variance hedger $\boldsymbol{\theta}^{MV}$ has the following properties:
(a) Any claim S and the time-1 value of its MV hedge are equal in $\mathbb{P}$-expectation:

$$\mathbb{E}^{\mathbb{P}}\left[S\right] = \mathbb{E}^{\mathbb{P}}\left[\boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y}\right], \quad \text{for any } S \in \mathcal{C}. \tag{2.50}$$

(b) The MV hedger is additive:

$$\boldsymbol{\theta}_{S_1+S_2}^{MV} = \boldsymbol{\theta}_{S_1}^{MV} + \boldsymbol{\theta}_{S_2}^{MV}, \quad \text{for any } S_1, S_2 \in \mathcal{C}. \tag{2.51}$$

(c) The MV hedger is positive homogeneous:

$$\boldsymbol{\theta}_{a\times S}^{MV} = a \times \boldsymbol{\theta}_S^{MV}, \quad \text{for any scalar } a > 0 \text{ and any } S \in \mathcal{C}. \tag{2.52}$$

(d) The MV hedge of the product of a hedgeable and an orthogonal claim:

$$\boldsymbol{\theta}_{S^h\times S^\perp}^{MV} = \boldsymbol{\nu} \times \mathbb{E}^{\mathbb{P}}\left[S^\perp\right], \quad \text{for any } S^h = \boldsymbol{\nu}\cdot\mathbf{Y} \in \mathcal{H} \text{ and } S^\perp \in \mathcal{O}. \tag{2.53}$$

Proof: The expression (2.50) follows immediately from the expression for $\theta_S^{(0)}$ in (2.49). The other expressions are easy to prove with the help of Theorem 5. ■

Based on the mean-variance hedger introduced above, we can define mean-variance hedge-based valuations.

Definition 15 (Mean-variance hedge-based valuation) The valuation $\rho : \mathcal{C} \to \mathbb{R}$ where for any claim S, $\rho[S]$ is determined by

$$\rho[S] = \boldsymbol{\theta}_S^{MV} \cdot \mathbf{y} + \pi \left[S - \boldsymbol{\theta}_S^{MV} \cdot \mathbf{Y} \right], \tag{2.54}$$

with $\boldsymbol{\theta}^{MV}$ is the mean-variance hedger and π is an actuarial valuation, is called a mean-variance hedge-based valuation (MVHB valuation).

As any MVHB valuation is a HB valuation, we immediately find the following result.

Corollary 3 Any MVHB valuation is a fair valuation.

Combining Theorems 2 and 6 leads to the following result.

Theorem 7 For any MVHB valuation ρ with underlying actuarial valuation π, the following properties hold:
(1) If π is positive homogeneous, then ρ is positive homogeneous.
(2) If π is subadditive, then ρ is subadditive.

In the following subsection, we illustrate the calculation of MVHB valuations with two examples.

2.4.4 Examples

Example 3
(a) Consider the financial-actuarial world in which a zero-coupon bond and a stock are traded. The current price of the zero-coupon bond equals $y^{(0)} = 1$, while its time-1 price is given by $Y^{(0)} = 1$. The stock trades at current price $y^{(1)} = 1/2$, whereas its value at time 1, notation $Y^{(1)}$, is either 0 or 1. In this world, we also observe a non-traded survival index. Its time-1 value $\mathcal{I}$ is either 0 (if few people of a given population survive) or 1 (in case many of them survive).
We model this financial-actuarial world in the probability space $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$, with the universe Ω given by

$$\Omega = \{(0,0), (0,1), (1,0)\,(1,1)\},$$

where each element denotes a possible scenario. The first component of any couple corresponds to a possible value of the stock price $Y^{(1)}$ at time 1, while the second component is a possible value of the survival index $\mathcal{I}$ at time 1. Suppose that the real-world probability measure $\mathbb{P}$ is characterized by

$$p_{00} = \frac{1}{6},\ p_{10} = \frac{2}{6},\ p_{01} = \frac{1}{6} \text{ and } p_{11} = \frac{2}{6},$$

where each p_{ij} stands for $\mathbb{P}\left[(i,j)\right]$. One can verify that the time-1 values $Y^{(1)}$ and $\mathcal{I}$ of the stock and the survival index are mutually independent under the physical measure $\mathbb{P}$, implying that the survival index is an orthogonal claim.
Let us now consider the valuation of the following non-traded hybrid claim:

$$S = \left(1 - Y^{(1)}\right) \times (1 - \mathcal{I}). \tag{2.55}$$

The MV hedge of S is given by

$$\boldsymbol{\theta}_S^{MV} = \arg\min_{\boldsymbol{\mu}\in\Theta} \mathbb{E}^{\mathbb{P}}\left[\left(S - \mu^{(0)} - \mu^{(1)}Y^{(1)}\right)^2\right] = \left(\frac{1}{2}, -\frac{1}{2}\right).$$

The MVHB value (2.54) of S is then equal to

$$\rho\left[S\right] = \frac{1}{4} + \pi\left[S - \frac{1}{2} + \frac{1}{2}Y^{(1)}\right].$$

Suppose that the actuarial valuation π is a cost-of-capital principle:

$$\pi\left[X\right] = \mathbb{E}^{\mathbb{P}}\left[X\right] + 0.06\left(\mathrm{VaR}_{0.995}\left[X\right] - \mathbb{E}^{\mathbb{P}}\left[X\right]\right), \qquad \text{for any } X \in \mathcal{C}. \tag{2.56}$$

As $\mathbb{E}^{\mathbb{P}}\left[S\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{1}{2} - \frac{1}{2}Y^{(1)}\right]$ and $\mathrm{VaR}_{0.995}\left[S - \frac{1}{2} + \frac{1}{2}Y^{(1)}\right] = 1/2$, we find that

$$\rho\left[S\right] = \frac{7}{25}.$$

(b) Next, we consider a market where in addition to the zero-coupon bond and the stock, also the survival index $\mathcal{I}$ is traded, with current price $y^{(2)} = \frac{2}{3}$ and value at time 1 given by $Y^{(2)} = \mathcal{I}$. The MV hedge of S is now given by

$$\begin{aligned}\boldsymbol{\theta}_S^{MV} &= \arg\min_{\boldsymbol{\mu}\in\Theta} \mathbb{E}^{\mathbb{P}}\left[\left(S - \mu^{(0)} - \mu^{(1)}Y^{(1)} - \mu^{(2)}Y^{(2)}\right)^2\right] \\ &= \left(\frac{2}{3}, -\frac{1}{2}, -\frac{1}{3}\right),\end{aligned}$$

while the MVHB valuation (2.54) of S takes the form

$$\rho[S] = \frac{7}{36} + \pi\left[S - \frac{2}{3} + \frac{1}{2}Y^{(1)} + \frac{1}{3}Y^{(2)}\right].$$

In case the actuarial valuation π is given by the cost-of-capital principle (2.56), taking into account that

$$\mathrm{VaR}_{0.995}\left[S - \frac{2}{3} + \frac{1}{2}Y^{(1)} + \frac{1}{3}Y^{(2)}\right] = 1/3,$$

we find that

$$\rho[S] = \frac{193}{900}.$$

(c) Let us now assume that, apart from the zero-coupon bond, the stock and the survival index $\mathcal{I}$, also the call option with current price $y^{(3)} = \frac{1}{6}$ and payoff at time 1 given by

$$Y^{(3)} = Y^{(2)} \times (Y^{(1)} - 0.5)_+ \tag{2.57}$$

is traded. The MV hedge of S now equals

$$\begin{aligned}\boldsymbol{\theta}_S^{MV} &= \arg\min_{\boldsymbol{\mu}\in\Theta} \mathbb{E}^{\mathbb{P}}\left[\left(S - \mu^{(0)} - \mu^{(1)}Y^{(1)} - \mu^{(2)}Y^{(2)} - \mu^{(3)}Y^{(3)}\right)^2\right] \\ &= (1, -1, -1, 2).\end{aligned}$$

The claim S is now perfectly hedged by its MV hedge:

$$S = Y^{(0)} - Y^{(1)} - Y^{(2)} + 2Y^{(3)}.$$

This is due to the fact that the introduction of the call option leads to a complete market, see forthcoming Example 5. In this case, the MVHB value (2.54) of S is given by the price of the replicating portfolio:

$$\rho[S] = \frac{1}{6}.$$

In this example, the fair value of S decreases by introducing additional traded assets. Notice however that this is not always necessarily the case. ◀

Example 4

(a) Consider a national population of N^{nat} members. For member

i, we introduce the Bernoulli r.v. I_i, which equals 0 if i dies in the coming year, while it equals 1 in the other case. The 'national survival index' I is given by

$$I = I_1 + I_2 + \ldots + I_{N^{\text{nat}}}.$$

Next, we consider an insured population, consisting of N^{ins} members, with J_i, $i = 1, 2, \ldots, N$, the Bernoulli r.v. which equals 1 in case insured i survives and 0 otherwise. Notice that the insured population is not necessarily a subset of the national population. The insurance claim at the end of the year is given by

$$S = J_1 + J_2 + \ldots + J_{N^{\text{ins}}}. \tag{2.58}$$

Suppose the financial market consists of 3 traded assets. The zero-coupon bond has current value $y^{(0)} = 1$, while its value at time 1 is given by $Y^{(0)} = e^r$. The second traded asset is a stock with current price $y^{(1)}$ and payoff at time 1 given by $Y^{(1)}$, which takes a value in the set $\mathcal{A}$. Finally, also the national survival index is traded. Its current value is $y^{(2)}$, while its payoff at time 1 is given by $Y^{(2)} = I$.

We model this financial-actuarial world by the probability space $\left(\Omega, 2^\Omega, \mathbb{P}\right)$, with

$$\Omega = \left\{(x_1, x_2, x_3) \mid x_1 \in \mathcal{A};\ x_2 = 0, 1..., N^{\text{nat}};\ x_3 = 0, 1, ..., N^{\text{ins}}\right\},$$

where any triplet (x_1, x_2, x_3) describes a possible outcome of the stock $Y^{(1)}$, the national survival index I and the insurance claim S, respectively. Throughout this example, we assume that mortality is independent of the stock price evolution. To be more precise, $Y^{(1)}$ and (I, S) are assumed to be mutually independent under the physical probability measure $\mathbb{P}$.

From (2.49) with $n = 2$, it follows that the mean-variance hedge $\boldsymbol{\theta}_S^{MV} = \left(\theta_S^{(0)}, \theta_S^{(1)}, \theta_S^{(2)}\right)$ of the insurance claim S is given by

$$\begin{cases} \theta_S^{(0)} = e^{-r}\left(\mathbb{E}^{\mathbb{P}}[S] - \mathbb{E}^{\mathbb{P}}[I]\ \frac{\text{cov}^{\mathbb{P}}[I,S]}{\text{var}^{\mathbb{P}}[I]}\right) \\ \theta_S^{(1)} = 0 \\ \theta_S^{(2)} = \frac{\text{cov}^{\mathbb{P}}[I,S]}{\text{var}^{\mathbb{P}}[I]}. \end{cases} \tag{2.59}$$

This MV hedging strategy for S does not contain an investment in the stock, due to its assumed $\mathbb{P}$- independence with mortality.

A higher correlation between the insurance claim and the national survival index leads, ceteris paribus, to a higher investment in the national survival index and a lower investment in zero coupon bonds.

(b) From here on, we assume that the insured population is a subset of the national population. More specifically, we assume that $N^{\text{ins}} \leq N^{\text{nat}}$ and $J_i = I_i$ for $i = 1, 2, \ldots, N^{\text{ins}}$. Furthermore, all I_i are assumed to be i.i.d. under $\mathbb{P}$, with $\mathbb{P}[I_i = 1] = p$. In this case we find that

$$\text{cov}^{\mathbb{P}}[I, S] = \text{var}^{\mathbb{P}}[S].$$

Taking into account the $\mathbb{P}$-i.i.d. assumption of the Bernoulli variables, one has that $\text{var}^{\mathbb{P}}[S] = Np(1-p)$, $\text{var}^{\mathbb{P}}[I] = Mp(1-p)$. These observations lead to the following MV hedge for S:

$$\begin{cases} \theta^{(0)} = 0 \\ \theta^{(1)} = 0 \\ \theta^{(2)} = \frac{N^{\text{ins}}}{N^{\text{nat}}}, \end{cases} \tag{2.60}$$

which corresponds with an investment in the national survival index only. The MVHB value (2.54) of S is then given by

$$\rho[S] = \frac{N^{\text{ins}}}{N^{\text{nat}}} y^{(2)} + \pi\left[S - \frac{N^{\text{ins}}}{N^{\text{nat}}} Y^{(2)}\right].$$

Suppose now that the actuarial valuation π is the standard-deviation principle:

$$\pi[X] = \mathbb{E}^{\mathbb{P}}[X] + \beta\sqrt{\text{var}[X]}, \qquad \text{for any } X \in \mathcal{C},$$

for some $\beta \geq 0$. In this case, we find that the MVHB value of S is given by

$$\rho[S] = \frac{N^{\text{ins}}}{N^{\text{nat}}} y^{(2)} + \beta\sqrt{\frac{N^{\text{ins}}}{N^{\text{nat}}}\left(N^{\text{nat}} - N^{\text{ins}}\right) p(1-p)}. \tag{2.61}$$

Obviously, when $N^{\text{ins}} = N^{\text{nat}}$, the insurance claim S is fully hedgeable, and we find that $\rho[S]$ is equal to the time-0 price of the national survival index. ◀

2.5 Two-step valuations

2.5.1 Conditional valuations and two-step valuations

In this section, we will introduce a class of valuations which is very closely related to, but slightly different from the two-step valuations proposed by Pelsser and Stadje (2014). Hereafter, a derivative of the vector of asset prices $\mathbf{Y}$ has to be understood as a claim that can be expressed in the form $f(\mathbf{Y})$, for some measurable function f. Hence, a derivative of $\mathbf{Y}$ is a r.v. defined on the measurable space $\left(\Omega, \mathcal{F}^{\mathbf{Y}}\right)$, where $\mathcal{F}^{\mathbf{Y}} \subseteq \mathcal{G}$ is the sigma-algebra generated by the asset price vector $\mathbf{Y}$. Examples of derivatives of $\mathbf{Y}$ are $\mathbb{E}^{\mathbb{P}}[S \mid \mathbf{Y}]$, $\mathrm{Var}^{\mathbb{P}}[S \mid \mathbf{Y}]$ and $\boldsymbol{\theta}_S \cdot \mathbf{Y}$, where S is a claim and $\boldsymbol{\theta}$ is a hedger. We denote the set of all derivatives of $\mathbf{Y}$ by $\mathcal{C}^{\mathbf{Y}}$.

Definition 16 (Conditional valuation) A conditional valuation is a mapping $\pi_{\mathbf{Y}} : \mathcal{C} \to \mathcal{C}^{\mathbf{Y}}$ attaching a derivative of $\mathbf{Y}$ to any claim S:

$$S \to \pi_{\mathbf{Y}}[S]$$

such that
(1) $\pi_{\mathbf{Y}}$ is normalized:

$$\pi_{\mathbf{Y}}[0] = 0$$

(2) $\pi_{\mathbf{Y}}$ is conditionally translation invariant:

$$\pi_{\mathbf{Y}}\left[S + S^h\right] = \pi_{\mathbf{Y}}[S] + e^{-r}\, S^h, \quad \text{for any } S \in \mathcal{C} \text{ and } S^h \in \mathcal{H}.$$

A conditional valuation is a mapping from the set of claims defined on $(\Omega, \mathcal{G})$ to the set of claims defined on $\left(\Omega, \mathcal{F}^{\mathbf{Y}}\right)$. For any conditional valuation, one has that

$$\pi_{\mathbf{Y}}[a] = e^{-r}\, a, \quad \text{for any scalar } a.$$

Notice that the derivative $\pi_{\mathbf{Y}}[S]$ may be hedgeable or not. Our definition of a conditional valuation is closely related but slightly different from the one proposed in Pelsser and Stadje (2014).

Definition 17 (Actuarial conditional valuation) An actuarial conditional valuation $\pi_{\mathbf{Y}}$ is a conditional valuation which reduces to an actuarial valuation on $\mathcal{O}$:

$$\pi_{\mathbf{Y}}\left[S^{\perp}\right] = \pi\left[S^{\perp}\right], \quad \text{for any } S^{\perp} \in \mathcal{O},$$

for some actuarial valuation π.

Hereafter, we will always denote the underlying actuarial valuation of an actuarial conditional valuation $\pi_{\mathbf{Y}}$ by π.

A first example of an actuarial conditional valuation is the *conditional standard deviation principle*:

$$\pi_{\mathbf{Y}}[S] = e^{-r}\left(\mathbb{E}^{\mathbb{P}}[S \mid \mathbf{Y}] + \beta\sqrt{\mathrm{Var}^{\mathbb{P}}[S \mid \mathbf{Y}]}\right), \tag{2.62}$$

for any $S \in \mathcal{C}$, where β is a non-negative real number.

As a second example of an actuarial conditional valuation, consider the *conditional cost-of-capital principle*:

$$\pi_{\mathbf{Y}}[S] = e^{-r}\left(\mathbb{E}^{\mathbb{P}}[S \mid \mathbf{Y}] + i\ \left(\mathrm{VaR}_p[S \mid \mathbf{Y}] - \mathbb{E}^{\mathbb{P}}[S \mid \mathbf{Y}]\right)\right), \tag{2.63}$$

for any $S \in \mathcal{C}$ and a given probability level p and cost-of-capital rate i, and where $\mathrm{VaR}_p[S \mid \mathbf{Y}]$ is the Value-at-Risk of S at confidence level p, conditional on the available information concerning the asset prices at time 1.

A third example of an actuarial conditional valuation is given by

$$\pi_{\mathbf{Y}}[S] = e^{-r}\ \boldsymbol{\theta}_S^f \cdot \mathbf{Y}, \tag{2.64}$$

where $\boldsymbol{\theta}^f$ is a fair hedger.

Definition 18 (Two-step valuation) A mapping $\rho : \mathcal{C} \to \mathbb{R}$ is a two-step valuation (TS valuation) if there exists an actuarial conditional valuation $\pi_{\mathbf{Y}}$ and an EMM $\mathbb{Q}$ such that for any claim S, $\rho[S]$ is determined by

$$\rho[S] = \mathbb{E}^{\mathbb{Q}}[\pi_{\mathbf{Y}}[S]]. \tag{2.65}$$

One can easily verify that the mapping defined in (2.65) is normalized and translation invariant, implying that a TS valuation is indeed a valuation as defined above. For any claim S, $\rho[S]$ is called the two-step value (TS value) of S. The two-step valuation is characterized by an actuarial conditional valuation $\pi_{\mathbf{Y}}$ and an EMM $\mathbb{Q}$. It is determined by first applying the actuarial conditional valuation $\pi_{\mathbf{Y}}$ to S, and then determining the market price of the derivative $\pi_{\mathbf{Y}}[S]$, based on a given pricing measure $\mathbb{Q}$.

As a first example of a TS valuation, consider the *two-step standard deviation valuation*, where the value of any claim S is determined by

$$\rho\left[S\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}\left[S \mid \mathbf{Y}\right] + \beta\ \sqrt{\mathrm{Var}^{\mathbb{P}}\left[S \mid \mathbf{Y}\right]}\right]. \tag{2.66}$$

This means that $\rho\left[S\right]$ is determined as the financial market price of the derivative that arises from applying the conditional standard deviation principle on the claim S, given the time-1 prices of traded assets.

A second example of a TS valuation is the *two-step cost-of-capital valuation*:

$$\rho\left[S\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}\left[S \mid \mathbf{Y}\right] + i\ \left(\mathrm{VaR}_p\left[S \mid \mathbf{Y}\right] - \mathbb{E}^{\mathbb{P}}\left[S \mid \mathbf{Y}\right]\right)\right]. \tag{2.67}$$

Finally, a third example of a TS valuation is given by

$$\rho\left[S\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\theta}_S^f \cdot \mathbf{Y}\right], \tag{2.68}$$

where $\boldsymbol{\theta}^f$ is a fair hedger.

Pelsser and Stadje (2014) assume that the financial market of the $(n+1)$ traded assets is complete in $\left(\Omega, \mathcal{F}^{\mathbf{Y}}, \mathbb{P}\right)$. Equivalently stated, they assume that any derivative $f\left(\mathbf{Y}\right)$ is hedgeable. In particular, any claim $\pi_{\mathbf{Y}}\left[S\right]$ is hedgeable, and hence, its market value is uniquely determined. The completeness condition means that there exists a mapping $\boldsymbol{\theta}^{TS} : \mathcal{C} \to \Theta$ such that

$$\boldsymbol{\theta}_S^{TS} \cdot \mathbf{Y} = e^{r}\ \pi_{\mathbf{Y}}\left[S\right], \qquad \text{for any } S \in \mathcal{C}. \tag{2.69}$$

We call $\boldsymbol{\theta}^{TS}$ the *two-step hedger of the two-step valuation* ρ. Due to the non-redundancy assumption (2.1), the time-1 value $\boldsymbol{\theta}_S^{TS} \cdot \mathbf{Y}$ uniquely determines $\boldsymbol{\theta}_S^{TS}$. It is straightforward to prove that $\boldsymbol{\theta}^{TS}$ is a fair hedger with

$$\boldsymbol{\theta}_{S^\perp}^{TS} = \left(\pi\left[S^\perp\right], 0, \ldots, 0\right), \qquad \text{for any } S^\perp \in \mathcal{O}. \tag{2.70}$$

Under the completeness assumption, the TS value $\rho\left[S\right]$ of S can be expressed as

$$\rho\left[S\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\theta}_S^{TS} \cdot \mathbf{Y}\right] = \boldsymbol{\theta}_S^{TS} \cdot \mathbf{y}, \tag{2.71}$$

which does not depend on the particular choice of the pricing measure $\mathbb{Q}$.

Hereafter, we will not make the completeness assumption, which implies that we have to choose a particular measure $\mathbb{Q}$ in the set of all feasible pricing measures and hence $\rho[S]$ might depend on this choice.

In the special case there is no financial market, except the risk-free bank account, any claim S is an orthogonal claim, and the two-step valuation reduces to an actuarial valuation:

$$\rho[S] = \pi[S].$$

In the following theorem, we prove that the class of two-step valuations is identical to the class of fair valuations.

Theorem 8 A mapping $\rho : \mathcal{C} \to \mathbb{R}$ is a TS valuation if and only if it is a fair valuation.

Proof: (a) Consider the TS valuation ρ with $\rho[S] = \mathbb{E}^{\mathbb{Q}}[\pi_{\mathbf{Y}}[S]]$ for any claim S. It is straightforward to prove that ρ is both market-consistent and actuarial valuation, which means that ρ is a fair valuation.
(b) Consider the fair valuation ρ. From Theorem 1, we know that there exists a fair hedger $\boldsymbol{\theta}^f$ such that

$$\rho[S] = e^{-r}\,\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\theta}_S^f \cdot \boldsymbol{Y}\right], \qquad \text{for any } S \in \mathcal{C}.$$

As $e^{-r}\,\boldsymbol{\theta}_S^f \cdot \boldsymbol{Y}$ is an actuarial conditional valuation, we can conclude that ρ is a TS valuation. ■

Consider the TS valuation ρ with underlying conditional valuation $\pi_{\boldsymbol{Y}}$ and EMM $\mathbb{Q}$. Let $S^h \in \mathcal{H}$ and $S^{\perp} \in \mathcal{O}$. In case

$$\pi_{\mathbf{Y}}\left[S^h \times S^{\perp}\right] = S^h \times \pi\left[S^{\perp}\right], \tag{2.72}$$

we find that the TS value of $S^h \times S^{\perp}$ is given by

$$\rho\left[S^h \times S^{\perp}\right] = \mathbb{E}^{\mathbb{Q}}\left[S^h\right] \times \pi\left[S^{\perp}\right]. \tag{2.73}$$

In the special case of complete diversification of the orthogonal claim, i.e. when $S^{\perp} = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]$, we have that the condition (2.72) is satisfied and (2.73) reduces to the Brennan-Schwartz formula

(2.21). From this result, it follows that the formula (2.73) is an intuitive generalization of the formula (2.21), proposed by Brennan and Schwartz (1976). It is a straightforward exercise to prove that the generalized Brennan-Schwartz formula (2.73) holds for the TS standard deviation valuation as well as for the TS Cost-of-Capital valuation, defined in (2.66) and (2.67), respectively, provided $S^h \geq 0$.

2.5.2 Examples

We end this section with two illustrative examples, which are the counterparts of the Examples 3 and 4 considered in Subsection 2.4.4.

Example 5
(a) Consider the financial-actuarial world $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$ as described in Example 3, with a non-traded survival index and a market of traded assets consisting of a zero-coupon bond and a stock. Suppose that we want to determine the fair value $\rho\left[S\right]$ of the hybrid claim S defined in (2.55) according to the two-step cost-of-capital valuation (2.67) with $r = 0$, $p = 0.995$ and $i = 0.06$. Taking into account that $\mathcal{I} \in \mathcal{C}^{\perp}$ and applying the generalized Brennan-Schwartz formula (2.73), we find that

$$\rho\left[S\right] = \mathbb{E}^{\mathbb{Q}}\left[1 - Y^{(1)}\right] \times \pi\left[1 - \mathcal{I}\right] = \frac{53}{200}.$$

Notice that in this setting, the vector of time-1 asset prices is given by $\boldsymbol{Y} = \left(Y^{(0)}, Y^{(1)}\right)$ and any derivative $f\left(\boldsymbol{Y}\right)$ is hedgeable. One can easily verify that the TS hedger $\boldsymbol{\theta}^{TS}$ is given by

$$\boldsymbol{\theta}^{TS} = \left(0.53, \ -0.53\right),$$

from which we find that

$$\rho\left[S\right] = \boldsymbol{\theta}^{TS}_{S} \cdot \boldsymbol{y} = \frac{53}{200}.$$

(b) Suppose now that, apart from the zero-coupon bond and the stock, also the survival index $\mathcal{I}$ is traded, with current price $y^{(2)} = 2/3$ and time-1 value $Y^{(2)} = \mathcal{I}$. In this case, the vector of time-1 asset prices is $\boldsymbol{Y} = \left(Y^{(0)}, Y^{(1)}, Y^{(2)}\right)$ and S is a non-hedgeable derivative of $\boldsymbol{Y}$. The two-step cost-of-capital valuation (2.67) transforms into

$$\rho\left[S\right] = \mathbb{E}^{\mathbb{Q}}\left[\left(1 - Y^{(1)}\right) \times \left(1 - Y^{(2)}\right)\right].$$

In order to fully characterize ρ, one has to choose a particular risk-neutral measure $\mathbb{Q}$ for the financial market. One can easily verify that $\mathbb{Q} \equiv (q_{00},\ q_{10},\ q_{01},\ q_{11})$ is an EMM if and only if there exists a $q \in \left(0, \frac{1}{3}\right)$ such that

$$q_{00} = q,\ q_{10} = \frac{1}{3} - q,\ q_{01} = \frac{1}{2} - q \text{ and } q_{11} = \frac{1}{6} + q. \tag{2.74}$$

Given that the payoff of S only differs from zero in the scenario $\left(Y^{(1)}, Y^{(2)}\right) = (0,0)$, we find that

$$\rho\left[S\right] = q.$$

The two-step value $\rho\left[S\right]$ can take any value in $\left(0, \frac{1}{3}\right)$, depending on the choice of the EMM. In case we require e.g. that $Y^{(1)}$ and $\mathcal{I}$ are independent under $\mathbb{Q}$, we find that the two-step value of S is equal to $1/6$.

(c) Let us now assume that apart from the zero coupon bond, the stock and the survival index, also the call option with current price $y^{(3)} = \frac{1}{6}$ and payoff at time 1 given by (2.57) is traded in the market. In this case, the set of EMM's is defined by (2.74) for some $q \in \left(0, \frac{1}{3}\right)$, complemented with the additional requirement

$$y^{(3)} = \mathbb{E}^{\mathbb{Q}}\left[Y^{(3)}\right].$$

This situation leads to a unique EMM $\mathbb{Q}$ characterized by (2.74) with $q = \frac{1}{6}$. We can conclude that in this complete market setting, the fair value of S is given by

$$\rho\left[S\right] = \frac{1}{6}.$$

Notice that under this unique EMM $\mathbb{Q}$, the payoffs $Y^{(1)}$ and $\mathcal{I}$ are independent. ◀

Example 6 Consider the financial-actuarial world described in Example 4(b). We model this financial-actuarial world by the probability space $\left(\Omega, 2^{\Omega}, \mathbb{P}\right)$, with

$$\Omega = \left\{(x_1, x_2, x_3) \mid x_1 \in \mathcal{A};\ x_2 = 0, 1..., N^{\text{nat}};\ x_3 = 0, 1, ..., N^{\text{ins}}\right\},$$

where any triplet (x_1, x_2, x_3) describes a possible outcome of the stock $Y^{(1)}$, the national survival index $Y^{(2)}$ and the insurance claim S, respectively. Suppose that the insurance claim S defined in (2.58) is valuated according to the TS standard deviation valuation (2.66), with $\boldsymbol{Y} = \left(Y^{(0)}, Y^{(1)}, Y^{(2)}\right)$. From the assumed $\mathbb{P}$-independence between mortality and the stock price, we find that

$$\rho\left[S\right] = e^{-r}\ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}\left[S \mid I\right] + \beta\sqrt{\mathrm{var}^{\mathbb{P}}\left[S \mid I\right]}\right]$$

Taking into account that

$$\mathbb{E}^{\mathbb{P}}\left[S \mid I\right] = \frac{N^{\mathrm{ins}}}{N^{\mathrm{nat}}} I$$

and

$$\mathrm{var}^{\mathbb{P}}\left[S \mid I\right] = \frac{N^{\mathrm{ins}}\left(N^{\mathrm{nat}} - N^{\mathrm{ins}}\right)}{N^{\mathrm{nat}}\left(N^{\mathrm{nat}} - 1\right)} I \frac{N^{\mathrm{nat}} - I}{N^{\mathrm{nat}}},$$

one finds that

$$\begin{aligned}\rho\left[S\right] = {} & \frac{N^{\mathrm{ins}}}{N^{\mathrm{nat}}} y^{(2)} \\ & + \beta e^{-r}\mathbb{E}^{\mathbb{Q}}\left[\sqrt{\frac{N^{\mathrm{ins}}\left(N^{\mathrm{nat}} - N^{\mathrm{ins}}\right)}{N^{\mathrm{nat}}\left(N^{\mathrm{nat}} - 1\right)} Y^{(2)} \frac{N^{\mathrm{nat}} - Y^{(2)}}{N^{\mathrm{nat}}}}\right].\end{aligned} \tag{2.75}$$

The incompleteness of the market requires the choice of an EMM $\mathbb{Q}$ for the valuation of S. In case the insurance and national populations coincide, it follows from (2.75) that the TS value $\rho\left[S\right]$ of S is equal to the price $y^{(2)}$ of the 'national survival index'. ◀

2.6 Final remarks

The fair value of a hybrid claim, which is by definition neither hedgeable nor orthogonal, is in general not uniquely determined. This is not only due to the involvement of actuarial judgement, but at an earlier stage in the valuation process also due to the ambiguity that exists in how to determine the hedgeable part of such a hybrid claim.

In this chapter we proposed a framework to combine market-consistency and actuarial considerations in a so-called *fair valuation*. In such a setting, a market-consistent valuation of claims is based on an extension of cash invariance to all hedgeable claims, such that all claims are valuated in agreement with current market prices. Under a market-consistent valuation, the valuation of hedgeable claims is consistent with risk-neutral pricing based on an EMM $\mathbb{Q}$. An actuarial valuation on the other hand, is typically performed with an actuarial premium principle, based on a physical probability measure $\mathbb{P}$, chosen by the actuary. In such a setting, the problem the actuary is solving is to value the claim such that the insurer will be able to pay the observed claim amount at the end of the period, ignoring the existence of a financial market. A fair valuation combines the financial approach of a market-consistent valuation and the actuarial approach of an actuarial valuation. Such a fair valuation makes use of $\mathbb{P}$- and $\mathbb{Q}$-measures and in this sense, it can be considered as the right setting to value claims which have financial and actuarial components.

We presented a fair valuation technique, baptized hedge-based valuation, where one first unbundles the hybrid claim in a hedgeable claim (determined from the original claim according to some well-defined hedging procedure) and the remaining claim (i.e. the original claim minus the payoff of the hedgeable claim). The fair value of the claim is then defined as the sum of the financial market price of the hedge and the actuarial value of the remaining claim.

We also investigated the class of two-step valuations. The definition of a two-step valuation proposed in this chapter is inspired by the two-step valuations definition of Pelsser and Stadje (2014). Our definition is slightly different, as opposed to the original approach of Pelsser and Stadje (2014), we do not require that the market of traded assets is complete.

We showed that the set of fair valuations coincides with the set of hedge-based valuations and also with the set of two-step valuations. The two-step and the hedge-based approaches are only two different ways of identifying the different members of the set of fair valuations. The hedge-based approach starts from the choice of a hedger and does not require the choice of an EMM. In general, the two-step approach starts from the choice of an EMM. In case the market of traded assets is complete, the two-step approach does not require the choice of an EMM and the underlying two-step hedger is determined via (2.69). The hedge-based approach has the advantage of providing an explicit additive decomposition of the fair value into a financial price and an actuarial value while in the two-step approach the decomposition is performed through a less intuitive conditional procedure.

In the next chapters, we will extend the notion of fair valuation which was defined here in a static one-period setting to the case of a multi-period setting where dynamic hedging strategies are allowed. Moreover, we will investigate the requirement of time-consistency of fair dynamic valuations.

Chapter 3

Fair valuation of insurance liabilities via mean-variance hedging in a multi-period setting

This chapter is based on
Barigou, K., & Dhaene, J. (2019). Fair valuation of insurance liabilities via mean-variance hedging in a multi-period setting. *Scandinavian Actuarial Journal*, **2019(2)**, 163-187.

3.1 Introduction

In the previous chapter, a general class of fair valuations which are both market-consistent (mark-to-market for any hedgeable part of a claim) and actuarial (mark-to-model for any claim that is independent of financial market evolutions) was introduced in a single period framework. In particular, we considered *mean-variance hedge-based (MVHB)* valuations where fair valuations of insurance liabilities are expressed in terms of mean-variance hedges and actuarial valuations. In this chapter, we generalize this MVHB approach to a multi-period dynamic investment setting. We will show that the classes of fair valuations and MVHB valuations are equivalent in this generalized setting. We will illustrate how in the

MVHB valuation framework applied to the valuation of equity-linked insurance claims in a stochastic mortality setting, the actuarial part of the valuation decomposes into a diversifiable and a non-diversifiable component. As another illustration, we will consider the fair valuation of a portfolio of equity-linked contracts where the self-financing trading strategy depends on the number of survivors in the insured population, a case which is rarely considered in the literature.

Throughout this chapter, we will give particular attention to time-T claims of the form

$$S = S^{\perp} \times S^{f},$$

where $S^{\perp}$ is a $T-$claim which is independent of the financial market evolutions, while S^{f} is a financial $T-$claim. Such *product claims* often arise in insurance as payoffs of equity-linked life-insurance contracts. For local risk minimization of such payoffs, see e.g. Pansera (2012) and Gaillardetz and Moghtadai (2017).

The rest of this chapter is structured as follows. In Section 3.2 we generalize the combined financial-actuarial world from the previous chapter to a multi-period setting. In this world, we will introduce the concepts of orthogonal claims, financial trading strategies and financially hedgeable claims. In Section 3.3 we consider mean-variance hedging in discrete time. We investigate the mean-variance hedge for product claims, as well as the mean-variance hedge for general claims in a linear subset of self-financing trading strategies available to the valuator. In Section 3.4, fair valuations and MVHB valuations in a multi-period setting are introduced. In particular, we show that these two classes of valuations are equivalent and provide some detailed illustrative examples. Section 3.5 concludes the chapter.

3.2 The combined financial-actuarial world

From now on, we consider a combined financial-actuarial world in a multi-period setting. The extension from one period to multi-period requires the introduction of new concepts (e.g. self-financing strategies and financially hedgeable claims) but also the generalization of previous notions such as orthogonal and hedgeable claims. The multi-period combined world is defined hereafter.

Consider a combined financial-actuarial world which is home to tradable as well as non-tradable claims. The time horizon is given by T, which is an element of the set $\{1, 2, \ldots\}$. The financial-actuarial world is modeled by the probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$, equipped with the finite and discrete-time filtration $\mathbb{G} = \{\mathcal{G}_t\}_{t \in \tau}$ with $\tau = \{0, 1, , \ldots, T\}$. The initial $\sigma-$algebra $\mathcal{G}_0$ is set equal to $\{\emptyset, \Omega\}$ while the $\sigma-$algebra $\mathcal{G}_T$ is identical to $\mathcal{G}$. The $\sigma-$algebra $\mathcal{G}_t$, $t \in \tau$, represents the general information available up to and including time t in the combined world. Further, $\mathbb{P}$ is the measure attaching physical probabilities to all events in that world. Like the previous chapter, we assume that all r.v.'s that we consider have finite second order moments under $\mathbb{P}$. Furthermore, we will denote the set of all *$t-$claims* defined on $(\Omega, \mathbb{G}, \mathcal{G})$, that is the set of all $\mathcal{G}_t-$measurable r.v.'s, by $\mathcal{C}_t$.

The combined financial-actuarial world hosts a number of insurance liabilities, which are due at time T. Any insurance liability is represented by a $T-$claim, which will be generally denoted by $S(T)$ or simply by S if no confusion is possible. A simple example of an insurance liability related to the remaining lifetime T_x of an insured (x) observed at time 0 is the indicator variable S defined by

$$S = \begin{cases} 0 & : T_x \leq T \\ 1 & : T_x > T \end{cases} \tag{3.1}$$

The combined financial-actuarial world $(\Omega, \mathcal{G}, \mathbb{P})$ is also home to a financial market of $n \in \{1, 2, \ldots\}$ tradable (non-dividend paying) risky assets and a risk-free bank account. For any $i = 1, 2, \ldots, n$, we introduce the notation $Y^{(i)}(t)$ for the market price of 1 unit of risky asset i at time $t \in \tau$. The risky assets can be stocks, bonds, mutual funds, etc. The time$-t$ value of an investment of amount 1 at time 0 in the risk-free bank account is given by $Y^{(0)}(t) = e^{rt}$, where $r \geq 0$ is the deterministic and constant risk-free interest rate. We assume that any tradable asset can be bought and/or sold in any quantities in a deep, liquid and transparent market with negligible transactions costs and other market frictions.

The price processes of the traded assets are described by the $(n+1)-$dimensional stochastic process $\boldsymbol{Y} = \{\boldsymbol{Y}(t)\}_{t \in \tau}$. Here, $\boldsymbol{Y}(t)$, $t \in \tau$, is the vector of time$-t$ prices of all tradable assets, i.e. $\boldsymbol{Y}(t) = \left(Y^{(0)}(t), Y^{(1)}(t), \ldots, Y^{(n)}(t)\right)$. We assume that the price process $\boldsymbol{Y}$ is adapted to the filtration $\mathbb{G}$:

$$\boldsymbol{Y}(t) \text{ is } \mathcal{G}_t - \text{measurable}, \qquad \text{for any } t \in \tau.$$

The filtration $\mathbb{G}$ may simply coincide with the filtration generated by the price process $\boldsymbol{Y}$. In this chapter however, we will consider a more general setting, where $\mathbb{G}$ is not only related to the price history of traded assets, but may also contain information related to non-tradable claims such as a survival index of a particular population.

A *trading strategy* (also called a *dynamic portfolio*) $\boldsymbol{\theta} = \{\boldsymbol{\theta}(t)\}_t$ is a *predictable* $(n+1)-$dimensional process with respect to the filtration $\mathbb{G}$:

$$\boldsymbol{\theta}(t) \text{ is } \mathcal{G}_{t-1} - \text{measurable}, \qquad \text{for any } t \in \{1, 2, \ldots, T\}.$$

The vector $\boldsymbol{\theta}(t) = \left(\theta^{(0)}(t), \theta^{(1)}(t) \ldots, \theta^{(n)}(t)\right)$ represents the number of units $\theta^{(i)}(t)$ invested in each asset i in time period t, that is in the time interval $(t-1, t]$. The $\mathcal{G}_{t-1}-$measurability requirement means that the portfolio composition $\boldsymbol{\theta}(t)$ for the period $(t-1, t]$ follows from the general information available up to and including time $t-1$, i.e. the information collected in time interval $[0, t-1]$. This information includes in particular the price history of traded assets in that time interval.

The value at time t of the trading strategy $\boldsymbol{\theta}$ is denoted by $V^{\boldsymbol{\theta}}(t)$:

$$V^{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}(t) \cdot \boldsymbol{Y}(t) = \sum_{i=0}^{n} \theta^{(i)}(t)\ Y^{(i)}(t), \qquad \text{for any } t = 1, 2, \ldots, T,$$

while

$$V^{\boldsymbol{\theta}}(0) = \boldsymbol{\theta}(1) \cdot \boldsymbol{Y}(0) = \sum_{i=0}^{n} \theta^{(i)}(1)\ Y^{(i)}(0).$$

Obviously, $V^{\boldsymbol{\theta}}(t)$ is $\mathcal{G}_t-$measurable. For any $t > 0$, we have that $V^{\boldsymbol{\theta}}(t)$ is the value of the trading strategy at time t, just before eventual rebalancing, whereas $V^{\boldsymbol{\theta}}(0)$ is the *initial investment* or the *endowment* of the trading strategy $\boldsymbol{\theta}$.

Fair valuation in the single period case $T = 1$ was investigated in detail in Chapter 2. Hereafter, we will always assume that

$T \geq 2$, implying that there is at least one rebalancing moment. A trading strategy $\boldsymbol{\theta}$ is said to be *self-financing* if

$$\boldsymbol{\theta}(t) \cdot \boldsymbol{Y}(t) = \boldsymbol{\theta}(t+1) \cdot \boldsymbol{Y}(t), \qquad \text{for any } t = 1, ..., T-1. \tag{3.2}$$

This means that no capital is injected or withdrawn at any rebalancing moment $t = 1, ..., T-1$. We denote the set of self-financing trading strategies by Θ. Taking into account (3.2), the time-T value of any self-financing strategy $\boldsymbol{\theta} \in \Theta$ with initial investment $V^{\boldsymbol{\theta}}(0)$ can be expressed as

$$V^{\boldsymbol{\theta}}(T) = \boldsymbol{\theta}(T) \cdot \boldsymbol{Y}(T) = V^{\boldsymbol{\theta}}(0) + \sum_{t=1}^{T} \boldsymbol{\theta}(t) \cdot \Delta \boldsymbol{Y}(t), \tag{3.3}$$

with $\Delta \boldsymbol{Y}(t) = \boldsymbol{Y}(t) - \boldsymbol{Y}(t-1)$. In this formula, $\boldsymbol{\theta}(t) \cdot \Delta \boldsymbol{Y}(t)$ is the change of the market value of the investment portfolio in the time period $(t-1, t]$, i.e. between time $t-1$ (just after rebalancing) and time t (just before rebalancing).

We will always assume that the market of traded assets is *arbitrage-free* in the sense that there is no self-financing strategy $\boldsymbol{\theta} \in \Theta$ with the following properties:

$$\mathbb{P}\left[V^{\boldsymbol{\theta}}(0) = 0\right] = 1, \ \mathbb{P}\left[V^{\boldsymbol{\theta}}(T) \geq 0\right] = 1 \text{ and } \mathbb{P}\left[V^{\boldsymbol{\theta}}(T) > 0\right] > 0. \tag{3.4}$$

In our discrete-time setting, the absence of arbitrage is equivalent to the existence of an equivalent martingale measure $\mathbb{Q}$, such that the price $Y^{(i)}(t)$ of any traded asset i at any trading date t can be expressed as

$$Y^{(i)}(t) = e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}\left[Y^{(i)}(T) \mid \mathcal{G}_t\right]. \tag{3.5}$$

For a proof of this equivalence, we refer to Chapter 6 in Delbaen and Schachermayer (2006).

Definition 19 (Hedgeable $\boldsymbol{T}-$claim) A hedgeable $T-$claim S is an element of $\mathcal{C}_T$ that can be replicated by a self-financing strategy $\boldsymbol{\theta} \in \Theta$:

$$S = V^{\boldsymbol{\theta}}(T).$$

We will denote the set of all hedgeable $T-$claims by $\mathcal{H}_T$. The no-arbitrage assumption guarantees that the time$-t$ price $S(t)$ of a hedgeable $T-$claim S is equal to the time$-t$ price of the underlying self-financing strategy $\boldsymbol{\theta}$:

$$S(t) = V^{\boldsymbol{\theta}}(t) = e^{-r(T-t)} \ \mathbb{E}^{\mathbb{Q}} \left[S \mid \mathcal{G}_t \right]. \tag{3.6}$$

In this chapter, we consider an incomplete market setting. This means that apart from the hedgeable $T-$claims, of which the valuation is straightforward, there are also $T-$claims that cannot be perfectly replicated. A possible example of an unhedgeable $T-$claim is the r.v. S defined in (3.1).

A self-financing strategy is by definition $\mathbb{G}-$predictable. Hence, the rebalancing of the portfolio at any time t may depend on all information available up to time t, not only including observed asset prices, but also actuarial information such as survival indices, earthquake indices, etc. Hereafter, we will often consider the smaller set of self-financing strategies which are predictable with respect to the financial information. For this purpose, we introduce the financial filtration $\mathbb{F}$. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in \mathcal{T}}$ contains all information about financial events. This filtration may coincide with the filtration $\mathbb{F}^{\boldsymbol{Y}}$ generated by the price process $\mathbf{Y}$ but may include additional financial information, such as economic barometers and/or information about non-traded securities. Hence, in general we have that

$$\mathbb{F}^{\boldsymbol{Y}} \subseteq \mathbb{F} \subseteq \mathbb{G}.$$

We will denote the set of all *financial* $t-$*claims*, that is the set of all $\mathcal{F}_t-$measurable r.v.'s, by $\mathcal{C}_t^{\mathcal{F}}$. It is obvious that

$$\mathcal{C}_t^{\mathcal{F}} \subseteq \mathcal{C}_t.$$

Furthermore, we introduce the notation $\Theta^{\mathcal{F}}$ for the set of self-financing strategies which are predictable with respect to $\mathbb{F}$ and call its elements *financial* self-financing trading strategies, as they are based on the financial filtration. We have that

$$\Theta^{\mathcal{F}} \subseteq \Theta.$$

For any financial self-financing strategy $\boldsymbol{\theta} \in \Theta^{\mathcal{F}}$, the investor selects his period $(t-1, t]$ portfolio, based on the financial information observed in the time period $[0, t-1]$, including asset prices and other additional financial information.

Next, we define the set of financially hedgeable $T-$claims.

Definition 20 (Financially hedgeable $T-$claim) A financially hedgeable $T-$claim S is an element of $\mathcal{C}_T^{\mathcal{F}}$ which can be replicated by a financial trading strategy $\boldsymbol{\theta} \in \Theta^{\mathcal{F}}$:

$$S = V^{\boldsymbol{\theta}}(T).$$

We introduce the notation $\mathcal{H}_T^{\mathcal{F}}$ for the set of all *financially* hedgeable $T-$claims. One has that

$$\mathcal{H}_T^{\mathcal{F}} \subseteq \mathcal{H}_T.$$

Finally, we introduce orthogonal $T-$claims. We will use the term $\mathbb{P}$-independence for independence between r.v.'s under the measure $\mathbb{P}$.

Definition 21 (Orthogonal claim) An orthogonal $T-$claim S is an element of $\mathcal{C}_T$ which is $\mathbb{P}$-independent of the financial filtration $\mathbb{F}$.

Hereafter, we will denote the set of all orthogonal T-claims by $\mathcal{O}_T$. Hence, $S \in \mathcal{O}_T$ means that S is $\mathbb{P}$-independent of any $\mathcal{F}_T-$measurable random variable. An example of an orthogonal $T-$claim is the indicator variable S defined in (3.1), provided T_x is independent of the financial market evolution. In case $\mathbb{F} \equiv \mathbb{F}^{\boldsymbol{Y}}$, one has that $S \in \mathcal{O}_T$ if and only if S is $\mathbb{P}$-independent of any r.v. which can be expressed as $f(\boldsymbol{Y})$ for some measurable function f.

We remark that

$$\mathbb{E}^{\mathbb{P}}\left[S^{\perp} \times S^{f}\right] = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] \times \mathbb{E}^{\mathbb{P}}\left[S^{f}\right],$$

for any $S^{\perp} \in \mathcal{O}_T$ and $S^f \in \mathcal{C}_T^{\mathcal{F}}$. In particular, we find that

$$\mathbb{E}^{\mathbb{P}}\left[S^{\perp} \times V^{\boldsymbol{\theta}}(T)\right] = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] \times \mathbb{E}^{\mathbb{P}}\left[V^{\boldsymbol{\theta}}(T)\right],$$

for any $S^{\perp} \in \mathcal{O}_T$ and $\boldsymbol{\theta} \in \Theta^{\mathcal{F}}$. This follows immediately from the fact that $V^{\boldsymbol{\theta}}(T) \in \mathcal{C}_T^{\mathcal{F}}$.

3.3 Mean-variance hedging of insurance liabilities

3.3.1 Some general results on mean-variance hedging

Mean-variance hedging (further abbreviated as MV hedging) is the technique of approximating, with minimal mean squared error, a given $T-$claim by the time$-T$ value of a self-financing trading strategy. The literature on MV hedging is extensive. We refer to Schweizer (2010) for a survey. Two main approaches are considered in the literature: the first one uses martingale measures and projection arguments, see e.g. Černý et al. (2007), while the second one describes the problem in terms of a linear backward stochastic differential equation, see e.g. Delong (2013).

In this section, we introduce MV hedging to determine the 'closest' hedge of a combined financial-actuarial claim. This hedge will constitute the first step of the mean-variance hedge-based valuation which will be considered in Section 3.4. Hereafter, whenever we consider a subset Θ' of the set of all self-financing trading strategies Θ, we assume that Θ' is a linear subspace (closed under addition and scalar multiplication) of Θ. This assumption implies, in particular, that the set $\left\{V^{\boldsymbol{\theta}}(T) \mid \boldsymbol{\theta} \in \Theta'\right\}$ is a linear subspace of $\mathcal{C}_T$.

Definition 22 (Mean-variance hedging) Consider a $T-$claim S. The MV hedge of S in $\Theta' \subseteq \Theta$ is the self-financing strategy $\boldsymbol{\theta}_S^{MV} \in \Theta'$ for which the expected quadratic hedging error at time T is minimized :

$$\boldsymbol{\theta}_S^{MV} = \arg\min_{\boldsymbol{\theta}\in\Theta'} \mathbb{E}^{\mathbb{P}}\left[\left(S - V^{\boldsymbol{\theta}}(T)\right)^2\right]. \tag{3.7}$$

The existence of a solution to the minimization problem (3.7) is tantamount to the condition that $\left\{V^{\boldsymbol{\theta}}(T) \mid \boldsymbol{\theta} \in \Theta'\right\}$ is a closed set. In this chapter, we will always assume that this condition is satisfied.[1] Uniqueness of this solution holds under the additional

[1] The closedness assumption is satisfied for $\Theta' = \Theta$ and for $\Theta' = \Theta^{\mathcal{F}}$, see Černý et al. (2007) for technical details. It is also satisfied for the set $\Theta^{(\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_m)} = \left\{\sum_{j=1}^m \alpha_j \boldsymbol{\theta}_j \mid (\alpha_1, \alpha_2, ..., \alpha_m) \in \mathbb{R}^m\right\}$, with $\boldsymbol{\theta}_j$, $j = 1, ..., m$, $\in \Theta$, which is considered in Section 3.3.3.

condition of non-redundancy of Θ'. Non-redundancy of Θ' means that for any $\boldsymbol{\theta} \in \Theta'$, one has that $V^{\boldsymbol{\theta}}(T) = 0$ implies that $\boldsymbol{\theta} = \mathbf{0}$, where $\mathbf{0}$ is the trivial zero investment strategy with all components equal to 0. Hereafter, we will not require non-redundancy of Θ'. This means that a $T-$claim S can have several mean-variance hedges. Notice however that the time$-T$ values of all these self-financing strategies are identical. In the remainder of the chapter, we will denote the unique time$-T$ value of all the mean-variance hedges of S by $V_S^{MV}(T)$.

The determination of the solution of the discrete time minimization problem (3.7) for the set Θ of $\mathbb{G}-$predictable self-financing strategies is considered in Černý et al. (2007), see also Schweizer (2010) and the references therein.

It is well-known that MV hedging in the linear subspace Θ' has the following properties:

$$V_{\alpha\times S}^{MV}(T) = \alpha \times V_S^{MV}(T), \qquad \text{for any scalar } \alpha \geq 0, \tag{3.8}$$

and

$$V_{S_1+S_2}^{MV}(T) = V_{S_1}^{MV}(T) + V_{S_2}^{MV}(T), \qquad \text{for any } S_1 \text{ and } S_2 \in \mathcal{C}_T. \tag{3.9}$$

A no-arbitrage argument leads to

$$V_{S_1+S_2}^{MV}(0) = V_{S_1}^{MV}(0) + V_{S_2}^{MV}(0), \qquad \text{for any } S_1 \text{ and } S_2 \in \mathcal{C}_T. \tag{3.10}$$

As a special case of the additivity property (3.9), we have that

$$V_{S+S^h}^{MV}(T) = V_S^{MV}(T) + S^h, \tag{3.11}$$

for any $S \in \mathcal{C}_T$ and $S^h = V^{\boldsymbol{\theta}}(T)$ with $\boldsymbol{\theta} \in \Theta'$.

In the following subsection, we will consider mean-variance hedging of claims which can be expressed as the product of an orthogonal claim and the time$-T$ value of a financial self-financing strategy.

3.3.2 MV hedging of product claims

The benefit payment of an insurance contract at contract termination date T can often be expressed as

$$S = S^{\perp} \times S^f, \qquad \text{with } S^{\perp} \in \mathcal{O}_T \text{ and } S^f \in \mathcal{C}_T^{\mathcal{F}}. \tag{3.12}$$

This situation occurs for unit-linked contracts in case the corresponding claim S is the product of an actuarial and a financial component, where the actuarial component is independent of the financial information flow over time. In the following theorem, we determine the MV hedge of $T-$claims of the form (3.12) in a subset of financial trading strategies: $\Theta' \subseteq \Theta^{\mathcal{F}}$.

Theorem 9 Consider the T-claim S defined in (3.12). The MV hedge $\boldsymbol{\theta}_S^{MV}$ of S in the subset Θ' of the set $\Theta^{\mathcal{F}}$ of financial self-financing strategies is given by

$$\boldsymbol{\theta}_S^{MV} = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] \times \boldsymbol{\theta}_{S^f}^{MV}, \tag{3.13}$$

where $\boldsymbol{\theta}_{S^f}^{MV}$ is the MV hedge of S^f in Θ'. Moreover, the time$-T$ value of the MV hedge of S equals

$$V_S^{MV}(T) = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] \times V_{S^f}^{MV}(T). \tag{3.14}$$

Proof: For any financial self-financing strategy $\boldsymbol{\mu} \in \Theta'$, we find that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[(S - V^{\boldsymbol{\mu}}(T))^2\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\left(\left(\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] S^f - V^{\boldsymbol{\mu}}(T)\right) + (S^{\perp} - \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right])S^f\right)^2\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\left(\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] S^f - V^{\boldsymbol{\mu}}(T)\right)^2\right] + \mathbb{E}^{\mathbb{P}}\left[\left((S^{\perp} - \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right])S^f\right)^2\right],
\end{aligned}$$

where the last step follows from taking into account that $S^{\perp} \in \mathcal{O}_T$, which is independent of S^f and $V^{\boldsymbol{\mu}}(T)$.
As $\mathbb{E}^{\mathbb{P}}\left[\left((S^{\perp} - \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right])S^f\right)^2\right]$ does not depend on $\boldsymbol{\mu}$, we find that the MV hedge $\boldsymbol{\theta}_S^{MV}$ in the set Θ' follows from

$$\boldsymbol{\theta}_S^{MV} = \arg\min_{\boldsymbol{\mu} \in \Theta'} \mathbb{E}^{\mathbb{P}}\left[\left(\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] S^f - V^{\boldsymbol{\mu}}(T))\right)^2\right].$$

Taking into account that Θ' is a linear space, we can conclude that $V_S^{MV}(T)$ is given by (3.14) and the self-financing strategy $\boldsymbol{\theta}_S^{MV}$ defined in (3.13) is a solution of the minimization problem (3.7). ∎

A special case of Theorem 9 arises when the financial part of the payoff at time T is equal to the time$-T$ value of a financial self-financing trading strategy in $\Theta^{\mathcal{F}}$. This case is considered in the following corollary, with $\Theta' = \Theta^{\mathcal{F}}$.

Corollary 4 Let $S^{\perp} \in \mathcal{O}_T$ and consider the following T-claim:

$$S = S^{\perp} \times V^{\boldsymbol{\theta}}(T), \qquad \text{with } S^{\perp} \in \mathcal{O}_T \text{ and } \boldsymbol{\theta} \in \Theta^{\mathcal{F}}. \tag{3.15}$$

The MV hedge of S in the set $\Theta^{\mathcal{F}}$ is given by

$$\boldsymbol{\theta}_S^{MV} = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] \times \boldsymbol{\theta},$$

while the time$-T$ value of this MV hedge equals

$$V_S^{MV}(T) = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] \times V^{\boldsymbol{\theta}}(T). \tag{3.16}$$

In the following corollary, we consider the special case of Corollary 4, where the financial part of the payoff at time T is equal to the time$-T$ price of a traded asset.

Corollary 5 Let $S^{\perp} \in \mathcal{O}_T$ and consider the following T-claim:

$$S = S^{\perp} \times Y^{(i)}(T), \qquad i = 0, 1, \ldots, n. \tag{3.17}$$

The MV hedge of S in the set $\Theta^{\mathcal{F}}$ is given by

$$\boldsymbol{\theta}_S^{MV} = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] \times \boldsymbol{\theta}^{(i)},$$

where $\boldsymbol{\theta}^{(i)} \in \Theta^{\mathcal{F}}$ is the static financial investment strategy consisting of buying 1 unit of asset i at time 0 and holding it until time T. The time$-T$ value of $\boldsymbol{\theta}_S^{MV}$ is given by

$$V_S^{MV}(T) = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] \times Y^{(i)}(T). \tag{3.18}$$

The MV hedge of an orthogonal claim is considered in the following corollary.

Corollary 6 The MV hedge of $S^{\perp} \in \mathcal{O}_T$ in the set $\Theta^{\mathcal{F}}$ is given by

$$\boldsymbol{\theta}_{S^{\perp}}^{MV} = e^{-rT}\, \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]\, \boldsymbol{\theta}^{(0)},$$

where $\boldsymbol{\theta}^{(0)} \in \Theta^{\mathcal{F}}$ is the static investment strategy consisting of a risk-free investment of 1 at time 0, which is maintained until time T. The time$-T$ value of $\boldsymbol{\theta}_{S^{\perp}}^{MV}$ is given by

$$V_{S^{\perp}}^{MV}(T) = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]. \tag{3.19}$$

The proof of this corollary follows immediately from Corollary 5.

3.3.3 MV hedging of general claims

3.3.3.1 MV hedging in the set of linear combinations of self-financing strategies

In this subsection, we consider a general $T-$claim S and search for the self-financing strategy with the minimal expected quadratic hedging error at time T, where we restrict our search to the set of all strategies which can be expressed as linear combinations of a number of given self-financing trading strategies which are available to the decision maker. More specifically, we consider a vector of m self-financing trading strategies $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_m)$, with any $\boldsymbol{\theta}_j \in \Theta$, and the following set of self-financing investment strategies:

$$\Theta^{(\boldsymbol{\theta}_1,...,\boldsymbol{\theta}_m)} = \left\{ \sum_{j=1}^{m} \alpha_j \boldsymbol{\theta}_j \mid (\alpha_1, \alpha_2, ..., \alpha_m) \in \mathbb{R}^m \right\}. \tag{3.20}$$

Notice that $\Theta^{(\boldsymbol{\theta}_1,...,\boldsymbol{\theta}_m)} \subseteq \Theta$, but it is not necessary a subset of $\Theta^{\mathcal{F}}$. In the following theorem, we determine the MV hedge of a general $T-$claim S in the set of trading strategies defined in (3.20). The MV hedge $\boldsymbol{\theta}_S^{MV}$ of S in $\Theta^{(\boldsymbol{\theta}_1,...,\boldsymbol{\theta}_m)}$ is determined from

$$\min_{\alpha \in \mathbb{R}^m} \mathbb{E}^{\mathbb{P}}\left[\left(S - \sum_{j=1}^{m} \alpha_j V^{\boldsymbol{\theta}_j}(T)\right)^2\right]. \tag{3.21}$$

Hereafter, we use the notation $\mathbf{A}^{\intercal}$ for the transpose of a matrix $\mathbf{A}$ and the notation $\times$ for the product of 2 matrices.

Theorem 10 Consider the vector $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_m)$ of self-financing investment strategies $\boldsymbol{\theta}_i \in \Theta$ and assume that their time$-T$ values

$V^{\boldsymbol{\theta}_i}(T)$, $i = 1, 2, , \ldots, m$, are linearly independent. The MV hedge $\boldsymbol{\theta}_S^{MV}$ of the $T-$claim S in the set $\Theta^{(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_m)}$ is given by

$$\boldsymbol{\theta}_S^{MV} = \sum_{j=1}^{m} \alpha_j \boldsymbol{\theta}_j,$$

where the $m-$vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is given by

$$\boldsymbol{\alpha}^{\intercal} = \mathbf{W}^{-1} \times \mathbf{V}^{\intercal}. \tag{3.22}$$

In this expression, $\mathbf{W}$ is the $(m \times m)-$matrix with elements (i, j) defined by

$$(\mathbf{W})_{ij} = \mathbb{E}^{\mathbb{P}}\left[V^{\boldsymbol{\theta}_i}(T)\ V^{\boldsymbol{\theta}_j}(T)\right], \tag{3.23}$$

while $\mathbf{V}$ is the $(1 \times m)-$matrix with $j-$th element given by

$$(\mathbf{V})_j = \mathbb{E}^{\mathbb{P}}\left[S\ V^{\boldsymbol{\theta}_j}(T)\right]. \tag{3.24}$$

Proof: Taking the derivatives of the objective function in (3.21) with respect to the α_i and setting them equal to 0 leads to the following set of equations:

$$\sum_{j=1}^{m} (\mathbf{W})_{ij}\ \ \alpha_j = \mathbb{E}^{\mathbb{P}}[S\ V^{\boldsymbol{\theta}_i}(T)], \qquad i = 1, \ldots, m,$$

where the elements $(\mathbf{W})_{ij}$ are defined in (3.23). This set of equations can be rewritten as follows:

$$\mathbf{W} \times \boldsymbol{\alpha}^{\intercal} = \mathbf{V}^{\intercal}.$$

The assumption of linear independence of the r.v.'s $V^{\boldsymbol{\theta}_i}(T)$, $i = 1, 2, , \ldots, m$, is equivalent to the non-singularity of the matrix $\mathbf{W}$. This proves (3.22). ■

The MV hedge of the $T-$claim S in the set $\Theta^{(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_m)}$ takes into account the mutual dependency structure between the time $T-$values of the m self-financing strategies via the components $\mathbb{E}^{\mathbb{P}}\left[V^{\boldsymbol{\theta}_i}(T)\ V^{\boldsymbol{\theta}_j}(T)\right]$ of the matrix $\mathbf{W}$, while the dependency between the time $T-$values of these m strategies and the claim S is captured by the components $\mathbb{E}^{\mathbb{P}}\left[S\ V^{\boldsymbol{\theta}_j}(T)\right]$ of the vector $\mathbf{V}$.

Remark that the optimization problem solved in Theorem 10 is very similar to the general MV hedging problem in the single period setting (see Theorem 5 in Section 2.4.3). This problem is strongly related to portfolio replication where one searches for a linear combination of traded assets which generates cash-flows that approximate the cash-flows of a given $T-$claim. For further details, we refer to Pelsser and Schweizer (2016) and Natolski and Werner (2017).

3.3.3.2 MV hedging in the set of linear combinations of a risk-free and risky self-financing strategies

In this subsection, we consider the special case of Theorem 10, where apart from a risk-free investment, there is a number of risky self-financing strategies. This case is considered in the following corollary.

Corollary 7 Consider the vector $(\boldsymbol{\theta}^{(0)}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_m)$, where $\boldsymbol{\theta}^{(0)}$ is the static strategy consisting of a risk-free investment of 1 at time 0, while $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_m$ are self-financing investment strategies in Θ. Assume that the $m+1$ time$-T$ values of the self-financing strategies are linearly independent. The MV hedge $\boldsymbol{\theta}_S^{MV}$ of any $T-$claim S in the set $\Theta^{(\boldsymbol{\theta}^{(0)}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_m)}$ is then given by

$$\boldsymbol{\theta}_S^{MV} = \alpha_0 \ \boldsymbol{\theta}^{(0)} + \sum_{j=1}^{m} \alpha_j \boldsymbol{\theta}_j,$$

with the α_j determined by the set of equations

$$\sum_{j=1}^{m} cov^{\mathbb{P}} \left[V^{\boldsymbol{\theta}_i}(T), V^{\boldsymbol{\theta}_j}(T) \right] \ \alpha_j = cov^{\mathbb{P}} \left[S, V^{\boldsymbol{\theta}_i}(T) \right], \qquad (3.25)$$

for $i = 1, 2, \ldots, m$, while α_0 follows from

$$\alpha_0 = e^{-rT} \left(\mathbb{E}^{\mathbb{P}}[S] - \sum_{j=1}^{m} \alpha_j \ \mathbb{E}^{\mathbb{P}} \left[V^{\boldsymbol{\theta}_j}(T) \right] \right) \qquad (3.26)$$

Moreover, we have that

$$\mathbb{E}^{\mathbb{P}} \left[V_S^{MV}(T) \right] = \mathbb{E}^{\mathbb{P}}[S]. \qquad (3.27)$$

Proof: The proof follows from (3.22). ■

As a special case of the previous corollary, consider the case where $m = 1$. Then we find that the MV hedge of S is given by

$$\boldsymbol{\theta}_S^{MV} = \alpha_0\ \boldsymbol{\theta}^{(0)} + \alpha_1 \boldsymbol{\theta}_1 \tag{3.28}$$

with

$$\alpha_1 = \frac{\mathrm{cov}^{\mathbb{P}}\left[S, V^{\boldsymbol{\theta}_1}(T)\right]}{\mathrm{Var}^{\mathbb{P}}\left[V^{\boldsymbol{\theta}_1}(T)\right]} \tag{3.29}$$

and

$$\alpha_0 = e^{-rT}\left(\mathbb{E}^{\mathbb{P}}[S] - \alpha_1\ \mathbb{E}^{\mathbb{P}}\left[V^{\boldsymbol{\theta}_1}(T)\right]\right). \tag{3.30}$$

For the particular case of a single-period setting, i.e. $T = 1$, these equations can be found e.g. in Tsanakas et al. (2013) and Černý and Kallsen (2009). In the particular case that S is $\mathbb{P}-$independent of the time$-T$ value $V^{\boldsymbol{\theta}_1}(T)$ of the risky self-financing strategy $\boldsymbol{\theta}_1$, we find that

$$\alpha_1 = 0$$

and

$$\alpha_0 = e^{-rT}\ \mathbb{E}^{\mathbb{P}}[S].$$

Hence, in this particular case, the MV hedge is given by e^{-rT} $\mathbb{E}^{\mathbb{P}}[S]\ \boldsymbol{\theta}^{(0)}$, which is a static investment strategy of amount e^{-rT} $\mathbb{E}^{\mathbb{P}}[S]$ in the risk-free asset at time 0.

3.3.3.3 MV hedging with a single self-financing strategy

In this subsection, we consider a self-financing strategy $\boldsymbol{\theta}_1 \in \Theta$ and determine the MV hedge of the $T-$claim S in the set

$$\Theta^{(\boldsymbol{\theta}_1)} = \{\alpha\ \boldsymbol{\theta}_1 \mid \alpha \in \mathbb{R}\}.$$

From Theorem 10, we immediately find the following corollary.

Corollary 8 Consider the self-financing investment strategy $\boldsymbol{\theta}_1 \in \Theta$. The MV hedge of the $T-$claim S in the set $\Theta^{(\boldsymbol{\theta}_1)}$ is given by

$$\boldsymbol{\theta}_S^{MV} = \alpha\ \boldsymbol{\theta}_1,$$

with α determined by

$$\alpha = \frac{\mathbb{E}^{\mathbb{P}}\left[S\ V^{\boldsymbol{\theta}_1}(T)\right]}{\mathbb{E}^{\mathbb{P}}\left[\left(V^{\boldsymbol{\theta}_1}(T)\right)^2\right]}. \tag{3.31}$$

Hereafter, we consider two special cases of this corollary.

First, suppose the strategy $\boldsymbol{\theta}_1$ coincides with the static risk-free investment strategy $\boldsymbol{\theta}^{(0)}$. In this case, we have that $V^{\boldsymbol{\theta}^{(0)}}(T) = e^{rT}$, which leads to

$$\alpha = \mathbb{E}^{\mathbb{P}}\left[S\right]\ e^{-rT}.$$

Hence, the MV hedge of S in $\Theta^{(\boldsymbol{\theta}^{(0)})}$ consists of buying $\mathbb{E}^{\mathbb{P}}\left[S\right]e^{-rT}$ zero-coupon bonds at time 0 and holding this portfolio until time T. Obviously, the time $T-$value of this hedge is given by

$$V_S^{\boldsymbol{\theta}^{(0)}}(T) = \mathbb{E}^{\mathbb{P}}\left[S\right].$$

Next, suppose that $\boldsymbol{\theta}_1 \in \Theta^{\mathcal{F}}$ and consider the $T-$claim $S = S^{\perp} \times V^{\boldsymbol{\theta}_1}(T)$, where $S^{\perp} \in \mathcal{O}_T$. In this case, we find that α is given by

$$\alpha = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right].$$

This means that the MV hedge of S in $\Theta^{(\boldsymbol{\theta}_1)}$ equals

$$\boldsymbol{\theta}_S^{MV} = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]\ \boldsymbol{\theta}_1.$$

This result was to be expected, taking into account Theorem 9.

3.3.4 Examples

In this subsection, we consider two examples illustrating the calculation of MV hedges of insurance liabilities. In a first example, we consider the MV hedge of an equity-linked life insurance contract with payment guarantee.

Example 7 (MV hedging of equity-linked liabilities)
Consider a portfolio of equity-linked life insurance contracts underwritten at time 0 on l_x persons of age x. Each contract specifies that at time T the financial $T-$claim $S^f \in \mathcal{C}_T^{\mathcal{F}}$ is paid out, provided the underlying insured is still alive at that time. Let T_i be the remaining lifetime of insured i, $i = 1, 2, \ldots, l_x$, at contract initiation. The time$-T$ payoff for policy i is given by

$$S_i = 1_{\{T_i > T\}} \times S^f, \qquad i = 1, 2, \ldots, l_x, \tag{3.32}$$

where $1_{\{T_i>T\}}$ is the indicator variable which equals 1 in case $T_i > T$ and 0 otherwise. We assume that the remaining lifetimes of all insureds follow the same distribution and introduce the notation ${}_Tp_x$ for the survival probability $\mathbb{P}[T_i > T]$. The average claim per policy at time T is given by the time$-T$ claim

$$S = \frac{L_{x+T}}{l_x} \times S^f, \tag{3.33}$$

with

$$L_{x+T} = \sum_{i=1}^{l_x} 1_{\{T_i > T\}}. \tag{3.34}$$

Furthermore, we assume that the policyholders' remaining lifetimes $T_1, ..., T_{l_x}$ are independent of the financial market evolution in the sense that any $T_i \in \mathcal{O}_T$. This implies that the indicator variables $1_{\{T_i>T\}} \in \mathcal{O}_T$ and also that $L_{x+T} \in \mathcal{O}_T$.
In case mortality is fully diversifiable and the portfolio is sufficiently large, we can substitute $\frac{L_{x+T}}{l_x}$ by ${}_Tp_x$ in (3.33) and we have that the claim S is a financial $T-$claim: $S = {}_Tp_x \times S^f \in \mathcal{C}_T^{\mathcal{F}}$, see Brennan and Schwartz (1976) and Boyle and Schwartz (1977).
An example of a payoff S^f is given by

$$S^f = \max\left(f\left(Y^{(1)}(T)\right), K\right). \tag{3.35}$$

Here, $Y^{(1)}(T)$ is the market price of 1 unit of risky asset 1 at time T, while f is a real-valued non-negative non-decreasing function, e.g. $f(x) = (1-\varepsilon)^T x$, where ε is an annual fee rate. Furthermore, $K \geq 0$ is the guaranteed minimal survival benefit. It is well-known that the payoff (3.35) can be split into a deterministic payment and a call option payoff:

$$\max\left(f\left(Y^{(1)}(T)\right), K\right) = K + \max(0, f\left(Y^{(1)}(T)\right) - K). \tag{3.36}$$

Hereafter, we investigate the valuation of the claim S defined in (3.33) in case the actuarial risk $\frac{L_{x+T}}{l_x}$ is not necessarily fully diversified.
(a) Let us first consider the case where the payoff upon survival is a financially hedgeable $T-$claim, i.e. $S^f \in \mathcal{H}_T^{\mathcal{F}}$. This means that

$$S^f = V^{\boldsymbol{\theta}}(T), \qquad \text{for some } \boldsymbol{\theta} \in \Theta^{\mathcal{F}}. \tag{3.37}$$

From Corollary 4 it follows that the MV hedge of the equity-linked payoff S in $\Theta^{\mathcal{F}}$ is given by

$$\boldsymbol{\theta}_S^{MV} = {}_Tp_x \times \boldsymbol{\theta}, \tag{3.38}$$

while the time-0 value of the MV hedge of S equals

$$V_S^{MV}(0) = {}_Tp_x \times V^{\boldsymbol{\theta}}(0). \tag{3.39}$$

(b) Usually, the time horizon for equity-linked life insurance policies (typically 5 to 10 years) is different from the time horizon of standard call options (less than a few years). This makes that the claim S^f is often unhedgeable. Therefore, let us now assume that $S^f \notin \mathcal{H}_T^{\mathcal{F}}$. In this case, one could determine the MV hedge of the claim S in the set $\Theta^{(\boldsymbol{\theta}^{(0)},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_m)}$, where $\boldsymbol{\theta}^{(0)}$ is the static zero-coupon bond investment and each $\boldsymbol{\theta}_i \in \Theta^{\mathcal{F}}$ is a financial self-financing strategy. We assume that the time$-T$ values of the $m+1$ self-financing strategies are linearly independent. Taking into account Theorem 9 and Corollary 7, we find that

$$\boldsymbol{\theta}_S^{MV} = {}_Tp_x \times \boldsymbol{\theta}_{S^f}^{MV} = {}_Tp_x \times \left(\alpha_0\ \boldsymbol{\theta}^{(0)} + \sum_{j=1}^{m} \alpha_j \boldsymbol{\theta}_j \right) \tag{3.40}$$

and

$$V_S^{MV}(0) = {}_Tp_x \times \left(\alpha_0\ V^{\boldsymbol{\theta}^{(0)}}(0) + \sum_{j=1}^{m} \alpha_j V^{\boldsymbol{\theta}_j}(0) \right), \tag{3.41}$$

with the α_j determined by the set of equations (3.25) and (3.26). From (3.27), if follows that

$$\mathbb{E}^{\mathbb{P}}\left[V_S^{MV}(T)\right] = \mathbb{E}^{\mathbb{P}}[S]. \tag{3.42}$$

Notice that in case S^f can be replicated by a hedge in the set $\Theta^{(\boldsymbol{\theta}^{(0)},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_m)}$, the two values (3.39) and (3.41) are equal. For instance, in the one-period binomial model (Cox et al. (1979)), one can verify that (α_0, α_1) is a perfect hedge for $\max\left(f\left(Y^{(1)}(T)\right), K\right)$. $\triangledown$

In the following example, we consider a two-period binomial setting and apply Theorem 10 to derive the MV hedge of a financial-actuarial liability with a survival benefit at time 2 equal to the maximum of two dynamic investment strategies.

Example 8
Suppose that the combined financial-actuarial world is home to a financial market where a risk-free and two risky assets are traded in a two-period setting. The current value of risky asset $i = 1, 2$, is given by $Y^{(i)}(0) = 1$, while its price dynamics follows a binomial tree over 2 periods. At each time $t = 1, 2$, the value of asset i can go up to $Y^{(i)}(t-1)\ u^{(i)}$ or down to $Y^{(i)}(t-1)\ \frac{1}{u^{(i)}}$, with $u^{(i)} > 1$. We assume that $u^{(1)} \neq u^{(2)}$.
First, consider a <u>constant-mix strategy</u>, which is defined as the self-financing strategy $\boldsymbol{\theta}_1 = \{\boldsymbol{\theta}_1(t)\}_{t=1,2}$ with

$$\begin{aligned}\boldsymbol{\theta}_1(1) &= (1, 1)\\ \boldsymbol{\theta}_1(2) &= \left(\frac{Y^{(1)}(1) + Y^{(2)}(1)}{2Y^{(1)}(1)}, \frac{Y^{(1)}(1) + Y^{(2)}(1)}{2Y^{(2)}(1)}\right).\end{aligned}$$

At time 0, 1 unit of each risky asset is bought. Hence, the time-0 market price of the strategy is 2. At time 1, the portfolio is rebalanced such that the initial proportions of 50% investment of the available capital in each risky asset, are restored. The time-2 value of this 50% / 50% mix portfolio is given by

$$\begin{aligned}V^{\boldsymbol{\theta}_1}(2) &= \boldsymbol{\theta}_1(2) \cdot \boldsymbol{Y}(2)\\ &= \sum_{i=1}^{2} \frac{Y^{(1)}(1) + Y^{(2)}(1)}{2Y^{(i)}(1)} Y^{(i)}(2).\end{aligned}$$

Next, consider the <u>buy-and-hold strategy</u> which keeps the number of units constant over time:

$$\boldsymbol{\theta}_2(t) = (1, 1), \qquad \text{for } t = 1, 2.$$

The time-0 price of this strategy is 2, while its time-2 value is given by

$$V^{\boldsymbol{\theta}_2}(2) = \sum_{i=1}^{2} Y^{(i)}(2).$$

Suppose that the combined financial-actuarial world is also home to the indicator variable $S^{\perp} \in \mathcal{O}_2$ defined by

$$S^{\perp} = \begin{cases} 0 & : T_x \leq 2, \\ 1 & : T_x > 2 \end{cases} \tag{3.43}$$

where T_x is the remaining lifetime of (x).
Consider an insurance liability $S \in \mathcal{C}_2$ which guarantees the maximum payoff of the two self-financing strategies defined above, provided the insured (x) is still alive at time 2:

$$S = S^{\perp} \times \max\left(V^{\boldsymbol{\theta}_1}(2), V^{\boldsymbol{\theta}_2}(2)\right). \qquad (3.44)$$

By Theorem 10, we know that the MV hedge of S in the set $\Theta^{(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)}$ is given by

$$\boldsymbol{\theta}_S^{MV} = \alpha_1\boldsymbol{\theta}_1 + \alpha_2\boldsymbol{\theta}_2,$$

with the α_i determined by

$$\alpha_i = \sum_{j=1}^{2}\left(\mathbf{W}^{-1}\right)_{ij}\ \mathbb{E}^{\mathbb{P}}\left[S\ V^{\boldsymbol{\theta}_j}(2)\right], \qquad i = 1, 2, \qquad (3.45)$$

where $\mathbf{W}$ is the $(2 \times 2)-$matrix with elements (i, j) defined by

$$(\mathbf{W})_{ij} = \mathbb{E}^{\mathbb{P}}\left[V^{\boldsymbol{\theta}_i}(2)\ V^{\boldsymbol{\theta}_j}(2)\right]. \qquad (3.46)$$

Apart from the claim S defined in (3.44), we also consider claims $\widetilde{S} \in \mathcal{C}_2$ of the form

$$\widetilde{S} = S^{\perp} \times \left(\beta_1 V^{\boldsymbol{\theta}_1}(2) + \beta_2 V^{\boldsymbol{\theta}_2}(2)\right),$$

for given real numbers β_1 and β_2.
From Theorem 9, we find that the MV hedge of $\widetilde{S}$ in the set $\Theta^{\mathcal{F}}$ is given by

$$\boldsymbol{\theta}_{\widetilde{S}}^{MV} = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]\ \left(\beta_1\boldsymbol{\theta}_1 + \beta_2\boldsymbol{\theta}_2\right).$$

As $\boldsymbol{\theta}_{\widetilde{S}}^{MV} \in \Theta^{(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)} \subseteq \Theta^{\mathcal{F}}$, it is obvious that the MV hedge of $\widetilde{S}$ in $\Theta^{\mathcal{F}}$ is equal to the MV hedge of $\widetilde{S}$ in $\Theta^{(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)}$.
Let us now suppose that $u^{(1)} = \frac{4}{3}$ and $u^{(2)} = \frac{8}{3}$, indicating that the second asset is more volatile. Furthermore, we suppose that the $\mathbb{P}-$probability of an up-movement equals $1/2$ for each risky asset and each time period. Finally, suppose that $\mathbb{P}\left[T_x > 2\right] = 0.9$, implying that $\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] = 0.9$. The numerical values of the MV

hedges of different time−2 claims are summarized in the following table.

Time−2 claim	**MV hedge**
$S^{\perp} \times V^{\boldsymbol{\theta}_1}(2)$	$0.9\ \boldsymbol{\theta}_1$
$S^{\perp} \times V^{\boldsymbol{\theta}_2}(2)$	$0.9\ \boldsymbol{\theta}_2$
$S^{\perp} \times \big(0.5V^{\boldsymbol{\theta}_1}(2) + 0.5V^{\boldsymbol{\theta}_2}(2)\big)$	$0.45\ \boldsymbol{\theta}_1 + 0.45\ \boldsymbol{\theta}_2$
$S^{\perp} \times \max(V^{\boldsymbol{\theta}_1}(2), V^{\boldsymbol{\theta}_2}(2))$	$0.52\ \boldsymbol{\theta}_1 + 0.46\ \boldsymbol{\theta}_2$

▽

The claims considered in the previous examples were in general not perfectly hedgeable. In the next section, we consider the mean-variance hedge-based approach which values unhedgeable claims as the sum of the time−0 price of their MV hedge and an actuarial value for the remaining (unhedged) part of the claim.

3.4 Fair valuation of insurance liabilities

In this section, we define the class of fair valuations as well as the class of mean-variance hedge-based (MVHB) valuations in a multi-period setting. These concepts were introduced and investigated in Chapter 2 in a single period framework and are now generalized. In Section 3.4.2, we show that the classes of fair valuations and MVHB valuations are equal. In Section 3.4.3, we provide some detailed examples illustrating the MVHB valuation.

3.4.1 Fair valuations

Solvency II, the European regulatory framework for insurance and reinsurance companies, focuses on the fair valuation of insurance liabilities. A distinction is made between hedgeable and non-hedgeable claims. For a hedgeable claim, the fair value equals the market value of the underlying hedging portfolio. The fair value of a non-hedgeable claim is defined as the sum of the expected present value (called best estimate) and a risk margin, see CEIOPS (2010). The application of this regulatory principle is not always straightforward as insurance liabilities are often partially replicable and it is usually not clear how the regulatory valuation principle should be applied in such a case.

In Chapter 2, we defined a general class of fair valuations which meet the fundamental regulatory requirements by merging actuarial judgement and market-consistency. Hereafter, we first define the class of valuations and then introduce the classes of market-consistent, actuarial and fair valuations in our multi-period setting.

Definition 23 (Valuation) A valuation is a mapping $\rho : \mathcal{C}_T \to \mathbb{R}$, attaching a real number to any claim $S \in \mathcal{C}_T$:

$$S \to \rho\left[S\right],$$

such that ρ is normalized:

$$\rho\left[0\right] = 0,$$

and ρ is translation invariant:

$$\rho\left[S + a\right] = \rho\left[S\right] + e^{-rT} a, \qquad \text{for any } S \in \mathcal{C}_T \text{ and } a \in \mathbb{R}.$$

Our convention of identifying r.v.'s which are equal in the $\mathbb{P}-$almost sure sense implies that $\rho\left[S_1\right] = \rho\left[S_2\right]$ in case S_1 and S_2 are equal in that sense.

Definition 24 (Market-consistent valuation)
A valuation $\rho : \mathcal{C}_T \to \mathbb{R}$ is market-consistent (MC) if any financially hedgeable part of any claim is marked-to-market:

$$\rho\left[S + S^h\right] = \rho[S] + V^{\boldsymbol{\theta}}(0), \tag{3.47}$$

for any $S \in \mathcal{C}_T$ and any $S^h = V^{\boldsymbol{\theta}}(T)$ with $\boldsymbol{\theta} \in \Theta^{\mathcal{F}}$.

In the literature, market-consistency is usually defined via a condition equal or similar to condition (3.47), see e.g. Artzner and Eisele (2010), Malamud et al. (2008) and Pelsser and Stadje (2014). The mark-to-market condition (3.47) postulates that any financial replicable part of a claim is valuated by the price of its hedge. The MC condition (3.47) can be seen as an extension of translation invariance from scalars to financially hedgeable claims. We remark that the condition (3.47) is closely related to the market-consistent property defined in Pelsser and Stadje (2014).

In order to define actuarial valuations, we first have to introduce the notions of $\mathbb{P}-$law invariant and market-invariant mappings on the set of orthogonal claims $\mathcal{O}_T$.

Definition 25 ($\mathbb{P}-$law invariant mapping)
A mapping $\rho : \mathcal{O}_T \to \mathbb{R}$ is $\mathbb{P}-$law invariant if for any $S_1^{\perp}$ and $S_2^{\perp} \in \mathcal{O}_T$ with the same $\mathbb{P}-$distribution, one has that $\rho\left[S_1^{\perp}\right] = \rho\left[S_2^{\perp}\right]$.

The $\mathbb{P}-$law invariance property stems from the fact that changing the r.v. $S_1^{\perp}$ into $S_2^{\perp}$ does not change the value of the mapping, provided both have the same $\mathbb{P}-$distribution. In other words, a $\mathbb{P}-$law invariant mapping $\rho : \mathcal{O}_T \to \mathbb{R}$ is in fact a mapping from the set of all $\mathbb{P}-$distributions of orthogonal claims to the real line. In this sense, one can say that $\rho\left[S^{\perp}\right]$ only depends on the $\mathbb{P}-$distribution of the orthogonal claim $S^{\perp}$.

Definition 26 (Market-invariant mapping)
A mapping $\rho : \mathcal{O}_T \to \mathbb{R}$ is market-invariant if for any $S^{\perp} \in \mathcal{O}_T$, the value $\rho\left[S^{\perp}\right]$ is independent of the current risky asset prices $Y^{(1)}(0), ..., Y^{(n)}(0)$.

In this case, the market-invariance property results from the observation that $\rho\left[S^{\perp}\right]$ is constant with respect to any change in the current risky asset prices.

Definition 27 (Actuarial valuation) A valuation $\rho : \mathcal{C}_T \to \mathbb{R}$ is actuarial if any orthogonal claim is marked-to-model:

$$\rho\left[S^{\perp}\right] = e^{-rT}\ \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] +\ \text{RM}\ \left[S^{\perp}\right], \qquad \text{for any } S^{\perp} \in \mathcal{O}_T, \tag{3.48}$$

where $\text{RM} : \mathcal{O}_T \to \mathbb{R}$ is a $\mathbb{P}-$law invariant and market-invariant mapping.

The mark-to-model (or actuarial) condition (3.48) introduces actuarial aspects in the valuation of claims by stating that for claims that are independent of the financial market information that will become available over time, the valuation does not depend on the current prices of traded risky assets and hence, also does not depend on $\mathbb{Q}$.

Notice that all results that we will derive hereafter in this paper remain valid if we define an actuarial valuation as a member of a given subset of the broad class of valuations considered in the definition above. For instance, we could define an actuarial valuation as a valuation of the form (3.48) where $\text{RM}\ \left[S^{\perp}\right] = e^{-rT}\ \beta$ $\text{var}^{\mathbb{P}}\left[S^{\perp}\right]$, for some deterministic $\beta \geq 0$.

Definition 28 (Fair valuation) A fair valuation is a valuation which is both market-consistent and actuarial.

Our definition of a fair valuation in a multi-period setting is in line with current insurance solvency regulations which impose mark-to-market as well as mark-to-model requirements for the fair valuation of assets and liabilities.[2] Definition 28 combines market-consistency considerations concerning financially hedgeable parts of claims with the traditional actuarial view involving actuarial judgement of insurance claims. We remark that our definition of a fair valuation is generic and does not necessarily fully correspond to any particular definition of fair value in a particular regulation.

3.4.2 Mean-variance hedge-based valuations

Valuating a $T-$claim S via MV hedging starts with finding the optimal self-financing trading strategy $\boldsymbol{\theta}_S^{MV}$ which hedges the claim S with minimal expected quadratic hedging error in a linear subspace Θ' of Θ. Defining the value of the claim S as the initial cost $V_S^{MV}(0)$ of the MV hedging strategy $\boldsymbol{\theta}_S^{MV}$ leads to the same value for the $T-$claim S and for $V_S^{MV}(T)$, neglecting the part of S which is not hedged, i.e. $S - V_S^{MV}(T)$. In order to solve this issue, we considered in Chapter 2 a class of fair valuations, the members of which are called mean-variance hedge-based (MVHB) valuations. Determining the MVHB value of a $T-$claim S departs from splitting this claim into the time$-T$ value of its MV hedge and the remaining claim:

$$S = V_S^{MV}(T) + \left(S - V_S^{MV}(T)\right).$$

The MVHB value of S is then defined as the sum of the financial market price of the MV hedge and an actuarial value of the remaining claim.

[2] In the 'Solvency II Glossary' of the 'Comité Européen des Assurances' and the 'Groupe Consultatif Actuariel Européen' of 2007, Fair Value is defined as 'the amount for which ... a liability could be settled between knowledgeable, willing parties in an arm's length transaction. This is similar to the concept of Market Value, but the Fair Value may be a mark-to-model price if no actual market price for the ... liability exists.'

Definition 29 (MVHB valuation) A mapping $\rho : \mathcal{C}_T \to \mathbb{R}$ is a mean-variance hedge-based (MVHB) valuation in case there exists a linear subspace Θ' of Θ and an actuarial valuation π such that

$$\rho\,[S] = V_S^{MV}(0) + \pi[S - V_S^{MV}(T)], \qquad \text{for any } S \in \mathcal{C}_T, \quad (3.49)$$

where $V_S^{MV}(0)$ and $V_S^{MV}(T)$ are the time-0 and time$-T$ values of the MV hedge $\boldsymbol{\theta}_S^{MV}$ of S in Θ', respectively.

It is straightforward to prove that a MVHB valuation is normalized and translation invariant, and hence, a valuation in the sense of Definition 23. Moreover, a MVHB valuation is positive homogeneous, provided the underlying actuarial valuation is positive homogeneous.

In the following lemma, a MVHB valuation formula is derived for product claims, taking into account Theorem 9.

Lemma 3 Consider the MVHB valuation with underlying MV hedging in the linear space of self-financing trading strategies $\Theta' \subseteq \Theta^{\mathcal{F}}$ and actuarial valuation π. For any $S^{\perp} \in \mathcal{O}_T$ and any $S^f \in \mathcal{C}_T^{\mathcal{F}}$, the MVHB value of $S = S^{\perp} \times S^f$ is given by

$$\rho\,[S] = \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]\, V_{S^f}^{MV}(0) + \pi\left[S^{\perp} \times S^f - \mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right] V_{S^f}^{MV}(T)\right]. \quad (3.50)$$

In the following theorem, we prove that the class of MVHB valuations is identical to the class of fair valuations.

Theorem 11 A mapping $\rho : \mathcal{C}_T \to \mathbb{R}$ is a MVHB valuation with underlying MV hedging in the set $\Theta^{\mathcal{F}}$ if and only if it is a fair valuation.

Proof: (a) Consider the MVHB valuation ρ defined in (3.49). In order to show that ρ is a fair valuation, we have to verify whether ρ is a market-consistent and actuarial valuation.

(i) Let $S \in \mathcal{C}_T$ and $S^h = V^{\boldsymbol{\theta}}(T)$ with $\boldsymbol{\theta} \in \Theta^{\mathcal{F}}$. We have that

$$V_{S+S^h}^{MV}(T) = V_S^{MV}(T) + S^h,$$

and hence, also

$$V_{S+S^h}^{MV}(0) = V_S^{MV}(0) + V^{\boldsymbol{\theta}}(0),$$

see (3.11). Taking into account these additivity relations, we find that

$$\begin{aligned}\rho\left[S+S^h\right] &= V^{MV}_{S+S^h}(0)+\pi\left[S+S^h-V^{MV}_{S+S^h}(T)\right]\\ &= V^{MV}_S(0)+V^{\boldsymbol{\theta}}(0)+\pi[S-V^{MV}_S(T)]\\ &= \rho[S]+V^{\boldsymbol{\theta}}(0).\end{aligned}$$

Hence, ρ is market-consistent.

(ii) Let $S^{\perp}\in\mathcal{O}_T$. From Corollary 6, we know that $\boldsymbol{\theta}^{MV}_{S^{\perp}} = e^{-rT}\,\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]\boldsymbol{\theta}^{(0)}$. Taking into account the translation-invariance of π leads to

$$\begin{aligned}\rho\left[S^{\perp}\right] &= V^{MV}_{S^{\perp}}(0)+\pi[S^{\perp}-V^{MV}_{S^{\perp}}(T)]\\ &= e^{-rT}\,\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]+\pi\left[S^{\perp}-\mathbb{E}^{\mathbb{P}}\left[S^{\perp}\right]\right]\\ &= \pi[S^{\perp}].\end{aligned}$$

Given that π is an actuarial valuation, we find that ρ is also an actuarial valuation.

(b) Consider the fair valuation ρ. Let $V^{MV}_S(T)$ be the time$-T$ value of the MV hedge of the $T-$claim S in $\Theta^{\mathcal{F}}$. By the market-consistency property, we immediately find that

$$\begin{aligned}\rho\left[S\right] &= \rho\left[V^{MV}_S(T)+\left(S-V^{MV}_S(T)\right)\right]\\ &= V^{MV}_S(0)+\rho\left[S-V^{MV}_S(T)\right].\end{aligned}$$

Given that ρ is fair, it is also actuarial. Hence, we can conclude that the fair valuation ρ is a MVHB valuation. ■

Theorem 11 holds for MVHB valuations with MV hedge determined in the set of financial self-financing strategies $\Theta^{\mathcal{F}}$ whereas the MC condition (3.47) in the definition of a fair valuation has to hold for all $S^h=V^{\boldsymbol{\theta}}(T)$ with $\boldsymbol{\theta}\in\Theta^{\mathcal{F}}$. Important to notice is that Theorem 11 remains to hold if we replace $\Theta^{\mathcal{F}}$ by a linear subspace Θ' of $\Theta^{\mathcal{F}}$ which includes $\boldsymbol{\theta}^{(0)}$ as one of its elements, provided we redefine a MC valuation as a valuation which satisfies the MC property only for claims which are hedgeable with a self-financing strategy in Θ', while we redefine a MVHB valuation as a valuation of the form (3.49), where the MV hedge is determined in the set Θ'. We remark that the self-financing strategy $\boldsymbol{\theta}^{(0)}$ is required to be an element of Θ' in order to guarantee that the MVHB valuation is actuarial.

Moreover, Theorem 11 is a generalization of Theorem 3 in Chapter 2 in the MV hedging case as it allows periodic rebalancing (for instance yearly) for long term $T-$claims. Obviously, this cannot be achieved within a single period model.

3.4.3 Examples

We end this section with two examples illustrating the fair valuation of insurance liabilities.

In Example 9, we consider the fair value of the liabilities related to a portfolio of equity-linked life insurance contracts by applying the MVHB valuation with a standard deviation actuarial valuation principle for the non-hedged part of the claims. Under the assumption of diversifiability of mortality, the actuarial value of the non-hedged part per policy converges to zero due to the law of large numbers (LLN). In case of conditional independence, instead of independence of the remaining lifetimes of the insureds, the LLN breaks down and the actuarial value in the MVHB valuation converges to a non-zero constant, giving rise to a risk margin for non-diversifiable mortality risk, see also Milevsky et al. (2006).

Example 9 (Valuation of equity-linked liabilities)
Consider the portfolio of l_x contracts underwritten at time 0 as described in Example 7. Each contract guarantees to its beneficiary the payment $S^f \in \mathcal{C}_T^{\mathcal{F}}$ at time T, provided the insured is still alive at that time. All insureds are assumed to be x years old at policy issue. As in Example 7, we assume that the policyholders' remaining lifetimes $T_1, ..., T_{l_x}$ are identically distributed and independent of the financial market evolution in the sense that any $T_i \in \mathcal{O}_T$. As before, we use the notation ${}_Tp_x$ for $\mathbb{E}^{\mathbb{P}}\left[\frac{L_{x+T}}{l_x}\right]$.
The average claim per policy at time T is given by (3.33):

$$S = \frac{L_{x+T}}{l_x} \times S^f,$$

with

$$L_{x+T} = \sum_{i=1}^{l_x} 1_{\{T_i > T\}}.$$

Suppose that we apply the MVHB valuation (3.49) with underlying MV hedging in the space of of self-financing trading strategies $\Theta^{(\boldsymbol{\theta}^{(0)},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_m)}$ defined in Example 7, and as actuarial valuation π the standard deviation principle, i.e.

$$\pi\left[S\right] = e^{-rT}\ \left(\mathbb{E}^{\mathbb{P}}\left[S\right] + \beta\ \sigma^{\mathbb{P}}[S]\right), \qquad \text{for any } S \in \mathcal{C}_T,$$

with β a given non-negative real number.
From (3.42), we know that

$$\boldsymbol{\theta}_S^{MV} = \ {}_Tp_x \times \boldsymbol{\theta}_{S^f}^{MV},$$

with

$$\boldsymbol{\theta}_{S^f}^{MV} = \left(\alpha_0\ \boldsymbol{\theta}^{(0)} + \sum_{j=1}^{m} \alpha_j \boldsymbol{\theta}_j\right),$$

where the coefficients α_j follow from (3.25) and (3.26).
Taking into account Lemma 3 and (3.42), we find that the MVHB value of S is given by

$$\rho\left[S\right] = \ {}_Tp_x \times V_{S^f}^{MV}(0) + e^{-rT}\ \beta\ \sigma^{\mathbb{P}}[\frac{L_{x+T}}{l_x} \times S^f - \ {}_Tp_x \times V_{S^f}^{MV}(T)]. \tag{3.51}$$

After some straightforward calculations, this value can be rewritten as follows:

$$\rho\left[S\right] = \ {}_Tp_x \times V_{S^f}^{MV}(0) + e^{-rT}\ \beta\ \sigma \tag{3.52}$$

with

$$\sigma^2 = ({}_Tp_x)^2 \times \mathrm{Var}^{\mathbb{P}}\left[S^f - V_{S^f}^{MV}(T)\right] + \mathbb{E}^{\mathbb{P}}\left[\left(S^f\right)^2\right] \times \mathrm{Var}^{\mathbb{P}}\left[\frac{L_{x+T}}{l_x}\right]. \tag{3.53}$$

The actuarial premium for the non-hedged part of the claim, i.e. $e^{-rT}\ \beta\ \sigma$, can be interpreted as a 'risk loading' composed of two components. The first component is related to the fact that S^f is not perfectly hedgeable, whereas the second component is due to the fact that the survival risk is not fully diversified. In case S^f is perfectly hedgeable in $\Theta^{(\boldsymbol{\theta}^{(0)},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_m)}$, the first term of σ^2

vanishes, whereas in case of full diversification of the survival risk, its second term disappears. Due to (3.27), we remark that

$$\begin{aligned} \mathrm{Var}^{\mathbb{P}}\left[S^f - V_{S^f}^{MV}(T)\right] &= \mathbb{E}^{\mathbb{P}}\left[\left(S^f - V_{S^f}^{MV}(T)\right)^2\right] \\ &= \min_{\boldsymbol{\mu}\in\Theta^{(\boldsymbol{\theta}^{(0)},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\boldsymbol{\theta}_m)}} \mathbb{E}^{\mathbb{P}}\left[\left(S^f - V^{\boldsymbol{\mu}}(T)\right)\right)^2\right]. \end{aligned}$$

(a) Let us additionally assume that $T_1, ..., T_{l_x}$ are i.i.d under $\mathbb{P}$. In this case, we find that

$$\mathrm{Var}^{\mathbb{P}}\left[\frac{L_{x+T}}{l_x}\right] = \frac{{}_Tp_x\ (1 - \ {}_Tp_x)}{l_x}.$$

The MVHB value $\rho[S]$ of the average claim per policy is then given by (3.52) with

$$\sigma^2 = ({}_Tp_x)^2 \times \mathrm{Var}^{\mathbb{P}}\left[S^f - V_{S^f}^{MV}(T)\right] + \mathbb{E}^{\mathbb{P}}\left[\left(S^f\right)^2\right] \times \frac{{}_Tp_x\ (1 - \ {}_Tp_x)}{l_x}.$$

Increasing the number of policies leads to a decrease of the value of the average claim per policy for the non-hedged part of the claim. Moreover, we have that

$$\lim_{l_x\to\infty} \rho[S] = \ {}_Tp_x \left(V_{S^f}^{MV}(0) + e^{-rT}\ \beta\ \sigma^{\mathbb{P}}\left[S^f - V_{S^f}^{MV}(T)\right]\right).$$

Therefore, when l_x goes to infinity, the actuarial value per policy for the non-hedged part of the claim is only due to the hedging error.

(b) Instead of assuming that $T_1, ..., T_{l_x}$ are $\mathbb{P}-$i.i.d, let us now assume that there exists a r.v. P with $\mathbb{P}-$cdf given by $F_P^{\mathbb{P}}(p)$, $p \in [0, 1]$, such that given $P = p$, the remaining lifetimes $T_1, ..., T_{l_x}$ are $\mathbb{P}-$i.i.d., with

$$\mathbb{P}[T_i > T \mid P = p] = p, \qquad p \in [0, 1].$$

P can be interpreted as the 'stochastic survival probability' and we find that

$$\mathbb{P}[T_i > T] = \mathbb{E}^{\mathbb{P}}[P].$$

Due to the random nature of P, the remaining lifetimes T_i are not mutually independent anymore. Instead they have become conditionally independent.
The expectation and the variance of $\frac{L_{x+T}}{l_x}$ are now given by

$$\mathbb{E}^{\mathbb{P}}\left[\frac{L_{x+T}}{l_x}\right] = \mathbb{E}^{\mathbb{P}}[P]$$

and

$$\mathrm{Var}^{\mathbb{P}}\left[\frac{L_{x+T}}{l_x}\right] = \frac{\mathbb{E}^{\mathbb{P}}[P(1-P)]}{l_x} + \mathrm{Var}^{\mathbb{P}}[P].$$

Inserting these expressions in (3.53), we find that the MVHB value $\rho[S]$ of the average claim per policy is given by (3.52) with

$$\begin{aligned}\sigma^2 &= ({}_T p_x)^2 \times \mathrm{Var}^{\mathbb{P}}\left[S^f - V_{S^f}^{MV}(T)\right] \\ &\quad + \mathbb{E}^{\mathbb{P}}\left[\left(S^f\right)^2\right] \times \left(\frac{\mathbb{E}^{\mathbb{P}}[P(1-P)]}{l_x} + \mathrm{Var}^{\mathbb{P}}[P]\right).\end{aligned}$$

Again, we can conclude that increasing the number of policies leads to a decrease of the actuarial value per policy for the non-hedged part of the claim. Moreover, we have that

$$\lim_{l_x\to\infty} \sigma^2 = ({}_T p_x)^2 \times \mathrm{Var}^{\mathbb{P}}\left[S^f - V_{S^f}^{MV}(T)\right] + \mathbb{E}^{\mathbb{P}}\left[\left(S^f\right)^2\right] \times \mathrm{Var}^{\mathbb{P}}[P].$$

Hence, in case $\mathrm{Var}^{\mathbb{P}}[P] \neq 0$, the survival risk is not fully diversifiable: even if the number of insureds becomes infinitely large, the actuarial premium of the unhedged risk contains a term related to the undiversifiable survival risk. ▽

Remark 1 In the previous example, we treated the remaining non-hedged part by a standard actuarial principle. Another alternative consists of splitting the non-hedged part into a diversifiable part and a residual part, that is neither hedgeable nor diversifiable, leading to a three-step valuation. We refer to Deelstra et al. (2018) and the references therein.

In the following example, we investigate the fair value of a product claim of the form $S = S^{\perp} \times V^{\boldsymbol{\theta}}(T)$ where the trading strategy $\boldsymbol{\theta}$ depends on $S^{\perp}$ (hence, $\boldsymbol{\theta} \notin \Theta^{\mathcal{F}}$). In particular, we consider the fair value of a pool of equity-linked contracts in which the investment portfolio depends on the number of survivors. We further quantify the impact on the fair value when the aggregate longevity risk is transferred from the pool to the insurer.

Example 10 (Transfer of the longevity risk)
(a) Consider a portfolio of $T-$year equity-linked policies underwritten at time 0 on a cohort of l_x insureds aged x at policy initiation. The random number of survivors a time t is denoted by L_{x+t}, $t = 0, 1, \ldots, T$. At time T, the value $V^{\boldsymbol{\theta}}(T)$ of a self-financing investment strategy $\boldsymbol{\theta} \in \Theta$, set up at time 0, is equally distributed among the survivors in the portfolio. Hence, the payoff per policy in force at time T is given by

$$S_i = \frac{V^{\boldsymbol{\theta}}(T)}{L_{x+T}} 1_{\{T_i > T\}}, \qquad i = 1, 2, \ldots, l_x, \tag{3.54}$$

where T_i is the remaining lifetime of insured i. Any policyholder i faces three sources of risk: investment risk (caused by the random nature of the final value $V^{\boldsymbol{\theta}}(T)$ of the investment strategy), individual longevity risk (due to the randomness of the remaining lifetime T_i of the insured) and aggregate longevity risk (because of the random nature of the number of survivors L_{x+T}). The aggregate portfolio liability at time T equals

$$S = \sum_{i=1}^{l_x} S_i = V^{\boldsymbol{\theta}}(T).$$

As S is a hedgeable claim, its fair value at time 0 is given by the cost of the initial investment of the trading strategy $\boldsymbol{\theta}$:

$$\rho[S] = V^{\boldsymbol{\theta}}(0). \tag{3.55}$$

The insurer who charges a single premium of $\frac{V^{\boldsymbol{\theta}}(0)}{l_x}$ per underwritten contract and sets up the investment strategy $\boldsymbol{\theta}$ at time 0 does not take any risk: all sources of risk are born by the pool of policyholders. The portfolio can be considered as a pool of tontine-like policies. For the reader interested in pooled funds and tontines, we refer to Milevsky and Salisbury (2015), Bräutigam et al. (2017) and the interesting book on the 1693 tontine by Milevsky (2015).

(b) Let us now in addition assume that $\boldsymbol{\theta} \in \Theta^{\mathcal{F}}$ and that all T_i, and hence also L_{x+t}, are orthogonal claims. Furthermore, we adapt the contract payoff (3.54) in the sense that the random number of survivors L_{x+T} is replaced by its deterministic estimate $l_{x+t} = \mathbb{E}^{\mathbb{P}}\left[L_{x+T}\right]$ in the payoff per policy:

$$S_i = \frac{V^{\boldsymbol{\theta}}(T)}{l_{x+T}} 1_{\{T_i > T\}}. \tag{3.56}$$

In this adapted contract, the aggregate longevity risk is transferred to the insurer, i.e. he bears the uncertainty on the number of survivors at maturity. The aggregate portfolio liability is now given by

$$S = V^{\boldsymbol{\theta}}(T) \frac{L_{x+T}}{l_{x+T}}. \tag{3.57}$$

As the aggregate liability S is no longer hedgeable, the insurer determines the fair value of S via a MVHB valuation.
From Corollary 4, it follows that the MV hedge of S in $\Theta^{\mathcal{F}}$ is given by

$$\boldsymbol{\theta}_S^{MV} = \boldsymbol{\theta}.$$

From Lemma 3, we find that the MVHB value of S equals

$$\rho\left[S\right] = V^{\boldsymbol{\theta}}(0) + \pi \left[\left(\frac{L_{x+T} - l_{x+T}}{l_{x+T}}\right) V^{\boldsymbol{\theta}}(T)\right].$$

Let us now choose, as actuarial valuation, the standard deviation principle, i.e.

$$\pi\left[S\right] = e^{-rT}\left(\mathbb{E}^{\mathbb{P}}\left[S\right] + \beta\ \sigma^{\mathbb{P}}\left[S\right]\right),$$

for some $\beta > 0$. Taking into account that

$$\mathrm{Var}^{\mathbb{P}}\left[\left(\frac{L_{x+T} - l_{x+T}}{l_{x+T}}\right) V^{\boldsymbol{\theta}}(T)\right] = \mathbb{E}^{\mathbb{P}}\left[\left(V^{\boldsymbol{\theta}}(T)\right)^2\right] \times \mathrm{Var}^{\mathbb{P}}\left[\frac{L_{x+T}}{l_{x+T}}\right],$$

we find that

$$\rho\left[S\right] = V^{\boldsymbol{\theta}}(0) + \beta\ e^{-rT}\ \sqrt{\mathbb{E}^{\mathbb{P}}\left[\left(V^{\boldsymbol{\theta}}(T)\right)^2\right]} \times \sigma^{\mathbb{P}}\left[\frac{L_{x+T}}{l_{x+T}}\right].$$

In case the insurer charges a premium of $\frac{\rho[S]}{l_x}$ per policy, we observe that the first part of the premium corresponds with the premium charged in (a), whereas the second part is the extra loading per policy for the transfer of the aggregate longevity risk to the insurer. This extra loading is caused by the volatility of both $V^{\boldsymbol{\theta}}(T)$ and L_{x+T}.

(c) As a special case of (b), suppose that $\boldsymbol{\theta} = l_{x+T}\ e^{-rT}\ \boldsymbol{\theta}^{(0)}$. The time$-T$ payoff per policy is then given by

$$S_i = 1_{\{T_i > T\}}. \tag{3.58}$$

In this case, the policyholder only bears the individual longevity risk. The aggregate portfolio liability is now given by

$$S = \sum_{i=1}^{l_x} S_i = L_{x+T}, \tag{3.59}$$

while the fair value of the portfolio liability is given by

$$\rho\left[S\right] = e^{-rT}\left(l_{x+T} + \beta\ \sigma^{\mathbb{P}}\left[L_{x+T}\right]\right).$$

Notice that the insurance contract considered in (c) corresponds to a classical pure endowment.

(d) Let us go back to (a) and consider the portfolio of l_x contracts with time$-T$ benefits given by (3.54), with trading strategy $\boldsymbol{\theta} \in \Theta$ defined in the following way:

At time 0, for any underwritten policy, an amount $\frac{A}{l_x}$ is fully invested in the risk-free bank account. Furthermore, any time an insured dies in any year $(j-1, j)$, the amount $\frac{A}{l_x}e^{rj}$ is withdrawn from the bank account at time j and is fully invested in asset 1, from time j until time T. The aggregate portfolio liability is then equal to the time$-T$ value of the investment strategy $\boldsymbol{\theta}$:

$$\begin{aligned} S = V^{\boldsymbol{\theta}}\left(T\right) &= \frac{A}{l_x}\left(L_{x+T}\ e^{rT} + \sum_{j=1}^{T} D_{x+j-1}\ e^{rj}\frac{Y^{(1)}(T)}{Y^{(1)}(j)}\right), \\ &= \frac{A}{l_x}L_{x+T}\ e^{rT} + S', \end{aligned}$$

where D_{x+j-1} is the number of people who died during the year $(j-1, j)$ and S' denotes the part of the aggregate survival benefits which was invested in the risky asset 1 (after the death of the respective insureds). Obviously, $\boldsymbol{\theta}$ is a self-financing trading strategy with $V^{\boldsymbol{\theta}}(0) = A$. Moreover, $\boldsymbol{\theta} \notin \Theta^{\mathcal{F}}$ as the investment strategy depends on the number of survivors at each time j. As in (a), we have that S is a hedgeable claim and the fair value of the portfolio is given by

$$\rho[S] = A.$$

From (3.56), it follows that the time$-T$ payoff per policy S_i is given by

$$S_i = \left(\frac{A}{l_x} e^{rT} + \frac{S'}{L_{x+T}}\right) 1_{\{T_i > T\}},$$

which clearly shows that the policyholder bears the risky investment risk, as well as the individual and the aggregate longevity risk.

(e) Let us consider the self-financing strategy $\boldsymbol{\theta}$ introduced in (d). Suppose now that the time$-T$ payoff per policy is defined by

$$S_i = \left(\frac{A}{l_x} e^{rT} + \frac{\overline{S}'}{l_{x+T}}\right) 1_{\{T_i > T\}}, \tag{3.60}$$

where

$$\overline{S}' = \frac{A}{l_x} \sum_{j=1}^{T} d_{x+j-1}\; e^{rj} \frac{Y^{(1)}(T)}{Y^{(1)}(j)},$$

with $d_{x+j-1} = \mathbb{E}^{\mathbb{P}}[L_{x+j-1} - L_{x+j}]$, the *expected* number of people who will die during the year $(j-1, j)$. The aggregate portfolio liability is now given by

$$S = \left(\frac{A}{l_x} e^{rT} + \frac{\overline{S}'}{l_{x+T}}\right) L_{x+T}. \tag{3.61}$$

From the expression (3.61), we observe that the aggregate longevity risk is transferred to the insurer and also that the aggregate portfolio liability is no longer hedgeable. Notice that S can be written as

$$\begin{aligned} S &= \frac{A}{l_x}\left(l_{x+T}\; e^{rT} + \sum_{j=1}^{T} d_{x+j-1}\; e^{rj} \frac{Y^{(1)}(T)}{Y^{(1)}(j)}\right) \frac{L_{x+T}}{l_{x+T}} \\ &= V^{\boldsymbol{\mu}}(T) \frac{L_{x+T}}{l_{x+T}}, \end{aligned}$$

where $\boldsymbol{\mu}$ is similar to the strategy $\boldsymbol{\theta}$ introduced in (d), but with the real numbers of deaths and survivors (D_{x+t} and L_{x+t}) replaced by their respective expectations (d_{x+t} and l_{x+t}). As $\boldsymbol{\mu} \in \Theta^{\mathcal{F}}$, we can follow the same approach as in (b) to determine the MVHB value of S.
From Corollary 4, the MV hedge of S in $\Theta^{\mathcal{F}}$ is given by

$$\boldsymbol{\theta}_S^{MV} = \boldsymbol{\mu},$$

and from Lemma 3, we find that the MVHB value of S equals

$$\rho\left[S\right] = V^{\boldsymbol{\mu}}(0) + \pi\left[\left(\frac{L_{x+T} - l_{x+T}}{l_{x+T}}\right) V^{\boldsymbol{\mu}}(T)\right].$$

Since $V^{\boldsymbol{\mu}}(0) = V^{\boldsymbol{\theta}}(0) = A$, the second term in this expression for $\rho\left[S\right]$ can be interpreted as the fair value for the transfer of the aggregate longevity risk to the insurer. $\bigtriangledown$

3.5 Concluding remarks

In this chapter, we investigated the fair valuation of insurance liabilities based on mean-variance hedging and extended the results of Chapter 2 to a multi-period dynamic investment setting. We focused on product claims, i.e. claims which can be expressed as the product of an actuarial and a financial claim. Under independence between the actuarial claim (typically a mortality-related claim) and the financial market, we derived the MV hedge in Theorem 9 and obtained tractable formulas for the fair valuation of such product claims. For general claims, we derived the MV hedge in the set of all strategies which can be expressed as linear combinations of a number of given self-financing trading strategies. The obtained results have been illustrated with numerous examples.

In Section 3.4, we showed that the class of fair valuations is identical to the class of mean-variance hedge-based valuations. Under the MVHB approach, we showed that the risk margin in equity-linked contracts can be decomposed into a risk loading for non-diversifiable mortality risk and a risk loading for non-hedgeable financial risk. Moreover, we determined the extra loading in the fair value when the longevity risk in pooled equity-linked contracts is transferred to the insurer.

As considered in Chapter 2, one can also define a two-step valuation based on a conditional actuarial valuation, which extends the two-step valuation of Pelsser and Stadje (2014) by introducing actuarial considerations. Like in the previous chapter, one can show that the set of two-step valuations coincides with the set of fair valuations and hence, also with the set of MVHB valuations. These results can be seen as generalizations of the equivalences which hold in a one-period static setting in Chapter 2.

Under the MVHB approach, the valuation gives an explicit hedge and an additive decomposition of the claim into a financial hedgeable part and an actuarial non-hedgeable part. Therefore, we believe that the MVHB valuation provides a relevant framework to determine the hedgeable part and the fair valuation of insurance liabilities which involve both actuarial and financial components.

Chapter 4

Fair dynamic valuation of insurance liabilities: Merging actuarial judgement with market- and time-consistency

This chapter is based on
Barigou, K., Chen, Z., & Dhaene, J. (2019). Fair dynamic valuation of insurance liabilities: Merging actuarial judgement with market-and time-consistency. *Insurance: Mathematics and Economics*, **88**, 19-29.

4.1 Introduction

In the previous chapters, we considered a static valuation problem: what is the value today of an insurance contract with payoff at time T?

To determine solvency capital requirements (SCR), insurers need to determine the value of their liabilities not only today but also at future points in time: what is the value of my insurance liabilities in five years from now if there is a shock in the financial market coupled with an increase in longevity?

An important question in a dynamic setting is how risk valuations at different times are interrelated. In this context, time-consistency is a natural approach to glue together static valuations. It means that the same value is assigned to a position regardless of whether it is calculated over two time periods at once or in two-steps backwards in time. Time-consistent valuations have been largely studied and we refer to Acciaio and Penner (2011) for an overview.

In this chapter, we investigate the fair valuation of insurance liabilities in a dynamic multi-period setting. We define a *fair dynamic valuation* as a valuation which is actuarial (mark-to-model for claims independent of financial market evolutions), market-consistent (mark-to-market for hedgeable parts of claims) and time-consistent, and study their properties. In particular, we provide a complete hedging characterization for fair dynamic valuations, extending the work of our previous chapters in a dynamic setting. Moreover, we show how we can implement fair dynamic valuations through a backward iterations scheme combining risk minimization methods from mathematical finance with standard actuarial techniques based on risk measures.

The chapter is organized as follows. In Section 4.2, we describe the combined financial-actuarial world and the notions of orthogonal and hedgeable claims. In Section 4.3, fair $t-$valuations and the related notion of fair $t-$hedgers are introduced. In particular, we show that any fair $t-$valuation can be characterized in terms of a fair $t-$hedger. In Section 4.4, we extend the results in a time-consistent setup and provide a time-consistent hedging characterization for time-consistent and market-consistent valuations. Section 4.5 presents a practical approach to apply our framework and some numerical illustrations. Section 4.6 concludes the paper.

4.2 The combined financial-actuarial world

In this chapter, we consider again the combined financial-actuarial world of Chapter 3. However, since we are considering dynamic valuations in this chapter, we need to adapt and generalize the notions of trading strategies, orthogonal and hedgeable claims that are defined hereafter.

Like in the previous chapter, the financial-actuarial world is modeled by the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, equipped with the finite and discrete time filtration $\mathbb{G} = \{\mathcal{G}_t\}_{t\in\tau}$, such that $\mathcal{G}_0$ is equal to $\{\emptyset, \Omega\}$ and $\mathcal{G}_T = \mathcal{G}$. We will denote the set of all $t-$claims defined on $(\Omega, \mathbb{G}, \mathcal{G})$, that is the set of all $\mathcal{G}_t-$measurable r.v.'s, by $\mathcal{C}_t$. Hereafter, when considering a $t-$claim, we will always silently assume that it is payable at time t, except if stated otherwise.

The combined financial-actuarial world $(\Omega, \mathbb{G}, \mathbb{P})$ is also home to a financial market of $n+1$ tradable (non-dividend paying[1] assets). The price processes of the traded assets are described by the $(n+1)-$dimensional stochastic process $\boldsymbol{Y} = \{\boldsymbol{Y}(t)\}_{t\in\tau}$. Here, $\boldsymbol{Y}(t)$, $t \in \tau$, is the vector of time$-t$ prices of all tradable assets, i.e. $\boldsymbol{Y}(t) = \left(Y^{(0)}(t), Y^{(1)}(t), \ldots, Y^{(n)}(t)\right)$. We assume that the price process $\boldsymbol{Y}$ is adapted to the filtration $\mathbb{G}$, which means that

$$\boldsymbol{Y}(t) \text{ is } \mathcal{G}_t - \text{measurable}, \qquad \text{for any } t = 0, 1, , \ldots, T.$$

A *time$-t$ trading strategy* (also called a *time$-t$ dynamic portfolio*), $t \in \{0, \ldots, T-1\}$, is an $(n+1)-$dimensional *predictable* process $\boldsymbol{\theta}_t = \{\boldsymbol{\theta}_t(u)\}_{u\in\{t+1,\ldots,T\}}$ with respect to the filtration $\mathbb{G}$. The predictability requirement means that

$$\boldsymbol{\theta}_t(u) \text{ is } \mathcal{G}_{u-1} - \text{measurable}, \qquad \text{for any } u = t+1, \ldots, T.$$

Notice that a time$-t$ trading strategy is only set up at time t by acquiring a portfolio $\boldsymbol{\theta}_t(t+1)$ at that time. Introducing the notations $\boldsymbol{\theta}_t(u) = \left(\theta_t^{(0)}(u), \theta_t^{(1)}(u) \ldots, \theta_t^{(n)}(u)\right)$ for the components of $\boldsymbol{\theta}_t(u)$, we interpret the quantity $\theta_t^{(i)}(u)$ as the number of units invested in asset i in time period u, that is in the time interval $(u-1, u]$. The $\mathcal{G}_{u-1}-$measurability requirement means that the portfolio composition $\boldsymbol{\theta}_t(u)$ for the time period u follows from the general information available up to and including time $u-1$. This information includes, but is broader than the price history of traded assets in that time interval.

The *initial investment* or the *endowment* at time t of the trading strategy $\boldsymbol{\theta}_t$ can be expressed as

$$\boldsymbol{\theta}_t(t+1) \cdot \boldsymbol{Y}(t) = \sum_{i=0}^{n} \theta_t^{(i)}(t+1) \times Y^{(i)}(t).$$

[1] Without loss of generality, we assume that there are no dividends. Otherwise, one can replace the traded asset by the gain process of the traded asset, which is the sum of its price process and the process describing its accumulated dividends.

The value of the trading strategy $\boldsymbol{\theta}_t$ at time u, just before rebalancing, is given by

$$\boldsymbol{\theta}_t(u) \cdot \boldsymbol{Y}(u) = \sum_{i=0}^{n} \theta_t^{(i)}(u) \times Y^{(i)}(u),$$

for any $u = t+1, \ldots, T$, whereas its value at time u, just after rebalancing, is given by

$$\boldsymbol{\theta}_t(u+1) \cdot \boldsymbol{Y}(u) = \sum_{i=0}^{n} \theta_t^{(i)}(u+1) \times Y^{(i)}(u),$$

for any $u = t+1, \ldots, T-1$.
Obviously, $\boldsymbol{\theta}_t(u) \cdot \boldsymbol{Y}(u)$ and $\boldsymbol{\theta}_t(u+1) \cdot \boldsymbol{Y}(u)$ are $\mathcal{G}_u-$measurable.

A time$-t$ trading strategy $\boldsymbol{\theta}_t$ is said to be *self-financing* if

$$\boldsymbol{\theta}_t(u) \cdot \boldsymbol{Y}(u) = \boldsymbol{\theta}_t(u+1) \cdot \boldsymbol{Y}(u), \qquad \text{for any } u = t+1, ..., T-1. \tag{4.1}$$

This means that no capital is injected or withdrawn at any rebalancing moment $u = t+1, ..., T-1$. We denote the set of self-financing time$-t$ trading strategies by Θ_t. Taking into account (4.1), the time$-T$ value of any self-financing time$-t$ strategy $\boldsymbol{\theta}_t \in \Theta_t$ can be expressed as

$$\boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T) = \boldsymbol{\theta}_t(t+1) \cdot \boldsymbol{Y}(t) + \sum_{u=t+1}^{T} \boldsymbol{\theta}_t(u) \cdot \Delta\boldsymbol{Y}(u), \tag{4.2}$$

with $\Delta\boldsymbol{Y}(u) = \boldsymbol{Y}(u) - \boldsymbol{Y}(u-1)$. In this formula, $\boldsymbol{\theta}_t(u) \cdot \Delta\boldsymbol{Y}(u)$ is the change of the market value of the investment portfolio in the time period u, i.e. between time $u-1$ (just after rebalancing) and time u (just before rebalancing).

We assume that the market of traded assets is *arbitrage-free* in the sense that there is no self-financing strategy $\boldsymbol{\theta}_0 \in \Theta_0$ with the following properties:

$$\begin{aligned} \boldsymbol{\theta}_0(1) \cdot \boldsymbol{Y}(0) &= 0, \\ \mathbb{P}\left[\boldsymbol{\theta}_0(T) \cdot \boldsymbol{Y}(T) \geq 0\right] &= 1, \\ \mathbb{P}\left[\boldsymbol{\theta}_0(T) \cdot \boldsymbol{Y}(T) > 0\right] &> 0. \end{aligned}$$

In our discrete-time setting, the absence of arbitrage is equivalent to the existence of an equivalent martingale measure $\mathbb{Q}$ (further abbreviated as EMM), under which the discounted price process $\boldsymbol{Y}$ is a $\mathcal{G}-$martingale:

$$\boldsymbol{Y}(t-1) = \mathbb{E}_{t-1}^{\mathbb{Q}}\left[e^{-\int_{t-1}^{t} r_s ds}\boldsymbol{Y}(t)\right], \qquad \text{for any } t = 1, ..., T, \quad (4.3)$$

for some (possibly stochastic) interest rate r_s.[2] For the rest of the chapter, we will use the notation $\mathbb{E}_t^{\mathbb{Q}}[\cdot] := \mathbb{E}^{\mathbb{Q}}[\cdot|\,\mathcal{G}_t]$. For a proof of this equivalence, we refer to Delbaen and Schachermayer (2006). Consider a time$-t$ self-financing strategy $\boldsymbol{\theta}_t \in \Theta_t$. From (4.3) it follows that its time$-u$ price is given by

$$\boldsymbol{\theta}_t(u+1)\cdot\boldsymbol{Y}(u) = \mathbb{E}_u^{\mathbb{Q}}\left[e^{-\int_u^T r_s ds}\boldsymbol{\theta}_t(T)\cdot\boldsymbol{Y}(T)\right], \qquad (4.4)$$

for any $u = t, ..., T-1$. In the remainder of the chapter, we assume that the asset 0 is the zero-coupon bond paying an amount of 1 at maturity T. Its price at time t, denoted by $B(t,T)$, is given by

$$Y^{(0)}(t) = B(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\right], \qquad \text{for any } t = 0, 1, ..., T-1.$$

A simple example of a self-financing time$-t$ trading strategy is the static trading strategy $\boldsymbol{\beta}_t$ consisting of buying one unit of the zero-coupon bond $B(t,T)$ at time t and holding it until maturity T. The value of this strategy at time u is given by

$$\boldsymbol{\beta}_t(u)\cdot\boldsymbol{Y}(u) = \mathbb{E}_u^{\mathbb{Q}}\left[e^{-\int_u^T r_s ds}\right], \qquad \text{for any } u = t+1, ..., T.$$

Definition 30 ($t-$hedgeable $T-$claim)
A $t-$hedgeable $T-$claim S^h is an element of $\mathcal{C}_T$ which can be replicated by a time$-t$ self-financing strategy $\boldsymbol{\theta}_t \in \Theta_t$:

$$S^h = \boldsymbol{\theta}_t(T)\cdot\boldsymbol{Y}(T),$$

where $\boldsymbol{\theta}_t(T)\cdot\boldsymbol{Y}(T)$ is the time$-T$ value of the hedging portfolio $\boldsymbol{\theta}_t$.

We introduce the notation $\mathcal{H}_T^t$ for the set of all time$-t$ hedgeable $T-$claims. For any time$-t$ hedgeable T-claim S^h, a time$-t$

[2] Even though we use the continuous-time notation for interest rates, in practice the discrete-time version will be used calibrated to a market yield curve (see for instance MacKay and Wüthrich (2015) and Wüthrich (2016)).

trading strategy which replicates S^h is called a *replicating* $t-$*hedge* of S^h.

The time$-t$ price of S^h is given by

$$\boldsymbol{\theta}_t(t+1)\cdot \boldsymbol{Y}(t) = \mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds}\boldsymbol{\theta}_t(T)\cdot \boldsymbol{Y}(T)],$$

where $\mathbb{Q}$ is a generic member of the class of EMM's and $\boldsymbol{\theta}_t$ is a replicating $t-$hedge of S^h.

Notice that $\mathcal{H}_T^t$ is increasing in t. The $T-$claim

$$S = Y^{(1)}(t)\; Y^{(2)}(T),$$

is an example of a $T-$claim which will in general not be an element of $\mathcal{H}_T^s$ for any $s = 0, 1, \ldots, t-1$, while $S \in \mathcal{H}_T^s$ for any $s = t, t+1, \ldots, T-1$.

Next, we introduce the notion of $t-$orthogonal $T-$claims.

Definition 31 ($t-$orthogonal $T-$claim)
A $t-$orthogonal $T-$claim $S^{\perp}$ is an element of $\mathcal{C}_T$ which is $\mathbb{P}$-independent of the stochastic process $\mathbf{Y}_{t+1} = \{\boldsymbol{Y}(u)\}_{u\in\{t+1,\ldots,T\}}$ describing the evolution of the traded assets from $t+1$ onwards:

$$S^{\perp} \perp \mathbf{Y}_{t+1}.$$

Hereafter, we will denote the set of all $t-$orthogonal T-claims by $\mathcal{O}_T^t$. We remark that the set $\mathcal{O}_T^t$ is also increasing in t. An example of a $T-$claim which does not belong to the initial set of orthogonal claims $\mathcal{O}_T^0$, but which is an element of $\mathcal{O}_T^t$ is given by

$$S = \frac{1}{t}\sum_{i=1}^{t} Y^{(1)}(i)\; 1_{(x)}$$

where $1_{(x)}$ is the indicator variable which equals 1 if (x) survives until time T and 0 otherwise. Hence, in case of survival, the claim guarantees the average price of asset 1 between time 1 and time t. Under independence between mortality and the traded assets, we have that $S \notin \mathcal{O}_T^u$, for $u = 0, 1, \ldots, t-1$, while $S \in \mathcal{O}_T^u$ for $u = t, t+1, \ldots, T$.

4.3 $t-$valuations

In this section, we define different classes of $t-$valuations. In a dynamic multiperiod setting, a $t-$valuation ρ_t assigns to each $T-$claim a $\mathcal{G}_t-$measurable random variable $\rho_t[S]$ that represents the value of the $T-$claim given the available information at time t. In Chapter 2 fair valuations of insurance claims in a static one-period setting were considered. We showed that any fair valuation can be characterized in terms of a fair hedger. In this section, we generalize this result in a dynamic setting by showing that any fair $t-$valuation can be characterized in terms of a fair $t-$hedger.

4.3.1 Fair $t-$valuations

In this subsection, we define the notion of $t-$valuation. Furthermore, we introduce the notions of actuarial, market-consistent and fair $t-$valuations, respectively.

Definition 32 ($t-$valuation) A $t-$valuation, $t = 0, 1, \ldots, T-1$, is a mapping $\rho_t : \mathcal{C}_T \to \mathcal{C}_t$, attaching a $t-$claim to any T-claim $S \in \mathcal{C}_T$:

$$S \to \rho_t[S],$$

such that

- ρ_t is normalized:
$$\rho_t[0] = 0.$$
- ρ_t is translation invariant:
$$\rho_t[S + a] = \rho_t[S] + B(t,T)a,$$
for any $S \in \mathcal{C}_T$ and $a \in \mathcal{C}_t$ payable at time T.

For any $T-$claim, the value $\rho_t[S]$ is a $t-$claim and hence, seen from the perspective of time 0, it is a random variable. On the other hand, having arrived at time t, $\rho_t[S]$ is clearly deterministic. In Pelsser and Stadje (2014), $t-$valuations are called $\mathcal{G}_t-$conditional evaluations.

Important subclasses of $t-$valuations include the class of actuarial and market-consistent $t-$valuations, which are defined hereafter.

Definition 33 (Actuarial and MC $t-$valuations)
Consider a $t-$valuation $\rho_t : \mathcal{C}_T \to \mathcal{C}_t$.

- ρ_t is actuarial if any $t-$orthogonal T-claim is marked-to-model:

$$\rho_t\left[S^{\perp}\right] = B(t,T)\pi_t\left[S^{\perp}\right], \qquad \text{for any } S^{\perp} \in \mathcal{O}_T^t, \tag{4.5}$$

  where the $t-$valuation $\pi_t : \mathcal{O}_T^t \to \mathcal{C}_t$ is $\mathbb{P}-$law invariant and $\mathbb{P}$-independent of time$-t$ and future asset prices $\mathbf{Y}_t = \{\boldsymbol{Y}(u)\}_{u\in\{t,\dots,T\}}$.

- ρ_t is market-consistent (MC) if any $t-$hedgeable part of any $T-$claim is marked-to-market:

$$\rho_t\left[S+S^h\right] = \rho_t[S] + \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}S^h\right], \tag{4.6}$$

  for any $S \in \mathcal{C}_T$ and $S^h \in \mathcal{H}_T^t$.

The mark-to-model condition (4.5) corresponds to the traditional valuation of orthogonal (i.e. non-equity-linked) claims in an insurance context. It postulates that any $t-$orthogonal claim is valuated by a $\mathbb{P}-$law invariant $t-$valuation π_t (e.g. standard deviation principle, mean-variance principle,...) multiplied by the time-t zero-coupon bond price $B(t,T)$. For instance, in case π_t is the standard deviation principle, we find that

$$\rho_t\left[S^{\perp}\right] = \left(\mathbb{E}_t^{\mathbb{P}}\left[S^{\perp}\right] + \alpha\sigma_t^{\mathbb{P}}\left[S^{\perp}\right]\right)B(t,T),$$

with $\sigma_t^{\mathbb{P}}\left[S^{\perp}\right] := \sqrt{Var^{\mathbb{P}}\left[S^{\perp} \mid \mathcal{G}_t\right]}$ and $\alpha > 0$.

Moreover, we make the technical requirement that $\pi_t\left[S^{\perp}\right]$ is $\mathbb{P}$-independent of asset prices at time t and beyond: $\mathbf{Y}_t = \{\boldsymbol{Y}(u)\}_{u\in\{t,\dots,T\}}$ for any $S^{\perp} \in \mathcal{O}_T^t$. Otherwise stated, the actuarial value of a claim independent of future asset prices is independent of time$-t$ *and* future asset prices. This intuitive requirement will be used in the proof of Theorem 13.

The mark-to-market condition (4.6) extends the notion of cash-invariance to all $t-$hedgeable claims by postulating that any $t-$hedgeable claim should be valuated at the price of its replicating

$t-$hedge. We remark that the mark-to-market condition can also be expressed as follows:

$$\rho_t \left[S + S^h \right] = \rho_t[S] + \boldsymbol{\theta}_t(t+1) \cdot \boldsymbol{Y}(t), \tag{4.7}$$

for any $S \in \mathcal{C}_T$ and $S^h \in \mathcal{H}_T^t$, with $\boldsymbol{\theta}_t$ a replicating $t-$hedge of S^h.

Combining these notions leads to the definition of a fair $t-$valuation.

Definition 34 (Fair $t-$valuation)
A fair $t-$valuation is a $t-$valuation which is both actuarial and market-consistent.

Hereafter, we provide a simple example of a fair $t-$valuation for equity-linked life-insurance contracts.

Example 11 (Fair $t-$valuation of product claims)
Consider a $T-$claim S for which we want to determine the fair valuation at time t. We assume that we can decompose the claim as follows

$$S = S^{\perp} \times S^h,$$

where $S^{\perp}$ is a $t-$orthogonal $T-$claim and S^h is a $t-$hedgeable $T-$claim.
Such *product* claims often arise in insurance as payoffs of equity-linked life-insurance contracts. In such payoffs, S^h is typically a hedgeable claim contingent on the price history of traded assets such as stock, mutual funds, options or bonds while $S^{\perp}$ is contingent on the survival or death of a policyholder. For any product $T-$claim S, we define the $t-$valuation

$$\rho_t\left[S\right] = \mathbb{E}_t^{\mathbb{P}}\left[S^{\perp}\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} S^h\right],$$

for any $S^{\perp} \in \mathcal{O}_T^t$ and $S^h \in \mathcal{H}_T^t$.
Hence, the $t-$valuation ρ_t appears as a product of two expectations. The non-equity linked part $S^{\perp}$ is valuated under the physical measure $\mathbb{P}$ modeling the non-hedgeable risks and the hedgeable part S^h is valuated under a risk-neutral measure $\mathbb{Q}$ modeling hedgeable risks.
One can easily verify that the $t-$valuation ρ_t is actuarial:

$$\rho_t\left[S^{\perp}\right] = \mathbb{E}_t^{\mathbb{P}}\left[S^{\perp}\right] B(t,T),$$

and market-consistent:

$$\rho_t\left[S+S^h\right]=\rho_t[S]+\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}S^h\right].$$

One can also extend the valuation by introducing a loading on the valuation of orthogonal claims via an appropriate distorted probability measure $\mathbb{P}^* \sim \mathbb{P}$ to take into account the uncertainty in the orthogonal claims, see e.g. Chapter 2.6 in Wüthrich (2016).

4.3.2 Fair $t-$hedgers

In this section, we introduce the class of $t-$hedgers, as well as the subclasses of actuarial, market-consistent and fair $t-$hedgers. These notions are generalizations of the time-0 hedgers which were defined in Chapter 2. In the forthcoming sections of this chapter, we will use these notions to express our main results.

Definition 35 ($t-$hedger)
A $t-$hedger is a function $\boldsymbol{\theta}_t : \mathcal{C}_T \to \Theta_t$ which maps any $T-$claim S into a self-financing time$-t$ trading strategy $\boldsymbol{\theta}_{t,S} \in \Theta_t$ such that

- $\boldsymbol{\theta}_t$ is normalized:

$$\boldsymbol{\theta}_{t,0}=\mathbf{0}_t,$$

  where $\mathbf{0}_t$ is the self-financing time$-t$ trading strategy corresponding to the null investment at time t, i.e. $\mathbf{0}_t(u) = (0,0,\dots,0)$ for all $u=t+1,\dots,T$.

- $\boldsymbol{\theta}_t$ is translation invariant:

$$\boldsymbol{\theta}_{t,S+a}=\boldsymbol{\theta}_{t,S}+a\boldsymbol{\beta}_t,$$

  for any $S \in \mathcal{C}_T$ and $a \in \mathcal{C}_t$ payable at time T, where $\boldsymbol{\beta}_t$ is the static trading strategy which consists in buying one unit of the zero-coupon bond $B(t,T)$ and holding it until maturity T.

The mapping $\boldsymbol{\theta}_t : \mathcal{C}_T \to \Theta_t$ is called a $t-$hedger, whereas for any T-claim S, the self-financing trading strategy $\boldsymbol{\theta}_{t,S}$ is called a $t-$hedge for S. The value of the hedge $\boldsymbol{\theta}_{t,S}$ of S at time $u = t+1,\dots,T$, before rebalancing, is given by $\boldsymbol{\theta}_{t,S}(u)\cdot\boldsymbol{Y}(u)$, while after rebalancing, it is $\boldsymbol{\theta}_{t,S}(u+1)\cdot\boldsymbol{Y}(u)$.

Hereafter, we introduce the subclasses of actuarial, market-consistent and fair $t-$hedgers.

Definition 36 (Actuarial and MC $t-$hedgers)
Consider a $t-$hedger $\boldsymbol{\theta}_t$.

- $\boldsymbol{\theta}_t$ is <u>actuarial</u> in case any $t-$orthogonal $T-$claim $S^{\perp}$ is hedged via an actuarial $t-$valuation ρ_t in zero-coupon bonds:

$$\boldsymbol{\theta}_{t,S^{\perp}} = \frac{\rho_t\left[S^{\perp}\right]}{B(t,T)}\boldsymbol{\beta}_t, \quad \text{for any } S^{\perp} \in \mathcal{O}_T^t. \tag{4.8}$$

- $\boldsymbol{\theta}_t$ is <u>market-consistent</u> (MC) in case any $t-$hedgeable part S^h of any T-claim S is hedged by a replicating hedge:

$$\boldsymbol{\theta}_{t,S+S^h} = \boldsymbol{\theta}_{t,S} + \boldsymbol{\theta}_{t,S^h}, \quad \text{for any } S \in \mathcal{C}_T \text{ and any } S^h \in \mathcal{H}_T^t, \tag{4.9}$$

  where $\boldsymbol{\theta}_{t,S^h}$ is a replicating $t-$hedge of S^h.

We remark that an actuarial $t-$hedger $\boldsymbol{\theta}_t$ is defined in terms of an actuarial $t-$valuation ρ_t. Hereafter, we will call ρ_t the underlying actuarial $t-$valuation of the actuarial $t-$hedger $\boldsymbol{\theta}_t$.

Combining the definitions of actuarial and market-consistent $t-$hedgers leads to the definition of fair $t-$hedgers.

Definition 37 (Fair $t-$hedger)
A $t-$hedger is <u>fair</u> in case it is actuarial and market-consistent.

In the remainder of the chapter, we often consider the trading strategy which consists in investing (at time t) $\rho_t[S]$ in the zero-coupon bond $B(t,T)$, for $t = 0, 1, ..., T-1$. It is clear that the initial investment at time t of this trading strategy is $\rho_t[S]$ and its time$-T$ value, denoted by $\widetilde{\rho}_t$, is given by

$$\widetilde{\rho}_t[S] = \frac{\rho_t[S]}{B(t,T)}. \tag{4.10}$$

Hereafter, we provide an example of a fair $t-$hedger. This will be used later in the proof of Theorem 12.

Example 12 Fix $t \in \{0, \ldots, T-1\}$ and define the $t-$hedger $\boldsymbol{\theta}_t$ as follows:

1. For any $t-$orthogonal $T-$claim $S^{\perp} \in \mathcal{O}_T^t$, we define the $t-$hedger $\boldsymbol{\theta}_t$ by
$$\boldsymbol{\theta}_{t,S^{\perp}} = \mathbb{E}_t^{\mathbb{P}}\left[S^{\perp}\right]\boldsymbol{\beta}_t.$$

2. For all other $T-$claims $S \notin \mathcal{O}_T^t$, the $t-$hedger $\boldsymbol{\theta}_t$ is defined as the mean-variance hedger:
$$\boldsymbol{\theta}_{t,S} = \arg\min_{\boldsymbol{\theta}\in\Theta_t} \mathbb{E}_t^{\mathbb{P}}\left[(S - \boldsymbol{\theta}_{t,S}(T)\cdot \boldsymbol{Y}(T))^2\right]. \tag{4.11}$$

   As we assume that the time$-T$ value of any time$-t$ trading strategy is square-integrable, a solution to the optimization problem (4.11) exists (see for instance Černỳ and Kallsen (2009)). It is then easy to verify that $\boldsymbol{\theta}_t$ is well defined and a fair $t-$hedger.

4.3.3 Characterization of $t-$valuations

In the following lemma, we consider properties of a $t-$hedger $\boldsymbol{\mu}_{t,S}$ which is defined as the sum of another $t-$hedger $\boldsymbol{\theta}_{t,S}$ and an investment in zero-coupon bonds of the remaining risk $S-\boldsymbol{\theta}_{t,S}(T)\cdot\boldsymbol{Y}(T)$. The proof of a forthcoming theorem is based on the construction of such hedgers.

Lemma 4 Consider a $t-$hedger $\boldsymbol{\theta}_t$ and a $t-$valuation ρ_t. Define the $t-$hedger $\boldsymbol{\mu}_t$ by
$$\boldsymbol{\mu}_{t,S} = \boldsymbol{\theta}_{t,S} + \widetilde{\rho}_t\left[S - \boldsymbol{\theta}_{t,S}(T)\cdot \boldsymbol{Y}(T)\right]\ \boldsymbol{\beta}_t, \tag{4.12}$$
for any $S \in \mathcal{C}_T$.

(a) If $\boldsymbol{\theta}_t$ is an actuarial $t-$hedger and ρ_t is an actuarial $t-$valuation, then $\boldsymbol{\mu}_t$ is an actuarial $t-$hedger with underlying actuarial $t-$valuation ρ_t.

(b) If $\boldsymbol{\theta}_t$ is a MC $t-$hedger, then $\boldsymbol{\mu}_t$ is a MC $t-$hedger and $\boldsymbol{\mu}_{t,S^h} = \boldsymbol{\theta}_{t,S^h}$ for any $t-$hedgeable $T-$claim S^h.

(c) If $\boldsymbol{\theta}_t$ is a fair $t-$hedger and ρ_t is an actuarial $t-$valuation, then $\boldsymbol{\mu}_t$ is a fair $t-$hedger with underlying actuarial $t-$valuation ρ_t.

Proof: It is a straightforward exercise to verify that $\boldsymbol{\mu}_t$ is a $t-$hedger.
(a) Suppose that $\boldsymbol{\theta}_t$ is an actuarial $t-$hedger with underlying actuarial $t-$valuation ψ_t. Further, suppose that ρ_t is an actuarial $t-$valuation. For any $t-$orthogonal $T-$claim $S^{\perp}$, we have

$$\begin{aligned}\boldsymbol{\mu}_{t,S^{\perp}} &= \boldsymbol{\theta}_{t,S^{\perp}} + \widetilde{\rho}_t\left[S^{\perp} - \boldsymbol{\theta}_{t,S^{\perp}}(T)\cdot \boldsymbol{Y}(T)\right]\ \boldsymbol{\beta}_t \\ &= \widetilde{\psi}_t\left[S^{\perp}\right]\ \boldsymbol{\beta}_t + \widetilde{\rho}_t\left[S^{\perp} - \widetilde{\psi}_t\left[S^{\perp}\right]\right]\ \boldsymbol{\beta}_t \\ &= \widetilde{\rho}_t\left[S^{\perp}\right]\ \boldsymbol{\beta}_t,\end{aligned}$$

where in the last step, we used the translation invariance of ρ_t. We can conclude that $\boldsymbol{\mu}_t$ is an actuarial $t-$hedger with underlying actuarial $t-$valuation ρ_t.
(b) Suppose that $\boldsymbol{\theta}_t$ is a MC $t-$hedger. By definition of $\boldsymbol{\mu}_t$, we have that

$$\boldsymbol{\mu}_{t,S+S^h} = \boldsymbol{\theta}_{t,S+S^h} + \widetilde{\rho}_t\left[S + S^h - \boldsymbol{\theta}_{t,S+S^h}(T)\cdot \boldsymbol{Y}(T)\right]\ \boldsymbol{\beta}_t,$$

for any $S^h \in \mathcal{H}_T^t$. Given that $\boldsymbol{\theta}_t$ is a MC $t-$hedger, we find

$$\begin{aligned}\boldsymbol{\mu}_{t,S+S^h} &= \boldsymbol{\theta}_{t,S} + \boldsymbol{\theta}_{t,S^h} + \widetilde{\rho}_t\left[S - \boldsymbol{\theta}_{t,S}(T)\cdot \boldsymbol{Y}(T)\right]\ \boldsymbol{\beta}_t \\ &= \boldsymbol{\mu}_{t,S} + \boldsymbol{\theta}_{t,S^h}.\end{aligned}$$

We can conclude that $\boldsymbol{\mu}_t$ is a MC $t-$hedger.
(c) Finally, suppose that $\boldsymbol{\theta}_t$ is a fair $t-$hedger with underlying actuarial $t-$valuation ψ_t, while ρ_t is an actuarial $t-$valuation. From (a) and (b) it follows immediately that $\boldsymbol{\mu}_t$ is a fair $t-$hedger with underlying actuarial $t-$valuation ρ_t. ∎

In the following theorem it is shown that any actuarial $t-$valuation ρ_t can be represented as the time$-t$ price of an actuarial $t-$hedger. Similar properties hold for market-consistent and fair $t-$valuations.

Theorem 12 Consider a $t-$valuation $\rho_t : \mathcal{C}_T \longrightarrow \mathcal{C}_t$.
(a) ρ_t is an actuarial $t-$valuation if and only if there exists an actuarial $t-$hedger $\boldsymbol{\theta}_t^a$ such that

$$\rho_t\left[S\right] = \boldsymbol{\theta}_{t,S}^a(t+1)\cdot \boldsymbol{Y}(t), \qquad \text{for any } S \in \mathcal{C}_T. \tag{4.13}$$

(b) ρ_t is a MC $t-$valuation if and only if there exists a MC $t-$hedger $\boldsymbol{\theta}_t^m$ such that

$$\rho_t\left[S\right] = \boldsymbol{\theta}^m_{t,S}(t+1)\cdot \boldsymbol{Y}(t), \qquad \text{for any } S\in\mathcal{C}_T. \tag{4.14}$$

(c) ρ_t is a fair $t-$valuation if and only if there exists a fair $t-$hedger $\boldsymbol{\theta}_t^f$ such that

$$\rho_t\left[S\right] = \boldsymbol{\theta}^f_{t,S}(t+1)\cdot \boldsymbol{Y}(t), \qquad \text{for any } S\in\mathcal{C}_T. \tag{4.15}$$

Proof: (a) Let ρ_t be an actuarial $t-$valuation. For any $S\in\mathcal{C}_T$, we can write $\rho_t\left[S\right]$ as

$$\begin{aligned}\rho_t\left[S\right] &= \widetilde{\rho}_t\left[S\right] B(t,T)\\ &= \boldsymbol{\theta}^a_{t,S}(t+1)\cdot \boldsymbol{Y}(t),\end{aligned}$$

with $\boldsymbol{\theta}^a_{t,S}$ defined by

$$\boldsymbol{\theta}^a_{t,S} = \widetilde{\rho}_t\left[S\right]\ \boldsymbol{\beta}_t.$$

Obviously, $\boldsymbol{\theta}^a_t$ is an actuarial $t-$hedger.
(a') Suppose that the $t-$valuation ρ_t is defined by (4.13) for some actuarial $t-$hedger $\boldsymbol{\theta}^a_t$ with underlying actuarial $t-$valuation π_t. For any $t-$orthogonal $T-$claim $S^\perp$, we have

$$\rho_t\left[S^\perp\right] = \boldsymbol{\theta}^a_{t,S^\perp}(t+1)\cdot \boldsymbol{Y}(t) = \pi_t\left[S^\perp\right].$$

We can conclude that the valuation ρ_t is an actuarial $t-$valuation.
(b) Let ρ_t be a MC $t-$valuation. Consider a MC $t-$hedger $\boldsymbol{\theta}_t$, e.g. the $t-$hedger defined in Example 12. For any $T-$claim S, we find from (4.6) that

$$\begin{aligned}\rho_t\left[S\right] &= \mathbb{E}^{\mathbb{Q}}_t\left[e^{-\int_t^T r_s ds}\boldsymbol{\theta}_{t,S}(T)\cdot\boldsymbol{Y}(T)\right] + \rho_t\left[S-\boldsymbol{\theta}_{t,S}(T)\cdot\boldsymbol{Y}(T)\right]\\ &= \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) + \rho_t\left[S-\boldsymbol{\theta}_{t,S}(T)\cdot\boldsymbol{Y}(T)\right]\\ &= \boldsymbol{\theta}^m_{t,S}(t+1)\cdot\boldsymbol{Y}(t),\end{aligned}$$

with

$$\boldsymbol{\theta}^m_{t,S} = \boldsymbol{\theta}_{t,S} + \widetilde{\rho}_t\left[S-\boldsymbol{\theta}_{t,S}(T)\cdot\boldsymbol{Y}(T)\right]\ \boldsymbol{\beta}_t. \tag{4.16}$$

From Lemma 4 we know that $\boldsymbol{\theta}^m$ is a MC $t-$hedger.
(b') Consider the $t-$valuation ρ_t defined by (4.14) for some MC

$t-$hedger $\boldsymbol{\theta}_t^m$. For any $T-$claim S and any $t-$hedgeable $T-$claim S^h, we find that

$$\begin{aligned}\rho_t\left[S+S^h\right] &= \boldsymbol{\theta}^m_{t,S+S^h}(t+1)\cdot\boldsymbol{Y}(t)\\ &= \boldsymbol{\theta}^m_{t,S}(t+1)\cdot\boldsymbol{Y}(t)+\boldsymbol{\theta}^m_{t,S^h}(t+1)\cdot\boldsymbol{Y}(t)\\ &= \rho_t\left[S\right]+\rho_t\left[S^h\right].\end{aligned}$$

We can conclude that ρ_t is a MC $t-$valuation.
(c) Let ρ_t be a fair $t-$valuation. Consider a fair $t-$hedger $\boldsymbol{\theta}_t$, e.g. the $t-$hedger defined in Example 12, with underlying actuarial $t-$valuation ψ_t. From (a) we know that for any $T-$claim S, $\rho_t\left[S\right]$ can be expressed as

$$\rho_t\left[S\right]=\boldsymbol{\theta}^m_{t,S}(t+1)\cdot\boldsymbol{Y}(t),$$

with the MC $t-$hedger $\boldsymbol{\theta}_t^m$ given by (4.16). For any $t-$orthogonal $T-$claim $S^\perp$, we find that

$$\begin{aligned}\boldsymbol{\theta}^m_{t,S^\perp} &= \boldsymbol{\theta}_{t,S^\perp}+\widetilde{\rho}_t\left[S^\perp-\boldsymbol{\theta}^m_{t,S^\perp}(T)\cdot\boldsymbol{Y}(T)\right]\ \boldsymbol{\beta}_t\\ &= \widetilde{\psi}_t\left[S^\perp\right]\boldsymbol{\beta}_t+\widetilde{\rho}_t\left[S^\perp-\widetilde{\psi}_t\left[S^\perp\right]\right]\ \boldsymbol{\beta}_t\\ &= \widetilde{\rho}_t\left[S^\perp\right]\ \boldsymbol{\beta}_t.\end{aligned}$$

As ρ_t is an actuarial valuation, we can conclude that the $t-$hedger $\boldsymbol{\theta}_t^m$ is not only market-consistent but also actuarial and hence, a fair $t-$hedger.
(c') Suppose that the $t-$valuation ρ_t is defined by (4.15) for some fair $t-$hedger $\boldsymbol{\theta}_t^f$. From (a) and (b) we can conclude that the $t-$valuation ρ_t is actuarial and market-consistent, which means that it is fair. ■

Taking into account (4.4), we have that the relation (4.15) for a fair $t-$valuation can be rewritten as follows:

$$\rho_t\left[S\right]=\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\boldsymbol{\theta}^f_{t,S}(T)\cdot\boldsymbol{Y}(T)\right],\quad\text{for any } S\in\mathcal{C}_T. \quad (4.17)$$

The fair valuation at time t of any $T-$claim can then be expressed as a conditional expectation of the time$-T$ value of a fair hedge for S, $\boldsymbol{\theta}^f_{t,S}$, under an equivalent martingale measure $\mathbb{Q}$. Actuarial considerations are implicitly involved since any fair valuation is an actuarial valuation, implying actuarial judgement on the valuation of orthogonal claims.

4.4 Dynamic valuations

In the previous section, we introduced the concept of $t-$valuations which assess a time$-t$ value for any $T-$claim, taking into account the available information at time t, for any time $t = 0, 1, \ldots, T-1$. This approach was static in the sense that we considered the value of a $T-$claim at different times $t < T$, without specifying the interconnection between the $t-$valuations. Bringing the $t-$valuations together leads to the concepts of *time-consistent* and *dynamic* valuations, which are defined hereafter.

4.4.1 Fair dynamic valuations

In the following definition, we introduce the notion of dynamic valuation. See for instance Acciaio and Penner (2011), Artzner et al. (2007) or Riedel (2004) for similar notions.

Definition 38 (Dynamic valuation) A dynamic valuation is a sequence $(\rho_t)_{t=0}^{T-1}$ where for each $t = 0, 1, ..., T-1$, ρ_t is a $t-$valuation.

After having introduced the concept of dynamic valuation, we now define actuarial, market-consistent and time-consistent dynamic valuations. Notice that a $t-$valuation ρ_t is defined for $T-$claims S payable at time T. In order to compare $t-$valuations at different times, we consider the $t-$valuation $\widetilde{\rho_t}\left[S\right]$ introduced in (4.10) which corresponds to the value at time T of the investment of the $t-$valuation $\rho_t\left[S\right]$ in the zero-coupon bond $B(t, T)$.

Definition 39 (Actuarial, MC and TC dynamic valuations)

Consider the dynamic valuation $(\rho_t)_{t=0}^{T-1}$.

- $(\rho_t)_{t=0}^{T-1}$ is <u>actuarial</u> in case any $t-$valuation ρ_t is actuarial.
- $(\rho_t)_{t=0}^{T-1}$ is <u>market-consistent</u> (MC) in case any $t-$valuation ρ_t is market-consistent.
- $(\rho_t)_{t=0}^{T-1}$ is <u>time-consistent</u> (TC) in case all $t-$valuations involved are connected in the following way:

$$\rho_t\left[S\right] = \rho_t\left[\widetilde{\rho}_{t+1}\left[S\right]\right], \tag{4.18}$$

  for any $S \in \mathcal{C}_T$ and $t = 0, 1, ..., T-2$.

Actuarial and market-consistent dynamic valuations are natural generalizations of actuarial and market-consistent $t-$valuations. Time-consistency is a concept that couples the different static $t-$valuations. It means that the same time$-t$ value is assigned to a $T-$claim regardless of whether it is calculated in one step or in two steps backwards in time. Some weaker notions of time-consistency have been proposed in the literature, see e.g. Roorda et al. (2005) and Kriele and Wolf (2014). The definition (4.18) is often named the "recursiveness" or "tower property" definition. In the literature, an alternative definition of time-consistency is often used: if a claim is preferred to another claim at time $t+1$ in almost all states of nature, then the same conclusions should be drawn at time t:

$$\rho_{t+1}\left[S_1\right] \leq \rho_{t+1}\left[S_2\right] \Longrightarrow \rho_t\left[S_1\right] \leq \rho_t\left[S_2\right], \tag{4.19}$$

for all $S_1, S_2 \in \mathcal{C}_T$ and $t < T$.
Under monotonicity of the dynamic valuation $(\rho_t)_{t=0}^{T-1}$, it is well-known that both notions of time-consistency are equivalent (see for instance Acciaio and Penner (2011)). Since (4.19) implies monotonicity, the advantage of using the definition (4.18) is that we can also apply time-consistency to non-monotone dynamic valuations.

Time-consistent valuations have been discussed extensively in recent years. For the discrete time case, we refer to Cheridito and Kupper (2011), Acciaio and Penner (2011) and Föllmer and Schied (2011). For the continuous case, we refer to Frittelli and Gianin (2004), Delbaen et al. (2010), Pelsser and Stadje (2014) and Feinstein and Rudloff (2015).

Merging the notions of actuarial, market-consistent and time-consistent valuations leads to the concept of fair dynamic valuations.

Definition 40 (Fair dynamic valuations) A fair dynamic valuation is a dynamic valuation which is actuarial, market-consistent and time-consistent.

4.4.2 Fair dynamic hedgers

After having defined the class of $t-$hedgers in the previous section, we introduce the notion of a dynamic hedger.

Definition 41 (Dynamic hedger) A dynamic hedger is a sequence $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$ where for each $t = 0, 1, \ldots, T-1$, $\boldsymbol{\theta}_t$ is a $t-$hedger.

Hereafter, we introduce natural definitions of actuarial, market-consistent and time-consistent dynamic hedgers in accordance with Definition 39.

Definition 42 (Actuarial, MC and TC dynamic hedgers) Consider the dynamic hedger $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$.

- $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$ is actuarial in case any $t-$hedger $\boldsymbol{\theta}_t$ is actuarial.
- $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$ is market-consistent (MC) in case any $t-$hedger $\boldsymbol{\theta}_t$ is market-consistent.
- $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$ is time-consistent (TC) in case all $t-$hedgers involved are connected in the following way:

$$\boldsymbol{\theta}_{t,S} = \boldsymbol{\theta}_{t,\widetilde{\rho}_{t+1}[S]}, \tag{4.20}$$

  for any $S \in \mathcal{C}_T$ and $t = 0, 1, ..., T-2$, where $\rho_{t+1}[S]$ is the initial investment of $\boldsymbol{\theta}_{t+1}$:

$$\rho_{t+1}[S] = \boldsymbol{\theta}_{t+1,S}(t+2) \cdot \boldsymbol{Y}(t+1).$$

The definition of a time-consistent dynamic hedger should be compared with the definition of a time-consistent dynamic valuation. It means that the same hedger is assigned to a $T-$claim regardless of whether it is hedged in one step (i.e. directly over $T-t$ periods) or in two steps backwards in time.

Similarly to the concept of fair dynamic valuations, we introduce the concept of fair dynamic hedgers.

Definition 43 (Fair dynamic hedgers) A fair dynamic hedger is a dynamic hedger which is actuarial, market-consistent and time-consistent.

4.4.3 Characterization of fair dynamic valuations

In the following theorem we show that a fair dynamic valuation can be characterized in terms of a fair dynamic hedger.

Theorem 13 A dynamic valuation $(\rho_t)_{t=0}^{T-1}$ is fair if and only if there exists a fair dynamic hedger $(\boldsymbol{\mu}_t)_{t=0}^{T-1}$ such that

$$\rho_t [S] = \boldsymbol{\mu}_{t,S}(t+1) \cdot \boldsymbol{Y}(t), \qquad \text{for any } S \in \mathcal{C}_T. \tag{4.21}$$

Proof: (a) Suppose that $(\rho_t)_{t=0}^{T-1}$ is a fair dynamic valuation. From Theorem 12, we have that for any $t = 0, 1, \ldots, T-1$, there exists a fair $t-$hedger $\boldsymbol{\theta}_t$ such that

$$\rho_t [S] = \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t), \qquad \text{for any } S \in \mathcal{C}_T. \tag{4.22}$$

The dynamic hedger $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$ is actuarial and market-consistent but is a priori not time-consistent. Based on the dynamic hedger $(\boldsymbol{\theta}_t)_{t=0}^{T-1}$, we construct a dynamic hedger $(\boldsymbol{\mu}_t)_{t=0}^{T-1}$ which is actuarial, market-consistent *and* time-consistent. First, we set $\boldsymbol{\mu}_{T-1} = \boldsymbol{\theta}_{T-1}$. Obviously, $\boldsymbol{\mu}_{T-1}$ is a fair $(T-1)-$hedger and

$$\rho_{T-1} [S] = \boldsymbol{\mu}_{T-1,S}(T) \cdot \boldsymbol{Y}(T-1), \qquad \text{for any } S \in \mathcal{C}_T.$$

Second, we define the $(T-2)-$hedger $\boldsymbol{\mu}_{T-2}$ via

$$\boldsymbol{\mu}_{T-2,S} = \boldsymbol{\theta}_{T-2,\widetilde{\rho}_{T-1}[S]}, \qquad \text{for any } S \in \mathcal{C}_T.$$

Let us prove that $\boldsymbol{\mu}_{T-2}$ is a fair $(T-2)-$hedger.

- Actuarial hedger: for any $(T-2)-$orthogonal $T-$claim $S^{\perp}$, we have
$$\boldsymbol{\mu}_{T-2,S^{\perp}} = \boldsymbol{\theta}_{T-2,\widetilde{\rho}_{T-1}\left[S^{\perp}\right]}.$$
Given that ρ_{T-1} is an actuarial $(T-1)-$valuation, $\widetilde{\rho}_{T-1}\left[S^{\perp}\right]$ equals $\pi_{T-1}\left[S^{\perp}\right]$, which is by definition a $(T-2)-$orthogonal $T-$claim. Given that $\boldsymbol{\theta}_{T-2}$ is actuarial, we have
$$\begin{aligned}\boldsymbol{\mu}_{T-2,S^{\perp}} &= \boldsymbol{\theta}_{T-2,\pi_{T-1}\left[S^{\perp}\right]} \\ &= \pi_{T-2}\left[\pi_{T-1}\left[S^{\perp}\right]\right] \boldsymbol{\beta}_{T-2} \\ &= \pi_{T-2}\left[S^{\perp}\right] \boldsymbol{\beta}_{T-2},\end{aligned}$$
where we used the time-consistency of $(\rho_t)_{t=0}^{T-1}$.
Hence, $\boldsymbol{\mu}_{T-2}$ is an actuarial $(T-2)-$hedger.

- Market-consistent hedger: for any $(T-2)-$hedgeable $T-$claim S^h, we have

$$\begin{aligned}\boldsymbol{\mu}_{T-2,S+S^h} &= \boldsymbol{\theta}_{T-2,\widetilde{\rho}_{T-1}\left[S+S^h\right]} \\ &= \boldsymbol{\theta}_{T-2,\widetilde{\rho}_{T-1}[S]+\widetilde{\rho}_{T-1}\left[S^h\right]} \\ &= \boldsymbol{\theta}_{T-2,\widetilde{\rho}_{T-1}[S]} + \boldsymbol{\theta}_{T-2,S^h} \\ &= \boldsymbol{\mu}_{T-2,S} + \boldsymbol{\theta}_{T-2,S^h},\end{aligned}$$

  where we used the fact that any $t-$hedgeable claim is $(t+1)-$hedgeable as well (remark that the inverse is not true) and the market-consistency of $\boldsymbol{\theta}_{T-2}$.
  Hence, $\boldsymbol{\mu}_{T-2}$ is a market-consistent $(T-2)-$hedger.

Moreover, by (4.22), we have

$$\begin{aligned}\rho_{T-2}\left[S\right] &= \boldsymbol{\theta}_{T-2,S}(T-1)\cdot \boldsymbol{Y}(T-2) \\ &= \boldsymbol{\theta}_{T-2,\widetilde{\rho}_{T-1}[S]}(T-1)\cdot \boldsymbol{Y}(T-2) \text{ by time-consistency} \\ &= \boldsymbol{\mu}_{T-2,S}(T-1)\cdot \boldsymbol{Y}(T-2) \;\text{ by definition of } \boldsymbol{\mu}_{T-2}.\end{aligned}$$

Iteratively, starting from a fair $t-$hedger $\boldsymbol{\theta}_t$, we construct the time-consistent adaptation

$$\boldsymbol{\mu}_{t,S} = \boldsymbol{\theta}_{t,\widetilde{\rho}_{t+1}[S]}, \qquad \text{for any } S\in\mathcal{C}_T.$$

Similarly to $\boldsymbol{\mu}_{T-2}$, one can verify that $\boldsymbol{\mu}_t$ is a fair $t-$hedger. Moreover, $(\boldsymbol{\mu}_t)_{t=0}^{T-1}$ is time-consistent by construction and we have

$$\begin{aligned}\rho_t\left[S\right] &= \boldsymbol{\theta}_{t,S}(t+1)\cdot \boldsymbol{Y}(t) \\ &= \boldsymbol{\theta}_{t,\widetilde{\rho}_{t+1}[S]}(t+1)\cdot \boldsymbol{Y}(t) \;\text{ by time-consistency} \\ &= \boldsymbol{\mu}_{t,S}(t+1)\cdot \boldsymbol{Y}(t) \;\text{ by definition of } \boldsymbol{\mu}_t,\end{aligned}$$

which ends the proof.
(b) Suppose that there exists a fair dynamic hedger $(\boldsymbol{\mu}_t)_{t=0}^{T-1}$ such that

$$\rho_t\left[S\right] = \boldsymbol{\mu}_{t,S}(t+1)\cdot \boldsymbol{Y}(t), \qquad \text{for any } S\in\mathcal{C}_T. \tag{4.23}$$

From Theorem 12, we know that any $t-$valuation ρ_t is fair. Moreover, we have

$$\begin{aligned}\rho_t\left[S\right] &= \boldsymbol{\mu}_{t,S}(t+1)\cdot \boldsymbol{Y}(t) \\ &= \boldsymbol{\mu}_{t,\widetilde{\rho}_{t+1}[S]}(t+1)\cdot \boldsymbol{Y}(t) \text{ given } (\boldsymbol{\mu}_t)_{t=0}^{T-1} \text{ is time-consistent} \\ &= \rho_t\left[\widetilde{\rho}_{t+1}\left[S\right]\right] \;\text{ by definition of } \rho_t,\end{aligned}$$

which ends the proof. ■

4.5 Fair dynamic valuations: A practical approach

This section is dedicated to the practical application of the concepts introduced above. In Section 4.5.1, we present a general procedure to determine the fair dynamic valuation of insurance liabilities. The procedure is based on a backward iterations scheme combining risk minimization methods from mathematical finance and standard actuarial techniques. In Section 4.5.2, we apply the procedure to a portfolio of equity-linked life insurance contracts via a Least Square Monte Carlo (LSMC) implementation.[3] We provide numerical results illustrating the impact of time-consistency on the fair valuation in Section 4.5.3.

4.5.1 Fair dynamic valuation problem

We study the problem of an insurer who needs to determine a *fair* (actuarial, market-consistent and time-consistent) dynamic valuation for an insurance liability S which matures at time T. We assume that the financial market consists of a risk-free asset $Y^{(0)}(t) = e^{rt}$ and a risky asset $Y^{(1)}(t)$, $t = 0, 1, ..., T$. This objective is achieved by a backward procedure in which the constructed hedger $\boldsymbol{\theta}_t$ is optimal (in the quadratic hedging sense) for the fair value $\rho_{t+1}[S]$ for any time $t = 0, ..., T-1$. Moreover, for each time step, the residual non-hedged risk is valuated via an actuarial $t-$valuation π_t, implying that the dynamic valuation is actuarial as well.

Consider a $T-$claim S. The optimal hedger at time $T-1$ is defined by

$$\begin{aligned}&\boldsymbol{\theta}_{T-1,S}(T)\\&= \arg \min_{\boldsymbol{\theta}\in\Theta_{T-1}} \mathbb{E}^{\mathbb{P}}_{T-1}\left[\left(S - \theta^{(0)}_{T-1,S}(T)\cdot e^{rT} - \theta^{(1)}_{T-1,S}(T)\cdot Y^{(1)}(T)\right)^2\right]\end{aligned}$$

[3] We remark that a similar approach was used to determine time-consistent valuations by a two-step operator. We refer to Ghalehjooghi and Pelsser (2017) and Pelsser and Ghalehjooghi (2016)

Hence, the hedging strategy is determined at time $T-1$ such that the value of the hedger at time T is as close as possible to S in the quadratic hedging sense. Once the hedging strategy is set up, we value the non-hedged risk via an actuarial $(T-1)-$valuation π_{T-1}. The fair value of S at time $T-1$ is then defined as the sum of the financial value of the optimal hedge and the actuarial value of the remaining risk:

$$\rho_{T-1}[S] = \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T-1) + \pi_{T-1}[S - \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T)].$$

Iteratively, the optimal hedge at time t for $\rho_{t+1}[S]$ is determined by

$$\begin{aligned}&\boldsymbol{\theta}_{t,S}(t+1)\\&= \arg\min_{\boldsymbol{\theta}\in\Theta_t} \mathbb{E}_t^{\mathbb{P}}\left[\left(\rho_{t+1}[S] - \theta_{t,S}^{(0)}(t+1)\cdot e^{r(t+1)} - \theta_{t,S}^{(1)}(t+1)\cdot Y^{(1)}(t+1)\right)^2\right]\end{aligned}$$

After some direct derivations, we find that

$$\theta_{t,S}^{(1)}(t+1) = \frac{Cov_t^{\mathbb{P}}\left[\rho_{t+1}[S], Y^{(1)}(t+1)\right]}{Var_t^{\mathbb{P}}\left[Y^{(1)}(t+1)\right]}, \tag{4.24}$$

$$\theta_{t,S}^{(0)}(t+1) = \left(\mathbb{E}_t^{\mathbb{P}}[\rho_{t+1}[S]] - \theta_{t,S}^{(1)}(t+1)\cdot \mathbb{E}_t^{\mathbb{P}}\left[Y^{(1)}(t+1)\right]\right)\cdot e^{-r(t+1)}. \tag{4.25}$$

Then, the fair value at time t is obtained via

$$\rho_t[S] = \boldsymbol{\theta}_{t,S}(t+1)\cdot \boldsymbol{Y}(t) + \pi_t[\rho_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1)\cdot \boldsymbol{Y}(t+1)],$$

with π_t an actuarial $t-$valuation.

The procedure is quite intuitive: for each time period, an optimal hedge is set up by quadratic hedging and the remaining risk is valuated via an actuarial valuation, combining actuarial judgement and market-consistency. Moreover, the scheme is iterated backward in time to make it time-consistent. Since the hedger $\boldsymbol{\theta}_t$ is fair, by Theorem 13, ρ_t is a fair dynamic valuation.

4.5.2 Application to a portfolio of equity-linked life-insurance contracts

The backward recursive scheme presented above is similar to the one solving the local quadratic hedging problem and can be implemented by dynamic programming. Since the optimal hedger is a function of conditional expectations, a popular technique consists of constructing a Markov grid with the use of a multinomial tree model (see e.g. Černý (2004), Coleman et al. (2006)). However, in order to decrease the calculation volume, we follow a LSMC approach.[4] This regression-based method was proposed by Carriere (1996) and Longstaff and Schwartz (2001) for the valuation of American-type options. The key idea is to regress the conditional expectations on the cross-sectional information of the underlying risk drivers (in our case, mortality and equity risks). The LSMC technique will be used in order to determine the dynamic hedger in the expressions (4.24)-(4.25).

For the remainder of this section, we assume that the insurance liability which matures at time T has the following form

$$S = N(T) \times \max\left(Y^{(1)}(T), K\right), \tag{4.26}$$

with $N(t)$ a mortality process, $Y^{(1)}(t)$ a risky asset process and K is a fixed guarantee level.

For simplicity of illustration[5], we assume that the stock follows a geometric Brownian motion:

$$dY^{(1)}(t) = Y^{(1)}(t)\left(\mu dt + \sigma dW_1(t)\right)$$

with parameters $\mu, \sigma > 0$. The conditional expectation and variance are then given by

$$\mathbb{E}_t^{\mathbb{P}}\left[Y^{(1)}(t+1)\right] = Y^{(1)}(t)e^{\mu+\frac{\sigma^2}{2}}, \tag{4.27}$$

$$Var_t^{\mathbb{P}}\left[Y^{(1)}(t+1)\right] = \left(Y^{(1)}(t)\right)^2 e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right). \tag{4.28}$$

We assume that the mortality process $N(t)$ counts the number of survivals among an initial population of l_x policyholders of age x. The mortality intensity is assumed to be stochastic and follows the dynamics under $\mathbb{P}$ given by

$$d\lambda_x(t) = c\lambda_x(t)dt + \xi dW_2(t),$$

[4] The LSMC approach is used to approximate conditional expectations that are needed to determine the optimal hedging strategy in (4.24) and (4.25).

[5] The presented approach can be easily adapted to other stock dynamics, e.g. stochastic volatility or Lévy models.

with $c, \xi > 0$ and $W_2(t)$ a standard Brownian motion, independent of $W_1(t)$. The survival function is then defined by

$$S_x(t) := \mathbb{P}(T_x > t) = \exp\left(-\int_x^{x+t} \lambda_x(s)ds\right),$$

where T_x is the remaining lifetime of an individual who is aged x at time 0.

Moreover, deaths of individuals are assumed to be independent events conditional on knowing population mortality (see Milevsky et al. (2006) for similar assumptions). Further, if we denote $D(t+1)$ the number of deaths during year $t+1$, the dynamics of the number of active contracts can be described as a nested binomial process as follows: $N(t+1) = N(t) - D(t+1)$ with $D(t+1)|N(t), q_{x+t} \sim Bin(N(t), q_{x+t})$. Here, q_{x+t} represents the one-year death probability

$$q_{x+t} := \mathbb{P}(T_x \leq t+1 | T_x > t) = 1 - \frac{S_x(t+1)}{S_x(t)}, \text{ for } t = 0, \dots, T-1.$$

Knowing the dynamics of $N(t)$ and $Y^{(1)}(t)$, one can simulate n scenarios for the mortality and the equity risk factors for $t = 1, \dots, T$. Finally, the conditional expectations at time t are regressed over the risk drivers at time t via a second-order[6] least-squares regression:

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}}\left[\rho_{t+1}[S]\right] \\
&\approx \beta_0 + \beta_1 N(t) Y^{(1)}(t) + \beta_2 \left(N(t) Y^{(1)}(t)\right)^2, \\
&\mathbb{E}_t^{\mathbb{P}}\left[\rho_{t+1}[S]\, Y^{(1)}(t+1)\right] \\
&\approx \beta_0 + \beta_1 N(t) \left(Y^{(1)}(t)\right)^2 + \beta_2 \left(N(t)\left(Y^{(1)}(t)\right)^2\right)^2.
\end{aligned}$$

For the one-year actuarial $t-$valuation[7], we consider a standard

[6] The choice of type and number of basis functions was based on an equilibrium between bias and complexity and the payoff structure in (4.26). For a discussion of the basis functions and its implications on robustness and convergence, we refer to Areal et al. (2008), Moreno and Navas (2003) and Stentoft (2012).

[7] We remark that one can also consider an actuarial valuation which depends on the maturity T and the number of time steps (for instance, a Value-at-Risk principle for which the level of confidence decreases with the number of steps, see for instance Devolder and Lebègue (2016)).

deviation principle

$$\pi_t[S] = e^{-r}\left[\mathbb{E}_t^{\mathbb{P}}[S] + \alpha\sigma_t^{\mathbb{P}}[S]\right],$$

with $\sigma_t^{\mathbb{P}}[S] := \sqrt{Var_t^{\mathbb{P}}[S]}$ and $\alpha > 0$, which is also approximated via the LSMC approach.

4.5.3 Numerical analysis

In this section, we provide a numerical analysis for the fair dynamic valuation of the insurance liability S introduced above. Our numerical results are obtained by generating 50000 sample paths for $N(t)$ and $Y^{(1)}(t)$, for $t = 1, \ldots, T$. The benchmark parameters for the financial market are $r = 0.01$, $\mu = 0.02$, $\sigma = 0.1$, $K = 1$ and $Y^{(1)}(0) = 1$. The mortality parameters $(\lambda_x(0) = 0.0087, c = 0.0750, \xi = 0.000597)$ follow from Luciano et al. (2017) and correspond to UK male individuals who are aged 55 at time 0. We assume that there are $l_x = 1000$ initial contracts at time 0 with a maturity of $T = 10$ years.

4.5.3.1 The effect of a time-consistent and actuarial dynamic valuation

First, we assess the effect of valuating the non-hedgeable risk in each step of our dynamic valuation. To do so, we compare two situations:

- Situation 1: We determine the optimal hedger in each step by quadratic hedging without adding an actuarial valuation for the remaining risk. In this case, the dynamic valuation is market-consistent and time-consistent but not actuarial in the sense that there is no risk margin for the mortality risk. Indeed, under this approach, one can prove that

$$\rho_t[N(T)] = \mathbb{E}_t^{\mathbb{P}}[N(T)] \cdot e^{-r(T-t)}.$$

- Situation 2: We determine the optimal hedger in each step as explained above by valuating the remaining risk through a dynamic standard deviation principle

$$\pi_t[S] = e^{-r}\left[\mathbb{E}_t^{\mathbb{P}}[S] + \alpha\sigma_t^{\mathbb{P}}[S]\right],$$

with $\alpha = 0.15$. In that case, the dynamic valuation is market-consistent, time-consistent and actuarial as well.

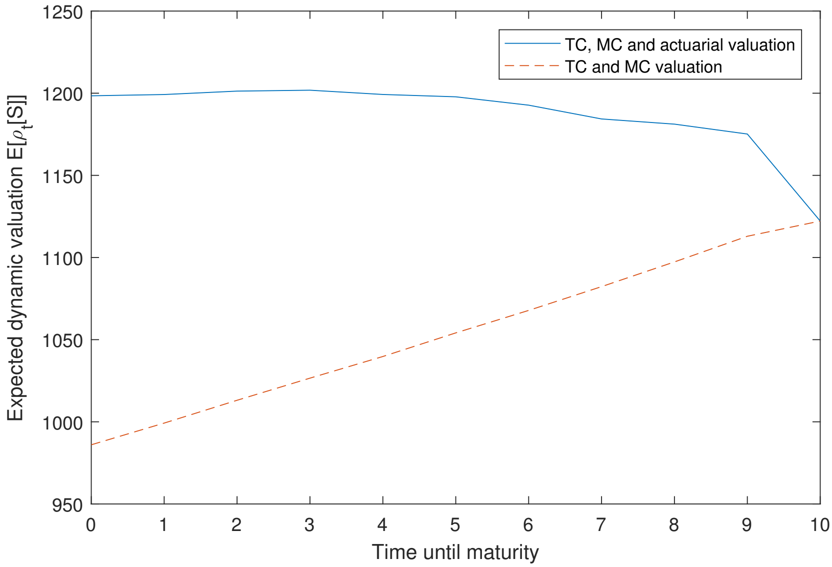

Figure 4.1: Expected dynamic valuation for the life-insurance portfolio with and without actuarial valuation for the non-hedgeable risk.

Figure 4.1 compares the dynamic valuations in situations 1 and 2 through time. Since $\rho_t[S]$ is random from the view point of time 0, we consider the evolution of the *expected* dynamic valuation $\mathbb{E}^{\mathbb{P}}[\rho_t[S]]$.

In situation 1, we observe that the dynamic valuation is steadily increasing over time to reach the expected payoff at maturity. This was expected since it is market-consistent, the dynamic valuation follows the trend of the risky asset. We remark that given there is no risk margin for the non-hedgeable risk (in particular the mortality risk), the insurer will suffer losses in case policyholders live longer than expected.

On the other hand, in situation 2, we observe a slightly decreasing trend of the dynamic valuation. This can be explained by two adverse effects: while the upward trend of the stock increases the dynamic valuation through time, the value of the non-hedgeable risk decreases over time (a shorter time horizon reduces the uncertainty). From Figure 4.1, we observe that this latter effect decreases at a higher rate than the increase of the former effect.

4.5.3.2 The effect of a static versus dynamic actuarial valuation for different maturities

Now, we take another perspective: instead of considering the evolution of the fair valuation until a fixed maturity, we consider the fair valuation at time 0 for different maturities. Moreover, compared to the previous case, we add an intermediate situation in which the non-hedgeable risk until maturity is valuated via a static actuarial valuation. The three situations can be summarized as follows:

- Situation 1: We follow the situation 1 above, i.e. the optimal hedger in each step is determined by quadratic hedging without adding an actuarial valuation for the remaining risk. Hence, there is no risk margin for the non-hedgeable risk.

- Situation 2: We introduce an intermediate situation in which we follow the situation 1 but add a *static* risk margin at time 0 for the non-hedgeable risk

$$RM\left[S\right] = \sum_{t=0}^{T-1} \pi\left[\rho_{t+1}\left[S\right] - \boldsymbol{\theta}_{t,S}(t+1)\cdot \boldsymbol{Y}\left(t+1\right)\right] \quad (4.29)$$

  with π is a static standard deviation principle.

- Situation 3: We consider the fair (actuarial, MC and TC) valuation in which the non-hedgeable risk is valuated via a *dynamic* standard deviation principle. This corresponds to the situation 2 in Section 4.5.3.1.

Figure 4.2 compares the three situations for different maturities $T = 1, ..., 15$ years. In situation 1, we observe that the valuation decreases with the maturity increase. This follows from two adverse effects: the longer the maturity, the fewer the number of survivals $N(T)$. But at the same time, the longer the maturity, the higher the financial guarantee $\max\left(Y^{(1)}(T), K\right)$. Figure 4.2

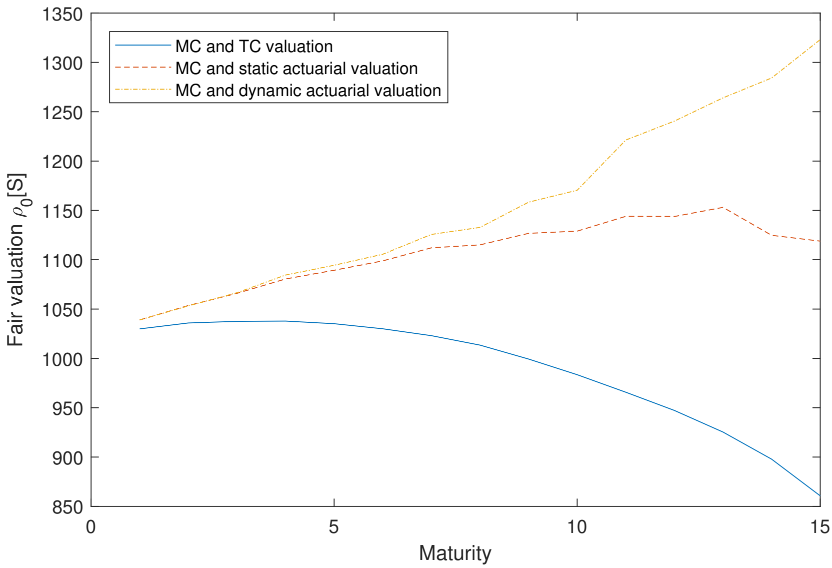

Figure 4.2: Fair valuation at time 0 with a static versus a dynamic actuarial valuation for the non-hedgeable risk.

shows that the mortality effect is stronger that the effect of the financial guarantee.

Not surprisingly, the fair valuation in situations 2 and 3 are higher than the pure market-consistent valuation because of the inclusion of a risk margin for the non-hedgeable risk. Moreover, the fair valuation with dynamic actuarial valuation dominates the one with static actuarial valuation. This difference is due to the iterating effect of the time-consistent valuation. While in situation 2, the one-year remaining risks are added up (see the relation (4.29)), the time-consistent valuation has a multiplicative effect since $\rho_t\left[S\right]$ contains all non-hedgeable risks from time t until maturity T.

Remark 2 Ghalehjooghi and Pelsser (2017) considered the implementation of time-consistent valuations via the two-step valuation of Pelsser and Stadje (2014). They compared the time-consistent valuations with the standard best-estimate and EIOPA valuations. The authors found, as in Figure 4.2, that the time-consistent valuations lead to higher prices due to the iterative effect of time-consistency. We also refer to Pelsser and Ghalehjooghi (2016) and Ghalehjooghi et al. (2016) for the application of the two-step operators to pension liabilities.

4.5.3.3 The effect of dependence between financial and actuarial risks

Finally, we study the impact of a dependence structure between mortality and equity risks on the fair dynamic valuation of the insurance liability S. We assume that under $\mathbb{P}$ the dynamics of the stock process and the population force of mortality are given by

$$dY^{(1)}(t) = Y^{(1)}(t)\left(\mu dt + \sigma dW_1(t)\right) \tag{4.30}$$

$$d\lambda_x(t) = c\lambda_x(t)dt + \xi dW_2(t), \tag{4.31}$$

with c, ξ, μ and σ are positive constants, and $W_1(t) = \rho W_2(t) + \sqrt{1-\rho^2}Z(t)$. Here, $W_2(t)$ and $Z(t)$ are independent standard Brownian motions.

We consider three levels of correlation, namely $\rho = \{-1, 0, 1\}$. The case $\rho = 0$ corresponds to the independence case of our previous analysis while the extreme cases $\rho = 1$ and $\rho = -1$ represents the comonotonic (respectively countermonotonic) situation in which stock and force of mortality are driven by the same random source in the same direction (respectively in the opposite direction). Intuitively, given the payoff

$$S = N(T) \times \max\left(Y^{(1)}(T), K\right),$$

we could expect that if $N(T)$ and $Y^{(1)}(T)$ move in the same direction, this is synonymous with a better hedging and hence a reduction of the non-hedgeable risk.

Figure 4.3 represents the expected value for the non-hedgeable risk until maturity, computed as the difference between the time-consistent valuation with and without inclusion of an actuarial valuation for the non-hedgeable risk. The figure confirms our intuition: if the number of survivals and the stock are moving in the same direction (i.e. force of mortality and stock are moving in the *opposite* direction), the non-hedgeable risk is reduced. Moreover, as expected, the non-hedgeable risk decreases when we

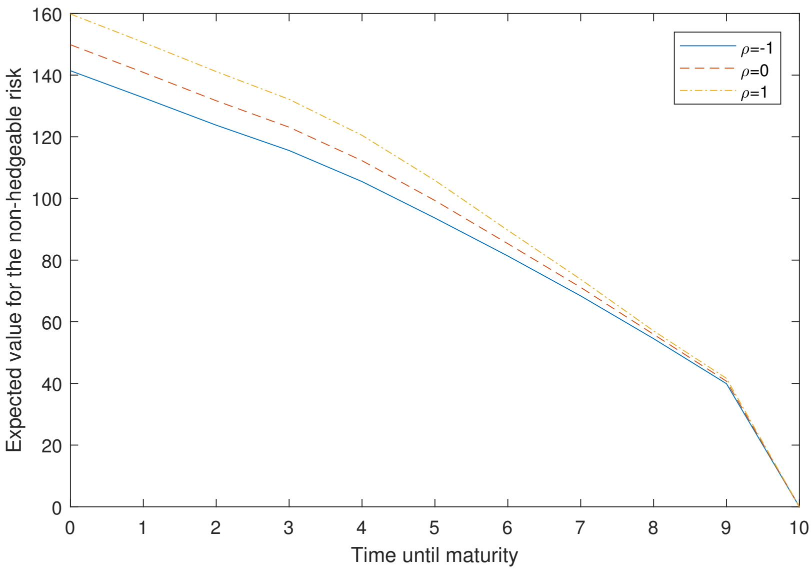

Figure 4.3: Expected value for the non-hedgeable risk under different dependence levels $\rho = \{-1, 0, 1\}$.

come closer to maturity. We remark that even in extreme cases, the non-hedgeable risk is not null given that the financial guarantee $\max\left(Y^{(1)}(T), K\right)$ and the number of survivals $N(T)$ are not completely hedgeable.

4.6 Concluding remarks

The determination of the fair valuation for insurance liabilities, which are often a combination of hedgeable and unhedgeable risks, has become a challenging task. Information about prices of traded assets provided by the financial market should be combined with information about mortality experience to provide a reliable market-consistent and actuarial valuation. Moreover, for the determination of future solvency capitals, the fair valuations have to be determined at future points in time in a consistent way, leading to time-consistent valuations.

In this chapter, we have studied the fair valuation of insurance liabilities in a dynamic discrete-time setting. We have proposed a new framework to merge actuarial, market-consistent and time-consistent considerations in a set of so-called *fair dynamic valuations*, extending the framework of Chapters 2 and 3. We have provided a complete hedging characterization in Theorem 13 and illustrated how these fair dynamic valuations can be implemented through a backward iterations scheme combining risk minimization techniques with standard actuarial principles.

Chapter 5

Two-step financial and actuarial valuations: Axiomatic characterization and applications

This chapter is based on
Barigou, K., Linders, D & Yang F. (2019). Two-step financial and actuarial valuations: Axiomatic characterization and applications. *Working paper.*

5.1 Introduction

Insurance liabilities depend in most cases on financial (e.g. interest rate risk, equity risk, etc) as well as non-financial risk (e.g. mortality risk). Pricing such payoffs involves combining standard actuarial valuation techniques performed under $\mathbb{P}$ for actuarial risks with risk-neutral valuation under $\mathbb{Q}$ for financial risks.

In an insurance context, in which risks are partially diversifiable and traded, building a valuation framework which combines actuarial and financial approaches is primordial. In the literature, different authors proposed such "hybrid" approaches. For instance, Pelsser and Stadje (2014) proposed a "two-step market valuation" which extends standard actuarial principles by conditioning on the financial information as we detailed in Chapter 2. Moreover, in the previous chapters, we proposed a new framework for the fair valuation of insurance liabilities in discrete time: we introduced the notion of a "fair valuation", which we defined as a valuation which is both market-consistent (mark-to-market for any hedgeable part of a claim) and actuarial (mark-to-model for any claim that is independent of financial market evolutions) and studied their properties. As it was pointed in Albrecher et al. (2018), these valuation frameworks end up with a two-step approach, where an actuarial valuation is applied after conditioning on the financial component.

In this chapter, we introduce two broad classes of valuations: two-step financial valuations that are market-consistent and two-step actuarial valuations that are actuarial-consistent. We provide a complete axiomatic characterization for the two-step valuations based on coherent valuations. The two-step valuations are general in the sense that they do not impose linearity constraints on the actuarial and financial valuations. Therefore, they allow to account for the diversification of actuarial risks and/or the incompleteness of the financial market (e.g. non-linear pricing with bid-ask prices).

While the two-step financial valuation is an extension of the two-step market valuation of Pelsser and Stadje (2014), the novelty of this paper is to study a two-step actuarial valuation which consists of reversing the valuation order: applying a financial valuation after conditioning on the actuarial component. We show that the two-step actuarial valuation can be decomposed into a best estimate (expected value) plus a risk margin to cover the uncertainty in the actuarial risks. The procedure will be illustrated on a portfolio of life insurance contracts with dependent financial and actuarial risks.

The rest of the chapter is structured as follows. In Section 5.2, we define the notions of two-step financial and actuarial valuations. In particular, we provide an axiomatic characterization of two-step valuations. Moreover, we investigate the notions of market- and actuarial-consistency and discuss if it is always possible to combine both notions. Section 5.3 presents a detailed numerical application of the two-step actuarial valuation on a portfolio of equity-linked contracts. Section 5.4 concludes the chapter.

5.2 Two-step valuations

All random variables introduced hereafter are defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Equalities and inequalities between r.v.'s have to be understood in the $\mathbb{P}$-almost sure sense. The space of square integrable random variables is denoted by $L^2(\mathbb{P}, \mathbb{F})$ and all risks (often called claims) are represented by random variables $S \in L^2(\mathbb{P}, \mathbb{F})$. The flow of information provided by the financial risks (stock prices, bond prices, etc.) is represented by the filtration $\mathbb{F}^{(1)} = \left\{\mathcal{F}_t^{(1)}\right\}_{t\in[0,T]}$ with $\mathcal{F}_T^{(1)} = \mathcal{F}^{(1)}$ and the flow of information provided by the actuarial risks (mortality intensity, death and survival of policyholders, etc.) by $\mathbb{F}^{(2)} = \left\{\mathcal{F}_t^{(2)}\right\}_{t\in[0,T]}$ with $\mathcal{F}_T^{(2)} = \mathcal{F}^{(2)}$. The general filtration $\mathbb{F}$ is then defined as the minimal σ-algebra containing all events of $\mathbb{F}^{(1)}$ and $\mathbb{F}^{(2)}$, i.e. $\mathbb{F} = \sigma\left(\mathbb{F}^{(1)} \cup \mathbb{F}^{(2)}\right)$. An overview of the different filtrations is depicted in Figure 5.1.

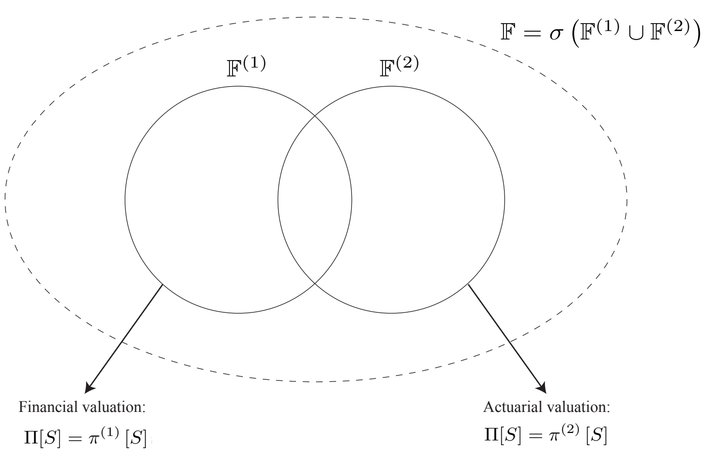

Figure 5.1: Overview of the different filtrations and valuations

We remark that the two filtrations are not disjoint for two main reasons:

- We allow for a general model with dependent financial and actuarial risks (e.g. a stock price and a force of mortality which can possibly be dependent). Then, by construction, both financial and actuarial filtrations are not independent.
- The payoffs of some traded assets can be mortality-linked (e.g. a longevity bond related to the longevity development of a particular population).

As far as the valuation is concerned, we assume there is no ambiguity on the valuation to use for pure financial or actuarial risks. For financial risks, we apply a financial valuation $\pi^{(1)}$ based on market prices while for actuarial risks, the valuation is determined based on historical information with an actuarial valuation $\pi^{(2)}$. The valuation issue comes from the observation that most insurance liabilities are a combination of financial and actuarial risks (i.e. elements in the ellipse in Figure 5.1). Therefore, different valuation frameworks can be considered depending on how financial and actuarial valuations are merged together.

In this section, we start by defining the family of linear and coherent valuations and the class of financial and actuarial valuations. In Section 5.2.2, we define a broad class of market-consistent valuations and two-step financial valuations inspired from Pelsser and Stadje (2014). In Section 5.2.3, we introduce a class of actuarial-consistent valuations and propose a new valuation framework, called the *two-step actuarial valuation.* We end this section by investigating whether it is always feasible to define a valuation that is market- and actuarial-consistent.

5.2.1 Valuations

5.2.1.1 Linear and coherent valuations

A contingent claim is a random liability that has to be paid at the future time T. Formally, a contingent claim is modeled by the random variable S which is $\mathcal{F}_T$-measurable and which is defined on the probability space $(\Omega, \mathbb{F}, \mathbb{P})$. In what follows we are interested in the *valuation* of contingent claims.

Definition 44 (Valuation) A valuation is a mapping $\Pi : L^2(\mathbb{P}, \mathbb{F}) \to \mathbb{R}$ satisfying the following properties:

- Normalization: $\Pi[0] = 0$.
- Translation-invariance: For any $S \in L^2(\mathbb{P}, \mathbb{F})$ and $a \in \mathbb{R}$, $\Pi[S + a] = \Pi[S] + a$.

We start by proceeding as in Buhlmann et al. (1992), Bühlmann (2000) and Wüthrich (2016) via the use of a linear, positive and continuous valuation (functional) on the set of contingent claims $S \in L^2(\mathbb{P}, \mathbb{F})$.

Definition 45 (Linear valuation) A mapping $\Pi : L^2(\mathbb{P}, \mathbb{F}) \to \mathbb{R}$ is a linear valuation if the following axioms hold:

- Linearity: For all $S_1, S_2 \in L^2(\mathbb{P}, \mathbb{F})$ and $a, b \in \mathbb{R}$ we have
$$\Pi[aS_1 + bS_2] = a\Pi[S_1] + b\Pi[S_2].$$
- Positivity: For any $S \in L^2(\mathbb{P}, \mathbb{F})$ with $S \geq 0$ we have $\Pi[S] > 0$.
- Continuity: For any sequence $(S_k)_k \subset L^2(\mathbb{P}, \mathbb{F})$ with $(S_k)_k \to S$ in L^2, we have $\Pi[S_k] \to \Pi[S]$ in $\mathbb{R}$ as k $\to \infty$.
- Translation-invariance: For any $S \in L^2(\mathbb{P}, \mathbb{F})$ and $a \in \mathbb{R}$, $\Pi[S + a] = \Pi[S] + a$.

We remark that in the positivity axiom, $S \geq 0$ means that $S \geq 0$ *a.s.* and $S > 0$ with positive probability. The valuation Π is a mapping which assigns to any claim $S \in L^2(\mathbb{P}, \mathbb{F})$ a monetary value $\Pi[S]$ which can be interpreted as the value of the claim at time 0.

Based on the Riesz' representation theorem[1], we have the following result.

Theorem 14 For any linear valuation Π, there exists a $\varphi \in L^2(\mathbb{P}, \mathbb{F})$ such that
$$\Pi[S] = \mathbb{E}\left[\varphi S\right]. \tag{5.1}$$
Moreover, φ is unique and $\varphi > 0$.

[1] Let Π be a linear continuous functional on a Hilbert space $\mathcal{H}$. Then there exists a unique $y \in \mathcal{H}$ such that $\Pi[S] = \langle y, S\rangle \; \forall S \in \mathcal{H}$ (see e.g. Rudin (1987)).

Proof: From the classical Riesz' representation theorem, for every linear and continuous functional, there exists a unique $\varphi \in L^2(\mathbb{P}, \mathbb{F})$ such that the relation (5.1) holds. The property $\varphi > 0$ *a.s.* follows from the positivity of Π. ∎

Since the linear case is too restrictive in an actuarial context, we also introduce the class of coherent valuations.

Definition 46 (Coherent valuation) A mapping $\Pi : L^2(\mathbb{P}, \mathbb{F}) \to \mathbb{R}$ is a coherent valuation if the following axioms hold:

- Monotonicity: For all $S_1, S_2 \in L^2(\mathbb{P}, \mathbb{F})$ with $S_1 \leq S_2$, $\Pi[S_1] \leq \Pi[S_2]$.
- Positive homogeneity: For any $S \in L^2(\mathbb{P}, \mathbb{F})$ and $a \in \mathbb{R}^+$, $\Pi[aS] = a\Pi[S]$.
- Subadditivity: For all $S_1, S_2 \in L^2(\mathbb{P}, \mathbb{F})$, $\Pi[S_1+S_2] \leq \Pi[S_1]+\Pi[S_2]$.
- Translation-invariance: For any $S \in L^2(\mathbb{P}, \mathbb{F})$ and $a \in \mathbb{R}$, $\Pi[S + a] = \Pi[S] + a$.

Coherent valuations can always be represented as upper expectations over a set of other probability measures $\widetilde{\mathbb{P}}$ than the real-world measure $\mathbb{P}$ such that the density function $\varphi = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}$ is well-defined. Such density functions form the set

$$\mathcal{P} = \{\varphi \in L^2(\mathbb{P}, \mathbb{F}) \mid \varphi \geq 0, \mathbb{E}^{\mathbb{P}}[\varphi] = 1\}.$$

Theorem 15 (Dual representation of coherent valuations) Π is a coherent valuation if and only if

$$\Pi[S] = \sup_{\varphi \in \mathcal{Q}} \mathbb{E}[\varphi S],$$

where $\mathcal{Q}$ is a unique, non-empty, closed convex subset of $\mathcal{P}$.

A coherent valuation can then be understood as a worst-case expectation with respect to some class of probability measures. This can be motivated by the desire for robustness: the valuator does not only want to rely on a single measure $\mathbb{P}$ for the occurrence of future events but prefers to test a set of plausible measures and value with the worst-case scenario. We also note that the set of linear valuations is a subset of the set of coherent valuations. In particular, Theorem 14 can be seen as a particular case of Theorem 15. The proof of Theorem 15 can be found in Artzner et al. (1999) for Ω finite and in Delbaen (2002) for general probability spaces. The special case of L^2 is proven in Rockafellar et al. (2006).

In the following subsections, we will consider two important types of claims, namely financial and actuarial claims, and their respective valuations.

5.2.1.2 Financial Valuation

The financial probability space is denoted by $\left(\Omega, \mathcal{F}^{(1)}, \mathbb{F}^{(1)}, \mathbb{P}\right)$. We assume there are n financial assets $Y = (Y_1, \ldots, Y_n)$ in the financial market generating the financial filtration $\mathbb{F}^{(1)} = \left\{\mathcal{F}_t^{(1)}\right\}_{t\in[0,T]}$ with $\mathcal{F}_T^{(1)} = \mathcal{F}^{(1)}$. The price of asset i at time t is denoted by $Y_i(t)$. Note that we do not assume that all assets can be bought and sold at a unique price $Y_i(t)$ at time t. There is a risk-free bank account and we assume the risk-free rate r to be deterministic and constant.[2] Examples of financial assets are: stocks, options, futures, government and mortality bonds.

A *financial claim* is a $\mathcal{F}_T^{(1)}$-measurable random variable defined on the financial probability space. Otherwise stated, a financial claim only depends on Y and its realization is completely known given the realization of the financial assets.

We assume that a valuation principle $\pi^{(1)}$ is available to price financial claims. The choice of the financial valuation principle $\pi^{(1)}$ depends on the additional assumption we make about the financial market. Assuming that markets are complete corresponds with assuming that any financial claim can be replicated. The law of one price implies that market participants can buy and sell an asset for the same price. Below, we consider how to determine the financial valuation for different market situations.

1. Completeness and the law of one price. Assume that all financial assets are traded in the market and, moreover, that the payoff of any financial claim S can be replicated. One can prove that completeness of the market and the law of

[2] For simplicity of presentation but our main results can be easily extended to stochastic interest rates.

one price is equivalent with the existence of an equivalent martingale measure (EMM) $\mathbb{Q}$ satisfying:

$$Y_i(s) = \mathrm{e}^{-r(t-s)}\mathbb{E}^{\mathbb{Q}}\left[Y_i(t)|\ \mathcal{F}_s^{(1)}\right], \text{ for } i = 1, 2, \ldots, n,$$

where $t > s$. In this complete financial market where one can buy and sell any asset at a unique price, the financial valuation principle is given by:

$$\pi^{(1)}[S] = \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}[S].$$

Any other price would lead to an arbitrage. The financial valuation is in this situation a linear valuation principle.

2. <u>Incompleteness and the law of one price.</u> Assume that not all assets in the financial market are traded, making the financial market incomplete. In terms of EMMs, this incompleteness implies that there is an infinite number of choices for the equivalent martingale measure $\mathbb{Q}$. Several techniques were proposed in the literature to "complete" the financial market by choosing the EMM which is the most "appropriate" one. Popular approaches include the minimal entropy martingale measure (Frittelli (2000) and Dhaene et al. (2015)), the minimal martingale measure (Föllmer and Schweizer (1991)) or the risk-neutral Esscher measure (Gerber and Shiu (1994)).

   Another approach is via the theory of utility indifference pricing. Under this approach, the value of the claim is set equal to the amount which makes the agent indifferent, in terms of expected utility, between holding the claim or not. In Musiela and Zariphopoulou (2004), the authors considered the indifference price under an exponential utility function with one traded and one non-traded asset. In this incomplete financial market, a financial claim consists of a hedgeable and an unhedgeable part. The filtration $\mathcal{F}^{(11)}$ captures the hedgeable information whereas $\mathcal{F}^{(12)}$ captures the unhedgeable information. The financial filtration $\mathcal{F}^{(1)}$ is then given by $\sigma\left(\mathbb{F}^{(11)} \cup \mathbb{F}^{(12)}\right)$. Musiela and Zariphopoulou (2004) showed that the indifference price is given by a two-step approach:

$$\pi^{(1)}[S] = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\gamma}\log\mathbb{E}^{\mathbb{P}}\left[e^{\gamma S}|\ \mathbb{F}^{(11)}\right]\right].$$

We note that the indifference price is not coherent because of its lack of positive homogeneity.

3. Incompleteness and bid-ask prices. In classical finance, markets are usually modelled as a counterparty for market participants. It is assumed that markets can accept any amount and direction of the trade (buy or sell) at the going market price. However, due to market imperfection, there is in practice a difference between the price the market is willing to buy (bid price) and the price the market is willing to sell (ask price). This difference, called the bid-ask spread, creates a two-price economy. In particular, the value $\pi^{(1)}[S]$ which corresponds with the price the market is willing to pay to take over the financial claim S will typically be higher than the risk-neutral price. Indeed, the asymmetry in the market allows that market to take a more prudent approach when determining the price $\pi^{(1)}[S]$. Instead of using a single risk-neutral probability measure, a set of "stress-test measures" is selected from the set of martingale measures and the price is determined as the supremum of the expectations w.r.t. the stress-test measures:

$$\pi^{(1)}[S] = \mathrm{e}^{-rT} \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}\left[S\right].$$

Due to the duality theorem (Theorem 15), this price can be expressed as follows:

$$\pi^{(1)}[S] = \rho_g\left[S\right],$$

where ρ_g is a distortion risk measure and g is an appropriately chosen concave distortion function. For more details on conic finance, we refer to Madan and Cherny (2010) and Madan and Schoutens (2016). We remark that in this case, the valuation of financial claims is *non-linear*.

In the remainder of the chapter, we consider coherent financial valuations to account for bid-ask spread and market incompleteness. Hereafter, we provide a simple example of the valuation of a call option with the law of one-price. This example will be used later in Example 17 to illustrate our two-step valuations.

Example 13 (a) Consider a financial market with a traded stock with spot price $Y(0) = 100$ and risk-free rate $r = 0$. The random price of the stock at time $T = 1$ is denoted by Y. The random variable Y is defined on the financial probability space $\left(\Omega, \mathcal{F}^{(1)}, \mathbb{P}\right)$. The set Ω of possible scenarios is given by

$$\Omega = \{\omega_1, \omega_2\}.$$

We assume that the future stock price can go up or down:

$$Y[\omega_1] = 50 \quad \text{and} \quad Y[\omega_2] = 200.$$

We consider a call option with strike 100 and maturity $T = 1$. The financial claim $S^{(1)}$ is the payoff of this call option. Then $S^{(1)}$ is $\mathcal{F}^{(1)}$-measurable and given by

$$S^{(1)}[\omega] = (Y[\omega] - K)_+ = \begin{cases} 0, & \text{if } \omega = \omega_1, \\ 100, & \text{if } \omega = \omega_2. \end{cases}$$

The financial market in this example is arbitrage-free and complete. Moreover, we assume that the stock can be bought and sold at the spot price (hence, the law of one price applies here). As the no-arbitrage condition is satisfied, the financial valuation principle $\pi^{(1)}$ should be given by the discounted risk-neutral expectation:

$$\pi^{(1)}\left[S^{(1)}\right] = \mathrm{e}^{-r}\mathbb{E}^{\mathbb{Q}}\left[S^{(1)}\right].$$

It is straightforward to prove that the risk-neutral probability measure $\mathbb{Q}$ is given by

$$\mathbb{Q}[\omega] = \begin{cases} 1-q, & \text{if } \omega = \omega_1, \\ q, & \text{if } \omega = \omega_2, \end{cases}$$

where $q = 1/3$. Therefore, the value of the call option is

$$\pi^{(1)}\left[S^{(1)}\right] = \frac{2}{3} \times 0 + \frac{1}{3} \times 100 = \frac{100}{3}. \tag{5.2}$$

This is the unique no-arbitrage price for $S^{(1)}$. Any different price would lead to an arbitrage, since one can buy and sell any quantity at the spot price.

(b) Let us now assume the future stock price can go up, down or stay at the same price:

$$Y[\omega_1] = 50, \quad Y[\omega_2] = 100 \quad \text{and} \quad Y[\omega_3] = 200.$$

In this case, the market becomes incomplete and there is a range of risk-neutral prices which is given by

$$\mathbb{Q}[\omega] = \begin{cases} \frac{2}{3}(1-q), & \text{if } \omega = \omega_1, \\ q, & \text{if } \omega = \omega_2, \\ \frac{1}{3}(1-q), & \text{if } \omega = \omega_3, \end{cases}$$

where $0 < q < 1$. Therefore, the call price is given by the interval:

$$0 < \pi^{(1)}\left[S^{(1)}\right] = \frac{100}{3}(1-q) < \frac{100}{3}.$$

The valuator should then make a subjective choice on the $\mathbb{Q}$-probability to be chosen.

5.2.1.3 Actuarial valuation

The actuarial probability space is denoted by $\left(\Omega, \mathcal{F}^{(2)}, \mathbb{F}^{(2)}, \mathbb{P}\right)$. We assume there are n actuarial risks $X = (X_1, \ldots, X_n)$ generating the actuarial filtration $\mathbb{F}^{(2)} = \left\{\mathcal{F}_t^{(2)}\right\}_{t\in[0,T]}$ with $\mathcal{F}_T^{(2)} = \mathcal{F}^{(2)}$. Examples of actuarial risks are: death, survival, lapse, medical expenses.

An *actuarial claim* is a $\mathcal{F}_T^{(2)}$-measurable random variable defined on the actuarial probability space. Equivalently stated, an actuarial claim only depends on X and its realization is completely known given the realization of the actuarial risks X.

We assume that a valuation principle $\pi^{(2)}$ is chosen to price actuarial claims. The actuarial valuation principle $\pi^{(2)}$ is based on the idea of pooling and diversification. The value of a completely diversifiable portfolio has to correspond with its expectation under the physical measure $\mathbb{P}$. However, there is always an amount of residual actuarial risk present because one can never fully diversify away all the risk. Moreover, there are also systematic actuarial risks (e.g. longevity risk) which cannot be diversified away. Below, we briefly discuss the most important actuarial valuation principles.

1. <u>Linear valuation:</u>
$$\pi^{(2)}[S] = \mathbb{E}_{\tilde{\mathbb{P}}}[S].$$
The risk margin is modelled by an appropriate change of measure from $\mathbb{P}$ to $\tilde{\mathbb{P}}$. In terms of life tables, the change

of measure can be interpreted as a switch from the second order life table (best-estimate survival or death probabilities) to a first order life table (survival or death probabilities that are chosen with a safety margin). For more details, see for instance Wüthrich (2016) and Laurent et al. (2016).

2. Standard deviation principle:

$$\pi^{(2)}[S] = \mathbb{E}^{\mathbb{P}}[S] + \beta\sqrt{\mathrm{Var}^{\mathbb{P}}[S]}, \tag{5.3}$$

with $\beta \geq 0$.
In this case, the loading equals β times the standard deviation. It is well-known that $\beta > 0$ is required in order to avoid getting ruin with probability 1 (see e.g. Kaas et al. (2008)).

3. Coherent valuation:

$$\pi^{(2)}[S] = \rho[S],$$

where ρ is a coherent valuation. We remark that the coherent valuation can also be expressed as

$$\pi^{(2)}[S] = \sup_{\mathbb{P}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}}[S].$$

Therefore, model risk can be taken into account by considering a family of different distributions and the actuarial claim is valuated with the most conservative one.

Hereafter, we provide an example of the valuation of a pure endowment liability by a standard deviation principle; example that will be used in Example 17 to illustrate the two-step valuations.

Example 14 (Standard deviation principle) Consider a pure endowment contract for a life (x), which pays 100 in case the policy holder is alive at time $T = 1$ and 0 otherwise. The actuarial risk of this contract is the survival of the policyholder in the interval $[0, 1]$. To model this risk, we first define the actuarial probability space $\left(\Omega, \mathcal{F}^{(2)}, \mathbb{P}\right)$. We assume that

$$\Omega = \{\omega_1, \omega_2\},$$

and

$$\mathbb{P}[\omega] = \begin{cases} 1-p, & \text{if } \omega = \omega_1, \\ p, & \text{if } \omega = \omega_2, \end{cases}$$

with $p = 0.95$. The random variable I is defined on the actuarial probability space:

$$I[\omega_1] = 0 \quad \text{and} \quad I[\omega_2] = 1.$$

The random variable I indicates if the policy holder survives ($I = 1$) or dies ($I = 0$) in the time interval $[0, 1]$. The actuarial contingent claim $S^{(2)}$ corresponds with the payout of the pure endowment, hence:

$$S^{(2)} = 100 \times I.$$

If $\pi^{(2)}$ is chosen to be the standard deviation principle (5.3), we find that

$$\begin{aligned} \pi^{(2)}\left[S^{(2)}\right] &= 100 \times \left(p + \beta\sqrt{p(1-p)}\right) \\ &= 95 + 0.218\beta. \end{aligned}$$

5.2.2 Market-consistency and two-step financial valuations

A hybrid claim is a contingent claim that is neither completely actuarial nor completely financial. Instead, a hybrid claim S is a $\mathcal{F}_T$-measurable random variable which is partly actuarial and financial. We assume that a financial valuation principle $\pi^{(1)}$ and an actuarial valuation principle $\pi^{(2)}$ are given. We search for valuations Π that combine the financial valuation $\pi^{(1)}$ and the actuarial valuation $\pi^{(2)}$ and study their properties. A first important class of valuations are the market-consistent valuations.

Definition 47 (Strong market-consistency) A valuation Π is called strong market-consistent (strong MCV) if for any financial claim $S^{(1)}$ the following holds:

$$\Pi\left[S + S^{(1)}\right] = \Pi[S] + \pi^{(1)}\left[S^{(1)}\right]. \tag{5.4}$$

In the literature, market-consistency is usually defined via a condition identical or similar to the condition (5.4) (see e.g. Pelsser and Stadje (2014) and the previous chapters). However, strong market-consistency implies linearity of the financial valuation principle. Indeed, we have that if Π is strong market-consistent, then for the financial claims X and Y, the following holds:

$$\pi^{(1)}\left[X+Y\right]=\pi^{(1)}\left[X\right]+\pi^{(1)}\left[Y\right]. \tag{5.5}$$

Combining (5.4) and (5.5) shows that strong market-consistency of the valuation operator Π restricts the financial valuation principle to linear valuations. Therefore, strong MCV is too restrictive when the law of one price does not prevail.

Definition 48 (Weak market-consistency) A valuation Π is called weak market-consistent (weak MCV) if for any financial claim $S^{(1)}$ the following holds:

$$\Pi\left[S^{(1)}\right]=\pi^{(1)}\left[S^{(1)}\right]. \tag{5.6}$$

Weak market-consistency only postulates that for all financial claims, a financial valuation is applied. This weaker notion of market-consistency does not impose linearity of the financial valuation and allows for two-price economy and market incompleteness. We remark that Assa and Gospodinov (2018) also investigated these two types of market-consistency, that they called market-consistency of type I and type II.

It is straightforward to show that strong MCV implies weak MCV. Following discussions from Pelsser and Stadje (2014), the following lemma proves that in case the financial valuation principle is linear, weak MCV does imply strong MCV.

Lemma 5 Consider a coherent valuation principle Π with financial valuation principle $\pi^{(1)}$ which is linear in the sense that

$$\pi^{(1)}\left[S^{(1)}\right]=\mathbb{E}\left[\varphi^{(1)}S^{(1)}\right],$$

for all financial claims $S^{(1)}$, with a positive $\mathcal{F}^{(1)}$-measurable density $\varphi^{(1)}$.
Then, the following statements are equivalent:

1. Π is strong MCV.

2. Π is weak MCV.

Proof: The proof of (1) → (2) is straightforward. In order to prove (2) → (1), we remark that since Π is coherent, we can write:

$$\Pi[S] = \sup_{\varphi\in\mathcal{Q}} \mathbb{E}\left[\varphi S\right]. \tag{5.7}$$

Weak market-consistency and linearity of the financial valuation principle $\pi^{(1)}$ implies that for all financial claims $S^{(1)}$, we have that

$$\begin{aligned}
\pi^{(1)}\left[S^{(1)}\right] &= \mathbb{E}\left[\varphi^{(1)}S^{(1)}\right] && (5.8)\\
&= \sup_{\varphi\in\mathcal{Q}} \mathbb{E}\left[\varphi S^{(1)}\right]\\
&= \sup_{\varphi\in\mathcal{Q}} \mathbb{E}\left[\mathbb{E}\left[\varphi S^{(1)}\middle|\ \mathcal{F}^{(1)}\right]\right]\\
&= \sup_{\varphi\in\mathcal{Q}} \mathbb{E}\left[S^{(1)}\mathbb{E}\left[\varphi\middle|\ \mathcal{F}^{(1)}\right]\right] && (5.9)
\end{aligned}$$

where we used that $S^{(1)}$ is $\mathcal{F}^{(1)}$-measurable. Since (5.8)=(5.9), it is sufficient to consider $\varphi \in \mathcal{Q} : \mathbb{E}\left[\varphi|\mathcal{F}^{(1)}\right] = \varphi^{(1)}$. Because $\varphi^{(1)}$ is positive, we can then write $\varphi = \varphi^{(1)}Z$ with $Z \in \mathcal{Q}_{(1)} = \{\varphi \in L^2(\mathbb{P},\mathbb{F}) \mid \mathbb{E}\left[Z|\ \mathcal{F}^{(1)}\right] = 1\}$. Thus, we have that:

$$\begin{aligned}
\Pi\left[S+S^{(1)}\right] &= \sup_{Z\in\mathcal{Q}_{(1)}} \mathbb{E}\left[\varphi^{(1)}Z\left(S+S^{(1)}\right)\right]\\
&= \sup_{Z\in\mathcal{Q}_{(1)}} \left\{\mathbb{E}\left[\varphi^{(1)}ZS\right] + \mathbb{E}\left[\varphi^{(1)}ZS^{(1)}\right]\right\}\\
&= \sup_{Z\in\mathcal{Q}_{(1)}} \left\{\mathbb{E}\left[\varphi^{(1)}ZS\right] + \mathbb{E}\left[\mathbb{E}\left[\varphi^{(1)}ZS^{(1)}\middle|\ \mathcal{F}^{(1)}\right]\right]\right\}\\
&= \sup_{Z\in\mathcal{Q}_{(1)}} \left\{\mathbb{E}\left[\varphi^{(1)}ZS\right] + \mathbb{E}\left[\varphi^{(1)}S^{(1)}\mathbb{E}\left[Z\middle|\ \mathcal{F}^{(1)}\right]\right]\right\}\\
&= \sup_{Z\in\mathcal{Q}_{(1)}} \left\{\mathbb{E}\left[\varphi^{(1)}ZS\right] + \mathbb{E}\left[\varphi^{(1)}S^{(1)}\right]\right\}\\
&= \sup_{Z\in\mathcal{Q}_{(1)}} \left\{\mathbb{E}\left[\varphi^{(1)}ZS\right] + \pi^{(1)}\left[S^{(1)}\right]\right\}\\
&= \sup_{Z\in\mathcal{Q}_{(1)}} \left\{\mathbb{E}\left[\varphi^{(1)}ZS\right]\right\} + \pi^{(1)}\left[S^{(1)}\right]\\
&= \Pi[S] + \pi^{(1)}\left[S^{(1)}\right],
\end{aligned}$$

which proves that Π is also strong MCV. ■

Now, we introduce a class of market-consistent valuations which we call *two-step financial valuations*, extending the two-step market valuation of Pelsser and Stadje (2014). First, we define the notion of $\mathcal{F}^{(i)}$-conditional valuation which maps any claim S into an $\mathcal{F}^{(i)}$-measurable r.v, for $i = 1, 2$. This operator allows us to transform a claim S into a financial or actuarial claim.

Definition 49 ($\mathcal{F}^{(i)}$-conditional valuation) A $\mathcal{F}^{(i)}$-conditional valuation is a mapping $\Pi : L^2(\mathbb{P}, \mathbb{F}) \to \Pi : L^2(\mathbb{P}, \mathbb{F}^{(i)})$ satisfying the following properties:

- Normalization: $\Pi[0|\mathcal{F}^{(i)}] = 0$.
- Translation-invariance: For any $S \in L^2(\mathbb{P}, \mathbb{F})$ and $S^{(i)} \in L^2(\mathbb{P}, \mathbb{F}^{(i)})$, we have

$$\Pi[S + S^{(i)}|\mathcal{F}^{(i)}] = \Pi[S] + S^{(i)}.$$

Remark 3 We remark that a $\mathcal{F}^{(1)}$-conditional valuation attaches the financial claim $\Pi[S|\mathcal{F}^{(1)}]$ to any claim S. Since $\mathbb{F}^{(1)}$ is the filtration generated by the financial assets $Y = (Y_1, \ldots, Y_n)$, conditioning on $\mathcal{F}_T^{(1)} = \mathcal{F}^{(1)}$ is equivalent to conditioning on the stochastic process $Y = (Y_1, \ldots, Y_n)$ from time 0 to time T. Similarly, a $\mathcal{F}^{(2)}$-conditional valuation $\Pi[S|\mathcal{F}^{(2)}]$ can be interpreted as conditioning on the actuarial risks $X = (X_1, \ldots, X_n)$ from time 0 to time T.

The two-step financial valuations do not arise from hedging but from operator splitting. Namely, in a first step we compute the actuarial value of S conditional on financial scenarios (the values of the financial assets Y), i.e. $\pi^{(2)}\left[S|\ \mathcal{F}^{(1)}\right]$. Then for every different future value of the financial assets we obtain a different actuarial value. However, since this payoff depends only on financial scenarios and is then $\mathcal{F}^{(1)}$-measurable, one could argue that the quantity $\pi^{(2)}\left[S|\ \mathcal{F}^{(1)}\right]$ should be valuated via a financial valuation $\pi^{(1)}$. This motivates the following definition.

Definition 50 (Two-step financial valuation) The valuation Π is called a two-step financial valuation principle if it can be expressed as follows:

$$\Pi\left[S\right] = \pi^{(1)}\left[\pi^{(2)}\left[S|\ \mathcal{F}^{(1)}\right]\right],$$

where $\pi^{(1)}$ is the financial valuation principle and $\pi^{(2)}$ is the $\mathcal{F}^{(1)}$-conditional actuarial valuation principle.

Definition 50 can be seen as a generalization of Pelsser and Stadje (2014) where the linear risk-neutral operator is replaced by a coherent valuation $\pi^{(1)}$. Pelsser and Stadje (2014) showed that, under appropriate assumptions, strong market-consistent valuation is equivalent to two-step financial valuation where $\pi^{(1)}$ is linear (see their Theorem 3.10). In the following theorem, this result is extended to the non-linear case: we show that coherent weak market-consistent valuation is equivalent to two-step financial coherent valuation.

Theorem 16 (Characterization of weak MCV) The following statements are equivalent:

1. Π is a coherent weak market-consistent valuation.
2. There exist an $\mathcal{F}^{(1)}$-conditional coherent valuation $\Pi^{(2)}$ and a coherent financial valuation $\Pi^{(1)}$ such that
$$\Pi[S] = \Pi^{(1)}\left[\Pi^{(2)}\left[S|\mathcal{F}^{(1)}\right]\right].$$

Proof: We start with (2) $\rightarrow$ (1). To prove that Π is weak market-consistent, it is sufficient to notice that

$$\begin{aligned}\Pi\left[S^{(1)}\right] &= \Pi^{(1)}\left[\Pi^{(2)}\left[S^{(1)}|\mathcal{F}^{(1)}\right]\right]\\ &= \Pi^{(1)}\left[S^{(1)}\right],\end{aligned}$$

where we have used that $S^{(1)}$ is $\mathcal{F}^{(1)}$-measurable. Moreover, Π is coherent since $\Pi^{(1)}$ and $\Pi^{(2)}$ are coherent.
Let us prove (1) $\rightarrow$ (2).[3] Because Π is coherent, we have that

$$\Pi[S] = \sup_{\varphi\in\mathcal{Q}} \mathbb{E}[\varphi S]$$

where $\mathcal{Q}$ is a unique, non-empty, closed convex subset of $\mathcal{P}$. By weak market-consistency, for any financial claim $S^{(1)}$, the following also holds:

$$\Pi\left[S^{(1)}\right] = \sup_{\varphi^{(1)}} \mathbb{E}[\varphi^{(1)}S^{(1)}],$$

[3] We acknowledge that some arguments of the proof are similar to Theorem 3.10 in Pelsser and Stadje (2014)

where the supremum is taken over a set of probability measures such that $\varphi^{(1)}$ is $\mathcal{F}^{(1)}$-measurable. Similar to the proof of Lemma 5, we can then write

$$\varphi = \varphi^{(1)} Z,$$

with $Z \in \mathcal{Q}_{(1)} := \{Z \in L^2(\mathbb{P}, \mathbb{F}) \mid \mathbb{E}^{\mathbb{P}}\left[Z|\mathcal{F}^{(1)}\right] = 1\}$. Therefore, we have that

$$\begin{aligned}
\Pi[S] &= \sup_{\varphi^{(1)}} \sup_{Z \in \mathcal{Q}_{(1)}} \mathbb{E}[\varphi^{(1)} Z S] \\
&= \sup_{\varphi^{(1)}} \sup_{Z \in \mathcal{Q}_{(1)}} \mathbb{E}\left[\varphi^{(1)} \mathbb{E}\left[ZS|\mathcal{F}^{(1)}\right]\right].
\end{aligned}$$

We remark that if we can prove that

$$\sup_{Z \in \mathcal{Q}_{(1)}} \mathbb{E}\left[\varphi^{(1)} \mathbb{E}\left[ZS|\mathcal{F}^{(1)}\right]\right] = \mathbb{E}\left[\varphi^{(1)} \sup_{Z \in \mathcal{Q}_{(1)}} \mathbb{E}\left[ZS|\mathcal{F}^{(1)}\right]\right], \tag{5.10}$$

then the proof is over since this relation implies that

$$\begin{aligned}
\Pi[S] &= \sup_{\varphi^{(1)}} \mathbb{E}\left[\varphi^{(1)} \sup_{Z \in \mathcal{Q}_{(1)}} \mathbb{E}\left[ZS|\mathcal{F}^{(1)}\right]\right] \\
&= \Pi^{(1)}\left[\Pi^{(2)}\left[S|\mathcal{F}^{(1)}\right]\right].
\end{aligned}$$

To prove (5.10), we first observe that

$$\sup_{Z \in \mathcal{Q}_{(1)}} \mathbb{E}\left[\varphi^{(1)} \mathbb{E}\left[ZS|\mathcal{F}^{(1)}\right]\right] \leq \mathbb{E}\left[\varphi^{(1)} \sup_{Z \in \mathcal{Q}_{(1)}} \mathbb{E}\left[ZS|\mathcal{F}^{(1)}\right]\right].$$

Let us prove the other inequality. By definition of the supremum, there exists a sequence $Z_n \in \mathcal{Q}_{(1)}$ with $\mathbb{E}\left[Z_1 S|\mathcal{F}^{(1)}\right] \leq \mathbb{E}\left[Z_2 S|\mathcal{F}^{(1)}\right] \leq \cdots$ such that

$$\lim_n \mathbb{E}\left[Z_n S|\mathcal{F}^{(1)}\right] = \sup_{Z \in \mathcal{Q}_{(1)}} \mathbb{E}\left[ZS|\mathcal{F}^{(1)}\right].$$

Thus, by the monotone convergence theorem,

$$\begin{aligned}
\mathbb{E}\left[\varphi^{(1)} \sup_{Z \in \mathcal{Q}_{(1)}} \mathbb{E}\left[ZS|\mathcal{F}^{(1)}\right]\right] &= \lim_n \mathbb{E}\left[\varphi^{(1)} \mathbb{E}\left[Z_n S|\mathcal{F}^{(1)}\right]\right] \\
&\leq \sup_{Z \in \mathcal{Q}_{(1)}} \mathbb{E}\left[\varphi^{(1)} \mathbb{E}\left[ZS|\mathcal{F}^{(1)}\right]\right],
\end{aligned}$$

which ends the proof. ■

5.2.3 Actuarial-consistency and two-step actuarial valuations

While the previous section focused on market-consistency and how the valuation should treat financial claims, this section is dedicated to *actuarial-consistency* and the valuation of actuarial claims. The essential idea is to replace the role played by a financial claim in market-consistency by an actuarial claim. One can then also define a two-step actuarial valuation where financial and actuarial valuations are interchanged.

Definition 51 (Strong actuarial-consistency) A valuation Π is called strong actuarial-consistent (strong ACV) if for any actuarial claim $S^{(2)}$ the following holds:

$$\Pi\left[S + S^{(2)}\right] = \Pi\left[S\right] + \pi^{(2)}\left[S^{(2)}\right]. \tag{5.11}$$

Similar to strong market-consistency, strong actuarial-consistency implies linearity of the actuarial valuation principle (this issue was also discussed in Chapter 2). Therefore, it would not be appropriate in an actuarial context in which diversification benefits are desired by pooling identical actuarial risks.

Definition 52 (Weak actuarial-consistency) A valuation Π is called weak actuarial-consistent (weak ACV) if for any actuarial claim $S^{(2)}$ the following holds:

$$\Pi\left[S^{(2)}\right] = \pi^{(2)}\left[S^{(2)}\right]. \tag{5.12}$$

Weak actuarial-consistency only postulates that an actuarial valuation is applied for all actuarial claims. We remark that in the previous chapters, we defined a similar notion of actuarial-consistency, but the condition only held for the claims which are independent of the financial filtration $\mathbb{F}^{(1)}$. Similar to Lemma 5, one can show that weak and strong actuarial-consistency of a valuation Π are equivalent when the actuarial valuation principle $\pi^{(2)}$ is linear.

Hereafter, we introduce a class of actuarial-consistent valuations which we call *two-step actuarial valuations*. These are the counterpart of the two-step financial valuations in which financial and actuarial valuations are reversed. More specifically, in a first step we compute the financial value of S conditional on actuarial scenarios (the values of the actuarial assets X), i.e. $\pi^{(1)}\left[S|\ \mathcal{F}^{(2)}\right]$. Since this conditional payoff depends only on actuarial scenarios and is then $\mathcal{F}^{(2)}$-measurable, the quantity $\pi^{(1)}\left[S|\ \mathcal{F}^{(2)}\right]$ should be valuated via a standard actuarial valuation $\pi^{(2)}$. This motivates the following definition.

Definition 53 (Two-step actuarial valuation) The valuation Π is called a two-step actuarial valuation principle if it can be expressed as follows:

$$\Pi\left[S\right] = \pi^{(2)}\left[\pi^{(1)}\left[S|\ \mathcal{F}^{(2)}\right]\right],$$

where $\pi^{(2)}$ is the actuarial valuation principle and $\pi^{(1)}$ is the $\mathcal{F}^{(2)}$-conditional financial valuation principle.

Hence, the two-step actuarial valuation consists of applying the market-adjusted valuation $\pi^{(1)}$ to the residual risk which remains after having conditioned on the future development of the actuarial risks, i.e. the filtration $\mathbb{F}^{(2)}$.

In the same vein as Theorem 16, we can prove that weak actuarial-consistent valuation is equivalent to two-step actuarial valuation for coherent valuations.

Theorem 17 (Characterization of weak ACV) The following statements are equivalent:

1. Π is a coherent weak actuarial-consistent valuation.
2. There exists an $\mathcal{F}^{(2)}$-conditional coherent valuation $\Pi^{(1)}$ and a coherent actuarial valuation $\Pi^{(2)}$ such that
$$\Pi[S] = \Pi^{(2)}\left[\Pi^{(1)}\left[S|\mathcal{F}^{(2)}\right]\right]. \tag{5.13}$$

Proof: The proof is similar to the one of Theorem 16 and is thus omitted. ■

5.2.4 Fair valuation: merging market- and actuarial-consistency

After having defined two broad classes of valuations: market-consistent and actuarial-consistent valuations, a natural question arises: Could we always define a *fair* valuation that is market-consistent *and* actuarial-consistent?

Definition 54 (Fair valuation) The valuation Π is fair if it is weak market-consistent and weak actuarial-consistent.

In general, it will not always be possible to define a fair valuation. Indeed, in a general probability space in which financial and actuarial risks are dependent, there is ambiguity on the valuation to be used: a market-consistent valuation calibrated to market prices *or* an actuarial-consistent valuation calibrated to historical actuarial data.

In the following lemma, we show that if the valuation is weak MCV and ACV and there exist a financial and actuarial claim that are equal *a.s.*, then the financial and actuarial valuations should coincide. In particular, this lemma implies that for a given financial valuation $\pi^{(1)}$ and actuarial valuation $\pi^{(2)}$, it is not always possible to define a fair valuation (i.e. a valuation that is weak MCV and ACV).

Lemma 6 Assume that there exist a financial claim $S^{(1)}$ and an actuarial claim $S^{(2)}$ such that

$$S^{(1)} \stackrel{a.s.}{=} S^{(2)}. \tag{5.14}$$

If the valuation Π is weak MCV and weak ACV, then the following holds:

$$\pi^{(1)}\left[S^{(1)}\right] = \pi^{(2)}\left[S^{(1)}\right].$$

Proof: Since Π is weak MCV, we can write:

$$\begin{aligned}\Pi\left[S^{(1)}\right] &= \pi^{(1)}\left[S^{(1)}\right] \\ &= \pi^{(1)}\left[S^{(2)}\right] \text{ by (5.14).}\end{aligned}$$

Moreover, weak ACV implies that

$$\begin{aligned}\Pi\left[S^{(2)}\right] &= \pi^{(2)}\left[S^{(2)}\right] \\ &= \pi^{(2)}\left[S^{(1)}\right] \text{ by (5.14).}\end{aligned}$$

Because we identify claims which are equal *a.s.*, we have that

$$\Pi\left[S^{(1)}\right] = \Pi\left[S^{(2)}\right],$$

which ends the proof. ■

To illustrate the previous lemma, we consider the financial claim of Example 13 and the actuarial claim of Example 14 and show that if the claims are comonotonic[4], a fair valuation cannot be properly defined.

Example 15 (Comonotonic financial and actuarial claims) Consider the financial claim $S^{(1)}$ and the actuarial claim $S^{(2)}$ which are given by

$$S^{(1)} = \begin{cases} 0, & \text{if } Y = 50, \\ 100, & \text{if } Y = 200. \end{cases} \text{ and } S^{(2)} = \begin{cases} 0, & \text{if } I = 0, \\ 100, & \text{if } I = 1. \end{cases} \tag{5.15}$$

Assume moreover that the claims are comonotonic:

$$\mathbb{P}[(Y, I) = (50, 0)] = p,$$
$$\mathbb{P}[(Y, I) = (200, 1)] = 1 - p.$$

Clearly, we have that $S^{(1)} \overset{a.s.}{=} S^{(2)}$. Based on the results of Examples 13 and 14, any fair valuation Π should satisfy:

$$\Pi\left[S^{(1)}\right] = 100q,$$
$$\Pi\left[S^{(2)}\right] = 100\left(p + \beta\sqrt{p(1-p)}\right),$$

if the financial valuation is the risk-neutral valuation and the actuarial valuation is the standard deviation principle. We then have two identical risks with two (possibly very different) values, creating inconsistency in the valuation mechanism.

With the emergence of the market for longevity derivatives, a valuator needs to make a choice between market-consistency and actuarial-consistency. For instance, consider a market with some traded longevity bonds and there is an issue of a new longevity product. One needs to decide to use either a market-consistent approach based on the traded longevity bonds in the market or an

[4] We recall that two claims $S^{(1)}$ and $S^{(2)}$ are comonotonic if they can be represented as increasing functions of the same random source: $(S^{(1)}, S^{(2)}) \overset{d}{=} (F^{-1}_{S^{(1)}}(U), F^{-1}_{S^{(2)}}(U))$ with $U \sim Uniform(0,1)$.

actuarial-consistent approach based on longevity trend assumptions.

In the following example, we illustrate this point and compare a market-consistent and an actuarial-consistent valuation in presence of a longevity bond.

Example 16 (Comparison between MCV and ACV)
(a) Consider a portfolio of pure endowment for l_x Belgian insureds of age x at time 0. The pure endowment guarantees a sum of 1 if the policyholder is still alive at maturity. The aggregate payoff can be written as

$$S = L_{x+T}$$

with L_{x+T} the number of policyholders who survive up to the maturity time T. Moreover, we assume that the financial market is composed of two assets: a risk-free asset $Y^{(0)}(t) = e^{rt}$ and a longevity bond for which the payoff at maturity is $Y^{(1)}(T) = \tilde{L}_{x+T}$, the equivalent of L_{x+T} but for the Dutch population. First, we determine the expected value (called *best-estimate*) by a two-step actuarial valuation:

$$\begin{aligned} BE[S] &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}L_{x+T}|\mathcal{F}^{(2)}\right]\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[e^{-rT}L_{x+T}\mathbb{E}^{\mathbb{Q}}\left[1|\mathcal{F}^{(2)}\right]\right] \\ &= e^{-rT}\mathbb{E}^{\mathbb{P}}\left[L_{x+T}\right]. \end{aligned}$$

The actuarial-consistent valuation would suggest a full investment in the risk-free asset. Secondly, assuming that the Belgian population live slightly shorter than the Dutch population[5]: $\mathbb{E}^{\mathbb{P}}\left[L_{x+T}|\tilde{L}_{x+T}\right] = \beta\tilde{L}_{x+T}$ with $\beta < 1$, we determine the best-estimate according to a two-step financial valuation:

$$\begin{aligned} BE^{*}[S] &= \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\mathbb{E}^{\mathbb{P}}\left[L_{x+T}|\mathcal{F}^{(1)}\right]\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\beta\tilde{L}_{x+T}\right] \\ &= \beta Y^{(1)}(0), \end{aligned}$$

[5] For the reader interested in Dutch and Belgian mortality projections, we refer to Antonio et al. (2017)

where $Y^{(1)}(0)$ is the current price of the longevity bond. The market-consistent valuation would then suggest a full investment in the Dutch longevity bond.

(b) In order to better grasp the difference between the actuarial-consistent and market-consistent valuations, we introduce some modelling assumptions.
Assume that the interest rate $r = 0$, and the bivariate Belgian-Dutch population follows the distribution: $(L_{x+T}, \tilde{L}_{x+T}) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

Hence, both Belgian and Dutch populations are normal distributed with correlation ρ. The best-estimate of $S = L_{x+T}$ by a two-step actuarial valuation is given by

$$\begin{aligned} BE[S] &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}L_{x+T}|L_{x+T}\right]\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[L_{x+T}\right] \\ &= \mu_1. \end{aligned} \tag{5.16}$$

To determine the best-estimate by a two-step financial valuation, we first notice that by standard results of normal distributions, we have

$$\mathbb{E}^{\mathbb{P}}\left[L_{x+T}|\tilde{L}_{x+T}\right] = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}\left(\tilde{L}_{x+T} - \mu_2\right).$$

Let us further assume that the distribution of $\tilde{L}_{x+T}$ under $\mathbb{Q}$ is

$$\tilde{L}_{x+T} \overset{\mathbb{Q}}{\sim} \mathcal{N}(\mu_2 + \sigma_2\kappa, \sigma_2^2),$$

where $\kappa > 0$ is the market price of risk for the longevity bond. Therefore, we find that

$$\begin{aligned} BE^*[S] &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}\left[L_{x+T}|\tilde{L}_{x+T}\right]\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\mu_1 + \rho\frac{\sigma_1}{\sigma_2}\left(\tilde{L}_{x+T} - \mu_2\right)\right] \\ &= \mu_1 + \rho\frac{\sigma_1}{\sigma_2}\left(\mathbb{E}^{\mathbb{Q}}\left[\tilde{L}_{x+T}\right] - \mu_2\right) \\ &= \mu_1 + \rho\sigma_1\kappa. \end{aligned} \tag{5.17}$$

We can compare the standard best-estimate for the Belgian population (5.16) with the market-consistent value taking into account the Dutch longevity bond (5.17). Intuitively, the difference should reflect two aspects:

1. The dependence between Belgian and Dutch populations.

2. The risk premium on the Dutch longevity bond.

The results confirm the intuition: the difference is given by

$$BE^*[S] - BE[S] = \rho\sigma_1\kappa. \tag{5.18}$$

We observe that the higher the correlation ρ, the higher the difference (this reflects the point 1.). Moreover, the difference is an increasing function of the market price of risk κ (this reflects the point 2.).

If the valuator can choose between the risk-free investment or the Dutch longevity bond, he will go for the longevity bond if the benefits are higher than the costs, i.e. if the risk reduction of investing in the longevity bond is higher than the extra price he has to pay (given by Equation (5.18)). The prices at time 0 of both approaches and the residual losses at maturity are given in the table below:

Price at time 0	Residual loss
$BE[S] = \mu_1$	$R_1 = L_{x+T} - \mathbb{E}^{\mathbb{P}}\left[L_{x+T}\right] \sim \mathcal{N}(0, \sigma_1^2)$
$BE^*[S] = \mu_1 + \rho\sigma_1\kappa$	$R_2 = L_{x+T} - \left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}\left(\tilde{L}_{x+T} - \mu_2\right)\right)$ $\sim \mathcal{N}(0, (1-\rho^2)\sigma_1^2)$

From the table, we observe that the investment in the longevity bond leads to a decrease in the volatility of the residual risk. Notice that in case of extreme dependence (i.e. $\rho \to \pm 1$), the claim S can be almost completely hedged with the longevity bond and the residual loss R_2 tends to 0. The valuator will typically go for the longevity bond if the risk reduction (computed in terms of Value-at-Risk for simplicity) is higher than the extra price to pay:

$$\rho\sigma_1\kappa < VaR_p[R_1] - VaR_p[R_2] = \sigma_1\Phi^{-1}(p)(1 - \sqrt{1-\rho^2}) \tag{5.19}$$

On the other hand, if the longevity bond price is too high in comparison with the risk reduction (e.g. if the market price of risk κ is set high), an actuarial-consistent valuation is then preferable. ◀

Remark 4 In the previous example, we compared the best-estimate values by choosing the actuarial valuation $\pi^{(2)}[S] = \mathbb{E}^{\mathbb{P}}[S]$ in the two-step operators (for simplicity of presentation). Similar conclusions can be drawn if the actuarial valuation includes a risk margin. For instance, if one considers a standard deviation principle as actuarial valuation, Equation (5.19) takes the form:

$$
\begin{aligned}
\rho^F[S] - \rho^A[S] &= \sigma_1 \left[\kappa\rho - \alpha(1 - \sqrt{1-\rho^2})\right] \\
&< VaR_p[R_1] - VaR_p[R_2] \\
&= \sigma_1(1 - \sqrt{1-\rho^2})\left(\Phi^{-1}(p) - \alpha\right),
\end{aligned}
$$

where $\rho^F[S]$ and $\rho^A[S]$ are the two-step financial and actuarial valuations, respectively. In that case, we also observe that, if the longevity bond price is high (meaning that κ is high) and the liability is not strongly correlated to the longevity bond (i.e. ρ close to zero), an actuarial-consistent valuation is desirable (the extra price to pay is higher than the risk reduction).

Remark 5 In this chapter, we do not want to argue that one method is better than another; each one has pros and cons. While the second method allows to transfer the risk to the financial market, it comes also with a price: the liabilities become totally dependent on the longevity bond. In particular, an adverse shock on the Dutch population or a counterparty's default will have a direct effect on the assets backing the liabilities.

More generally, as pointed out by Vedani et al. (2017), market-consistent valuations are directly subject to market movements, and can lead to excess volatility, depending on the calibration sets chosen by the actuary. We also refer to Rae et al. (2018) for different concerns around the appropriateness of market-consistency to the insurance business.

In the next example, we consider the valuation of a hybrid claim via a two-step financial and actuarial valuation. More specifically, we combine Example 13 for the financial valuation and Example 14 for the actuarial valuation into the two-step valuations and investigate the difference between the two-step operators.

Example 17 (Two-step valuations for hybrid claims) Consider an equity-linked contract for a life (x), which pays the call option $(Y-K)_+$ in case the policyholder is alive at time $T=1$ and 0 otherwise. We recall that the stock Y can go up to 200 or down to 50, the strike $K=100$ and the policyholder survival is modelled by the indicator I. Therefore, the payoff of this contract is given by

$$S = (Y-K)_+ \times I = \begin{cases} 100, & \text{if } Y = 200, I = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{5.20}$$

We assume that the financial valuation is the risk-neutral expectation and the actuarial valuation is the standard deviation principle:

$$\begin{aligned} \pi^{(1)}[S] &= \mathbb{E}^{\mathbb{Q}}[S], \\ \pi^{(2)}[S] &= \mathbb{E}^{\mathbb{P}}[S] + \beta\sqrt{\mathrm{Var}^{\mathbb{P}}[S]}. \end{aligned}$$

We consider the two-step valuations for the hybrid payoff (5.20):

1. Two-step financial valuation: The value of S is given by

$$\begin{aligned} \Pi^{(1)}[S] &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}\left[S|\mathcal{F}^{(1)}\right] + \beta\sqrt{\mathrm{Var}^{\mathbb{P}}\left[S|\mathcal{F}^{(1)}\right]}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[(Y-K)_+\left(\mathbb{E}^{\mathbb{P}}[I|Y] + \beta\sigma^{\mathbb{P}}[I|Y]\right)\right]. \end{aligned}$$

   If we note that

$$\mathbb{E}^{\mathbb{P}}[I|Y] = \begin{cases} \mathbb{P}[I=1|Y=50], & \text{if } Y = 50, \\ \mathbb{P}[I=1|Y=200], & \text{if } Y = 200, \end{cases}$$

   then we find that the two-step financial value of S is

$$\Pi^{(1)}[S] = 100\, q_Y\left(p_{I|Y=200} + \beta\sqrt{p_{I|Y=200}(1-p_{I|Y=200})}\right), \tag{5.21}$$

   where q_Y is the $\mathbb{Q}$-probability that Y goes up:

$$q_Y = \mathbb{Q}[Y=200]$$

   and $p_{I|Y=200}$ is the $\mathbb{P}$-probability that the policyholder is alive given that the stock goes up:

$$p_{I|Y=200} = \mathbb{P}[I=1|Y=200].$$

2. Two-step actuarial valuation: The value of S is given by

$$\Pi^{(2)}[S] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right] + \beta\sigma^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right]$$
$$= \mathbb{E}^{\mathbb{P}}\left[I\mathbb{E}^{\mathbb{Q}}\left[(Y-K)_+|I\right]\right] + \beta\sigma^{\mathbb{P}}\left[I\mathbb{E}^{\mathbb{Q}}\left[(Y-K)_+|I\right]\right].$$

Noting that

$$\mathbb{E}^{\mathbb{Q}}\left[(Y-K)_+|I\right] = \begin{cases} 100\ \mathbb{Q}[Y=200|I=0], & \text{if } I=0, \\ 100\ \mathbb{Q}[Y=200|I=1], & \text{if } I=1, \end{cases}$$

then we find that the two-step actuarial value of S is

$$\Pi^{(2)}[S] = 100\, q_{Y|I=1}\left(p_I + \beta\sqrt{p_I(1-p_I)}\right), \tag{5.22}$$

where p_I is the $\mathbb{P}$-probability that the policyholder is alive: $p_I = \mathbb{P}[I=1]$ and $q_{Y|I=1}$ is the $\mathbb{Q}$-probability that the stock goes up given that the policyholder is alive:

$$q_{Y|I=1} = \mathbb{Q}[Y=200|I=1].$$

If we compare the two-step financial and actuarial values (5.21) and (5.22), the structure is similar but the dependence between financial and actuarial risks is taken into account differently. In the first case, it is via the $\mathbb{P}$-probability of actuarial risks given financial scenarios, i.e. $p_{I|Y=200}$ while in the second case, it is via the $\mathbb{Q}$-probability of financial risks given actuarial scenarios, i.e. $q_{Y|I=1}$. In case of independence under $\mathbb{P}$ and $\mathbb{Q}$, both valuations are equal. In case of dependence, both valuations (5.21) and (5.22) will in general be different as we illustrate below:

$$\Pi^{(2)}[S] - \Pi^{(1)}[S] = 100\, q_{Y|I=1}\, p_I - 100\, q_Y\, p_{I|Y=200}$$
$$+ 100\, q_{Y|I=1}\, \beta\sqrt{p_I(1-p_I)}$$
$$- 100\, q_Y\, \beta\sqrt{p_{I|Y=200}(1-p_{I|Y=200})}.$$

Let us further assume that the difference between $\mathbb{P}$ and $\mathbb{Q}$ is given by a constant market price of risk κ:

$$\kappa = q_Y - p_Y,$$
$$= q_{Y|I=1} - p_{Y|I=1}.$$

Therefore, by Bayes' Theorem, we find that

$$\begin{aligned}\Pi^{(2)}[S] - \Pi^{(1)}[S] = 100 &\left(\frac{p_{I|Y=200}p_Y}{p_I} + \kappa\right) p_I \\ &- 100\,(p_Y + \kappa)\, p_{I|Y=200} \\ &+ 100\,(p_{Y|I=1} + \kappa)\,\beta\sqrt{p_I(1-p_I)} \\ &- 100\,(p_Y + \kappa)\,\beta\sqrt{p_{I|Y=200}(1-p_{I|Y=200})}.\end{aligned}$$

After simplifications, we find that

$$\begin{aligned}&\Pi^{(2)}[S] - \Pi^{(1)}[S] \\ &= 100\kappa\left(p_I - p_{I|Y=200}\right) \\ &+ 100\kappa\beta\left(\sqrt{p_I(1-p_I)} - \sqrt{p_{I|Y=200}(1-p_{I|Y=200})}\right) \\ &+ 100 p_{Y|I=1}\beta\sqrt{p_I(1-p_I)} \\ &- 100 p_Y\beta\sqrt{p_{I|Y=200}(1-p_{I|Y=200})}.\end{aligned}$$

Similar to Example 16, we observe that the difference between the two-step valuations relies mainly on

- The risk premium κ which reflects the difference between the real-world measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$.
- The dependence between actuarial and financial risks (expressed as the difference between p_I and $p_{I|Y=200}$ as well as the difference between p_Y and $p_{Y|I=1}$). ◀

We remark that in the literature, it is common to assume that financial and actuarial claims are independent (either under $\mathbb{P}$ or $\mathbb{Q}$[6]). In that case, one can define a valuation that is MCV and ACV since the valuation is decoupled into two independent valuations, one for financial claims and one for actuarial claims. In the following lemma, we show that if the conditional actuarial valuation of actuarial claims does not depend on the financial filtration, the two-step financial valuation is fair. Similar result holds for the two-step actuarial valuation.

[6] Note that independence under $\mathbb{P}$ does not necessarily imply independence under $\mathbb{Q}$, see Dhaene et al. (2013).

Lemma 7 Consider hybrid claims of the form $S = S^{(1)} \times S^{(2)}$ where $S^{(1)}$ is financial and $S^{(2)}$ is actuarial. If one of the two following conditions holds:

1. Π is a two-step financial valuation and $\pi^{(2)}\left[S^{(2)} \middle| \mathcal{F}^{(1)}\right] = \pi^{(2)}\left[S^{(2)}\right]$.
2. Π is a two-step actuarial valuation and $\pi^{(1)}\left[S^{(1)} \middle| \mathcal{F}^{(2)}\right] = \pi^{(1)}\left[S^{(1)}\right]$.

Then, we have that

$$\Pi\left[S^{(1)} \times S^{(2)}\right] = \pi^{(1)}\left[S^{(1)}\right] \times \pi^{(2)}\left[S^{(2)}\right].$$

In particular, the valuation Π is fair:

$$\begin{aligned}\Pi\left[S^{(1)}\right] &= \pi^{(1)}\left[S^{(1)}\right] \\ \Pi\left[S^{(2)}\right] &= \pi^{(2)}\left[S^{(2)}\right].\end{aligned}$$

Proof: The proof follows directly from the definition of the two-step valuations. ■

From Lemma 7, we observe that we can define a fair valuation under appropriate independence assumptions. The first condition will typically require independence under $\mathbb{P}$ (e.g. $\mathbb{E}\left[S^{(2)}|\mathcal{F}^{(1)}\right] = \mathbb{E}[S^{(2)}]$) and the second condition independence under $\mathbb{Q}$ (e.g. $\mathbb{E}^{\mathbb{Q}}\left[S^{(1)}|\mathcal{F}^{(2)}\right] = \mathbb{E}^{\mathbb{Q}}[S^{(1)}]$). Another possibility is to restrict the notion of actuarial-consistency to the risks which are independent of the financial market as we did in the previous chapters.

Example 17 (continued) If we assume independence under $\mathbb{P}$ in the two-step financial valuation or independence under $\mathbb{Q}$ in the two-actuarial valuation, both valuations lead to a fair valuation. This is in line with Lemma 7.

5.3 Numerical illustration

Based on the two-step actuarial valuation introduced in the previous section, we show how the valuation can be decomposed into a best estimate and a risk margin as required by solvency regulations. Moreover, we illustrate the valuation on a portfolio of equity-linked life insurance contracts with dependent financial and actuarial risks.

5.3.1 Best estimate, risk margin and fair valuation

5.3.1.1 Best estimate

In Article 77 of the DIRECTIVE 2009/138/EC (European Commission (2009)), the best estimate is defined as "the probability-weighted average of future cash-flows taking account of the time value of money" (expected present value of future cash-flows). Hence, the best estimate of an insurance liability can be interpreted as an appropriate estimation of the expected present value based on actual available information.

Based on our two-step actuarial valuation, we can define a broad notion of best estimate for a general claim S.

Definition 55 (Best estimate) For any claim $S \in L^2(\mathbb{P}, \mathbb{F})$, the best estimate is given by

$$BE[S] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right]. \qquad (5.23)$$

It turns out that the best estimate appears as a two-step actuarial valuation for which there is no distortion of the different measures, i.e. $\pi^{(1)}[S] = \mathbb{E}^{\mathbb{Q}}[S]$ and $\pi^{(2)}[S] = \mathbb{E}^{\mathbb{P}}[S]$. In general, the expression (5.23) could be hardly tractable since we can possibly have an infinite number of actuarial scenarios. For practical purposes, we will often consider the *approximated* best estimate $\widehat{BE}$ defined by

$$\widehat{BE}[S] = \sum_{i=1}^{n} \mathbb{P}[A_i]\, \mathbb{E}^{\mathbb{Q}}[S|\, A_i], \qquad (5.24)$$

for a finite number n of actuarial scenarios: $A_1, A_2, ..., A_n \in \mathcal{F}^{(2)}$.

The best estimate in Definition 55 appears as an average of risk-neutral valuations which are applied to the risk which remains after having conditioned on the actuarial filtration. Hereafter, we consider some special cases:

- For any actuarial risk $S^{(2)}$, we find that

$$BE[S^{(2)}] = \mathbb{E}^{\mathbb{P}}\left[S^{(2)}\right].$$

- For any product claim S with independent actuarial and financial risks (under $\mathbb{Q}$), we find that

$$\begin{aligned} BE[S] &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[S^{(1)} \times S^{(2)}|\mathcal{F}^{(2)}\right]\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[S^{(2)} \times \mathbb{E}^{\mathbb{Q}}\left[S^{(1)}|\mathcal{F}^{(2)}\right]\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[S^{(2)} \times \mathbb{E}^{\mathbb{Q}}\left[S^{(1)}\right]\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[S^{(2)}\right] \times \mathbb{E}^{\mathbb{Q}}\left[S^{(1)}\right]. \end{aligned}$$

5.3.1.2 Risk margin

In order to motivate the risk margin, we recall that the best estimate is centered around the risk

$$\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right].$$

This risk represents the risk-neutral financial price of S conditional on the actuarial information. Looking at the tail of this (actuarial) risk will provide information on the actuarial scenarios which yield the worst financial price. Hence, applying a coherent actuarial valuation on this conditional financial price allows to measure the impact of the actuarial uncertainty on the risk-neutral price. This motivates the following definition.

Definition 56 (SCR for actuarial risk)
For any claim $S \in L^2(\mathbb{P}, \mathbb{F})$ and any actuarial coherent valuation $\pi^{(2)}$, the SCR for actuarial risk is given by

$$SCR\,[S] = \pi^{(2)}\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right] - \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right]. \qquad (5.25)$$

It turns out that the SCR for actuarial risk appears as a two-step actuarial coherent valuation for which we deducted the best estimate. Thanks to the representation theorem (see Theorem 15), the SCR for actuarial risk can be represented as

$$SCR\,[S] = \sup_{\tilde{\mathbb{P}}}\left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right] - \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right]\right\}$$

where the supremum is taken over a set of probability measures $\tilde{\mathbb{P}}$ absolutely continuous to $\mathbb{P}$. Hence, the SCR for actuarial risk can be interpreted as a worst case scenario: we can consider a family of stressed actuarial models (e.g. different mortality dynamics) and define the SCR as the value under the worst-case model.

For instance, in case the actuarial valuation $\pi^{(2)}$ is the TVaR measure at a confidence level p, we find that

$$SCR\left[S\right] = TVaR_p\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right] - \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right]. \tag{5.26}$$

The expression (5.26) can be interpreted as follows: If we assume that the insurer receives the best estimate from the policyholder and a capital buffer equal to SCR provided by investors, the insurer will be able to cover all losses with a confidence level equal to p since

$$\begin{aligned} BE\left[S\right] + SCR\left[S\right] &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right] + TVaR_p\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right] \\ &\quad - \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right], \\ &= TVaR_p\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right] \geq VaR_p\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right]. \end{aligned}$$

For any product claim with independent actuarial and financial risks, the SCR for actuarial risk takes the form

$$\begin{aligned} SCR\left[S\right] &= \pi^{(2)}\left[\mathbb{E}^{\mathbb{Q}}\left[S^{(1)} \times S^{(2)}|\mathcal{F}^{(2)}\right]\right] - \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[S^{(1)} \times S^{(2)}|\mathcal{F}^{(2)}\right]\right] \\ &= \left(\pi^{(2)}\left[S^{(2)}\right] - \mathbb{E}^{\mathbb{P}}\left[S^{(2)}\right]\right)\mathbb{E}^{\mathbb{Q}}\left[S^{(1)}\right]. \end{aligned} \tag{5.27}$$

From the expression (5.27), we observe that the SCR appears as a capital buffer to add on top of the best estimate in order to cover losses due to the uncertainty in the actuarial risk.

5.3.1.3 Cost-of-capital fair valuation of insurance liabilities

In Solvency II, the fair value of insurance liabilities is defined as the sum of the best estimate and the risk margin in which the latter is defined as the cost of capital needed to cover the unhedgeable risks.

Similarly to regulatory directives, we define a cost-of-capital fair valuation as the sum of the best estimate (expected present value) plus the risk margin (cost to cover unhedgeable risks) where the latter represents the cost of providing the SCR for actuarial risk.

Definition 57 (Cost-of-capital fair valuation) For any claim $S \in L^2(\mathbb{P}, \mathbb{F})$ and any coherent actuarial valuation $\pi^{(2)}$, the fair value of S is defined by

$$\rho[S] = BE[S] + iSCR[S] \tag{5.28}$$

with

$$SCR[S] = \pi^{(2)}\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right] - \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right],$$

where i is the cost-of-capital rate and SCR is the SCR for actuarial risk defined in Definition 56.

Remark 6 The fair valuation from Definition 57 has some similarities with the fair valuation of Solvency II in the sense that the valuation is decomposed into an expected present value plus a cost-of-capital risk margin. However, Definition 57 is based on a one-period perspective while Solvency II takes a one-year view and the risk margin for long-term liabilities is defined as the sum of all future one-year cost-of-capital premiums, see e.g. Moehr (2011).

5.3.2 Numerical application

In this subsection, we show how to determine the fair value (5.28) for a portfolio of guaranteed minimum maturity benefit (GMMB) contracts underwritten at time 0 on l_x persons of age x. In particular, we detail the numerical procedure for the best estimate and the SCR for actuarial risk. Moreover, we compare the fair valuation with the setting of Brennan and Schwartz (1976) in which complete diversification of mortality is assumed.

The GMMB contract offers at maturity the greater of a minimum guarantee K and a stock value if the policyholder is still alive at that time. Let T_i be the remaining lifetime of insured i, $i = 1, 2, \ldots, l_x$, at contract initiation. The payoff per policy can be written as

$$S = \frac{L_{x+T}}{l_x} \times \max\left(S^{(1)}(T), K\right), \tag{5.29}$$

with

$$L_{x+T} = \sum_{i=1}^{l_x} 1_{\{T_i > T\}}.$$

Here, L_{x+T} is the number of policyholders who survived up to time T and $S^{(1)}(T)$ is the value of the stock at time T.

We consider a continuous time setting for the stock and the force of mortality dynamics. Let us assume that the dynamics of the stock process and the population force of mortality are given by

$$dS^{(1)}(t) = S^{(1)}(t)\left(\mu dt + \sigma dW_1(t)\right) \tag{5.30}$$
$$d\lambda(t) = c\lambda(t)dt + \xi dW_2(t), \tag{5.31}$$

with c, ξ, μ and σ are positive constants, and $W_1(t) = \rho W_2(t) + \sqrt{1-\rho^2}Z(t)$. Here, $W_2(t)$ and $Z(t)$ are independent standard Brownian motions. The specification of a non-mean reverting Ornstein-Uhlenbeck (OU) process (5.31) for the mortality model allows negative mortality rates. However, Luciano and Vigna (2008) and Luciano et al. (2017) showed that the probability of negative mortality rates is quite negligible with calibrated parameters. The great benefit of such specification is to allow tractability of mortality rates. Indeed, under Equation (5.31), $\lambda(t)$ is a Gaussian process and $\int_0^T \lambda(v)dv$ is normal distributed.

First we start by computing the best estimate. Since we want to determine the best estimate mortality, we assume that there is no risk premium in the actuarial market or, equivalently, that Equation (5.31) holds under $\mathbb{P}$ and $\mathbb{Q}$. Therefore, the calibration of the mortality intensity is performed by estimating its dynamic under the real-world measure, and then using it under the risk-neutral measure.[7] For the stock process, we define

$$dW_1^{\mathbb{Q}}(t) = \frac{\mu - r}{\sigma}dt + dW_1^{\mathbb{P}}(t),$$

where $\frac{\mu-r}{\sigma}$ represents the market price of equity risk. We can then write the dynamics under $\mathbb{Q}$ as follows

$$dS^{(1)}(t) = S^{(1)}(t)\left(rdt + \sigma dW_1(t)\right) \tag{5.32}$$
$$d\lambda(t) = c\lambda(t)dt + \xi dW_2(t). \tag{5.33}$$

The best estimate for the aggregate payoff (5.29) is given by

$$BE[S] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\frac{L_{x+T}}{l_x} \times \max\left(S^{(1)}(T), K\right) | \mathcal{F}^{(2)}\right]\right].$$

[7] A similar approach is considered in Luciano et al. (2017)

Under the independence assumption between the force of mortality and the stock dynamics, one can easily show that the best estimate simplifies into

$$BE[S] = \mathbb{E}^{\mathbb{P}}\left[\frac{L_{x+T}}{l_x}\right] \times \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\max\left(S^{(1)}(T), K\right)\right] \tag{5.34}$$

$$= {}_Tp_x\left[S^{(1)}(0)N(d_1) + Ke^{-rT}\left(1 - N(d_2)\right)\right] \tag{5.35}$$

$$= \underbrace{\mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^T \lambda(v)dv}\right]}_{{}_Tp_x}\left[S^{(1)}(0)N(d_1) + Ke^{-rT}\left(1 - N(d_2)\right)\right] \tag{5.36}$$

with

$$d_1 = \frac{\ln\left(\frac{S^{(1)}(0)}{K}\right) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

We remark that the survival probability ${}_Tp_x$ can be obtained in closed-form (for details, see for instance Mamon (2004)):

$${}_Tp_x = \mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^T \lambda(v)dv}\right] = e^{A\lambda(0)+\frac{B}{2}},$$

with

$$A = \frac{1}{c}\left(1 - e^{cT)}\right)$$
$$B = \frac{\xi^2}{c^3}\left(cT + \frac{3}{2} - 2e^{cT} + \frac{1}{2}e^{2cT}\right). \tag{5.37}$$

Under the dependence assumption, we provide in the next proposition the approximated best estimate for the portfolio of GMMB contracts.

Proposition 1 If we denote by ${}_Tp_x^i$ $(i = 1, ..., n)$ the survival rates for each actuarial scenario[8], the approximated best estimate for the aggregate payoff of GMMB contracts:

$$\widehat{BE}[S] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\frac{L_{x+T}}{l_x} \times \max\left(S^{(1)}(T), K\right)\middle| L_{x+T} = l_x\ {}_Tp_x^i\right]$$

[8] We assume that the actuarial scenarios are generated by a Monte-Carlo sample of i.i.d. observations

is given by

$$\widehat{BE}[S] = \frac{1}{n}\sum_{i=1}^{n} {}_Tp_x^i \left(\widetilde{S}^{(1)}(0)N(d_1) + e^{-rT}K\left(1 - N(d_2)\right)\right), \tag{5.38}$$

with

$$\widetilde{S}^{(1)}(0) = S^{(1)}(0)e^{\frac{-\sigma\rho_0\sqrt{T}}{\sqrt{\frac{1}{2c}e^{2cT}-\frac{2}{c}e^{cT}+T+\frac{3}{2c}}}\left(\frac{c}{\xi}\ln\ {}_Tp_x^i+\frac{\lambda(0)}{\xi}\left(e^{cT}-1\right)\right)}e^{-\frac{1}{2}\sigma^2\rho_0^2 T},$$

$$\rho_0 = \frac{\rho\left(\frac{1}{c}e^{cT} - \frac{1}{c} - T\right)}{\sqrt{T\left(\frac{1}{2c}e^{2cT} - \frac{2}{c}e^{cT} + T + \frac{3}{2c}\right)}},$$

$$d_1 = \frac{\ln\left(\frac{\widetilde{S}^{(1)}(0)}{K}\right) + \left(r + \frac{1}{2}\sigma^2(1-\rho_0^2)\right)T}{\sigma\sqrt{\left(1-\rho_0^2\right)T}},$$

$$d_2 = d_1 - \sigma\sqrt{\left(1-\rho_0^2\right)T}.$$

Proof: The proof based on classical arguments of stochastic calculus can be found in the Appendix Section 5.5. ■

The approximated best estimate (5.38) appears as an average of Black-Scholes call option prices which are adjusted for the dependence between the population force of mortality and the stock price processes. In each call option, there is an adjustment of the current stock price $S^{(1)}(0)$ to $\widetilde{S}^{(1)}(0)$, taking into account the realized survival rate ${}_Tp_x^i$ in each actuarial scenario. It is also worth noticing that in case of independence ($\rho = 0$), the approximated best estimate (5.38) converges to the best estimate (5.36).

To determine the best estimate (5.38), we only need to generate survival rates ${}_Tp_x^i$ $(i = 1, ..., n)$ and plug them into the Black-Scholes option pricing formulas. Since the force of mortality dynamics is given by

$$d\lambda(t) = c\lambda(t)dt + \xi dW_2(t),$$

one can prove (for details, see Appendix 5.5) that

$$\ln\ {}_Tp_x = -\int_0^T \lambda(s)ds \sim N\left(\mu, \sigma^2\right)$$

with

$$\mu = \frac{\lambda(0)}{c}\left(e^{cT} - 1\right),$$
$$\sigma^2 = \frac{\xi^2}{c^3}\left(\frac{1}{2}e^{2cT} - 2e^{cT} + cT + \frac{3}{2}\right).$$

We generate $n = 100000$ mortality paths. The benchmark parameters for the stock and the force of mortality are given in Table 5.1. The mortality parameters follow from Luciano et al. (2017) while the financial parameters are based on Bernard and Kwak (2016). The mortality parameters correspond to UK male individuals who are aged 55 at time 0.

Parameter set for numerical analysis
Force of mortality model: $c = 0.0750, \xi = 0.000597, \lambda(0) = 0.0087$.
Financial model: $r = 0.02, T = 10, S^{(1)}(0) = 1, K = 1, \sigma = 0.2$.

Table 5.1: Parameter values used in the numerical illustration.

Table 5.2 displays the best estimate per policy obtained using Equation (5.38) for a range of correlation coefficients: $\rho \in [-1, 1]$. We observe that the best estimate slightly decreases with the increase of the correlation parameter. This can be justified by a compensation effect between the mortality and the stock dynamics:

- In case of positive dependence, high mortality scenarios (respectively low mortality scenarios) are linked with high stock values (respectively low stock values). In consequence, the expected value of the claim

$$S = \frac{L_{x+T}}{l_x} \times \max\left(S^{(1)}(T), K\right)$$

  will be reduced since high values of survivals L_{x+T} will be associated with low financial guarantees, $\max\left(S^{(1)}(T), K\right)$, and vice-versa.

- On the other hand, in case of negative dependence, high survival rates will be linked with high financial guarantees, which implies a higher uncertainty and an increase of the best estimate.

ρ	Best estimate
-1.0	1.01132
-0.9	1.01086
-0.8	1.01041
-0.7	1.00995
-0.6	1.0095
-0.5	1.00904
-0.4	1.00858
-0.3	1.00811
-0.2	1.00764
-0.1	1.00716
0	1.00667
0.1	1.00618
0.2	1.00568
0.3	1.00517
0.4	1.00466
0.5	1.00414
0.6	1.0036
0.7	1.00307
0.8	1.00252
0.9	1.00196
1.0	1.00141

Table 5.2: Best estimate for the GMMB contract using Equation (5.38).

The SCR for actuarial risk is given by

$$SCR\,[S] = \pi^{(2)}\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right] - \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[S|\mathcal{F}^{(2)}\right]\right]$$

for some coherent actuarial valuation $\pi^{(2)}$. For this numerical illustration, we consider the TVaR with a confidence level $p = 0.95$.

Figure 5.2 displays the SCR for actuarial risk under the TVaR with a confidence level $p = 0.95$ for a range of correlation coefficients: $\rho \in [-0.5, 0.5]$. We observe that the SCR tends to increase when the correlation coefficient ρ increases in absolute value. This observation expresses the increase of the tails and the variance of

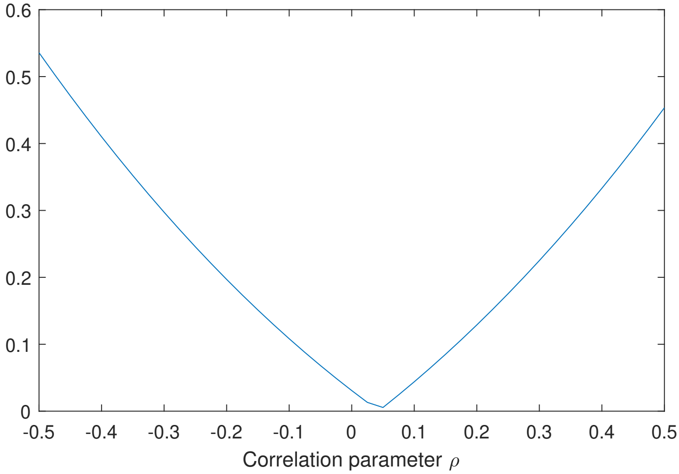

Figure 5.2: SCR for actuarial risk under the TVaR measure ($p = 0.95$)

the conditional financial price

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{L_{x+T}}{l_x}\times\max\left(S^{(1)}(T),K\right)|\mathcal{F}^{(2)}\right]$$

in case of an increase of the dependence parameter $|\rho|$. To clarify this point, we can distinguish two situations:

- In case of positive dependence, scenarios in which mortality is higher than expected increase the trend of the stock and the price of the financial guarantee.
- On the contrary, in case of negative dependence, scenarios in which mortality is lower than expected increase the trend of the stock and the price of the financial guarantee.

The fair value of the insurance liability S is then determined by

$$\rho\left[S\right]=BE\left[S\right]+iSCR\left[S\right],$$

where the cost-of-capital rate i is fixed at 6%.

Figure 5.3 represents the fair value of S for a range of correlation coefficients: $\rho\in[-1,1]$. Overall, we observe an increase of

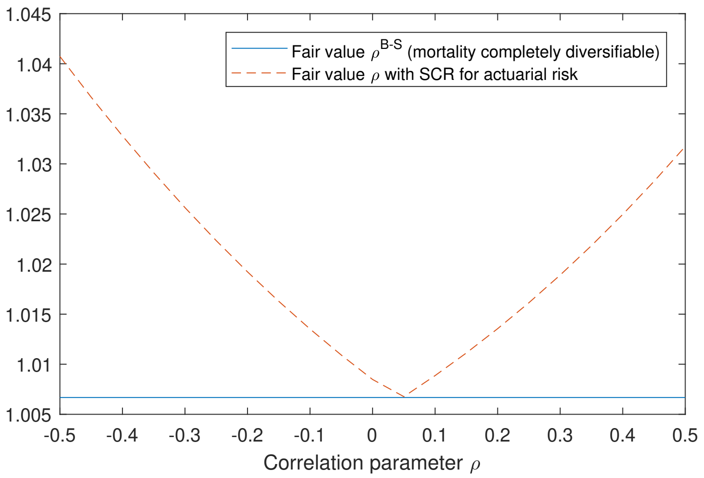

Figure 5.3: Fair value for the GMMB contract under the two-step actuarial approach and the approach of Brennan and Schwartz (1976).

the fair value of the GMMB contract under dependent mortality and equity risks. However, this effect is less pronounced for positive dependence. By comparison, the fair value of S under the assumption that mortality can be completely diversified (denoted by $\rho^{\text{B-S}}$ for Brennan and Schwartz (1976)), is given by Equation (5.36):

$$\rho^{\text{B-S}}[S] = \mathbb{E}^{\mathbb{P}}\left[\frac{L_{x+T}}{l_x}\right] \times \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\max\left(S^{(1)}(T), K\right)\right] \tag{5.39}$$

$$= {}_Tp_x\left[S^{(1)}(0)N(d_1) + Ke^{-rT}\left(1 - N(d_2)\right)\right] \tag{5.40}$$

$$= 1.0067.$$

From Figure 5.3, we clearly observe that this assumption underestimates the fair value of the contract since it does not take into account the actuarial uncertainty and the possible dependence with the financial market.

5.4 Concluding remarks

In this chapter, we have proposed a general valuation framework for insurance liabilities based on two-step valuation procedures. First, we introduced the family of two-step financial valuations that are weak market-consistent and the family of two-step actuarial valuations that are weak actuarial-consistent. For each family, we provided a complete axiomatic characterization of the two-step operators. In particular, we showed that any weak market-consistent or actuarial-consistent valuation has a two-step representation. We also studied under which conditions it is feasible to define a fair valuation, i.e. a valuation that is weak market-consistent and actuarial-consistent.

Based on our two-step actuarial valuation, we have defined a cost-of-capital fair valuation in which the valuation is defined as the sum of a best estimate (expected value) and a risk margin (cost of providing the SCR for actuarial risk). The detailed numerical illustration has shown the important impact on risk management when relaxing the independence assumption between actuarial and financial risks. In an extended B-S financial market, we determined the fair value of a GMMB contract where the force of mortality dynamics is a Vasicek-type model as considered in Luciano et al. (2017). It turns out that the dependency structure has an important impact on the fair valuation and the related SCR.

As pointed out by Liu et al. (2014), Solvency II Directive highly recommends the testing of capital adequacy requirements on the assumption of mutual dependence between financial markets and life insurance markets. In that respect, we believe that our two-step framework provides a plausible setting for the valuation of insurance liabilities with dependent financial and actuarial risks.

5.5 Appendix A: Proof of Proposition 1

Proof: We recall that the dynamics of the stock process and the population force of mortality under $\mathbb{Q}$ are given by

$$dS^{(1)}(t) = S^{(1)}(t)\left(rdt + \sigma dW_1(t)\right) \tag{5.41}$$
$$d\lambda(t) = c\lambda(t)dt + \xi dW_2(t) \tag{5.42}$$

with c, ξ, μ and σ_1 are positive constants, and $W_1(t) = \rho W_2(t) + \sqrt{1-\rho^2}Z(t)$. Here, $W_2(t)$ and $Z(t)$ are independent standard Brownian motions under $\mathbb{Q}$. From (5.42), we note that

$$\begin{aligned} d\left(e^{-ct}\lambda(t)\right) &= -ce^{-ct}\lambda(t)dt + e^{-ct}d\lambda(t) \\ &= \xi e^{-ct}dW_2(t). \end{aligned}$$

Hence, the force of mortality is a Gaussian process:

$$\lambda(t) = \lambda(0)e^{ct} + \xi \int_0^t e^{-c(u-t)}dW_2(u).$$

Moreover, we find that

$$\begin{aligned} \int_0^T \lambda(s)ds &= \frac{\lambda(0)}{c}\left(e^{cT}-1\right) + \xi \int_0^T \int_0^s e^{-c(u-s)}dW_2(u)ds \\ &= \frac{\lambda(0)}{c}\left(e^{cT}-1\right) + \xi \int_0^T \int_u^T e^{-c(u-s)}dsdW_2(u) \\ &= \frac{\lambda(0)}{c}\left(e^{cT}-1\right) + \frac{\xi}{c}\int_0^T \left(e^{-c(u-T)}-1\right)dW_2(u) \\ &= \frac{\lambda(0)}{c}\left(e^{cT}-1\right) + \frac{\xi}{c}X_T, \end{aligned}$$

with

$$X_T = \int_0^T \left(e^{-c(u-T)}-1\right)dW_2(u) \sim N\left(0, \frac{1}{2c}e^{2cT} - \frac{2}{c}e^{cT} + T + \frac{3}{2c}\right).$$

We can also remark that

$$\begin{aligned} \mathbb{E}\left(W_1(T)X_T\right) &= \mathbb{E}\left(\int_0^T dW_1(u)\int_0^T \left(e^{-c(u-T)}-1\right)dW_2(u)\right) \\ &= \rho\left(\frac{1}{c}e^{cT} - \frac{1}{c} - T\right), \end{aligned}$$

which leads to

$$corr\left(W_1(T), X_T\right) = \frac{\rho\left(\frac{1}{c}e^{cT} - \frac{1}{c} - T\right)}{\sqrt{T\left(\frac{1}{2c}e^{2cT} - \frac{2}{c}e^{cT} + T + \frac{3}{2c}\right)}} \equiv \rho_0.$$

We can then assume that

$$W_1(T) = \frac{\rho_0\sqrt{T}}{\sqrt{\frac{1}{2c}e^{2cT} - \frac{2}{c}e^{cT} + T + \frac{3}{2c}}}X_T + \sqrt{T\left(1-\rho_0^2\right)}Z,$$

where Z is a standard normal r.v. independent of X_T.
From

$$e^{-\int_0^T \lambda(s)ds} = {}_Tp_x^i,$$

we find that

$$X_T = -\frac{c}{\xi}\ln\ {}_Tp_x^i - \frac{\lambda(0)}{\xi}\left(e^{cT}-1\right).$$

The stock price at time T can be written as

$$\begin{aligned}
S^{(1)}(T) &= S^{(1)}(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_1(T)} \\
&= S^{(1)}(0)e^{\frac{-\sigma\rho_0\sqrt{T}}{\sqrt{\frac{1}{2c}e^{2cT}-\frac{2}{c}e^{cT}+T+\frac{3}{2c}}}\left(\frac{c}{\xi}\ln\ {}_Tp_x^i+\frac{\lambda(0)}{\xi}\left(e^{cT}-1\right)\right)} \\
&\times e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{1-\rho_0^2}\sqrt{T}Z}, \\
&= \widetilde{S}^{(1)}(0)e^{\left(r-\frac{1}{2}\sigma^2\left(1-\rho_0^2\right)\right)T+\sigma\sqrt{1-\rho_0^2}\sqrt{T}Z},
\end{aligned}$$

with

$$\widetilde{S}^{(1)}(0) = S^{(1)}(0)e^{\frac{-\sigma\rho_0\sqrt{T}}{\sqrt{\frac{1}{2c}e^{2cT}-\frac{2}{c}e^{cT}+T+\frac{3}{2c}}}\left(\frac{c}{\xi}\ln\ {}_Tp_x^i+\frac{\lambda(0)}{\xi}\left(e^{cT}-1\right)\right)}e^{-\frac{1}{2}\sigma^2\rho_0^2T}.$$

Finally, we find that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}L_{x+T}\times\max\left(S^{(1)}(T),K\right)\Big|\, e^{-\int_0^T\lambda(s)ds} = {}_Tp_x^i\right] \\
&= l_x\ {}_Tp_x^i\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}K + e^{-rT}\max\left(S^{(1)}(T)-K,0\right)\Big|\, e^{-\int_0^T\lambda(s)ds} = {}_Tp_x^i\right] \\
&= l_x\ {}_Tp_x^i\left(\widetilde{S}^{(1)}(0)N(d_1) + e^{-rT}K\left(1-N(d_2)\right)\right),
\end{aligned}$$

which ends the proof. ∎

Chapter 6

Outlook

During the past decades, major changes have taken place in the way risk is assessed and managed in the insurance industry. One of the most critical risks to which insurers writing long-term business are exposed is the risk of a mismatch between assets and liabilities. Therefore, it is essential to determine a valuation of insurance liabilities which is consistent to the financial market for the traded risks on the one hand and to actuarial judgement for the non-traded risks on the other hand.

As pointed out in Albrecher et al. (2018), determining such valuation framework is challenging in practice for many practical reasons: long maturity of insurance liabilities, incompleteness of the insurance market, negative interest rates, etc. This valuation issue leads to a variety of interesting academic and practical questions. In all modesty, this thesis studied some valuation frameworks which can contribute to solve these valuation problems.

Hereafter, we provide a brief overview of the general results of this dissertation. Moreover, we discuss some possibilities for future research.

6.1 General Contribution

This dissertation introduced and studied different valuation frameworks for the determination of a fair valuation of insurance liabilities. We started in Chapter 2 by defining a fair valuation as a valuation which is market-consistent (marked-to-market for hedgeable claims) and actuarial (marked-to-model for claims independent of financial market evolutions). We also introduced two families of valuations: hedge-based valuations and two-step valuations, and we showed that fair, hedge-based and two-step valuations are identical classes of valuations. From a practical perspective, this means that if one valuates insurance liabilities either via a hedge-based approach or a two-step approach, this will lead to a fair valuation. Moreover, we obtained a one-to-one hedging characterization of fair valuations.

Chapter 3 studied more in detail a specific case of the hedge-based valuations in a multi-period setting, namely the mean-variance hedge-based (MVHB) valuations where in the first step, the hedge is determined by mean-variance hedging. We obtained some characterization results and explicit decomposition formulas for product claims (e.g. unit-linked contracts).

In Chapter 4, we generalized our fair valuation framework to incorporate time-consistency in a full dynamic setting. We provided a complete hedging characterization for fair dynamic valuations (i.e. valuations which are market-consistent, actuarial and time-consistent). Moreover, we presented a backward iterations scheme based on Least-Squares Monte Carlo in order to determine explicitly the fair dynamic valuation.

Finally, Chapter 5 investigated two-step valuation operators. We studied the two-step financial valuations as considered in Chapter 2 and two-step actuarial valuations which essentially consist of reversing the valuation order. We provided an axiomatic characterization for the two-step operators, compared the two types of two-step operators and illustrated the two-step actuarial valuations on a portfolio of equity-linked life insurance.

6.2 Future Research

There are still many open questions related to the valuation of insurance liabilities. Hereafter, we raise some questions which are directly related to the analysis of this thesis:

- **Independence assumption between financial and actuarial risks:**
  The actuarial condition of a fair valuation is based on claims that are independent of financial market evolutions, for in-

stance a survival index if we assume that mortality is independent of the financial market. If we consider dependent financial and actuarial risks, the actuarial condition becomes restricted and valuation formulas are more difficult to obtain. Future research is needed to study the impact of the dependence between financial and actuarial risks on the fair valuation and hedging of insurance liabilities (see e.g. Deelstra et al. (2016), Zhao and Mamon (2018)).

- **The time-consistent property is too strong:**
  The time-consistent property used in Chapter 4 was the most used definition of time-consistency. However, different authors argue that this definition is too strong and proposed alternative weaker notions of time-consistency, see e.g. Roorda and Schumacher (2007) and Chapter 2 in Kriele and Wolf (2014).

- **Machine learning for pricing and hedging life insurance liabilities:**
  We covered the valuation and hedging problem from a probabilistic rather than a statistical approach. However, machine learning techniques can be useful to solve these problems, see e.g. Buehler et al. (2019) and Han et al. (2018). These methods can help to reduce the computational burden of standard Monte Carlo methods.

- **Non-linear financial pricing rule:**
  Standard no-arbitrage financial pricing is based on the risk-neutral expectation that is linear. In practice, the presence of bid-ask spread, transaction costs, model ambiguity and limited liquidity implies that one needs to be careful with a linear pricing rule. Further research on the impact of market imperfection on the fair valuation is necessary.

- **Fair valuation in non-life insurance:**
  We focused on the fair valuation in life insurance. There remain several open problems on the determination of a market-consistent valuation in non-life insurance which need to be investigated further, see Wüthrich (2016).

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