This chapter appeared in: B. Van Kerkhove, K. François, S. Ducheyne & P. Allo (eds.) *Laat ons niet ernstig blijven: Huldeboek voor Jean Paul Van Bendegem*, Academia Press (2018) pp. 321–326.

On infinitesimals and the world below

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This short paper comments on Van Bendegem's (1987) proposal for a discrete and finitary structure underlying Euclidean geometry. The first two sections contrast his guiding idea with a Leibnizean heuristic and liken it to renormalization group methods in physics. The last two sections show how infinitesimals, from non-standard models of the reals, may help to flesh out the proposal and offer a viewpoint that may make this approach more acceptable to a finitist.

Jean Paul Van Bendegem wrote, in a paper in which he considers whether geometric space might be discrete (1987, p. 302): ""Down below" the world may appear to be quite different." First, I will contrast this statement with Leibniz's law of continuity, which says the opposite. Then, I will liken it to the renormalization group of theoretical physics. Finally, I will try to argue that infinitesimals (in the sense of Robinson's hyperreals) are both useful to model discrete space and congenial to the idea that the microworld is different from the macroworld, although they clearly require methods that may not be acceptable to a finitist.

1. As above, so below

First, let me briefly introduce Leibniz's law of continuity: this heuristic principle played an important role in Leibniz's formulation of the calculus. It says that whatever succeeds for the finite, also succeeds for the infinite, and vice versa. This clearly cannot hold in full generality, but skillful seventeenth century mathematicians certainly managed to use this principle to develop new results, while circumventing any inconsistencies. In the 1960s, Abraham Robinson succeeded in formalizing Leibniz's approach to the calculus in a way that is rigorous according to today's standards. He developed non-standard analysis (Robinson 1961, 1966), in which the Transfer principle plays the role of Leibniz's heuristic law of continuity. The Transfer principle says that every first-order logic statement that holds for the standard real numbers can be transferred to a corresponding statement about a non-standard model of a real closed fields, the elements of which are called 'hyperreal numbers'.

The basic idea is: "As above, so below". This idea – mystical in origin – sits well with various mathematical structures, which are self-similar at every scale, such as the real numbers and some fractals (e.g., the Koch curve). It also harmonizes with the hope of a truly rational world: if the same principles hold at every level, we don't need to explore all of it in order to understand everything. If, in contrast, every level has its own rules, we cannot hope to develop a theory of everything, since at any point in time there is more that we haven't explored than what we have.

2. Small-distance effects may weaken at longer scales

Now, let us read the statement about the world "down below" appearing to be different in the light of the history of physics: it seems that history teaches us that, indeed, at different scales different laws may dominate. Not all fractals are fully self-similar at different length scales; in fact, most are not. This is related to renormalization and rescaling methods, which play an important role in modern physics, for instance in field theories. As Butterfield (2014, §5.2.1) remarked: "[The scaling equation shows that] we can be agnostic about whether – indeed, we can even deny that – our theory describes physics in a continuous spacetime.” Notably, renormalization group methods are also used to get rid of (or at least to tame) infinities, so a likening to Van Bendegem’s idea seems apt.

As a result, the sentence quoted at the beginning of this article may strike us as an utterly sensible remark to make – for a physicist, at least. Since it was written by a philosopher of mathematics about geometry, however, it shows a creative mind at work.

3. Discrete models with actual infinitesimals

Although I contrasted Van Bendegem’s idea with Leibniz’s law of continuity, there is an interesting connection between the two. As mentioned, Leibniz’s version of the calculus can be formalized with Robinson’s hyperreals, which – unlike the standard reals – contain actual infinitesimals: those can be used to discretize continuous models.

To avoid misunderstandings, let me first point out that infinitesimals are not atoms or indivisibles: they are infinitely divisible, just like any standard real number is. The hyperreals form an alternative formalization of the continuum – in particular, a non-Archimedean one. A finitist might lose interest at this point: in a later paper, indeed, Van Bendegem (2011) comments: “From the viewpoint of discreteness, this is a move in the opposite direction as we are adding elements to a structure that is already uncountable.” But this dismissal may have been too quick, for the plan is not to use the whole structure of the hyperreals.

When we build a standard discrete set, we first pick a non-zero real number as the unit. Likewise, by picking a fixed infinitesimal, a hyperfinite grid can be built that is finer than any continuous line or area, yet discrete. And because hyperfinite grids are internal objects (immediately available by Transfer from a suitable family of finite grids), they are as simple to use as standard finite grids.

Let’s illustrate what happens to the proposal of Van Bendegem (1987) if we take the tiles to have a size that can be expressed by a hyperreal infinitesimal. As long as there are only finitely many infinitesimals in the width of a line, the total width is infinitesimal. Hence, the standard part (that is the unique nearest standard real value) of the width is zero. For a line to have a standard non-zero length, it needs to consist of an infinite hypernatural number of infinitesimal tiles.

This approach would solve one shortcoming of the proposal by Van Bendegem (1987). He assumes that (p. 297): “the size of the squares is small compared to the macroscopical width of the lines.” ‘Small’ is a vague predicate. Possibly in an attempt to remedy this, he adds: “the elementary units must be very small.” This does not help, of course, since ‘very small’ is vague too. This suggests that the notion of infinitesimals is welcome here: it gives a precise meaning to the term ‘very small’, though not one that can be picked out by a first-order statement. Moreover, there is no largest infinitesimal, so like a truly vague notion, lack of a sharp border remains.

Infinitesimals are reciprocal to infinite numbers. (We already mentioned the need of an infinite collection of tiles to form a line of macroscopic dimensions.) And the construction of the hyperreals certainly requires methods – such as free ultrafilters – that are not available to a finitist.

4. Making infinitesimals palatable to a finitist

To my non-finitist eyes, it seems that the notion of infinitesimals is just what is needed here. So, can we make this solution more appealing to a finitist? I believe the answer is ‘yes’.

First of all, Hrbáček has developed an approach to non-standard analysis that relativizes the standardness predicate, so that there can be many levels of relatively infinitesimal or 'ultrasmall' structures below the first tiles. They can be used to build microworlds that are all different, too. This should be appealing to anyone who is open to the microworld being different from the macroworld.

Secondly, we should assess carefully just what the grounds for rejecting infinitesimals are. They have been unpopular for most of the history of mathematics: see Alexander (2014) for an account of the seventeenth-century attempts by Jesuits and others to ban them and Wenmackers (forthcoming) for a brief historical overview. So, the possible reasons for rejecting them certainly go beyond finitism.

Van Bendegem (2011) briefly considered infinitesimals in the context of a discussion on discrete time. He mentions that one problem with infinitesimals is that they appear in the end result. Berkeley (1734) famously criticized the way they were used in the early calculus: one can divide by infinitesimals (so they are clearly non-zero), but at the end of a computation they are crossed out (as though they were zero). It has long been believed that they were used inconsistently or that they constituted an inherently inconsistent concept: an assessment that makes it rather mysterious how they could be used to develop the calculus in the first place. Leibniz's transcendental law of homogeneity allowed him to round off all infinitesimals at the end of a computation, much in the same way as physicists round off their values up to significant digits only in the last step. As a result, not all contemporary scholars agree that there ever was a "fundamental problem of infinitesimal reasoning" (as Van Bendegem, 2011 calls it): some authors maintain that it has merely been misunderstood (Katz & Sherry, 2012). In any case, within non-standard analysis, this heuristic principle can be replaced by taking the aforementioned standard part function: by applying it to both sides of the equation both informality and inconsistency are avoided.

Van Bendegem (2011) seems to agree with this. So, what is his reason for rejecting the hyperreal approach, then? I already cited it in §3: adding more numbers to the standard reals seems a move in the wrong direction. Let me also briefly repeat the reply: it is correct that the hyperreals form an alternative continuum, but they do contain interesting, discrete subsets. These hyperfinite grids contain a hypernatural number of grid points. The internal cardinality of such a set is hyperfinite, which is the non-standard analogue of finite. Admittedly, the external cardinality is uncountable, but that is not 'visible' within the first-order model itself.

This brings me to the third and final move: infinite hypernatural numbers are similar to indefinitely large numbers and infinitesimals are akin to indefinitely small numbers. Indefinitely large and small numbers can already be found in the standard reals, and plausibly already in finite mathematics. This comes close to Lavine's (1995) ideas: suppose we take a set $\{1, \ldots, N\}$ with N a natural number. As long as we don't specify its value, N might be an indefinitely large number. If we are working with a non-standard model, the question becomes: how would we know that N is *not* an infinite number? These ideas can be modelled by Hrbáček's approach: N might be ultralarge compared to the numbers that are standard in the context at hand. I have written on the formal analogy between indefinitely large and infinite numbers and the relation to vagueness in the context of probability theory (Wenmackers, 2013). Also see Hamkins' (2016) classification of three different perspectives on non-standard methods.

This shows that "from the inside" the world may appear to be quite different. And it is my hope that such a perspective might be appealing to Jean Paul Van Bendegem.

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