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Do Infinitesimal Probabilities Neutralize the Infinite Utility in Pascal's Wager?

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> Of these two infinites of science, that of greatness is much more obvious, … the infinitely small is much harder to see. [Pascal, 1966 [1670], L199/S230]

In the "Infinity – nothing" passage of *Pensées* (1966 [1670], L418/S680), Pascal considers wagering for or against the existence of God, taking into account both probability ("chance") and utility ("happiness," "gain"). Jordan (2006) reconstructs the passage as presenting four related arguments: the first is based on weak dominance, the second is based on maximizing expected utility assuming the same probability for or against the existence of God, the third allows these probabilities to be different, and the fourth is based on strong dominance. Jordan calls the third part, which is the most discussed in contemporary philosophy, the "Canonical Wager."

It is only in the context of this Canonical Wager (henceforth simply referred to as "the Wager") that bringing up infinitesimal probabilities is relevant. Pascal explicitly excluded this possibility, assuming that there is "one chance of winning against a finite number of chances of losing," so that "there are not infinite chances of losing against that of winning" (1966 [1670], L418/S680). In this chapter, we investigate whether Pascal was right in excluding infinitesimal probabilities. In other words: can infinitesimal probabilities

[1] I am grateful to Paul Bartha and Lawrence Pasternack for very helpful feedback on this chapter.

be used to neutralize the infinite utility in the Wager, blocking the conclusion stating that it's prudent to wager for the existence of God? We will study this using a formal framework that enables us to represent both infinitesimal probabilities and infinite utilities and to combine them algebraically.

1 Infinities in the Wager

For our purposes, we can build on Hájek (2003)'s "perfectly standard interpretation" of the Wager passage in the *Pensées*, which is "cast in the anachronistic terminology of modern Bayesian decision theory (and that casting too is standard)" (p. 52).

1.1 Decision Matrix for the Wager

Hájek (2003, pp. 27–28) takes the argument to have three premises. Slightly rewriting:

(1) Rationality requires the agent to assign a positive, non-infinitesimal probability to God's existence, p.

(2) The decision matrix is as laid out in Table 15.1 with S the infinite utility associated with salvation and f_1, f_2, and f_3 finite utility values.

Table 15.1
Pascal's Wager

	Probabilities:	***p***	**1 – *p***
		Possible state 1:	*Possible state 2:*
		God exists	God does not exist
Action 1:	Wager for God	S	f_2
Action 2:	Wager against God	f_1	f_3

(3) Rationality requires the agent to opt for the action that has the highest expected utility (when there is one).

From the decision matrix, it follows (Hájek, 2003, p. 30) that the expected utility associated with wagering for God's existence is $S \times p + f_2 \times (1 - p)$, which is infinite given that S is. By comparison, the expected utility of wagering for God's non-existence is $f_1 \times p + f_3 \times (1 - p)$, a finite value. Maximizing the expected utility, this leads to the conclusion that it is rational to wager for God's existence.

The utilities in the decision matrix are based on Pascal's description in L418/S680 that "there is an infinity of an infinitely happy life to be won" (if you wager for the existence of God and this turns out to be correct), whereas "what you are staking is finite." To decide the best wager, we have to compare a finite probability value times an infinite utility value to a finite probability value times a finite utility value. Pascal writes (L418/S680): "The finite is annihilated in the presence of the infinite and becomes pure nothingness."

Hájek (2003)'s decision matrix leaves the finite utility values unspecified, which is consistent with L418/S680 as well as another fragment, in which Pascal writes (1966 [1670], L199/S230): "In the perspective of these infinites, all finites are equal and I see no reason to settle our imagination on one rather than another." Although this is written in a different context (pertaining to the size of the human body as compared to the whole of the universe), this may be applied to the utility values in the wager argument as well, making the argument very robust (f_1, f_2, and f_3 can be any real numbers). As long as neither probability value (for or against God's existence) is zero or some non-zero infinitesimal, the contrast between an infinite product and a finite product persists either way. So, also in this sense, the argument is very robust (p can be any real number in the open interval (0,1), i.e., $0 < p < 1$).

Since the Wager mentions infinite utility (a feature shared by all versions of the argument), that raises the question how to represent it formally. There are multiple options for doing so. In addition, we want to investigate the effect of allowing infinitesimal probabilities, so the formal model should allow us to represent infinitesimals as well. And since this argument involves expected utility, we should adopt a formal system that allows us to algebraically combine infinitesimal probabilities and infinite utilities. Otherwise the product and addition necessary for computing the expected utility associated with wagering for God (i.e., $S \times p + f_2 \times (1 - p)$, where we will allow p to be an infinitesimal) may turn out to be undefined.

Before looking into this, we first define infinite and infinitesimal numbers and consider the Wager in the context of real analysis.

1.2 Infinitely Large and Infinitely Small Numbers

In this chapter, we use the following definitions for infinite numbers and infinitesimals.

Infinite numbers are numbers of which the absolute value is larger than n for any natural number n. Within the fields of rational and real numbers, there are no infinite numbers.

Infinitesimals are numbers of which the absolute value is smaller than $1/n$ for any natural number n. Within the fields of rational or real numbers, the only infinitesimal is zero; such fields are called Archimedean. Fields of numbers that do contain non-zero infinitesimals are called non-Archimedean: we will see examples below. Sometimes the word infinitesimal is reserved for non-zero numbers, but here we will stick to the definition just given, which includes zero.

1.3 Infinity in Real Analysis

Hájek (2003)'s formalization of the Wager uses the lemniscate symbol for infinity, ∞, rather than S. Although it isn't meant to refer to a particular mathematical theory, the symbol is most commonly used in real analysis. Since both the symbol and the theory are well known, let's briefly elaborate. The values of real-valued sequences and functions that diverge become larger (or smaller) than any given finite value: we say that the limit "at infinity" is plus (or minus) infinity. This is represented in the affinely extended real number system by two symbols added to the set of reals, $\mathbb{R} \cup \{-\infty, +\infty\}$. This allows us to extend some arithmetic operations on the reals to sums and products involving infinite and finite limits (making $+\infty \times p + f_2 \times (1 - p) = +\infty$ true for non-zero p and any f_2). Yet, some algebraic combinations cannot be defined in any consistent way, such as the indeterminate form $+\infty - +\infty$. So, unlike the real numbers, the extended reals do not form a field and these lemniscates should be recognized for what they are: shorthand notations that do not signify real numbers. By definition, divergent sequences and functions do not have a limit in the real numbers.

Pascal's L418/S680 description of the utility of eternal bliss as "an infinity of infinitely happy life" suggests that the infinite utility is a product of two infinities (infinite happiness per time and infinite duration) and also in L199/S230 Pascal discusses the “double infinity” of nature and science. To formalize this part of the argument, we have to take into account that standard measure theory doesn’t deal with infinite quantities. It does allow us to state that a function diverges or, in other words, has a limit of infinity. It also allows us to express the rate of divergence, by considering the derivative of the function. A product of diverging functions diverges faster than its factors: the product function approaches infinity faster. Although the original argument is very robust, in the sense that any rate of divergence will do, it will be crucial to return to this issue once we consider infinitesimal probabilities.

If we look outside the field of real numbers of the standard calculus, there are multiple ways to represent infinitesimals and infinite numbers formally, although not all of them have been used explicitly to represent probabilities and utilities. We elaborate on this in the next section.

2 Harmonious Number Systems

In the context of a discussion on Pascal’s Wager, Oppy (1991, p. 163) considered the epistemic possibility “that the probability that God exists is infinitesimal,” in which case “the calculation of the expected return of a bet on [the existence of] God is no longer as straightforward as the initial argument suggested.” Oppy (1991, p. 163) writes that “it is incorrect to suppose that ‘non-zero’ and ‘finite’ are coextensive; for it is epistemically possible that the probability that God exists is infinitesimal.” Thereby, he raises the possibility that the agent’s degree of belief might be infinitesimal rather than some non-zero real number. (Unlike Oppy, I will not use “finite” as a contrary to zero and/or infinitesimal numbers.)

Following up on this, Hájek (2003, p. 38) wrote: “Once infinitesimal probabilities are allowed, the reformulated argument no longer goes through automatically: the infinitesimal probability can ‘cancel’ the infinite utility so as to yield a finite expectation for wagering for God; and this may be exceeded by the expectation of wagering against God.”

In this section, we investigate a necessary condition that the formal system needs to satisfy in order to allow for this “cancelling” of infinite quantities by infinitesimals. Two systems that satisfy it will be briefly introduced and some reasons for preferring one over the other will be evaluated.

2.1 “Cancelling” Requires a Harmonious Number System

Some well-known theories that include infinity do not include non-zero infinitesimals. Cantor’s theory of cardinality, for instance, contains an infinite hierarchy of transfinite cardinalities, but not a single non-zero infinitesimal. Likewise for Cantor’s theory of ordinal numbers. In fact, Cantor was strongly opposed to the idea of infinitesimals. He tried to prove their inconsistency (Cantor, 1966 [1932], pp. 407–9), but Zermelo pointed out that Cantor’s purported proof relies on the multiplication of an infinitesimal by a transfinite ordinal number, which is undefined (Cantor, 1966 [1932], pp. 439). It seems more accurate to say that infinitesimals are simply incommensurable with transfinite cardinal and ordinal numbers (Benci, Horsten, and Wenmackers forthcoming, section 4.1.1). Also Sobel (1996, endnote 8) observed that the multiplicatory inverse of Cantorian cardinalities are undefined and hence cannot be used to represent infinitesimals.

So, some ways of formalizing Pascal's Wager with infinitesimal probabilities simply will not work. If we use a theory that doesn't allow us to multiply transfinite numbers by non-zero infinitesimals, then we cannot investigate the question whether infinitesimal probabilities block the Wager's conclusion. Other approaches are more promising: there are number systems in which each infinite number is the multiplicatory inverse of a particular infinitesimal number, and vice versa. We will call number systems that have this property "harmonious." This property is suggested by the etymology of "infinitesimal": the word is formed in analogy to "decimal" (meaning tenth or one divided by ten), which suggests it is equal to one divided by infinity. For our purposes, we require this relation to be symmetric: each infinitesimal is equal to the inverse of an infinite number and vice versa. So, neither the infinite nor the infinitesimal numbers are conceptually prior to or privileged over the other in any way.

We now introduce two types of harmonious number systems. A first type of harmonious number system is given by non-standard models of rational or real closed fields, which are used in non-standard analysis. A second type of harmonious number system is given by Conway's surreal numbers. Both systems have also been discussed by Hájek (2003) in the context of the Wager, e.g., to tell apart various infinite utilities.

2.2 Robinson's Hyperrational and Hyperreal Numbers

A well-developed route for handling infinitesimal and infinite numbers is that of non-standard models of rational or real closed fields, due to Robinson (1966). The numbers in such non-Archimedean, ordered fields are known as hyperrational ($\in {}^*\mathbb{Q}$) and hyperreal numbers ($\in {}^*\mathbb{R}$). While I will not go into the technical details, the next subsection gives a sketch of the construction. If you decide to skip section 2.2.1, the main thing you need to know to follow the discussion in section 4 is this: all algebraic operations that are defined on the real numbers are also defined on the hyperreal numbers.

Non-standard analysis is a branch of mathematics geared towards proving results in the standard calculus via the non-standard domain: this is the main, but not the only application. Rounding off the infinitesimal part of a finite hyperreal number k yields a standard real and is called taking the standard part, written st(k). Authors like Edward Nelson and Vieri Benci have defended taking infinitesimals more seriously; in particular, by investigating hyperreal-valued probability functions in their own right (Nelson, 1987; Benci, Horsten, and Wenmackers, 2013).

2.2.1 Harmony of Non-Standard Models

I will briefly sketch how to construct a non-standard model of the real numbers, to illustrate how the aforementioned property of harmony comes into play. One way of constructing the hyperrational or hyperreal numbers is by taking the ultrapower of the rationals or reals with respect to a free ultrafilter or, equivalently, by taking the quotient ring of the rationals or reals under a maximal ideal. To explain this, let's start with Peano arithmetic, which gives the properties of natural numbers in first-order logic. Starting from a model of Peano arithmetic (which includes an infinite alphabet, with a name for each natural number), one builds a non-standard model by introducing a new symbol, for instance N, declaring it larger than any number already in the model. The proof relies on the Compactness theorem; for details see Boolos et al. (2007, pp. 302–18). N is the first example of an infinite number in the non-standard model (by the definition in section 1.2). Since the first-order properties are required to hold in this non-standard model, like any standard natural number, N will have a successor, which itself has a successor, and so on. So, by introducing N, we allow an infinite number of infinite numbers. All the numbers in a non-standard model of Peano arithmetic (finite and infinite ones) are called hypernatural numbers ($\in {}^*\mathbb{N}$).

Building non-standard models of rational or real closed fields can be done in similar fashion: starting from a model of the first-order properties of the rational or real field, one adds a new symbol. In this case we have two options for how to proceed, but the results will turn out to be equivalent. Either we add a symbol R for a new number larger than any rational or real number, or we add a symbol ε for a number larger than zero but smaller than 1/r for any positive rational or real number r. Since the non-standard model has to be closed under the same first-order operations as the standard model, it has to be closed under taking the reciprocal, so adding the infinite number R ensures that the non-zero infinitesimal $1/R$ is in the model as well, whereas adding the non-zero infinitesimal ε ensures that the infinite number 1/ε is in the non-standard model. This illustrates the harmony of the number system.

It is possible to embed the standard rationals and reals in a non-standard model in a canonical way. However, there is not just a single or even a canonical non-standard model of the real numbers. There are infinitely many fields of hyperreal numbers, with different properties. It is even possible to iterate the procedure of building a non-standard model starting from a non-standard model: simply add yet a new symbol and declare it bigger than any previous one. This yields hyperhyperreal numbers (∈**ℝ).

2.2.2 Hyperreals and the Wager

Herzberg (2011) uses hyperreal numbers to represent utilities in the Wager, but he does not deal with infinitesimal probabilities in the main text. In a footnote, he does remark (p. 71): "Zero probabilities would make the argument invalid straightaway, and infinitesimal probabilities would require a sufficiently high utility of salvation in order to preserve the validity of the argument, cf. Oppy (1991) and Hájek (2003)." Oppy's and Hájek's ideas of cancelling is indeed what a field of hyperrationals or hyperreals allows us to formalize.

This was discussed by Sobel (1996): though the main text deals with representing the infinite utility in Pascal's Wager by Cantorian infinities, he included an appendix on Robinsonian hyperreals. In the endnotes, Sobel discusses the interplay between infinite utility and infinitesimal probabilities, when both are taken to be hyperreal numbers. In endnote 6, Sobel considers using an infinite hyperreal number to represent the utility of eternal bliss in a "hyperreal decision theory." He says this is relevant only if one is "supposed to prefer only every non-infinitesimal chance of eternal bliss even to a certainty of every worldly loss" (p. 55). According to Sobel, it makes no sense to apply hyperreals, however, if one is "supposed to prefer absolutely every chance of eternal bliss, even to a certainty of every worldly loss" (p. 55). This observation is related to the fact that for a given infinite hyperreal value, S (used here to represent the infinite utility of salvation), there is exactly one infinitesimal that is its inverse, $1/S$, but for any infinitesimal hyperreal there at least continuum many smaller ones (and similarly for larger ones). We will return to this is section 4.1, to see for which values of $p(S)$ expected utility does not favor belief over disbelief and the Wager fails.

2.3 Conway's Surreal Numbers

A decade after Robinson, Conway developed a system aimed to represent all numbers, now known as the surreal numbers (Knuth, 1974). The construction of the proper class of surreal numbers is done inductively, in stages. The first infinite surreal number is constructed at the same stage as the first non-zero infinitesimal and they are each other's multiplicatory inverse. Hence, harmony is built into the number system.

In a reconstruction of Pascal's Wager argument, Hájek (2003, p. 35) favors the surreal numbers over hyperreals, citing their ingenuity and user-friendliness. Also Easwaran (2014) favors Conway's system.

To the best of my knowledge, however, there is not yet a well-developed probability theory or decision theory based on the surreal numbers.[2] Hence my pragmatic choice for the hyperreals. Since there exists an isomorphism between the field of Conway numbers

[2] Shortly before this book was finalized, I discovered that Chen and Rubio (forthcoming) have developed such a theory and that they applied it to Pascal's Wager. The interested reader is encouraged to read their excellent article.

and a field of hyperreals that is maximal in a particular sense, this choice need not be an exclusive one.

2.4 Towards a Harmonious Decision Theory

Combining standard probability theory (that uses the real numbers, which do not include any non-zero infinitesimals) with infinite utilities is notorious for producing puzzling and even paradoxical results, of which the St. Petersburg paradox is the most famous example.

Using a harmonious number system seems to be a necessary but not a sufficient condition to formalize Pascal's Wager with infinitesimal probabilities: we need to apply it in the context of a theory that is capable of representing probabilities as well as utilities. I take this as a reason to prefer the non-standard approach – at least at this point in time. The situation may be assessed differently once a decision theoretic framework is built around the Conway numbers, or maybe a different system altogether.

There are several ways to base a probability theory on non-standard fields of numbers. Some approaches are geared towards obtaining standard probability functions (without infinitesimals), such as Loeb's (1975), but others embrace infinitesimal probabilities and other values in the hyperreal unit interval. An important example is due to Nelson (1987); more recently, Benci, Horsten, and Wenmackers (2013). There have been counterarguments against the notion of infinitesimal probabilities. Criticisms include worries about the non-uniqueness of the underlying hyperreal fields, the observation that infinitesimal chances are incompatible with symmetry considerations (such as temporal translation and rotation), and an argument that suggests physical agents cannot hold hyperreal credences (Easwaran, 2014). We will not engage with these criticisms here; for an overview and replies, see Benci, Horsten, and Wenmackers (forthcoming).

Non-Archimedean probabilities do not mix well with real-valued utilities. Hence, to deal adequately with infinitesimal probabilities in the context of decision theory, a non-Archimedean utility theory is needed, such as the one developed by Pivato (2014): a recent theory that both represents probabilities and utilities based on such non-standard fields of numbers. This can be used to model choice under uncertainty, as is the case in Pascal's Wager.

We will not present the theory in detail here, but the crucial part is that, in order to ensure harmony, the construction of a non-Archimedean field of numbers has to be one and the same to represent the infinitesimal probabilities as well as the infinite numbers. In what follows, we assume this theory, without spelling it out here.

On the other hand, we could also wonder if it is necessary to model infinite utilities and infinitesimal probabilities at all. Bartha (2007) focuses on relative utilities as a way to represent preferences without having to represent infinite utilities explicitly. Advantages are that he doesn't have to "pre-suppose a definite number of dimensions or to attach a definite value to the utility of salvation" (p. 29). He merely raises the option of infinitesimal probabilities at the end, suggesting that they may be helpful to deal with more complex problems. Indeed, there are non-Archimedean orders that do not presuppose a field of numbers, which suffice for a theory of qualitative probability (Narens, 1980), but for computing expected utilities some field of numbers is required.

3 But Is It Sufficiently Pascalian?

Before we continue to bring these recent formal systems to bear on the case of infinitesimal probabilities in a Wager argument, we briefly pause to reflect on the question of whether these approaches are sufficiently Pascalian.

The formal systems described in sections 2.2 and 2.3 were not developed in Pascal's time. What is more, one should keep in mind that the same holds for standard decision theory. One of its components, contemporary probability theory, has its roots in the correspondence between Pascal and Fermat, but also encompasses many mathematical notions not available in Pascal's own time, such as functions and the field of real numbers, of which the unit interval is now used to represent probability values. Due to the axiomatization by Kolmogorov (1933), probability theory has since been incorporated into standard measure theory: the theory that uses real numbers to assign values to lengths, areas, and volumes is also used to express probabilities. Hence, our standard decision theory built on this framework can only be applied to Pascal's Wager anachronistically.

3.1 Pascal on Infinities and Infinitesimals

Some seminal ideas concerning infinity and infinitesimals were definitely known to Pascal.

3.1.1 Pascalian Harmony

In light of what I called the harmony of some number systems, it is interesting that Pascal (1966 [1670], L199/S230) clearly saw a symmetry between the infinitely large and the infinitely small, writing: “These extremes touch and join by going in opposite directions, and they meet in God and God alone.” He regarded human bodies as placed between these two extremes, “between these two abysses of infinity and nothingness,” neither of which we can perceive or truly grasp. For instance, our finite size is nothing compared to the whole of the infinite universe, but on the other hand he suggested that small things or animal bodies aren’t simply infinitely divisible, but infinitely structured into the small (suggesting a fractal structure *avant-la-lettre*).

3.1.2 Pascalian Infinity

A generation before Pascal, Galileo (1954 [1638]) had written on a paradox that arises if we try to compare infinite sizes (in particular, the collection of whole numbers and that of perfect squares; see also Mancosu, 2009). Galileo had concluded from this that it is meaningless to determine (in-)equality of sizes of infinite collections. Yet, in his *Pensées*, Pascal (1966 [1670], L199/S230) did write of infinites (in plural) and said that “mathematics, for instance, has an infinity of infinities of propositions to expound,” which may suggest that he did allow distinctions among infinite sizes.

Numerosity theory (Benci and Di Nasso, 2003; see also Mancosu, 2009) is a particular motivation and axiomatic approach for the hypernatural numbers, which is closely related to the non-Archimedean probability (NAP) theory mentioned in section 2.4. According to numerosity theory, on which NAP is based, there is a numerosity of natural numbers, and this infinite number – usually called α – is either even or odd. Adding a unit to α, or any other numerosity, yields another value. Although NAP theory can be derived from axioms and need not be derived from numerosity theory or any other form of non-standard analysis, it is this precisely this Euclidean part–whole property that powers the regularity of the NAP-function.

Textual evidence of the passage just before the Wager argument suggests Pascal might have objected to such a theory (1966 [1760], L418/S680): "[W]e know that it is untrue that numbers are finite. Thus it is true that there is an infinite number, but we do not know what it is. It is untrue that it is even, untrue that it is odd, for by adding a unit it does not change its nature." This suggests that Pascal had a Galilean concept of infinity, or possibly proto-Cantorian (given the cited part of L199/S230), both of which are incompatible with infinite hypernatural (and by extension, hyperreal) numbers. Pascal (1966 [1760], L418/S680) also wrote: "Unity added to infinity does not increase it at all, any more than a foot added to an infinite measurement: the finite is annihilated in the presence of the infinite and becomes pure nothingness." Although this goes in the same direction, it can be understood as a merely comparative claim, unlike the previously cited passage.

Hájek (2003) worries that assigning an infinite hyperreal to the value of salvation is insufficiently Pascalian, because salvation is supposed to be the best thing possible, whereas for each infinite hypernatural number, there are infinitely many larger ones. But it is not clear that the existence of larger numbers on the utility scale entails the existence of states or rewards corresponding to them. Rota (2017, endnote 4) also comments that this need not be a decisive reason to reject the use of hyperreals in a reconstruction. In addition, he suggests that Hájek's underlying worry may be that there is arbitrariness "in selecting a particular [infinite] number to represent the value." Although Hájek wrote with the surreal numbers in mind, the same worry may be flagged for hyperreal numbers. This interpretation seems plausible to me, since similar worries have been raised in arguments geared against infinitesimal probabilities: there the worry is related to assigning a particular infinite number to represent the size of an infinite sample space (for instance by numerosities; Benci and Di Nasso, 2003), rather than to an infinite utility as is the case for the Wager. Rota (2017) does not find the objection a strong one, however, since any choice for the infinite value yields a valid argument.

3.1.3 Pascalian Infinitesimal Probabilities

All formalizations that include infinitesimal probabilities are more recent than the foundations of standard probability theory. Nevertheless, the notion of infinitesimals in general, and of infinitesimal probabilities in particular, is certainly not alien to Pascal's time. The notion of infinitesimals was prominent in the European mathematics of the time. Bonaventura Cavalieri had rediscovered forgotten ideas of Archimedes: areas and volumes can be computed as sums of infinitesimally thin cross-sections. This idea was crucial for the development of the calculus by Newton and, even more explicitly so, by Leibniz. This use of infinitesimals was famously criticized by Berkeley, who called them "Ghosts of departed Quantities" (Berkeley, 1734; section XXXV, p. 59). But it is undeniable that they were at the roots of our calculus, which has been called 'infinitesimal calculus' for a long time. It was only after the nineteenth-century formalization of the calculus in terms of real numbers and epsilon-delta definitions that the notion of infinitesimals was banished from mathematics. To this day, mathematicians continue to use the word "infinitesimal," but not in the sense defined in section 1.2. As a result, the notion of infinitesimal probabilities in the original sense became suspect as well – especially after the twentieth-century incorporation of probability theory in measure theory (itself firmly rooted in the calculus).

As the quote in the second paragraph of this chapter shows, Pascal explicitly mentions infinitesimal probabilities, albeit only to exclude this case from his argument. Interestingly, Hájek (2003) writes that Pascal "deserves considerable credit for apparently having a notion of infinitesimal probability years ahead of his time" (p. 39). In light of the history of the concept, which I just briefly sketched, however, I believe this should not be so surprising. Nevertheless, when we apply the very recent formalizations of the concept to Pascal's argument, we should be aware that we approach it very differently from anyone in Pascal's time.

See section 5.1 for another aspect that may be non-Pascalian about any reconstruction with infinitesimal probabilities.

3.2 Pascalian Probability and Cromwell's Rule

3.2.1 Epistemic versus Personal Probabilities

It may also be worthwhile to reflect on Pascal's interpretation of probabilities in general. It is (pre-)classical: probabilities are related to (lack of) knowledge, but they are epistemic, not personal probabilities. They are much closer to Laplacian chances (on which we can all agree), than to the contemporary (neo-)Bayesian interpretation of probabilities as credences: rational degrees of belief that depend on a person's knowledge, not all of which needs to be common knowledge. Regarding the existence or non-existence of God, Pascal remarks that neither possibility can be ruled out *a priori*. In the classical setting, both agents would have to agree on the (prior) probabilities.

In the contemporary Bayesian approach to probability, probability assignments are personal. In the objective Bayesian approach, priors have to be shared, whereas they may differ for subjective Bayesians. For all Bayesians, probabilities should be updated upon receiving new information (which may be different across agents). Based on different life experiences, someone may have become an atheist and someone else a theist. In a Bayesian formalization, this is reflected in these persons having different posterior probabilities. If we grant Pascal that there is no decisive knowledge to be gained either way, then we may assume that the posterior probability that the atheist assigns to "God exists" is not *zero*, neither is that of the theist *one*.

3.2.2 Cromwell's Rule

It is in this context that the requirement of "regularity" (Carnap, 1950b) or "strict coherence" (Shimony, 1955) is usually discussed, which requires us to assign non-zero (prior) probability to all elementary possibilities. (This requirement is violated in standard probability for uniform probability distributions over infinite sample spaces.) Avoiding the extreme probability values (zero and one) allows the agents to remain within reach of each other's arguments and of evidence contrary to their current highest degree of belief: they may be "converted" by Bayesian conditionalization. Pascal's Wager argument doesn't take this dynamical aspect in account, but for his argument to be effective, he too needs to rule out probability zero (the only standard infinitesimal) as well as other infinitesimals.

Hájek (2003, section 3.2) discussed regularity in the context of Pascal's Wager. In the literature on the foundations of statistics, Lindley (1991, p. 104) coined the term "Cromwell's rule" for essentially the same idea: probability unity should only be assigned to tautologies, and probability zero should be reserved for contradictions. The name refers to Oliver Cromwell, who in 1650 wrote a letter to the General Assembly of the Church of Scotland in which he urged: "I beseech you, in the bowels of Christ, think it possible that you may be mistaken." However, Cromwell doesn't seem to have applied this plea to himself: he did not consider the possibility that his own belief could be mistaken, and the suggestion was rejected. In the Wager argument, Pascal tries to start from a more open-minded position: although he is a theist, he considers the option favored by an atheist *a priori* possible too. In doing so, he puts himself in a vulnerable position, within reach of arguments from atheists, but of course, his goal is opposite: to make them consider and embrace theism.

We already quoted Hájek commenting that if we allow infinite utilities, we should also allow infinitesimal probabilities in the formalism. Oppy made a similar remark, that also touches upon regularity:

> Note, by the way, that it is no objection to observe that infinitesimals are somewhat dubious entities. The same point could be made in the language of measure theory – i.e., in the mathematical theory which is appropriate for dealing with probabilities in the case in which there are infinitely many options. The set of worlds in which the Christian God exists may have measure zero, and yet be non-empty. [1991, p. 168]

In other words, standard probabilities violate the requirement of regularity; infinitesimals provide a way to restore it (Benci, Horsten, and Wenmackers, 2013; albeit at the cost of other problems, see Benci, Horsten, and Wenmackers forthcoming).

4 Applying Hyperreal Decision Theory to the Wager

Recall from section 2.2.1 that all algebraic operations that are defined on the real numbers are also defined on the hyperreal numbers. One effect of this is that formulas involving infinite utility values and infinitesimal probabilities will look exactly the same as formulas that only involve finite values. Using the decision matrix of section 1.1, but allowing both probabilities and utilities to take values in some fixed non-Archimedean field of hyperreal numbers (p, f_1, f_2, f_3, $S \in {}^*\mathbb{R}$), the Wager is successful (as an argument for the existence of God) if

$$S \times p + f_2 \times (1 - p) > f_1 \times p + f_3 \times (1 - p)$$

and unsuccessful if

$$S \times p + f_2 \times (1 - p) \leq f_1 \times p + f_3 \times (1 - p).$$

Assuming the utilities are fixed, this can be rewritten as a condition on the value of p. The wager is successful iff (if and only if):

$$p > (f_3 - f_2) / (S - f_1 - f_2 + f_3). \qquad (1)$$

Now we can reformulate the main question of this chapter as follows: does it suffice to demand regularity in the probability assignments of the agent in order for the argument to keep being successful? As is known from the literature, the answer is: it depends.

For instance, against Rescher's (1985, pp. 16–17) claim that the Wager is favourable "as long as the chance of winning is nonzero," Sobel (1996, fn. 10) offered a counterexample, which we can now check against our success condition. Sobel's counterexample corresponds to setting $f_3 - f_2 = 1$ and $p = 1/(2(S - f_1))$. Substituting this in condition (1), the wager is successful iff $f_1 > S - 1$. This cannot hold, since we assume f_1 to be finite and S infinite. Hence, Sobel (1996) indeed provided a counterexample to Rescher's claim that regularity suffices.

The fraction in the right-hand side of inequality (1) is small in the order of $1/S$, so there is some infinitesimal threshold on p above which the Wager is successful. We will investigate this dependence.

4.1 Dependence of the Wager's Success on the Incredulity

In general, the larger the infinite utility S is compared to $1/p$, the easier it is for the Wager to be successful. We illustrate this by considering two special cases.

4.1.1 Probability Inversely Proportional to Utility of Salvation

Let us first consider a skeptical agent whose degree of belief is inversely proportional to the grandness of the promised salvation. In such a case, we have that the infinitesimal probability p is equal to f/S, with f some positive hyperreal number that is neither infinitesimal nor infinite, such that $p \times S$ is finite and non-infinitesimal. This case is of special interest, not because it is especially well motivated, but because (i) it encompasses the switch between successful and unsuccessful wagering and (ii) it seems not well motivated to rule it out.

Because of (i), the case where $p = f/S$ is very similar to a decision problem in which all utilities and probabilities are finite. In particular, the values of the three finite utilities in the decision matrix do become crucial for the conclusion based on utility maximalization. If someone could argue, based on the interpretation of p and S, that this case is irrelevant, we would only have to deal with cases for which it is clear whether the wager is successful or not, irrespective of the values of f_1, f_2, and f_3. Moreover, if we could rule out this case, we would probably not need hyperreal analysis at all (see also section 4.2). However, I see no obvious way of ruling it out, so let's proceed with the detailed analysis.

Substituting $p = f/S$ in inequality (1) and rewriting, we see that the Wager is successful iff

$$f_3 < f_2 - 1/(S/f - 1) \times f_1 + f/(1 - f/S).$$

Taking the standard part (cf. section 2.2), we obtain $\mathrm{st}(f_2) \geq \mathrm{st}(f_3) - \mathrm{st}(f)$ as a necessary (though not a sufficient) condition for a successful wager. In any case, the seemingly irrelevant matter of how to assign the finite utilities in the wager (cf. section 1.1), turns out to be crucial – even sensitive up to infinitesimal differences – for determining the successfulness of the argument once p is allowed to be an infinitesimal. This result is consistent with Oppy (1991), who already wrote that the expected utility will depend upon the exact values of the infinite utility and the infinitesimal probability.

Of course, whether any valuation of f_1, f_2, and f_3 that makes a crucial difference between successful and unsuccessful wagering makes sense given the meaning of the utilities can still be debated, but I will not pursue this matter too far. After all, setting p

proportional to $1/S$ in the first place was just meant as an illustration. But it is interesting to observe that in the fourth part of the Wager (in Jordan 2006's reconstruction), Pascal goes on to argue that one gains even in this life by living piously (argument from strong dominance). In other words, he argues that $f_2 > f_3$, which is indeed a sufficient condition for a successful wager in the presence of infinitesimal probabilities.

4.1.2 Probability Inversely Proportional to Either Duration or Intensity of Salvation

In section 1.3 we already indicated that Pascal characterized S as "an infinity of an infinitely happy life" suggesting it to be a product of two infinities. Now, it is true for *any* infinite hyperreal, that it can be regarded as a product of two infinite hyperreals (e.g., the initial number's square root). But if $1/p$ were to scale with either infinite dimension of salvation (either its duration, or its intensity, but not both), the Wager would still be successful. Could this "doubly infinite" utility still be an ace up Pascal's sleeve, even in the presence of some "moderate" infinitesimal probabilities?

Consider an agent to which the existence of a being with an infinite capacity is infinitely implausible, but additional infinite capacities do not increase the implausibility so strongly. This suggests substituting $p \sim 1/\sqrt{S}$. Let's be slightly more general, however, by substituting in inequality (1): $p = f/S^a$, with f and $a<1$ positive hyperreal numbers that are neither infinitesimal nor infinite. (Hence, $p \times S$ is infinite, as is the case in a wager without infinitesimal probabilities.) After rewriting, we see that the wager is successful iff:

$$f_3 < f_2 - 1/(S^a/f - 1) \times f_1 + f/(1 - f/S^a) \times S^{1-a}.$$

Observe that the righthand-side has a positive infinite value (the first and second term are finite and the third term is positive infinite). Given that f_3 is finite, this inequality always holds, so the Wager is always successful when, e.g., $p \sim 1/\sqrt{S}$.

4.2 Keeping It Real

In section 1.3 we already remarked that sequences and functions may diverge at different rates. Here we can develop this observation and connect it to the hyperreal framework. It will also help us to answer the following question: did we really need decision theory with hyperreal numbers to get to this result? Or can we regard non-standard tools, once again, merely as a tool to find a result within the standard domain?

To address this question, it is helpful to consider the following interpretation of infinitesimals: the limit of a function that converges to zero also represents information about the rate of convergence. An example may be helpful: consider the functions $1/x$ and $1/x^2$. Both converge to zero if we take the limit of x to infinity, but they do so at different rates. The non-standard limit of these functions are infinitesimals: let's call them ε_1 and ε_2. Both infinitesimals are infinitely close to the same real number: zero. In addition, the fact that $1/x^2$ converges to zero much faster than $1/x$ is represented in the relationship between them: it holds that $\varepsilon_2 = \varepsilon_1^2 < \varepsilon_1$. Infinite hyperreal numbers can be understood in a similar vein: as limits of diverging functions, encoding additional information about the rate of divergence. For instance, the functions x and x^2 both diverge, but at different rates. The non-standard limits associated with them are, respectively: $1/\varepsilon_1$ and $1/\varepsilon_2 = 1/\varepsilon_1^2 > 1/\varepsilon_1$.

Applying this interpretation to Pascal's Wager is not straightforward, however, since it involves no limit processes associated with S and p. For the special cases in the previous section, where we assumed a functional dependence of p on S, we can apply this interpretation.

Consider inequality (1), but now interpret all quantities appearing in it as real-valued functions of S. We are interested in the limit as S goes to infinity. The boundary case for an unsuccessful wager implies that the left- and right-hand side of (1) are "equivalent infinitesimals" (with infinitesimal *not* in the sense of section 1.2): this means that the limit of their ratio equals one. (Computing this limit will typically involve applying L'Hôpital's rule.) This is equivalent to taking the standard part of both sides of (1) in its original interpretation (i.e., with symbols representing hyperreal values).

So, the move to real numbers is slightly less precise (yielding a necessary but not a sufficient success condition) and presupposes a functional dependence of p on S. Considering whether such an assumption is always sensible leads to a further complication, to which we turn next.

4.3 Independent Disbelief and Practical Limitations on Incredulity?

Oppy (2006, p. 257) wrote: "While one now has a neat mathematical apparatus for handling infinities and infinitesimals, it is not clear that we have any way of understanding how this apparatus can be applied to the case at hand." Although he wrote this with Conway's numbers in mind, it seems applicable here, too.

We stressed the importance of employing a decision theory that is built upon a harmonious number system. This harmony is necessary to ensure that the relevant expected utilities can be computed. Now, the special cases discussed in section 4.1 may raise a worry: unless the size of the utility of eternal bless S directly influences the agent's probability assignment p, it may not be clear what the relation between the two values is. Yet, as long as it is transparent to the agent what the values of p and S are, they are representable by some tractable relation $p(S)$ and it can be determined by inequality (1) whether the Wager fails or not.

On the other hand, we might deny the existence of any such equation, expressible within the number system the agent uses, to represent the relation between p and S. In other words, we may deny the harmony of the number system to the relevant quantities. For instance, if S belongs to a non-standard model of the non-standard model that the agent uses ($S \in {}^{**}\mathbb{R}$), it is larger than any number representable by the agent and the wager goes through (because $p \times S$ is infinite). Since presumably S is fixed at the dawn of time, this would require foreknowledge of what are the largest and smallest positive numbers ever constructed in human mathematics. Without such foreknowledge, S has to be some kind of absolute infinity, to which none of the formal systems herein discussed applies.

4.4 Ruling Out Mixed Strategies

Hájek (2003) holds that the Wager is an invalid argument and considers four valid reformulations, finding fault with all of them. To argue that Pascal's Wager is invalid, Hájek points out that mixed strategies have infinite utility too and hence are not to be preferred over the pure strategy of wagering for God's existence directly. For example, letting one's wager depend on the outcome of a coin toss, a die toss, or some other random event, no matter how unlikely, will still yield infinite expected utility.

Using hyperreals to represent the expected utilities, however (and assuming larger than infinitesimal probabilities), it can be shown that that of the pure strategy is the largest. For example, if an agent considers a chance process to decide between wagering for or against God, which leads to wagering for God with some probability q, the expected utility of this mixed strategy is $q \times (S \times p + f_2 \times (1 - p)) + (1 - q) \times (f_1 \times p + f_3 \times (1 - p))$, which is strictly smaller than the expected utility of the pure strategy of wagering for God. If either p or q or both are allowed to take infinitesimal values, as before, the argument's success may co-depend on the finite utilities.

5 Final Thoughts

The Wager fragment doesn't motivate why Pascal excludes infinitesimal probabilities: Is it because he (correctly) intuited that the argument may fail or would he (also) deny that a rational agent can have such credences in the first place? We already discussed in section 3.2.1 that Pascal's proto-classical interpretation would probably require the probabilities to be equal for all agents, but we set aside that aspect here.

5.1 Assigning Probability to the Inconceivable

Since Pascal wrote that humans can neither grasp the infinitely large nor the infinitely small, he could indeed deny that a finite human can hold an infinitesimal credence in any proposition. Whereas God is capable to provide salvation of any utility, there is a limit to the fineness of difference a human can perceive and reason about: we are "infinitely remote from an understanding of the extremes" (1966 [1760], L199/S230). In this sense, the harmony of the numbers may speak *against* allowing infinitesimal probabilities in a wager argument, for compared to God's, human capacities are imperceptibly small. Also, from a naturalistic perspective, Easwaran (2014, section 5.4) has argued that actual agents cannot hold hyperreal credences (though I worry that this argument is too strong, for a similar argument would also rule out irrational probabilities).

Of course, some people might set their degree of belief inversely proportional to the estimated utility value of salvation, but then it seems Pascalian to point out that any such estimation of salvation will fall short as a finite value ($p = 1/S'$ with S' some finite estimation of salvation, such that section 4.1 does not apply). The Wager may then fail, not because the probability assigned to the hypothesis of God's existence is infinitesimal, but simply because the utility value is very large but finite (S underestimates the true value of salvation by an infinite factor).

Both observations can motivate why Pascal excluded infinitesimal probabilities in the first place.

5.2 Infinitely Many Alternative Hypotheses

As a final point, we remark that infinitesimal probabilities may be introduced into the wager in a different way. In order to apply our probability theory, we first need to check whether the possibilities are mutually exclusive and jointly exhaustive. It isn't clear that this is the case: the negation of the proposition “God exists” may include possibilities other than “God doesn’t exist,” such as the concept of God being contradictory or meaningless, but Pascal doesn’t mention these. Moreover, it is clear that Pascal only discussed the existence or non-existence of the God of Christianity, but there are infinitely many alternative deities conceivable. If any non-zero probability is assigned to the non-elementary premise that a supernatural being exists, a uniform and regular probability assignment will assign an infinitesimal probability to each elementary premise, including to “Pascal’s Christian God exists.” Hence, the many-gods-objection to the Wager may pose a context in which infinitesimal probabilities arise (more) naturally.

Oppy (1991, p. 166) also considered an agent who deliberates on “an infinite range of possible deities about whose existence he acknowledges that his reason is impotent to decide” and writes that such an agent “ought to assign no more than an infinitesimal value to the subjective probability that any one of these deities exists.” Likewise, Oppy (2006, pp. 249–50) asks us to “[c]onsider the hypothesis that there is a source of infinite utility that one will obtain just in case one forms a correct belief about the identity of a natural number that is intimately associated with admission to this source of infinite utility.” For instance, the agent has to guess a deity’s favorite natural number to receive salvation. Distributing probability over the natural numbers in a regular and uniform way requires infinitesimal probabilities (Wenmackers and Horsten, 2013).

Alternatively, and similarly to section 4.1, we could consider agents whose degree of belief is a function of the utility provided by each deity. Then, distinguishing between a deity that can grant an eternal life or one infinitely happy day, on the one hand, and Pascal’s God who can grant eternal infinite happiness, on the other hand, becomes relevant. The double infinity of salvation may be a double-edged sword: if infinitesimal probabilities are allowed, it raises the utility *S*, but it might lower the associated degree of belief *p* proportionately. Yet, excluding infinitesimal probabilities in such a context may guarantee success once more.

However, as Oppy (2006, pp. 250–51) remarked, for each hypothesis, there exist alternatives involving a malevolent deity that assigns salvation the opposite way: rewarding disbelievers (switching the rows in the decision matrix of a benevolent deity) or punishing believers (allotting utility -*S* instead of *S*). Successful wagering in the presence of symmetric hypotheses would require an additional argument for assigning higher probability to a benevolent deity than a malevolent one, but then the infinite utility of salvation is no longer at the heart of the argument. So, the symmetry of the hypotheses neutralizes Pascal's argument completely, irrespective of whether we allow infinitesimal probabilities.

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