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From Wallis and Forsyth to Ramanujan

Paul Levrie · Amrik Singh Nimbran

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Abstract We show how a couple of Ramanujan's series for $1/\pi$ can be deduced directly from Forsyth's series and from Wallis's product formula for π. The same method is used to obtain Bauer's alternating series.

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1 Introduction

In 1883 the Cambridge mathematician A.R. Forsyth derived the following formula for $\frac{1}{\pi}$ using Legendre polynomials [5]:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n-1)^2 2^{4n}} = \frac{4}{\pi}. \tag{1}$$

J.W.L. Glaisher [6] used elliptic functions to prove in 1905 a number of similar formulas including the following one:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(n+1)2^{4n}} = \frac{4}{\pi}. \tag{2}$$

Paul Levrie
Faculty of Applied Engineering, University of Antwerp, Groenenborgerlaan 171, 2020 Antwerpen; Department of Computer Science, KU Leuven, P.O. Box 2402, B-3001 Heverlee, Belgium
E-mail: paul.levrie@cs.kuleuven.be

Amrik Singh Nimbran
B3-304, Palm Grove Heights, Ardee City, Gurgaon, Haryana, INDIA 122003
E-mail: amrikn622@gmail.com

And there is also this series:

$$-\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n-1)2^{4n}} = \frac{2}{\pi}. \tag{3}$$

Both (2) and (3) are equivalent with Euler's version of Wallis's famous product formula for π derivable from the infinite product expansion of the sine function [14]:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(n+1)2^{4n}} &= 1 + \frac{1}{2{\cdot}2^2} + \frac{1^2 3^2}{2{\cdot}4^2 6} + \frac{1^2 3^2 5^2}{2{\cdot}4^2 6^2 8} + \frac{1^2 3^2 5^2 7^2}{2{\cdot}4^2 6^2 8^2 10} + \cdots \\ &= \frac{3{\cdot}3}{2{\cdot}4} + \frac{1^2 3^2}{2{\cdot}4^2 6} + \frac{1^2 3^2 5^2}{2{\cdot}4^2 6^2 8} + \frac{1^2 3^2 5^2 7^2}{2{\cdot}4^2 6^2 8^2 10} + \cdots \\ &= \frac{3{\cdot}3}{2{\cdot}4}{\cdot}\frac{5{\cdot}5}{4{\cdot}6} + \frac{1^2 3^2 5^2}{2{\cdot}4^2 6^2 8} + \frac{1^2 3^2 5^2 7^2}{2{\cdot}4^2 6^2 8^2 10} + \cdots \\ &= \frac{3{\cdot}3}{2{\cdot}4}{\cdot}\frac{5{\cdot}5}{4{\cdot}6}{\cdot}\frac{7{\cdot}7}{6{\cdot}8} + \frac{1^2 3^2 5^2 7^2}{2{\cdot}4^2 6^2 8^2 10} + \cdots \\ &= \frac{3{\cdot}3}{2{\cdot}4}{\cdot}\frac{5{\cdot}5}{4{\cdot}6}{\cdot}\frac{7{\cdot}7}{6{\cdot}8}\cdots = \frac{4}{\pi}\end{aligned}$$

and

$$\begin{aligned}-\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n-1)2^{4n}} &= 1 - \frac{1}{2^2} - \frac{1^2 3}{2^2 4^2} - \frac{1^2 3^2 5}{2^2 4^2 6^2} - \frac{1^2 3^2 5^2 7}{2^2 4^2 6^2 8^2} - \cdots \\ &= \frac{1{\cdot}3}{2{\cdot}2} - \frac{1^2 3}{2^2 4^2} - \frac{1^2 3^2 5}{2^2 4^2 6^2} - \frac{1^2 3^2 5^2 7}{2^2 4^2 6^2 8^2} - \cdots \\ &= \frac{1{\cdot}3}{2{\cdot}2}{\cdot}\frac{3{\cdot}5}{4{\cdot}4} - \frac{1^2 3^2 5}{2^2 4^2 6^2} - \frac{1^2 3^2 5^2 7}{2^2 4^2 6^2 8^2} - \cdots \\ &= \frac{1{\cdot}3}{2{\cdot}2}{\cdot}\frac{3{\cdot}5}{4{\cdot}4}{\cdot}\frac{5{\cdot}7}{6{\cdot}6} - \frac{1^2 3^2 5^2 7}{2^2 4^2 6^2 8^2} - \cdots \\ &= \frac{1{\cdot}3}{2{\cdot}2}{\cdot}\frac{3{\cdot}5}{4{\cdot}4}{\cdot}\frac{5{\cdot}7}{6{\cdot}6}\cdots = \frac{2}{\pi}.\end{aligned}$$

The following series is due to G. Bauer (1859) [10]:

$$\sum_{n=0}^{\infty} (-1)^n \frac{(4n+1)\binom{2n}{n}^3}{2^{6n}} = \frac{2}{\pi}. \tag{4}$$

All the previous series converge very slowly. The last one can be found in S. Ramanujan's first letter to G. Hardy, dated January 31, 1913. In a paper published in 1914 [16] Ramanujan lists some more rapidly converging series for $1/\pi$ which he found using modular equations. These are two of them:

$$\sum_{n=0}^{\infty} \frac{(6n+1)\binom{2n}{n}^3}{2^{8n}} = \frac{4}{\pi}, \tag{5}$$

$$\sum_{n=0}^{\infty} \frac{(42n+5)\binom{2n}{n}^3}{2^{12n}} = \frac{16}{\pi}. \tag{6}$$

In this paper we show how Ramanujan's series (5) and a number of other series for $\frac{1}{\pi}$ can be derived from Forsyth's in a straightforward way. The same method can also be applied to Glaisher's series (2) and to (3), leading to a proof of (4) and (6).

Series for $\frac{1}{\pi}$ have received much attention lately, see for instance the survey paper by Baruah, Berndt and Chan [2] and the work of Guillera [7,8] who uses the Wilf-Zeilberger method to find series of this form. Chu [3] and Liu [11, 12] have obtained similar results by manipulating hypergeometric series using Dougall's and Gauss's summation formulae.

2 A first recurrence

Note that all the series in the introduction contain the central binomial coefficients which can be rewritten using the Pochhammer symbol:

$$\binom{2n}{n} = 2^{2n}\frac{\left(\frac{1}{2}\right)_n}{n!} = 2^{2n}\frac{\left(\frac{1}{2}\right)_n}{(1)_n}.$$

Using this in the general term of (1) we find that:

$$\frac{\binom{2n}{n}^2}{(2n-1)^2 2^{4n}} = \frac{1}{(2n-1)^2}\frac{\left(\frac{1}{2}\right)_n^2}{(1)_n^2} = \frac{1}{4}\frac{\left(\frac{1}{2}\right)_{n-1}^2}{(1)_n^2} = \frac{1}{4}\frac{\left(\frac{1}{2}\right)_{n-1}^2}{(2)_{n-1}^2}.$$

Hence Forsyth's series can be written in the following form:

$$\frac{4}{\pi} = 1 + \frac{1}{4}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(2)_n^2}. \tag{7}$$

This last series is a special case of the more general series

$$\sum_{n=0}^{\infty}\frac{(a)_n^2}{(a+b)_n^2} = {}_3F_2\left[\begin{matrix} a & a & 1 \\ a+b & a+b & \end{matrix}; 1\right]$$

with $a = \frac{1}{2}$ and $b = \frac{3}{2}$. In the rest of the paper we will use the abbreviation:

$$s(a,b) = \sum_{n=0}^{\infty}\frac{(a)_n^2}{(a+b)_n^2}. \tag{8}$$

This expression satisfies the following recurrence relation:

Theorem 1 **($+0, +1$–scheme)**

$$s(a,b) = \frac{2a+3b-2}{2(2b-1)} + \frac{b^3}{2(2b-1)(a+b)^2}\cdot s(a,b+1). \tag{9}$$

Proof Manipulation of the general term in the series $s(a, b+1)$ leads to the required result. We start by writing:

$$\sum_{n=0}^{\infty} \frac{(a)_n^2}{(a+b+1)_n^2} = \frac{1}{b^2} \sum_{n=0}^{\infty} \frac{(a)_n^2 (a+b+n-(a+n))^2}{(a+b+1)_n^2}.$$

If we work out the numerator, we get the following 3 series:

$$\sum_{n=0}^{\infty} \frac{(a)_n^2 (a+b+n)^2}{(a+b+1)_n^2} = (a+b)^2 \sum_{n=0}^{\infty} \frac{(a)_n^2}{(a+b)_n^2} = (a+b)^2 s(a,b)$$

$$\sum_{n=0}^{\infty} \frac{(a)_n^2 (a+n)^2}{(a+b+1)_n^2} = (a+b)^2 \sum_{n=0}^{\infty} \frac{(a)_{n+1}^2}{(a+b)_{n+1}^2} = (a+b)^2 (s(a,b)-1)$$

$$\sum_{n=0}^{\infty} \frac{(a)_n^2 (a+b+n)(a+n)}{(a+b+1)_n^2} = (a+b)^2 \sum_{n=0}^{\infty} \frac{(a)_n^2 (a+n)}{(a+b)_n^2 (a+b+n)}.$$

We now deal with this last series. Using a similar trick as before, we get:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(a)_n^2 (a+n)}{(a+b)_n^2 (a+b+n)} &= \frac{1}{b} \sum_{n=0}^{\infty} \frac{(a)_n^2 (a+n)(a+b+n-(a+n))}{(a+b)_n^2 (a+b+n)} \\
&= \frac{1}{b} \left(\sum_{n=0}^{\infty} \frac{(a)_n^2 (a+n)}{(a+b)_n^2} - \sum_{n=0}^{\infty} \frac{(a)_{n+1}^2}{(a+b)_n^2 (a+b+n)} \right) \\
&= \frac{1}{b} \left(a + \sum_{n=1}^{\infty} \frac{(a)_n^2 (a+n)}{(a+b)_n^2} - \sum_{n=0}^{\infty} \frac{(a)_{n+1}^2}{(a+b)_n^2 (a+b+n)} \right) \\
&= \frac{1}{b} \left(a + \sum_{n=0}^{\infty} \frac{(a)_{n+1}^2 (a+n+1)}{(a+b)_{n+1}^2} \right. \\
&\qquad \left. - \sum_{n=0}^{\infty} \frac{(a)_{n+1}^2}{(a+b)_n^2 (a+b+n)} \right) \\
&= \frac{a}{b} + \frac{1}{b} \sum_{n=0}^{\infty} \frac{(a)_{n+1}^2 (a+n+1-(a+b+n))}{(a+b)_{n+1}^2} \\
&= \frac{a}{b} + \frac{1-b}{b} \sum_{n=0}^{\infty} \frac{(a)_{n+1}^2}{(a+b)_{n+1}^2} = \frac{a}{b} + \frac{1-b}{b} (s(a,b)-1).
\end{aligned}$$

We bring everything together:

$$s(a, b+1) = \frac{(a+b)^2}{b^2} \left(s(a,b) + s(a,b) - 1 - 2 \left(\frac{a}{b} + \frac{1-b}{b} (s(a,b)-1) \right) \right).$$

Rearranging leads to (9).

Note that this theorem can be found in a slightly less general form in Knopp [9, p. 261-262]. It is a special case of Kummer's transformation of series.

Using Theorem 1 we are now able to rewrite $s(a, b)$:

Theorem 2

$$s(a,b)=\sum_{n=0}^{\infty}\frac{2a+3(b+n)-2}{2(2(b+n)-1)}\cdot\frac{(b)_n^3}{2^{2n}(b-\frac{1}{2})_n(a+b)_n^2}. \tag{10}$$

Proof From (9) it follows that:

$$s(a,b+n)=\frac{2a+3(b+n)-2}{2(2(b+n)-1)}+\frac{(b+n)^3}{2^2(b+n-\frac{1}{2})(a+b+n)^2}\cdot s(a,b+n+1).$$

Iterating this formula starting from $n=0$ proves the result.

Ramanujan's formula (5) is an immediate consequence of Theorem 2:

Corollary 1

$$\sum_{n=0}^{\infty}\frac{(6n+1)\binom{2n}{n}^3}{2^{8n}}=\frac{4}{\pi}.$$

Proof Forsyth's formula is equivalent with (7):

$$\frac{16}{\pi}-4=s(\tfrac{1}{2},\tfrac{3}{2}).$$

We rewrite the right-hand side using Theorem 2:

$$\begin{aligned}
s(\tfrac{1}{2},\tfrac{3}{2})&=\sum_{n=0}^{\infty}\frac{2+3(3+2n)-4}{2(2(3+2n)-2)}\cdot\frac{(\frac{3}{2})_n^3}{2^{2n}(1)_n(2)_n^2}\\
&=\sum_{n=0}^{\infty}\frac{6n+7}{8}\cdot\frac{8(\frac{1}{2})_{n+1}^3}{2^{2n}(1)_{n+1}^3}\\
&=4\sum_{n=1}^{\infty}\frac{(6n+1)\binom{2n}{n}^3}{2^{8n}}.
\end{aligned}$$

This leads to:

$$\frac{16}{\pi}=4+4\sum_{n=1}^{\infty}\frac{(6n+1)\binom{2n}{n}^3}{2^{8n}}\quad\Rightarrow\quad\frac{4}{\pi}=\sum_{n=0}^{\infty}\frac{(6n+1)\binom{2n}{n}^3}{2^{8n}}.$$

3 Other recurrences

Note that the following recurrence relation:

$$s(a,b)=1+\frac{a^2}{(a+b)^2}s(a+1,b) \tag{11}$$

is related to the series (8) in the same way that (9) is related to (10). By combining (9) and (11) we get new recurrences.

Theorem 3 **(**$+1, +1$**–scheme)**

$$s(a,b) = \frac{(2a+3b-2)(a+b)^2+b^3}{2(2b-1)(a+b)^2} + \frac{a^2b^3}{2(2b-1)(a+b)^2(a+b+1)^2}\,s(a{+}1,b{+}1).$$

The corresponding series is:

$$s(a,b) = \sum_{n=0}^{\infty} \frac{(5n+2a+3b-2)(a+b+2n)^2+(b+n)^3}{2(2(b+n)-1)(a+b+2n)^2} \cdot \frac{(a)_n^2 (b)_n^3}{2^{2n}(b-\frac{1}{2})_n (a+b)_{2n}^2}.$$

Proof We replace $s(a, b+1)$ in the right-hand side of (9) by the corresponding formula from (11):

$$\begin{aligned} s(a,b) &= \frac{2a+3b-2}{2(2b-1)} + \frac{b^3}{2(2b-1)(a+b)^2} \cdot s(a,b+1) \\ &= \frac{2a+3b-2}{2(2b-1)} + \frac{b^3}{2(2b-1)(a+b)^2} \cdot \left(1 + \frac{a^2}{(a+b+1)^2} s(a+1,b+1)\right). \end{aligned}$$

The series follows immediately from this recurrence.

If we take $a = \frac{1}{2}$, $b = \frac{3}{2}$, we get the following series:

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{168n^3-36n^2+6n+1}{(2n-1)^2} \frac{\binom{2n}{n}^3}{2^{12n}}. \tag{12}$$

By combining the two recurrences (9) and (11) we can derive other series. For instance, if we use (11) again in the recurrence of Theorem 3, we get this series ($+2, +1$–scheme):

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{18560n^5-20736n^4+8160n^3-992n^2+30n+9}{(4n-1)^2(4n-3)^2} \frac{\binom{2n}{n}\binom{4n}{n}^2}{2^{16n}}.$$

And this is the series we obtain if we use (9) again in the recurrence of Theorem 3 ($+1, +2$–scheme):

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{186368n^5-128000n^4+22304n^3-152n^2+2n+1}{(2n-1)(4n-1)^3} \frac{\binom{4n}{2n}\binom{4n}{n}^2}{2^{20n}}.$$

4 Recurrences related to Glaisher's and Wallis's series

Glaisher's series (2) can be written in the following form:

$$\frac{4}{\pi} = 1 + \frac{1}{4} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n^2}{(2)_n (2)_{n+1}} \ .$$

The series on the right-hand side is the special case $a = \frac{3}{2}$, $b = \frac{1}{2}$ of this series:

$$t(a,b) = \sum_{n=0}^{\infty} \frac{(a)_n^2}{(a+b)_n (a+b)_{n+1}}.$$

In a similar way as in the previous sections we can prove the two recurrences equivalent with (9) and (11) for t:

$$t(a,b) = \frac{1}{a+b} + \frac{a^2}{(a+b)^2} t(a+1,b),$$

$$t(a,b) = \frac{a+2ab+b+3b^2}{2(2b+1)b(a+b)} + \frac{b(b+1)^2}{2(2b+1)(a+b)^2} t(a,b+1).$$

Using only the last recurrence (+0, +1–scheme) with $a = \frac{3}{2}$ and $b = \frac{1}{2}$ we get the following new series:

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{12n^2+4n+1}{(2n-1)^2(n+1)} \frac{\binom{2n}{n}^3}{2^{8n}}.$$

With the $+1, +1$-scheme we again obtain the series (12).

The series (3) can be rewritten like this:

$$\frac{2}{\pi} = 1 - \frac{1}{8} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_{n-1} \left(\frac{3}{2}\right)_n}{(2)_n^2} \ .$$

where the series on the right is a special case of

$$u(a,b) = \sum_{n=0}^{\infty} \frac{(a)_{n-1}(a)_n}{(a+b)_n^2}$$

with corresponding recurrences:

$$u(a,b) = \frac{1}{a-1} + \frac{a^2}{(a+b)^2} u(a+1,b),$$

$$u(a,b) = \frac{a+2ab+3b^2-1}{2(2b+1)b(a-1)} + \frac{b(b+1)^2}{2(2b+1)(a+b)^2} u(a,b+1).$$

The $+0, +1$–scheme leads in this case to the series:

$$\frac{2}{\pi} = -\sum_{n=0}^{\infty} \frac{12n^2-1}{(2n-1)^2} \frac{\binom{2n}{n}^3}{2^{8n}}.$$

With the $+1, +1$ –scheme we find this series:

$$\frac{2}{\pi} = -\sum_{n=0}^{\infty} \frac{168n^3+20n^2-2n-1}{(2n-1)^2} \frac{\binom{2n}{n}^3}{2^{12n}}. \qquad (13)$$

With these series we can prove (6):

Corollary 2

$$\sum_{n=0}^{\infty} \frac{(42n+5)\binom{2n}{n}^3}{2^{12n}} = \frac{16}{\pi}.$$

Proof If we add (12) and (13), we get a new series:

$$\frac{3}{\pi} = \sum_{n=0}^{\infty} \frac{-28n^2+4n+1}{(2n-1)^2} \frac{\binom{2n}{n}^3}{2^{12n}}.$$

If we substract (13) from 6 times the previous series, the sum of the new series is $\frac{16}{\pi}$, and the polynomial in the numerator is given by:

$$6(-28n^2+4n+1) + 168n^3 + 20n^2 - 2n - 1 = (42n+5)(2n-1)^2.$$

Hence the resulting series is Ramanujan's series (6).

Note that (3) can also be written in this form:

$$\frac{2}{\pi} = 1 - \frac{1}{4} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n}{(2)_n^2} \tag{14}$$

and hence is a special case ($a=\frac{1}{2}$, $b=\frac{3}{2}$) of this series:

$$v(a,b) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(a+b)_n^2} = {}_3F_2\left[\begin{matrix} a \;\; b \;\; 1 \\ a+b \;\; a+b \end{matrix}; 1\right].$$

The corresponding recurrences are:

Theorem 4

$$v(a,b) = \frac{a+2b-1}{a+b-1} - \frac{b^3}{(a+b)^2(a+b-1)} \cdot v(a,b+1),$$

$$v(a,b) = \frac{2a+b-1}{a+b-1} - \frac{a^3}{(a+b)^2(a+b-1)} \cdot v(a+1,b).$$

Proof We prove the first one. The second one follows by symmetry. Note that the identity we want to prove can be rewritten like this:

$$v(a,b+1) = \frac{(a+b)^2}{b^2}\left(1 - \frac{a+b-1}{b}(v(a,b)-1)\right)$$

or

$$\sum_{n=0}^{\infty} \frac{(a)_n (b+1)_n}{(a+b+1)_n^2} = \frac{(a+b)^2}{b^2}\left(1 - \frac{a+b-1}{b}(v(a,b)-1)\right).$$

We use the definition of the pochhammer symbol to rewrite the left-hand side and at the same time we add a factor:

$$\sum_{n=0}^{\infty} \frac{(a)_n (b+1)_n}{(a+b+1)_n^2} = \frac{(a+b)^2}{b^2} \sum_{n=0}^{\infty} \frac{(a)_n (b)_{n+1}}{(a+b)_{n+1}^2}(a+b+n-(a+n)).$$

After simplifying and using the definition of v, what we have to prove becomes:

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_{n+1}}{(a+b)_{n+1}^2}(a+b+n)-\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(a+b)_{n+1}^2}=1-\frac{a+b-1}{b}\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(a+b)_{n+1}^2}.$$

Note that two sums cancel out, and we are left with:

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_{n+1}}{(a+b)_{n+1}^2}(a+b+n)=1-\frac{a-1}{b}\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(a+b)_{n+1}^2}.$$

We multiply by b and rewrite the left-hand side:

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_{n+1}}{(a+b)_{n+1}^2}(a+b+n)(a+b+n-(a+n))=b-(a-1)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(a+b)_{n+1}^2}$$

$$\Leftrightarrow\quad \sum_{n=0}^{\infty}\frac{(a)_n(b)_{n+1}}{(a+b)_{n}^2}-\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(a+b)_{n+1}^2}(a+b+n) =b-(a-1)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(a+b)_{n+1}^2}$$

$$\Leftrightarrow\quad \sum_{n=0}^{\infty}\frac{(a)_n(b)_{n+1}}{(a+b)_{n}^2}-\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(a+b)_{n+1}^2}(b+n)=b+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(a+b)_{n+1}^2}.$$

If we change the index of summation in the first term on the left, the b at the right cancels out:

$$\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}(b+n+1)}{(a+b)_{n+1}^2}-\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}(b+n)}{(a+b)_{n+1}^2}=\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(a+b)_{n+1}^2}.$$

It is now easy to see that both sides are equal.

Using the first recurrence in the same way as in Theorem 2, we get the following result:

Theorem 5

$$v(a,b)=\sum_{n=0}^{\infty}(-1)^n\frac{a+2(b+n)-1}{a+b+n-1}\frac{(b)_n^3}{(a+b)_n^2(a+b-1)_n}. \qquad (15)$$

An immediate consequence is (4):

Corollary 3

$$\sum_{n=0}^{\infty}(-1)^n\frac{(4n+1)\binom{2n}{n}^3}{2^{6n}}=\frac{2}{\pi}.$$

Proof We rewrite the right-hand side of (14) using Theorem 5:

$$\begin{aligned} v(\tfrac{1}{2},\tfrac{3}{2}) &= \sum_{n=0}^{\infty}(-1)^n \frac{1+2(3+2n)-2}{2n+2}\cdot\frac{(\frac{3}{2})_n^3}{(1)_n(2)_n^2} \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{4n+5}{2}\cdot\frac{8(\frac{1}{2})_{n+1}^3}{(1)_{n+1}^3} \\ &= 4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(4n+1)\binom{2n}{n}^3}{2^{6n}}. \end{aligned}$$

This leads to:

$$\frac{2}{\pi} = 1-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(4n+1)\binom{2n}{n}^3}{2^{6n}} = \sum_{n=0}^{\infty}(-1)^n\frac{(4n+1)\binom{2n}{n}^3}{2^{6n}}.$$

Other consequences of Theorem 5 are:

Corollary 4

$$\frac{3\sqrt{3}}{2\pi} = \sum_{n=0}^{\infty}(-1)^n(6n+1)\frac{(\frac{1}{3})_n^3}{n!^3} \qquad \frac{3}{\pi} = \sum_{n=0}^{\infty}(-1)^n(12n+1)\frac{(\frac{1}{6})_n^3}{n!^3}$$

$$\frac{2\sqrt{2}}{\pi} = \sum_{n=0}^{\infty}(-1)^n(8n+1)\frac{(\frac{1}{4})_n^3}{n!^3} \qquad \frac{4\sqrt{2-\sqrt{2}}}{\pi} = \sum_{n=0}^{\infty}(-1)^n(16n+1)\frac{(\frac{1}{8})_n^3}{n!^3}$$

$$\frac{5\sqrt{5-\sqrt{5}}}{2\sqrt{2}\pi} = \sum_{n=0}^{\infty}(-1)^n(10n+1)\frac{(\frac{1}{5})_n^3}{n!^3} \qquad \frac{5(\sqrt{5}-1)}{2\pi} = \sum_{n=0}^{\infty}(-1)^n(20n+1)\frac{(\frac{1}{10})_n^3}{n!^3}$$

Proof These series follow from the Wallis-type products for π which can be found in [1] (see also [13], formulas (10), (11) and (19)):

$$\frac{\sin(\pi m/k)}{\pi m/k} = \prod_{n=0}^{\infty}\frac{nk+k-m}{nk+k}\,\frac{nk+k+m}{nk+k}$$

for m,k positive integers with $m<k$. The choice $m=1,k=2$ leads to Wallis's product. If we take $m=1,k=3$, the product takes this form:

$$\frac{3\sqrt{3}}{2\pi} = \frac{2\cdot 4}{3\cdot 3}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{8\cdot 10}{9\cdot 9}\cdots.$$

As we did in the introduction with Wallis's product, we can rewrite this product as a series:

$$\begin{aligned} \frac{3\sqrt{3}}{2\pi} &= 1-\frac{1}{3^2}-\frac{2\cdot 4}{3^2\cdot 6^2}-\frac{2\cdot 4\cdot 5\cdot 7}{3^2\cdot 6^2\cdot 9^2}-\cdots \\ &= 1-\frac{1}{3^2}\sum_{n=0}^{\infty}\frac{(\frac{2}{3})_n(\frac{4}{3})_n}{(n+1)!^2} \\ &= 1-\frac{1}{3^2}v(\tfrac{2}{3},\tfrac{4}{3}). \end{aligned}$$

Using Theorem 5, we get the first formula. The second formula is found by taking $m = 1, k = 4$, the third one by taking $m = 1, k = 5$, and so on.

Concluding remarks.

1. The method used above to convert a product to a series can be applied directly to Euler's product formula for the sine-function:

$$\sin \pi x = \pi x \prod_{m=1}^{\infty} \left(1 - \frac{x^2}{m^2}\right).$$

The result is the following series:

$$\frac{\sin \pi x}{\pi x} = \sum_{n=0}^{\infty} \frac{(-x)_n (x)_n}{n!^2}. \tag{16}$$

which converges (by Raabe's test) for all $x \neq 0$.

2. Applying Theorem 5 to (16) results in this series:

$$\frac{\sin \pi x}{\pi x} = \sum_{n=0}^{\infty} (-1)^n \frac{2n + x}{x} \frac{(x)_n^3}{n!^3}.$$

It can be found in Dougall's paper [4, p. 124 formula (16)]. All series in Corollary 4 are special cases of this general formula.

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