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Analysis of Heavy-Tailed Risks

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Preface

The purpose of this Ph.D. thesis is twofold. Firstly, we concentrate on mathematical properties of reinsurance risks and the analysis of related risk measures. A particular emphasis is laid on asymptotic results. Secondly, we deal with the asymptotic behavior of a specific ratio of random variables and closely related estimators of risk measures. All our asymptotic results make heavy use of the theory of regular variation and classical extreme value theory.

Chapter 1 concerns mathematical preliminaries. We start with the introduction of basic concepts and notation. Next, we specify the classes of distributions which are used in this thesis, namely specific heavy-tailed classes and also the extremal classes that naturally appear in applications of extreme value theory. In particular, the distributions with a regularly varying right tail constitute a prominent class of heavy-tailed distributions. Important and useful properties about functions of regular variation are then collected. We also discuss the notions of the domain of attraction of an extreme value distribution and the domain of attraction of a stable distribution which are closely connected to regular variation. Furthermore, we introduce the concept of a counting process and related notions are given. The specific counting processes encountered in this dissertation are then defined. Among them, the mixed Poisson process turns out to play a major role.

The first part of the thesis (Chapters 2, 3, 4 and 5) is devoted to reinsurance. Chapter 2 gives an introduction to reinsurance. After the definition of the collective risk model, we review usual reinsurance treaties depending on whether they are of random walk type or of extreme value type. Then, we specify the modeling assumptions that are frequently made to study these treaties. For the claim number process, three major models are actually used. For the claim size distribution, appropriate models obviously differ according as small claims or large claims are involved. In Chapters 3 and 5, we deal with treaties of extreme value type since we have the coverage of large claims in mind. Chapter 3 is devoted to the study of asymptotic properties mainly for the reinsured amount in the ECOMOR treaty and to a lesser extent for the reinsured amount in the LC treaty. Depending on the heavy-tailed character of the claim size distribution, asymptotic equivalences and bounds are established for the tail of the distribution of the reinsured amount in the ECOMOR treaty. For a mixed Poisson claim number process and a claim size distribution that belongs to an extremal class, weak convergence results are derived for the suitably normalized reinsured amounts in the ECOMOR and LC treaties. Finally, an asymptotic comparison of the ECOMOR-reinsured amount with the total claim amount is given. In Chapter 5, we study asymptotic and finite time properties for the reinsured amount in the QDXL treaty. After establishing the distribution of the QDXL-reinsured amount, a weak convergence result is derived for a properly normalized version of it. Next, the exact expression of the moments of the QDXL-reinsured amount is obtained. The asymptotic behavior of these moments is then given. Also, the asymptotic expression of the moments of the appropriately normalized QDXL-reinsured amount is derived. All the asymptotic

results are established under the assumptions that the claim number process is mixed Poisson and that the claim size distribution belongs to an extremal class. Finally, we look at some risk measures for the QDXL-reinsured amount in a fixed time setting. In Chapter 4, we analyze finite time properties of several risk measures for reinsurance layers. First, risk measures are considered for a general layer defined in terms of some proportionality factor and lower and upper retentions. Next, we assume a reinsurance chain for a proportion of the insurer's total claim amount where each layer depends on different values of the retentions. The effect of reinsurance on the partners involved in the risk sharing is then investigated by comparing risk measures for the different layers. Let us remark that although Chapter 4 does not focus on asymptotic properties, it is nevertheless placed before Chapter 5 because the reinsured amount in the QDXL treaty is a particular version of the general layer considered in Chapter 4.

The second part of the thesis (Chapters 6 and 7) is devoted to the asymptotic analysis of the ratio of the random sum of squares of a.s. positive i.i.d. random variables over the square of the random sum of these random variables whose common distribution is assumed to have a regularly varying right tail. The number of summands is then itself a random variable that is governed by a counting process. The ratio is actually closely related to the usual estimator of the coefficient of variation of a randomly indexed sample of a.s. positive i.i.d. random variables. Extending earlier results in the literature for a non-random number of summands, we derive asymptotic expressions for the moments of the ratio when the counting process is mixed Poisson in Chapter 6 and we establish weak convergence results for the properly normalized ratio under various asymptotic assumptions on the counting process in Chapter 7. The results heavily depend on the range of the index of regular variation and hence on the number of existing moments of the underlying distribution. In Chapter 7, weak convergence results are also derived for the appropriately normalized sample coefficient of variation from those obtained for the ratio. Moreover, because the usual estimator of the dispersion of a randomly indexed sample of a.s. positive i.i.d. random variables can be expressed in terms of a slight modification of the ratio, weak convergence results are obtained for the suitably normalized sample dispersion by appropriate adaptations. Both the coefficient of variation and the dispersion are popular risk measures that found many applications within the context of insurance and reinsurance.

Chapter 1

Mathematical Preliminaries

1.1 Notation and Basic Definitions

In this section, we specify some notational conventions and basic definitions that will be used throughout this dissertation.

We say *positive* for > 0 and *nonnegative* for $\geqslant 0$. Let $\mathbb{R} := (-\infty, \infty)$ denote the real line. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *increasing* (respectively, *nondecreasing*) if $f(x) < f(y)$ (respectively, $f(x) \leqslant f(y)$) for all real numbers x, y with $x < y$. The meaning of *negative*, *nonpositive*, *decreasing*, *nonincreasing* is easily deduced. For any $x_0 \in \mathbb{R}$, the notation $x \uparrow x_0$ (respectively, $x \downarrow x_0$) means $x < x_0$, $x \to x_0$ (respectively, $x > x_0$, $x \to x_0$). Now, $f(x_0-) := \lim_{x \uparrow x_0} f(x)$ and $f(x_0+) := \lim_{x \downarrow x_0} f(x)$. Let f and g be two measurable functions. For any $x_1 \in \mathbb{R} \cup \{-\infty, \infty\}$, we write $f(x) = o(g(x))$ as $x \to x_1$ if $\lim_{x \to x_1} f(x)/g(x) = 0$ and $f(x) \sim g(x)$ as $x \to x_1$ if $\lim_{x \to x_1} f(x)/g(x) = 1$. Moreover, $\mathbb{N} := \{1, 2, 3, \dots\}$ denotes the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ the set of nonnegative integers. The greatest lower and least upper bounds of a set $A \subseteq \mathbb{R}$ are respectively denoted by $\inf A$ and $\sup A$, with the convention that $\inf \varnothing = \infty$ and $\sup \varnothing = -\infty$.

For a nondecreasing function $f\colon \mathbb{R} \to \mathbb{R}$, we define its *left-continuous inverse* as:

$$f^{\leftarrow}(y) := \inf\{x \in \mathbb{R}\colon f(x) \geqslant y\}, \quad y \in \mathbb{R}.$$

For any set or event Δ, the associated *indicator function* is defined as:

$$\mathbb{1}_{\Delta}(x) := \begin{cases} 1 & \text{if } x \in \Delta \\ 0 & \text{if } x \notin \Delta. \end{cases}$$

Some special functions will also be used. The *gamma function* is denoted by $\Gamma(x) := \int_0^\infty t^{x-1}\,\mathrm{e}^{-t}\,\mathrm{d}t$ for $x > 0$ and the *beta function* by $\mathrm{B}(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ for $x > 0$ and $y > 0$. Finally, the *incomplete gamma function* is denoted by $\Gamma_y(x) := (1/\Gamma(y)) \int_x^\infty t^{y-1}\,\mathrm{e}^{-t}\,\mathrm{d}t$ for $x \geqslant 0$ and $y > 0$.

The following standard abbreviations of probability theory will be used: a.s. for *almost surely* and i.i.d. for *independent and identically distributed*.

Let F be a distribution function on the real line. The *left endpoint* of the support of F is denoted by $x_\ell := \inf\{x \in \mathbb{R}\colon F(x) > 0\}$ and the *right endpoint* by $x_r := \sup\{x \in \mathbb{R}\colon F(x) < 1\}$. The *tail quantile function* U of F is defined by:

$$U(y) := F^{\leftarrow}(1 - 1/y) = \inf\{x \in \mathbb{R}\colon F(x) \geqslant 1 - 1/y\}, \quad y > 1.$$

Note that $1-F(U(y)) = 1/y$ for $y > 1$ if F is continuous while $U(1/(1-F(x))) = x$ for $x > x_\ell$ if U is continuous. Note also that $U(1+) = x_\ell \geqslant -\infty$ and $U(\infty) := \lim_{y\to\infty} U(y) = x_r \leqslant \infty$. The *n-fold convolution* of F is denoted by F^{*n} for all $n \in \mathbb{N}_0$. It is defined iteratively as $F^{*0}(x) := \mathbb{1}_{[0,\infty)}(x)$, $x \in \mathbb{R}$, and $F^{*n} := F^{*(n-1)} * F = F * \cdots * F$ (n times) for $n \geqslant 1$.

Let X be a random variable with distribution function F, i.e. $F(x) := \mathbb{P}[X \leqslant x]$ for $x \in \mathbb{R}$. The *characteristic function* of X (and of F) is given by:

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}tX}\big] = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}F(x), \quad t \in \mathbb{R}.$$

If X is a.s. nonnegative, i.e. F has support in $[0,\infty)$, the *Laplace-Stieltjes transform* of X (and of F) is given by:

$$\mathbb{E}\big[\mathrm{e}^{-sX}\big] = \int_{0}^{\infty} \mathrm{e}^{-sx}\,\mathrm{d}F(x), \quad s \geqslant 0.$$

Consider two random variables X and Y. Whenever it exists, the value $\mathbb{E}[X^n]$ denotes the nth *moment* of X ($n \in \mathbb{N}_0$). In particular, $\mathbb{E}[X]$ is the *mean* or *expectation* of X and $\mathbb{E}[X^0] = 1$. The *variance* of X is $\mathbb{V}[X] := \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ and $\sqrt{\mathbb{V}[X]}$ is *the standard deviation* of X. The *covariance* of X and Y is defined as $\mathrm{Cov}(X,Y) := \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$ provided that $\mathbb{E}[X^2] < \infty$ and $\mathbb{E}[Y^2] < \infty$.

Let X, Y be two random variables defined on possibly different probability spaces. We write $X \overset{\mathcal{D}}{=} Y$ if X and Y have the same distribution. Further, Y is said to be a *copy* of X if $X \overset{\mathcal{D}}{=} Y$ holds. For random variables $X, X_1, X_2, \ldots$ defined on a common probability space, we respectively write $X_n \xrightarrow{a.s.} X$, $X_n \xrightarrow{p} X$ or $X_n \xrightarrow{r\text{th}} X$ as $n \to \infty$ if X_n converges *almost surely*, *in probability* or *in* rth *mean* ($r > 0$) to X. When the random variables are not necessarily defined on a common probability space, we write $X_n \xrightarrow{\mathcal{D}} X$ as $n \to \infty$ if X_n converges *in distribution* (or *weakly*) to X.

In the next section, we introduce classes of distributions which will be relevant for our purposes.

1.2 Classification of Distributions

We introduce the classes of distributions which will be encountered in this thesis. Each class is characterized by properties that look relevant within the framework of extremes. Subsection 1.2.1 concerns classes of *heavy-tailed distributions*. Subsection 1.2.2 deals with the *extremal classes of distributions* that naturally appear in applications of extreme value theory.

1.2.1 Classes of Heavy-Tailed Distributions

We start by defining the concept of a *heavy-tailed distribution*. Notice however that no definition is universally accepted.

Definition 1.1. *The distribution function F of a random variable X is said to be* heavy-tailed (on the right) *if for all $s > 0$:*

$$\mathbb{E}\big[\mathrm{e}^{sX}\,\mathbb{1}_{\{X>0\}}\big] = \int_{(0,\infty)} \mathrm{e}^{sx}\,\mathrm{d}F(x) = \infty.$$

The first class of heavy-tailed distributions that we consider is the class of *long-tailed distributions* defined as follows.

Definition 1.2. *A distribution function F on $(0,\infty)$ such that $F(x) < 1$ for all $x > 0$ belongs to the* long-tailed class $\mathcal{L}$ *if for all $y \in \mathbb{R}$:*

$$\lim_{x\to\infty} \frac{1-F(x+y)}{1-F(x)} = 1.$$

Remark 1.1. The long-tailed class may actually be defined for distributions on the real line. Such distributions are then said to be *long-tailed on* $\mathbb{R}$.

Remark 1.2. For any distribution function F, the assumption $F(x) < 1$ for all $x > 0$ means that the support of F is unbounded above (i.e., $x_r = \infty$).

The class $\mathcal{L}$ seems to be the largest class of heavy-tailed distributions for which one can derive asymptotic results.

The second and very important class of heavy-tailed distributions that we specify now is called the class of *subexponential distributions*.

Definition 1.3. *A distribution function F on $(0,\infty)$ such that $F(x) < 1$ for all $x > 0$ belongs to the* subexponential class $\mathcal{S}$ *if:*

$$\lim_{x\to\infty} \frac{1-F^{*2}(x)}{1-F(x)} = 2.$$

Remark 1.3. The notion of subexponentiality may be extended to distributions on the real line as follows. A distribution function G on $\mathbb{R}$ is called *subexponential on* $\mathbb{R}$ if there exists $F \in \mathcal{S}$ such that $1 - G(x) \sim 1 - F(x)$ as $x \to \infty$.

The class $\mathcal{S}$ is a proper subset of the class $\mathcal{L}$. Examples for a distribution in $\mathcal{L}$ but not in $\mathcal{S}$ can be found in Embrechts and Goldie [22] and Pitman [52]. The class of subexponential distributions has been introduced by Chistyakov [18]. It enjoys a large number of interesting properties and some of them are collected below.

Lemma 1.1. *Let $F \in \mathcal{S}$. Then the following properties hold:*

(*i*) *For all $\epsilon > 0$, $\lim_{x\to\infty} \mathrm{e}^{\epsilon x}(1-F(x)) = \infty$.*

(*ii*) *Given any $\epsilon > 0$, there exists a constant $K := K(\epsilon) < \infty$ such that for all $x \geqslant 0$ and $n \in \mathbb{N}_0$:*

$$\frac{1-F^{*n}(x)}{1-F(x)} \leqslant K(1+\epsilon)^n.$$

(*iii*) *For every* $n \in \mathbb{N}_0$*:*

$$\lim_{x\to\infty} \frac{1-F^{*n}(x)}{1-F(x)} = n. \tag{1.1}$$

Lemma 1.1(*i*) justifies the name *subexponential* since $1-F(x)$ decays to 0 as $x \to \infty$ slower than any exponential function $\mathrm{e}^{-\epsilon x}$ for $\epsilon > 0$. Lemma 1.1(*ii*) is due to Kesten (for a proof, see Athreya and Ney [4]). Lemma 1.1(*iii*) goes back to Chistyakov [18]. He actually proved that (1.1) holds for all $n \geqslant 2$ if and only if it holds for $n = 2$. Further, Embrechts and Goldie [23] showed that (1.1) holds for $n = 2$ if it holds for some $n \geqslant 2$.

Let $\{X_i;\, i \in \mathbb{N}\}$ be a sequence of a.s. positive i.i.d. random variables with distribution function F. For each $n \in \mathbb{N}$, denote the sum of $X_1, \ldots, X_n$ by $S_n := X_1 + \cdots + X_n$ and their maximum by $M_n := \max(X_1, \ldots, X_n)$. Lemma 1.1(*iii*) shows that the condition $F \in \mathcal{S}$ entails $\mathbb{P}[S_n > x] \sim n\,\mathbb{P}[X_1 > x]$ as $x \to \infty$ for every $n \in \mathbb{N}$, i.e. the tail distribution of the sum S_n behaves asymptotically as n times the tail distribution of X_1. Moreover, $\mathbb{P}[M_n > x] = (1-F(x)) \sum_{k=0}^{n-1} F^k(x) \sim n\,\mathbb{P}[X_1 > x]$ as $x \to \infty$ for every $n \in \mathbb{N}$. Combining the two arguments then leads to:

$$F \in \mathcal{S} \iff (\mathbb{P}[S_n > x] \sim \mathbb{P}[M_n > x] \text{ as } x \to \infty \text{ for some } n \geqslant 2)\,.$$

A distribution that belongs to $\mathcal{S}$ or $\mathcal{L}$ is heavy-tailed and as such has upper extreme values that are very dominant among sample values. The subexponential distributions are often suggested as appropriate models for heavy-tailed distributions. The importance of $\mathcal{S}$ as a useful class of heavy-tailed distributions in the context of applied probability, in general, and insurance mathematics, in particular, was realized by Teugels [60]. The potential role of subexponential distributions within queueing theory was recognized by Pakes [50] and Borovkov [14, 15]. For textbook treatments of subexponential distributions, we refer to Embrechts et al. [24] and Rolski et al. [54]. A survey paper on $\mathcal{S}$ and related classes of distributions is given in Goldie and Klüppelberg [28].

In this dissertation, we will often consider distributions whose right tail is *regularly varying*. They are also referred to as distributions with *power law tail behavior*. Let us start by defining the concept of a *regularly varying function (at infinity)*.

Definition 1.4. *A measurable function* $f\colon (0,\infty) \to (0,\infty)$ *is called* regularly varying (at infinity) *with* index $\alpha \in \mathbb{R}$ *(written* $f \in \mathrm{RV}_\alpha$*) if for all* $u > 0$*:*

$$\lim_{x\to\infty} \frac{f(ux)}{f(x)} = u^\alpha. \tag{1.2}$$

In the case $\alpha = 0$*, the function* f *is said to be* slowly varying (at infinity) *and the notation* ℓ *is customarily used for such a function.*

Remark 1.4. The definition of regular variation can be relaxed in various ways. For example, it suffices to require that the limit in (1.2) exists, is positive and finite for all $u > 0$.

Note that one can speak about *regular variation at* 0 by replacing $x \to \infty$ by $x \downarrow 0$ in (1.2). This concept will be formally defined later on in Definition 1.10.

Remark 1.5. By *regular variation*, we will always mean regular variation at infinity. The qualifier "at 0" will be added when regular variation at 0 is meant.

We refer to Bingham et al. [13] for a textbook treatment on the theory and applications of regular variation. A useful survey of regular variation is given in Seneta [58]. Some important and useful properties of functions of regular variation will be given in Section 1.3.

In Definition 1.4, regular variation of positive functions on $(0,\infty)$ is considered. However, various extensions exist including *regular variation on* $\mathbb{R}$ as remarked below.

Remark 1.6. By extension, a measurable and ultimately positive function g on the real line is said to be *regularly varying on* $\mathbb{R}$ with *index* $\alpha \in \mathbb{R}$ if $g(x) \sim f(x)$ as $x \to \infty$ for some $f \in \mathrm{RV}_\alpha$. We write $g \in \mathrm{RV}_\alpha$ on $\mathbb{R}$, adding the qualifier "on $\mathbb{R}$" to distinguish from regular variation on $(0,\infty)$.

Remark 1.7. Every function f that varies regularly with index $\alpha \in \mathbb{R}$ can be represented for $x > 0$ as $f(x) = x^\alpha \ell(x)$ with $\ell \in \mathrm{RV}_0$.

Distributions with a regularly varying right tail constitute a natural class of heavy-tailed distributions that plays a major role in applications of extreme value theory. It is defined as follows.

Definition 1.5. *A distribution function F on $\mathbb{R}$ belongs to the* regularly varying class $\mathcal{R}$ *if there exists $\alpha \geqslant 0$ such that $1 - F \in \mathrm{RV}_{-\alpha}$ on $\mathbb{R}$, or equivalently such that:*

$$1 - F(x) \sim x^{-\alpha}\ell(x) \quad \textit{as} \quad x \to \infty \tag{1.3}$$

for some $\ell \in \mathrm{RV}_0$.

The notion of regular variation shows up in various fields of applied probability such as in extreme value theory, point process theory, queueing theory, renewal theory and theory of summation of random variables. In particular, regular variation of the (left and right) tails of a distribution appears as a natural condition in various theoretical results of probability theory. Note that the work of de Haan [30] on regular variation has been of great importance for the development of modern extreme value theory.

The heavy-tailed family obtained by restricting the class $\mathcal{R}$ to distributions on $(0,\infty)$ may be split according as $\alpha = 0$ or $\alpha > 0$. When $\alpha = 0$ in (1.3), we have the following definition.

Definition 1.6. *A distribution function F on $(0,\infty)$ is said to be* super heavy-tailed *if $1-F \in \mathrm{RV}_0$.*

Remark 1.8. A super heavy-tailed distribution F has the representation $1-F(x) \sim \ell(x)$ as $x \to \infty$ for some slowly varying function ℓ that therefore has to satisfy $\ell(x) = o(1)$ as $x \to \infty$. A simple example is given by $1 - F(x) = (\log x)^{-1}$ for $x \geqslant \mathrm{e}$. Moreover, an a.s. positive random variable whose distribution function is super heavy-tailed has no finite moments of any positive order.

The case $\alpha > 0$ in (1.3) leads to the famous family of *Pareto-type distributions* that will play a prominent part in this thesis. The formal definition is as follows.

Definition 1.7. *A distribution function F on $(0,\infty)$ is said to be of* Pareto-type *with* index $\alpha > 0$ *if $1 - F \in \mathrm{RV}_{-\alpha}$.*

Remark 1.9. Any a.s. positive random variable X whose distribution function F is of Pareto-type with index $\alpha > 0$ has a moment of order $\delta > 0$ that satisfies:

$$\mathbb{E}\big[X^\delta\big] = \delta \int_0^\infty x^{\delta-1}\,(1 - F(x))\,\mathrm{d}x \begin{cases} < \infty & \text{if } \delta < \alpha \\ = \infty & \text{if } \delta > \alpha. \end{cases}$$

It is well-know that any super heavy-tailed or Pareto-type distribution is subexponential (e.g., Corollary 2(i) of Teugels [60]). Examples of distributions in $\mathcal{S}$ but not in $\mathcal{R}$ are the lognormal distribution $\mathrm{LN}(\mu, \sigma^2)$ with $\mu \in \mathbb{R}$ and $\sigma > 0$ and the Weibull distribution $\mathrm{W}(\tau, c)$ with $\tau \in (0, 1)$ and $c > 0$ (see Table 2.1). The class $\mathcal{R}$ restricted to distributions on $(0, \infty)$ is therefore a proper subset of the class $\mathcal{S}$ and as such provides a wealth of members of $\mathcal{S}$. Examples of subexponential distributions will be given in Section 2.5.

1.2.2 The Extremal Classes of Distributions

Distributions in the *extremal classes* are very useful in applications of extreme value theory. In fact, the set of such distributions coincides with the set of all distributions that belong to the *domain of attraction of an extreme value distribution* (or to a *maximum domain of attraction*). In particular, this means that the suitably normalized partial maximum of i.i.d. random variables from a distribution in any extremal class has a nondegenerate limit distribution.

Let $\{X_i;\, i \in \mathbb{N}\}$ be a sequence of i.i.d. random variables with distribution function F whose tail quantile function is denoted by U. Let $x_\ell := U(1+) \geqslant -\infty$ and $x_r := U(\infty) \leqslant \infty$ denote the left and right endpoints of the support of F respectively (see Section 1.1). For each $n \in \mathbb{N}$, let $X_{1:n}, \ldots, X_{n:n}$ denote the n random variables $X_1, \ldots, X_n$ arranged in nondecreasing order of magnitude, i.e.:

$$X_{1:n} \leqslant \cdots \leqslant X_{n:n} \quad \text{a.s.}$$

where $X_{n-i+1:n}$ is called the ith *largest order statistic* ($i = 1, \ldots, n$).

Remark 1.10. For each $n \in \mathbb{N}_0$, we define $X_{i:n} := x_\ell$ if $i \leqslant 0$ and $X_{i:n} := x_r$ if $i \geqslant n + 1$.

To study the asymptotic behavior of quantities involving upper order statistics of the i.i.d. random variables $X_1, \ldots, X_n$, it may be essential to assume that their common distribution function F belongs to an *extremal class* as defined below.

Definition 1.8. *A distribution function F on $\mathbb{R}$ with tail quantile function U belongs to the* extremal class $\mathcal{C}_\gamma$, $\gamma \in \mathbb{R}$, *if there exists a measurable function $a\colon (0, \infty) \to (0, \infty)$ such that for all $u > 0$:*

$$\lim_{x \to \infty} \frac{U(ux) - U(x)}{a(x)} = h_\gamma(u) := \begin{cases} \frac{u^\gamma - 1}{\gamma} & \text{if } \gamma \neq 0 \\ \log u & \text{if } \gamma = 0. \end{cases} \tag{1.4}$$

The function a appearing in (1.4) is called an *auxiliary function* for U. Note that any function $a_1\colon (0, \infty) \to (0, \infty)$ is an auxiliary function for U if and only if $a_1(x) \sim a(x)$ as $x \to \infty$, showing that a is unique up to asymptotic equivalence. We write $F \in \mathcal{C}_\gamma(a)$ if (1.4) holds with the auxiliary function a.

Remark 1.11. The limit on the right of (1.4) does not have to be specified. Indeed, if we only assume that $\lim_{x\to\infty}\{U(ux) - U(x)\}/a(x)$ exists for all $u > 0$ and is not constant in u, then $a \in \mathrm{RV}_\gamma$ for some $\gamma \in \mathbb{R}$ and the limit is given by the function h_γ (e.g., Theorem 1.9 of Geluk and de Haan [27]).

De Haan [31] proved that for every $\gamma \in \mathbb{R}$, the condition $F \in \mathcal{C}_\gamma$ is necessary and sufficient for F to belong to the *domain of attraction of the extreme value distribution* G_γ, by which we mean that

there exist normalizing constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that:

$$a_n^{-1}(X_{n:n} - b_n) \xrightarrow{\mathcal{D}} Y_\gamma \quad \text{as} \quad n \to \infty \tag{1.5}$$

where the limiting random variable Y_γ has the following distribution function:

$$G_\gamma(x) := \begin{cases} \exp\{-(1+\gamma x)^{-1/\gamma}\} & \text{if } \gamma \neq 0 \text{ for } 1+\gamma x > 0 \\ \exp\{-\mathrm{e}^{-x}\} & \text{if } \gamma = 0 \text{ for } x \in \mathbb{R}. \end{cases} \tag{1.6}$$

Remark 1.12. Since G_0 may be seen as the limit of G_γ when $\gamma \to 0$, the distribution function $G_\gamma(x) = \exp\{-(1+\gamma x)^{-1/\gamma}\}$, $1+\gamma x > 0$, serves as a representation for all $\gamma \in \mathbb{R}$ and provides as such a convenient unifying representation of the three extreme value types, namely *Fréchet-Pareto* ($\gamma > 0$), *extremal Weibull* ($\gamma < 0$) and *Gumbel* ($\gamma = 0$).

Remark 1.13. The domain of attraction of the Fréchet-Pareto distribution and that of the extremal Weibull distribution are both characterized by regular variation conditions. However, an extension of regular variation is required to characterize the domain of attraction of the Gumbel distribution.

The normalizing constants in (1.5) prove to be defined as $a_n = a(n)$ and $b_n = U(n)$, where a is an auxiliary function for U. Note that a possible choice for a is given by $a(x) = (U(\mathrm{e}x) - U(x))/h_\gamma(\mathrm{e})$, $x > 1$. Furthermore, it is well-known that the only possible nondegenerate limit distributions in (1.5) are given up to affine type by the one-parameter family $\{G_\gamma;\, \gamma \in \mathbb{R}\}$ of extreme value distributions (e.g., de Haan [30]).

Remark 1.14. The partial maximum $X_{n:n}$ satisfies $X_{n:n} \xrightarrow{a.s.} x_r \leqslant \infty$ as $n \to \infty$, according as the support of F is bounded above or not. The normalization of $X_{n:n}$ with a_n and b_n is therefore crucial to end up with a nondegenerate limit distribution.

The condition $F \in \mathcal{C}_\gamma$, $\gamma \in \mathbb{R}$, actually entails the existence of constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that for each $i \in \mathbb{N}$, the normalized ith largest order statistic $(X_{n-i+1:n} - b_n)/a_n$ converges in distribution as $n \to \infty$ to a nondegenerate random variable. This holds in particular with the same normalizing constants a_n and b_n as for the partial maximum $X_{n:n}$ in (1.5). The choice for the extremal classes of distributions is therefore not surprising when one deals with limiting results for upper order statistics.

The extremal class condition (1.4) rephrases the extreme value condition (1.5) in terms of the tail quantile function U of F, showing as such the prominent role played by U in extreme value theory. Furthermore, it turns out that the relation (1.4) fits within the framework of *generalized regular variation (of first order)*, see Chapter 3 of Bingham et al. [13] and Section I.2 of Geluk and de Haan [27]. In particular, the case $\gamma = 0$ leads to the so-called *de Haan class* Π that proves to be relevant to the description of the domain of attraction of the Gumbel distribution G_0. Being useful in Chapter 5, we now define this class of functions.

Definition 1.9. *A measurable function* $U\colon (0,\infty) \to \mathbb{R}$ *belongs to the* de Haan class Π *if there exists an auxiliary function* a *such that* (1.4) *holds with* $\gamma = 0$. *Such a function* U *is also said to be* Π_a-varying (at infinity) *and we write* $U \in \Pi_a$.

Let $U \in \Pi_a$. As in Remark 1.11, it automatically follows that $a \in \mathrm{RV}_0$. Moreover, $|U| \in \mathrm{RV}_0$ and $\lim_{x\to\infty} |U(x)|/a(x) = \infty$ (e.g., Theorem 3.7.4 of Bingham et al. [13]). Furthermore, any function

$a_1 \colon (0,\infty) \to (0,\infty)$ is such that $U \in \Pi_{a_1}$ if and only if $a_1(x) \sim a(x)$ as $x \to \infty$.

Finally, notice that the tail quantile function U of any distribution $F \in \mathcal{C}_0(a)$ is Π_a-varying and as such satisfies the last mentioned properties.

In parts of this dissertation, the extremal classes will be extensively used. In the following, we then collect properties in connection with these classes of distributions that will turn out to be essential in proving some of our results.

The relationship between U and a as appearing in (1.4) is very different according to the sign of γ. This is clear from the following result that can be found partly in Theorems 3.2.6, 3.2.7 and 3.7.4 of Bingham et al. [13].

Proposition 1.2. *Assume that $F \in \mathcal{C}_\gamma(a)$ for some $\gamma \in \mathbb{R}$. Then $a \in \mathrm{RV}_\gamma$ and:*

(i) If $\gamma > 0$: $a(x)/U(x) \to \gamma$ as $x \to \infty$ and $U \in \mathrm{RV}_\gamma$ or equivalently $1 - F \in \mathrm{RV}_{-1/\gamma}$.

(ii) If $\gamma < 0$: $U(\infty) < \infty$, $a(x)/(U(\infty) - U(x)) \to |\gamma|$ as $x \to \infty$ and $U(\infty) - U \in \mathrm{RV}_\gamma$.

(iii) If $\gamma = 0$: $a(x)/U(x) \to 0$ as $x \to \infty$ and in case $U(\infty)$ is finite, also $a(x)/(U(\infty)-U(x)) \to 0$ as $x \to \infty$; moreover $|U| \in \mathrm{RV}_0$.

Note that the regular variation property of the auxiliary function a has already been pointed out in Remark 1.11. Moreover, we see that for $\gamma > 0$ and distributions on $(0,\infty)$, we fall back on the family of Pareto-type distributions.

Remark 1.15. When the left endpoint of the support of F satisfies $x_\ell < 0$, the regular variation property of both functions $1 - F$ and U in Proposition 1.2(i) has to be considered on $\mathbb{R}$ since U then assumes negative values at the beginning of its domain.

Remark 1.16. One can show by using Proposition 1.2 that:

$$F \in \mathcal{C}_\gamma(a) \implies \begin{cases} a/U \in \mathrm{RV}_\gamma & \text{if } \gamma < 0 \text{ and } U(\infty) \neq 0 \\ a/U \in \mathrm{RV}_0 & \text{if } \gamma \geqslant 0 \text{ or } U(\infty) = 0. \end{cases}$$

The next result due to de Haan [30] provides alternative conditions for a distribution function F to belong to a specific extremal class. In particular, it offers the possibility to switch from extremal properties of F to that of its tail quantile function U and conversely. See also Section 2.6 of Beirlant et al. [5].

Theorem 1.3. *Let $\gamma \in \mathbb{R}$ and $a \colon (0,\infty) \to (0,\infty)$ be a measurable function. Then the following assertions are equivalent:*

(i) $F \in \mathcal{C}_\gamma(a)$;

(ii) For all $u \in \mathbb{R}$ for which $1 + \gamma u > 0$:

$$\lim_{x \uparrow x_r} \frac{1 - F(x + h(x)u)}{1 - F(x)} = \frac{1}{h_\gamma^{\leftarrow}(u)} =: \eta_\gamma(u)$$

where $h \circ U = a$;

(*iii*) *For all* $u \in \mathbb{R}$ *for which* $1 + \gamma u > 0$*:*

$$\lim_{x\to\infty} x\{1 - F(U(x) + a(x)u)\} = \eta_\gamma(u).$$

It follows from the definition of the function h_γ that $\eta_\gamma(u) = (1+\gamma u)^{-1/\gamma}$ for $1 + \gamma u > 0$ if $\gamma \neq 0$ while $\eta_0(u) = \mathrm{e}^{-u}$ for $u \in \mathbb{R}$ if $\gamma = 0$.

If there exists $\gamma \in \mathbb{R}$ such that (1.5) holds with normalizing constants $a_n = a(n)$ and $b_n = U(n)$, where $a\colon (0,\infty) \to (0,\infty)$ is a measurable function, then $F \in \mathcal{C}_\gamma(a)$ since Theorem 1.3(*ii*) proves to be true. It is readily shown that the converse is also valid.

Remark 1.17. It follows from Theorem 3 of de Haan and Stadtmüller [32] that for every $\gamma \in \mathbb{R}$, the convergence in Theorem 1.3(*ii*)-(*iii*) holds *locally uniformly* in $(-1/\max(0,\gamma), 1/\max(-\gamma,0))$, i.e. uniformly on each compact subset of $(-1/\max(0,\gamma), 1/\max(-\gamma,0))$, with the convention $1/0 = \infty$.

A last useful result connected to uniform convergence for the asymptotic relation (1.4) is now given (e.g., Theorem 1.14 of Geluk and de Haan [27] for the case $\gamma = 0$ and Theorems 3.2.6 and 3.2.7 of Bingham et al. [13] for the cases $\gamma < 0$ and $\gamma > 0$ respectively).

Theorem 1.4. *Assume that* $F \in \mathcal{C}_\gamma(a)$ *for some* $\gamma \in \mathbb{R}$*. Then the convergence in* (1.4) *holds uniformly in* u*:*

(*i*) *on each* $[b,c]$ $(0 < b \leqslant c < \infty)$ *if* $\gamma = 0$*;*

(*ii*) *on each* $(0,c]$ $(0 < c < \infty)$ *if* $\gamma > 0$*;*

(*iii*) *on each* $[b,\infty)$ $(0 < b < \infty)$ *if* $\gamma < 0$*.*

Remark 1.18. Theorem 1.4 shows in particular that for every $\gamma \in \mathbb{R}$, the convergence in (1.4) holds locally uniformly in $(0,\infty)$.

1.3 Properties of Regularly Varying Functions

Functions of regular variation will show up very often in this dissertation. We then collect in this section some important and useful results about them. The properties concern regular variation at infinity (see Definition 1.4). However, corresponding properties can be derived for *regularly varying functions at* 0 whose concept will be defined in Definition 1.10. For proofs and further properties, we refer to Seneta [58], Bingham et al. [13] and Geluk and de Haan [27].

We first present two basic but fundamental theorems since most properties of regularly varying functions follow readily from them. The first result determines exactly which functions can be of slow variation (e.g., Theorem 1.3.1 of Bingham et al. [13]).

Theorem 1.5. *A function* $\ell\colon (0,\infty) \to (0,\infty)$ *is slowly varying if and only if it can be written as:*

$$\ell(x) = c(x)\exp\left\{\int_{x_0}^{x} \frac{\varepsilon(u)}{u}\,\mathrm{d}u\right\}, \quad x \geqslant x_0$$

for some $x_0 > 0$*, where* $\lim_{x\to\infty} c(x) = c \in (0,\infty)$ *and* $\lim_{x\to\infty} \varepsilon(x) = 0$*.*

Typical examples of slowly varying functions are positive constants or functions converging to a positive constant, logarithms, powers of logarithms and iterated logarithms.

Remark 1.19. It follows from Remark 1.7 and Theorem 1.5 that a function $f\colon (0,\infty) \to (0,\infty)$ is regularly varying with index $\alpha \in \mathbb{R}$ if and only if it has the representation:

$$f(x) = d(x) \exp\left\{ \int_{x_0}^{x} \frac{\alpha(u)}{u}\, \mathrm{d}u \right\}, \quad x \geqslant x_0$$

for some $x_0 > 0$, where $\lim_{x\to\infty} d(x) = d \in (0,\infty)$ and $\lim_{x\to\infty} \alpha(x) = \alpha$.

It follows at once from Remark 1.19 that if $f \in \mathrm{RV}_\alpha$, $\alpha \neq 0$, then:

$$\lim_{x\to\infty} f(x) = \begin{cases} \infty & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha < 0. \end{cases}$$

This shows in particular that if $\ell \in \mathrm{RV}_0$, then for every $\epsilon > 0$:

$$\lim_{x\to\infty} x^{-\epsilon}\ell(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} x^{\epsilon}\ell(x) = \infty.$$

Some intuitive meaning to the notion of *slow variation* is then given since a slowly varying function is "small" compared to any power function.

The second useful result is that the convergence in (1.2) holds locally uniformly in $(0,\infty)$. However, some refinements exist when $\alpha \neq 0$ as stated below (e.g., Theorem 1.5.2 of Bingham et al. [13]).

Theorem 1.6. *Assume that $f \in \mathrm{RV}_\alpha$, $\alpha \in \mathbb{R}$. In the case $\alpha > 0$, assume further that f is bounded on each interval $(0, c]$, $c > 0$. Then:*

$$\lim_{x\to\infty} \frac{f(ux)}{f(x)} = u^\alpha \quad \textit{uniformly in } u$$

(*i*) *on each $[a, b]$ $(0 < a \leqslant b < \infty)$ if $\alpha = 0$;*

(*ii*) *on each $(0, b]$ $(0 < b < \infty)$ if $\alpha > 0$;*

(*iii*) *on each $[a, \infty)$ $(0 < a < \infty)$ if $\alpha < 0$.*

The next result follows immediately from Theorem 1.5. It concerns *Potter-type bounds* (e.g., Theorem 1.5.6 of Bingham et al. [13]).

Theorem 1.7.

(*i*) *If $\ell \in \mathrm{RV}_0$ then for any chosen constants $A > 1$, $\delta > 0$, there exists $c := c(A, \delta)$ such that:*

$$\frac{\ell(y)}{\ell(x)} \leqslant A \max\left\{ \left(\frac{y}{x}\right)^{\delta}, \left(\frac{y}{x}\right)^{-\delta} \right\}, \quad x \geqslant c,\ y \geqslant c.$$

(*ii*) *If $\ell \in \mathrm{RV}_0$ is bounded away from 0 and ∞ on every compact subset of $[0,\infty)$ then for every $\delta > 0$, there exists $A' := A'(\delta) > 1$ such that:*

$$\frac{\ell(y)}{\ell(x)} \leqslant A' \max\left\{ \left(\frac{y}{x}\right)^{\delta}, \left(\frac{y}{x}\right)^{-\delta} \right\}, \quad x > 0,\ y > 0.$$

(*iii*) *If* $f \in \mathrm{RV}_\alpha$, $\alpha \in \mathbb{R}$, *then for any chosen constants* $A > 1$, $\delta > 0$, *there exists* $c := c(A, \delta)$ *such that:*

$$\frac{f(y)}{f(x)} \leqslant A \max\left\{ \left(\frac{y}{x}\right)^{\alpha+\delta}, \left(\frac{y}{x}\right)^{\alpha-\delta} \right\}, \quad x \geqslant c,\, y \geqslant c.$$

The following theorem on asymptotic conjugacy is due to de Bruijn [16]. See also Theorem 1.5.13 of Bingham et al. [13].

Theorem 1.8. *If* $\ell \in \mathrm{RV}_0$, *there exists* $\ell^\# \in \mathrm{RV}_0$, *unique up to asymptotic equivalence, satisfying:*

$$\lim_{x\to\infty} \ell(x)\, \ell^\#(x\ell(x)) = 1 \quad \textit{and} \quad \lim_{x\to\infty} \ell^\#(x)\, \ell(x\ell^\#(x)) = 1.$$

Furthermore, $\ell^{\#\#}(x) \sim \ell(x)$ *as* $x \to \infty$.

The function $\ell^\#$ is called a *de Bruijn conjugate* of ℓ. This concept arises in problems connected with the asymptotic inversion of regularly varying functions with positive index. A typical area of applications involves maximum and additive domain of attraction problems in probability theory. The notion of an *additive domain of attraction* will be introduced in Section 1.4.

Regularly varying functions possess a number of closure properties. Some of the most useful are collected below (e.g., Proposition 1.5.7 of Bingham et al. [13]).

Proposition 1.9.

(*i*) *If* $f \in \mathrm{RV}_\alpha$ *(*$\alpha \in \mathbb{R}$*) then* $f^\rho \in \mathrm{RV}_{\rho\alpha}$ *for any* $\rho \in \mathbb{R}$.

(*ii*) *If* $f_1 \in \mathrm{RV}_{\alpha_1}$ *(*$\alpha_1 \in \mathbb{R}$*),* $f_2 \in \mathrm{RV}_{\alpha_2}$ *(*$\alpha_2 \geqslant 0$*) and* $\lim_{x\to\infty} f_2(x) = \infty$ *then* $f_1 \circ f_2 \in \mathrm{RV}_{\alpha_1\alpha_2}$.

(*iii*) *If* $f_1 \in \mathrm{RV}_{\alpha_1}$ *(*$\alpha_1 \in \mathbb{R}$*) and* $f_2 \in \mathrm{RV}_{\alpha_2}$ *(*$\alpha_2 \in \mathbb{R}$*) then* $f_1 f_2 \in \mathrm{RV}_{\alpha_1+\alpha_2}$ *and* $f_1 + f_2 \in \mathrm{RV}_\alpha$ *where* $\alpha := \max(\alpha_1, \alpha_2)$.

(*iv*) *If* $f_i \in \mathrm{RV}_{\alpha_i}$ *with* $\alpha_i \in \mathbb{R}$ *(*$i = 1, \ldots, k$*) and* $r(x_1, \ldots, x_k)$ *is a rational function with positive coefficients then* $r(f_1(x), \ldots, f_k(x))$ *is regularly varying.*

It is sometimes convenient to transfer attention from infinity to the origin. With obvious changes, we may speak about *regular variation at* 0. The concept of a *regularly varying function at* 0 is formally defined as follows.

Definition 1.10. *A measurable function* $f\colon (0,\infty) \to (0,\infty)$ *is called* regularly varying (on the right) at 0 *with* index $\alpha \in \mathbb{R}$ *(written* $f \in \mathrm{RV}_\alpha^0$*) if for all* $u > 0$*:*

$$\lim_{x\downarrow 0} \frac{f(ux)}{f(x)} = u^\alpha.$$

In the case $\alpha = 0$, *the function* f *is said to be* slowly varying at 0.

Note that for every $\alpha \in \mathbb{R}$, $f \in \mathrm{RV}_\alpha^0$ if and only if $g \in \mathrm{RV}_{-\alpha}$ where $g(x) := f(1/x)$ for all $x > 0$. The theories related to regular variation at infinity and regular variation at 0 are thus equivalent. So far, we have considered properties of regularly varying functions at infinity. Corresponding properties can then be derived for regularly varying functions at 0.

The next section deals with the notion of the *domain of attraction of a stable distribution* which is closely connected to regular variation.

1.4 Stable Distributions and Their Domains of Attraction

We start by defining the concept of a *stable distribution* (e.g., Sections VI.1 and XIII.6 of Feller [25] or Section 8.3 of Bingham et al. [13]).

Definition 1.11. *The distribution function G of a nondegenerate random variable Y is said to be* stable *if for every $n \in \mathbb{N}$, there exist constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that:*

$$Y_1 + \cdots + Y_n \stackrel{\mathcal{D}}{=} c_n Y + d_n \tag{1.7}$$

where $Y_1, \ldots, Y_n$ are mutually independent copies of Y. If $d_n = 0$ for all $n \in \mathbb{N}$, the distribution function G is called strictly stable.

Alternatively, we will see below that a nondegenerate distribution is stable if and only if it is the limit distribution of a suitably normalized partial sum of i.i.d. random variables.

For every stable distribution, there exists a constant $\alpha \in (0, 2]$ such that (1.7) holds with $c_n = n^{1/\alpha}$. This constant is called the *exponent (of stability)* and stable distributions with exponent α are also referred to as *α-stable distributions*. A distribution function is said to be *non-normal stable* if it is stable with exponent $\alpha < 2$. Finally, a random variable is *stable* if its distribution function is stable.

The stable distributions are usually described analytically through their characteristic functions. A distribution on $\mathbb{R}$ is α-stable, $\alpha \in (0, 2]$, if and only if its characteristic function has the form:

$$\phi(t) = \exp\{\mathrm{i}\mu t - \sigma|t|^\alpha(1 - \mathrm{i}\beta \operatorname{sign}(t)\, z(t, \alpha))\}, \quad t \in \mathbb{R}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, $\beta \in [-1, 1]$ and:

$$z(t, \alpha) = \begin{cases} \tan\left(\frac{\alpha\pi}{2}\right) & \text{if } \alpha \neq 1 \\ -\frac{2}{\pi}\log|t| & \text{if } \alpha = 1. \end{cases}$$

In the above parametrization, μ is a location parameter, σ is a scaling constant and β describes the skewness of the α-stable distribution. The stable distributions with exponent $\alpha = 1$ are the Cauchy distributions, symmetric if $\beta = 0$ and asymmetric otherwise. For $\alpha = 2$, we clearly have the normal distributions since β drops out. The density function of an α-stable distribution is not known in closed form except for the normal distributions, the symmetric Cauchy distributions or the Lévy distributions ($\alpha = 1/2$, $\beta = 1$). A stable distribution is strictly stable if and only if ($\alpha \neq 1$, $\mu = 0$) or ($\alpha = 1$, $\beta = 0$). A stable random variable X with exponent $\alpha < 2$ satisfies $\mathbb{E}[|X|^\delta] = \infty$ for $\delta \geqslant \alpha$. In particular, $\mathbb{V}[X] = \infty$ if $\alpha < 2$ and $\mathbb{E}[|X|] = \infty$ if $\alpha \leqslant 1$. Finally, note that every stable distribution is *infinitely divisible* (e.g., Chapter XVII of Feller [25]).

Now, we introduce the notion of an *additive domain of attraction*.

Definition 1.12. *The distribution function F of i.i.d. random variables $X_1, \ldots, X_n$ belongs to the* additive domain of attraction *of a nondegenerate distribution function G if there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that:*

$$a_n^{-1}\left(\sum_{i=1}^{n} X_i - b_n\right) \xrightarrow{\mathcal{D}} Y \quad \text{as} \quad n \to \infty \tag{1.8}$$

where the limiting random variable Y has distribution function G.

It is well-known that a distribution function possesses an additive domain of attraction if and only if it is stable. In other words, the class of stable distributions coincides with the class of all possible limit distributions in (1.8). Furthermore, it is interesting to remark that the stable distributions play the same role for sums as the extreme value distributions do for extremes (see Subsection 1.2.2).

The following result characterizes the (additive) domain of attraction of a stable distribution completely in terms of regular variation (e.g., Sections IX.8 and XVII.5 of Feller [25] or Theorem 8.3.1 of Bingham et al. [13]).

Theorem 1.10.

(*i*) *A distribution function F on $\mathbb{R}$ belongs to the domain of attraction of a normal distribution ($\alpha = 2$) if and only if the truncated second moment function $V_2(x) := \int_{-x}^{x} y^2 \, \mathrm{d}F(y)$ is slowly varying, or equivalently if and only if:*

$$\lim_{x \to \infty} \frac{x^2\{1 - F(x) + F(-x)\}}{V_2(x)} = 0.$$

(*ii*) *A distribution function F on $\mathbb{R}$ belongs to the domain of attraction of a non-normal stable distribution with exponent $\alpha \in (0, 2)$ if and only if the tail sum $1 - F(x) + F(-x)$ is regularly varying with index $-\alpha$ and the following tail-balance condition holds:*

$$\lim_{x \to \infty} \frac{1 - F(x)}{1 - F(x) + F(-x)} = p, \quad \lim_{x \to \infty} \frac{F(-x)}{1 - F(x) + F(-x)} = q$$

with $p \geqslant 0$, $q \geqslant 0$ and $p + q = 1$.

Note that in the non-normal case, the value of the skewness parameter β is determined from the above tail-balance condition by $\beta = p - q$. Also, we know from the central limit theorem that the distribution function of a random variable X belongs to the domain of attraction of a normal distribution if $\mathbb{E}[X^2] \in (0, \infty)$. This condition is actually included in Theorem 1.10(i) as the case $\lim_{x \to \infty} V_2(x) = \mathbb{E}[X^2] \in (0, \infty)$.

Any random variable X whose distribution function belongs to the domain of attraction of an α-stable distribution satisfies $\mathbb{E}[|X|^\delta] < \infty$ for $\delta < \alpha$ while if $\alpha < 2$, $\mathbb{E}[|X|^\delta] = \infty$ for $\delta > \alpha$ (e.g., Lemma XVII.5 of Feller [25]). In particular, $\mathbb{V}[X] = \infty$ if $\alpha < 2$, $\mathbb{E}[|X|] < \infty$ if $\alpha > 1$ and $\mathbb{E}[|X|] = \infty$ if $\alpha < 1$.

Remark 1.20. Recall from Section XIII.6 of Feller [25] that the stable distributions with Laplace-Stieltjes transform e^{-s^α} for $\alpha \in (0, 1)$ are to within scale the only ones concentrated on $(0, \infty)$. Also, it follows from Theorem 1.10(ii) that a distribution function F on $(0, \infty)$ belongs to the domain of attraction of a non-normal stable distribution with exponent $\alpha \in (0, 2)$ if and only if $1 - F \in \mathrm{RV}_{-\alpha}$ (i.e., F is of Pareto-type with index α).

1.5 Counting Processes

Let us specify that all the random variables involved here are a.s. finite-valued. Also, the greatest lower bound of a set $A \subseteq [0, \infty)$ is denoted by $\inf A$ and the least upper bound by $\sup A$, with the conventions $\inf \varnothing = \infty$ and $\sup \varnothing = 0$. A *counting process* is formally defined as follows.

Definition 1.13. *A continuous-time process* $\{N(t);\, t \geqslant 0\}$ *is called a* counting process *if a.s., its sample paths are* $\mathbb{N}_0$*-valued, nondecreasing and right-continuous, start at* 0 *and have jumps of size* 1.

To a given counting process $\{N(t);\, t \geqslant 0\}$, we associate the sequence $\{T_n;\, n \in \mathbb{N}_0\}$ of *jump times* and the sequence $\{W_n;\, n \in \mathbb{N}_0\}$ of *inter-jump times* by defining:

$$\begin{aligned} T_n &:= \inf\{t \geqslant 0 \colon N(t) = n\}, \quad n \in \mathbb{N}_0 \\ W_n &:= T_{n+1} - T_n, \quad n \in \mathbb{N}_0. \end{aligned}$$

Note that $T_0 = 0$ a.s. is not considered as a jump time. Since $\mathbb{P}[T_n < \infty] = 1$ for $n \in \mathbb{N}_0$, we have $T_n < T_{n+1}$ a.s. for $n \in \mathbb{N}_0$ and, in particular, $T_1 > 0$ a.s. so that $\{T_n;\, n \in \mathbb{N}\}$ is an increasing sequence of a.s. positive random variables. It follows that $\mathbb{P}[N(t) < \infty] = 1$ for each $t \geqslant 0$. Also, we have $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$. One can interpret the random variables T_n, $n \in \mathbb{N}$, as the *arrival times* of some events while for any $t \geqslant 0$, the random variable $N(t)$ *counts* the number of events which occurred by time t. This interpretation justifies the terminology *counting process*. Note that for any $0 \leqslant s < t$, the random variable $N(t) - N(s)$ counts the number of events that have occurred in the time interval $(s, t]$. The random variables W_n, $n \in \mathbb{N}_0$, are then considered as the *inter-arrival times* between two consecutive events.

Remark 1.21. For every $t \geqslant 0$, it is clear that we have $\{N(t) = n\} = \{T_n \leqslant t < T_{n+1}\}$ and then $\{N(t) \geqslant n\} = \{T_n \leqslant t\}$ for all $n \in \mathbb{N}_0$.

Conversely, one can construct a counting process $\{N(t);\, t \geqslant 0\}$ from any sequence $\{T_n;\, n \in \mathbb{N}_0\}$ of random variables satisfying $T_0 = 0$ a.s. and $T_n < T_{n+1}$ a.s. for $n \in \mathbb{N}_0$ by putting:

$$N(t) := \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n \leqslant t\}} = \sup\{n \in \mathbb{N} \colon T_n \leqslant t\}, \quad t \geqslant 0. \tag{1.9}$$

It is clear that $N(0) = 0$ a.s. (since $T_1 > 0$ a.s.) and, by construction, that the process $\{N(t);\, t \geqslant 0\}$ is $\mathbb{N}_0$-valued, nondecreasing and *càdlàg* (since a.s., the sample paths are right-continuous in $t \geqslant 0$ with left limits in $t > 0$). Finally, the jumps are of size 1 because the sequence $\{T_n;\, n \in \mathbb{N}_0\}$ is increasing.

It is readily seen that any sequence $\{W_n;\, n \in \mathbb{N}_0\}$ of a.s. positive random variables also generates a counting process $\{N(t);\, t \geqslant 0\}$ by defining the sequence $\{T_n;\, n \in \mathbb{N}_0\}$ involved in (1.9) as $T_n := W_0 + \ldots + W_{n-1}$ for $n \in \mathbb{N}$ and $T_0 := 0$. If further the W_i's are i.i.d., the counting process $\{N(t);\, t \geqslant 0\}$ is called a *renewal counting process* as formally defined below.

Definition 1.14. *Let* $\{W_n;\, n \in \mathbb{N}_0\}$ *be a sequence of a.s. positive i.i.d. random variables. Then the counting process* $\{N(t);\, t \geqslant 0\}$ *defined by* (1.9) *when the sequence* $\{T_n;\, n \in \mathbb{N}_0\}$ *is such that* $T_n := W_0 + \ldots + W_{n-1}$ *for* $n \in \mathbb{N}$ *and* $T_0 := 0$ *is said to be a* renewal counting process *generated by the distribution function of* W_0.

The first explicit example of a counting process is trivial and refers to discrete time. It is called *non-random* since for each $t \geqslant 0$:

$$\mathbb{P}[N(t) = n] = \begin{cases} 1 & \text{if } n = t \\ 0 & \text{if } n \neq t \end{cases}, \quad n \in \mathbb{N}_0.$$

Another example is given by the important *homogeneous Poisson process*. It can be defined as a particular renewal counting process in the following way.

Definition 1.15. *A renewal counting process* $\{N(t);\, t \geqslant 0\}$ *is called a* homogeneous Poisson process *with* intensity $\lambda \in (0,\infty)$ *if it is generated by the exponential distribution* $\mathrm{Exp}(\lambda)$. *Also, a homogeneous Poisson process with intensity* $\lambda = 1$ *is said to be* standard.

For the definition of the exponential distribution $\mathrm{Exp}(\lambda)$ with parameter $\lambda > 0$, see Table 2.2. The homogeneous Poisson process has various attractive theoretical properties. In particular, it is the only counting process with independent and stationary increments. If $\{N(t);\, t \geqslant 0\}$ is a homogeneous Poisson process with intensity $\lambda \in (0,\infty)$, then the random variable $N(t)$ has the Poisson distribution $\mathrm{Poi}(\lambda t)$ for every $t > 0$, i.e.:

$$\mathbb{P}[N(t) = n] = \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n \in \mathbb{N}_0.$$

Clearly, it follows that $\mathbb{V}[N(t)] = \mathbb{E}[N(t)] = \lambda t$ for any $t > 0$.

In many practical situations, it may be of prime interest to suppose that the distribution of a counting process is given by a mixture of homogeneous Poisson processes. This means that, conditioning on an extraneous random variable Λ, the counting process behaves like a homogeneous Poisson process. A *mixed Poisson process* is then constructed by randomizing the intensity λ of a homogeneous Poisson process so that it can be derived from a standard homogeneous Poisson process by a random time change. We refer to the monograph by Grandell [29] for a very thorough treatment of mixed Poisson processes and their properties. A formal definition is as follows.

Definition 1.16. *Let an a.s. positive random variable* Λ *and a standard homogeneous Poisson process* $\{\tilde{N}(t);\, t \geqslant 0\}$ *be independent of each other. Then the counting process* $\{N(t);\, t \geqslant 0\} = \{\tilde{N}(\Lambda t);\, t \geqslant 0\}$ *is called a* mixed Poisson process *with* mixing random variable Λ.

The distribution function of the mixing random variable Λ is called the *mixing distribution* and denoted by H. For each fixed $t \geqslant 0$, the random variable $N(t)$ then has a *mixed Poisson distribution* given by:

$$\mathbb{P}[N(t) = n] = \mathbb{E}\left[\mathrm{e}^{-\Lambda t}\frac{(\Lambda t)^n}{n!}\right] = \int_0^\infty \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!}\,\mathrm{d}H(\lambda), \quad n \in \mathbb{N}_0.$$

The role of the mixing random variable Λ can be highlighted by observing that:

$$\frac{N(t)}{t} \xrightarrow{a.s.} \Lambda \quad \text{as} \quad t \to \infty. \tag{1.10}$$

It is clear that for any $\lambda \in (0,\infty)$, the conditional distribution of $N(t)$ given $\Lambda = \lambda$ is the Poisson distribution $\mathrm{Poi}(\lambda t)$. When the mixing distribution H is degenerate at a single point $\lambda \in (0,\infty)$, i.e. $\Lambda = \lambda$ a.s., we retrieve the homogeneous Poisson process with intensity λ. Furthermore, note that the homogeneous Poisson process is the only mixed Poisson process that is simultaneously a renewal counting process.

The mixed Poisson process has stationary increments but loses some of the properties of the homogeneous Poisson process when the mixing random variable Λ is nondegenerate. Indeed, it then has dependent increments and is *over-dispersed* since $\mathbb{V}[N(t)] = t\,\mathbb{E}[\Lambda] + t^2\mathbb{V}[\Lambda] > t\,\mathbb{E}[\Lambda] = \mathbb{E}[N(t)]$ for any $t > 0$ when $\mathbb{V}[\Lambda] < \infty$.

A very popular example of a mixed Poisson process is the *Pólya process* with parameters $\alpha > 0$ and $\beta > 0$ which is obtained by choosing the gamma distribution $\Gamma(\alpha, \beta)$ for the mixing random variable Λ (see Table 2.2). It then follows that the random variable $N(t)$ has the negative binomial distribution $\mathrm{NB}(\alpha, t/(t+\beta))$ for every $t > 0$, i.e.:

$$\mathbb{P}[N(t) = n] = \frac{\Gamma(\alpha+n)}{n!\,\Gamma(\alpha)} \left(\frac{\beta}{t+\beta}\right)^{\alpha} \left(\frac{t}{t+\beta}\right)^{n}, \quad n \in \mathbb{N}_0.$$

Note that for $\alpha \to \infty$, $\beta \to \infty$ while $\alpha/\beta \to \lambda \in (0, \infty)$, the Pólya process turns into the homogeneous Poisson process with intensity λ.

Weakening the convergence in (1.10) to convergence in probability leads to an obvious generalization of the mixed Poisson process.

Definition 1.17. *Let Λ be an a.s. positive random variable. A counting process $\{N(t);\, t \geqslant 0\}$ is said to* p-average in time to Λ *if:*

$$\frac{N(t)}{t} \xrightarrow{\;p\;} \Lambda \quad \textit{as} \quad t \to \infty.$$

We can even request the weaker condition that the convergence holds in distribution rather than in probability.

Definition 1.18. *Let Λ be an a.s. positive random variable. A counting process $\{N(t);\, t \geqslant 0\}$ is said to* $\mathcal{D}$-average in time to Λ *if:*

$$\frac{N(t)}{t} \xrightarrow{\;\mathcal{D}\;} \Lambda \quad \textit{as} \quad t \to \infty.$$

The convergence in distribution is sufficient for the pointwise convergence of the corresponding Laplace-Stieltjes transforms. It is also necessary provided the limiting Laplace-Stieltjes transform is continuous on the right at the origin. A counting process $\{N(t);\, t \geqslant 0\}$ that $\mathcal{D}$-averages in time or p-averages in time to the random variable Λ therefore satisfies:

$$\lim_{t\to\infty} \mathbb{E}\big[\mathrm{e}^{-\theta \frac{N(t)}{t}}\big] = \mathbb{E}\big[\mathrm{e}^{-\theta\Lambda}\big], \quad \theta \geqslant 0. \tag{1.11}$$

Remark 1.22. For every $\theta \geqslant 0$, define $u_\theta(x) := \mathrm{e}^{-\theta x}$ for $x \geqslant 0$. The family of functions $\{u_\theta\}_{\theta\geqslant 0}$ being equicontinuous provided θ is restricted to a finite interval, the convergence in (1.11) holds uniformly in every finite θ-interval (e.g., Corollary VIII.1 of Feller [25]).

Very popular counting processes $\mathcal{D}$-average in time. We quote a number of examples. The non-random case provides a first trivial example where $\Lambda = 1$ a.s. (since $N(t) = t$ a.s. for each $t \in \mathbb{N}_0$). Because almost sure convergence entails convergence in distribution, any mixed Poisson process $\mathcal{D}$-averages in time to its mixing random variable. Any renewal counting process generated by a positive distribution with mean $\mu < \infty$ also $\mathcal{D}$-averages in time with $\Lambda = 1/\mu$ a.s. (the convergence actually holds almost surely, see Theorem 5.5.2 of Chung [19]).

Consider a general counting process $\{N(t);\, t \geqslant 0\}$. We now introduce some useful notation and specify a number of its characteristics.

Fix a time $t \geqslant 0$. The distribution of the random variable $N(t)$ that counts the number of events up to t is denoted by:

$$p_n(t) := \mathbb{P}[N(t) = n], \quad n \in \mathbb{N}_0$$

and the probability of less than r events up to t is given by:

$$\Pi_r(t) := \mathbb{P}[N(t) \leqslant r-1] = \sum_{n=0}^{r-1} p_n(t), \quad r \in \mathbb{N}.$$

Remark 1.23. Since $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$, it follows that $\lim_{t\to\infty} p_n(t) = 0$ for any $n \in \mathbb{N}_0$ and then that $\lim_{t\to\infty} \Pi_r(t) = 0$ for any $r \in \mathbb{N}$.

Also, the *probability generating function* of $N(t)$ is defined by:

$$Q_t(z) := \mathbb{E}\big[z^{N(t)}\big] = \sum_{n=0}^{\infty} p_n(t)\, z^n, \quad |z| \leqslant 1.$$

For every $r \in \mathbb{N}_0$, the rth derivative of $Q_t(z)$ with respect to z is given by:

$$Q_t^{(r)}(z) := \frac{\partial^r Q_t(z)}{\partial z^r} = \sum_{n=r}^{\infty} \frac{n!}{(n-r)!}\, p_n(t)\, z^{n-r} = r!\, \mathbb{E}\left[\binom{N(t)}{r} z^{N(t)-r}\right], \quad |z| < 1. \tag{1.12}$$

The function $Q_t^{(r)}$ can be used to evaluate consecutive moments of $N(t)$ since:

$$Q_t^{(r)}(1-) = \mathbb{E}[N(t)(N(t)-1)\dots(N(t)-r+1)] = r!\, \mathbb{E}\left[\binom{N(t)}{r}\right], \quad r \in \mathbb{N}.$$

For instance, we get $\mathbb{E}[N(t)] = Q_t^{(1)}(1-)$ and $\mathbb{V}[N(t)] = Q_t^{(2)}(1-) + Q_t^{(1)}(1-)\big[1 - Q_t^{(1)}(1-)\big]$ for the mean and the variance of $N(t)$, respectively.

Now, consider a counting process $\{N(t);\, t \geqslant 0\}$ that $\mathcal{D}$-averages in time to the random variable Λ. For every $r \in \mathbb{N}_0$, we then define the auxiliary function $q_r(w) := \mathbb{E}\big[\mathrm{e}^{-w\Lambda}\Lambda^r\big]$ for $w \geqslant 0$. Notice that q_0 is the Laplace-Stieltjes transform of Λ. Also, $q_r(0) = \mathbb{E}[\Lambda^r] \leqslant \infty$.

For any $\beta > 0$ and $r \in \mathbb{N}_0$, we readily obtain:

$$\int_0^{\infty} w^{\beta-1} q_r(w)\, \mathrm{d}w = \Gamma(\beta)\, \mathbb{E}\big[\Lambda^{r-\beta}\big]. \tag{1.13}$$

In particular, it follows that for any $r \in \mathbb{N}$:

$$\frac{1}{(r-1)!} \int_0^{\infty} w^{r-1} q_r(w)\, \mathrm{d}w = 1. \tag{1.14}$$

In the special case where $\{N(t);\, t \geqslant 0\}$ is a mixed Poisson process with mixing random variable Λ, we compute for any $r \in \mathbb{N}_0$:

$$Q_t^{(r)}(z) = t^r\, \mathbb{E}\big[\mathrm{e}^{-t(1-z)\Lambda}\Lambda^r\big], \quad |z| < 1$$

so that the following equality holds for all $w \in (0, t]$:

$$\frac{1}{t^r}\, Q_t^{(r)}\Big(1 - \frac{w}{t}\Big) = q_r(w) \tag{1.15}$$

where the right-hand side does no longer depend upon t.

Part I

Large Claims Reinsurance

Chapter 2

Introduction to Reinsurance

2.1 Aims of Reinsurance

A *reinsurance treaty* is a contractual agreement between an insurance company (*insurer*) and a reinsurance company (*reinsurer*) stipulating which share of future losses suffered by the insurer will be borne by the reinsurer. *Reinsurance* is therefore seen as one way of *risk sharing*. Among the main reasons to ponder reinsurance, the insurer can think about the appearance of *excessively large claims* (airplane crashes, medical malpractice, terrorism, ...) and an unusually *large number of claims* (earthquakes, fires, storms and floods, hurricanes, volcano eruptions, ...). These aspects will be considered in this dissertation.

In the present chapter as well as in the other chapters of Part I, we assume that the insurance portfolio is governed by the *collective risk model* that combines *claim sizes* and *claim arrival times*. This basic model of collective risk theory is introduced in Section 2.2. The choice of a specific reinsurance treaty may be motivated by one or more of the arguments above. In Section 2.3, we provide the mathematical formulation of some standard reinsurance treaties. They can be classified in two types, namely treaties of *random walk type* (see Subsection 2.3.1) and treaties of *extreme value type* (see Subsection 2.3.2). In Section 2.4, we specify major models for the *claim number process*, i.e. the counting process of the claim arrival times. Finally, realistic *claim size distributions* are discussed in Section 2.5.

2.2 The Collective Risk Model

Consider an insurance portfolio consisting of a number of anonymous policies which we do not observe separately. Assume that the *number of claims* arising from the portfolio is represented by a counting process $\{N(t);\, t \geqslant 0\}$ (see Definition 1.13). This means that for any $t \geqslant 0$, the random variable $N(t)$ counts the number of claims up to time t. It follows from Section 1.5 that the sequence $\{T_n;\, n \in \mathbb{N}\}$ of the *claim arrival times* (i.e., the epochs at which claims occur) consists of a.s. positive random variables satisfying $T_n < T_{n+1}$ a.s. for $n \in \mathbb{N}$. The ith claim arriving at time T_i causes the *claim size* X_i for every $i \in \mathbb{N}$. The sequence $\{X_i;\, i \in \mathbb{N}\}$ of consecutive claim sizes is assumed to consist of a.s. positive i.i.d. random variables with *claim size distribution* F. Moreover, the claim size sequence $\{X_i;\, i \in \mathbb{N}\}$ is assumed to be independent of the claim number process $\{N(t);\, t \geqslant 0\}$. The object of main interest for the insurance company is the *total claim amount*

process $\{S_{N(t)};\, t \geqslant 0\}$ defined as:

$$S_{N(t)} := \sum_{i=1}^{\infty} \mathbb{1}_{\{T_i \leqslant t\}} X_i = \begin{cases} 0 & \text{if } N(t) = 0 \\ \sum_{i=1}^{N(t)} X_i & \text{if } N(t) \geqslant 1 \end{cases}, \quad t \geqslant 0. \tag{2.1}$$

The i.i.d. property of the claim sizes reflects the fact that there is a *homogeneous probabilistic structure* in the portfolio. The assumption that the claim number process $\{N(t);\, t \geqslant 0\}$ and the claim size sequence $\{X_i;\, i \in \mathbb{N}\}$ be independent is very natural intuitively but also mathematically convenient.

We take a closer look at the total claim amount process $\{S_{N(t)};\, t \geqslant 0\}$. The sequence $\{S_n;\, n \in \mathbb{N}_0\}$ of *partial sums* associated with the claim size sequence $\{X_i;\, i \in \mathbb{N}\}$ is defined by $S_n := X_1 + \cdots + X_n$ for $n \in \mathbb{N}$ and $S_0 := 0$. It follows that $\{S_{N(t)};\, t \geqslant 0\}$ is a *random partial sum process*, referring to the fact that the deterministic index n of the partial sum S_n is replaced by the random variable $N(t)$. Depending on the choice of the claim number process $\{N(t);\, t \geqslant 0\}$, we get different models for the total claim amount process $\{S_{N(t)};\, t \geqslant 0\}$. For example, the famous *Cramér-Lundberg model* is obtained when $\{N(t);\, t \geqslant 0\}$ is a homogeneous Poisson process (see Definition 1.15). Note that it is one of the most popular and useful models in non-life insurance. Another prominent example is the *Sparre Andersen model* which is obtained by assuming that $\{N(t);\, t \geqslant 0\}$ is a renewal counting process (see Definition 1.14). Note that it generalizes the Cramér-Lundberg model since the homogeneous Poisson process is a particular renewal counting process. Finally, let us mention that the independence of $\{N(t);\, t \geqslant 0\}$ and $\{X_i;\, i \in \mathbb{N}\}$ can be exploited to obtain tractable formulas for quantities related to the total claim amount $S_{N(t)}$ at time $t \geqslant 0$. For instance, its mean and variance are respectively given by:

$$\mathbb{E}[S_{N(t)}] = \mathbb{E}[X_1]\,\mathbb{E}[N(t)] \tag{2.2}$$

$$\mathbb{V}[S_{N(t)}] = \mathbb{V}[X_1]\,\mathbb{E}[N(t)] + \big(\mathbb{E}[X_1]\big)^2\,\mathbb{V}[N(t)] \tag{2.3}$$

provided that the relevant moments exist.

2.3 Reinsurance Treaties

Fix a time $t \geqslant 0$. Recall that $X_1, \ldots, X_{N(t)}$ acts as the sequence of successive claim sizes that occurred up to t and consider the total claim amount $S_{N(t)}$ defined in (2.1).

Stimulated by one or more arguments under Section 2.1, the insurer redesigns the portfolio so that he only keeps a certain amount $D(t)$ of the total claim amount $S_{N(t)}$ called the *deductible* while he looks for reinsurance for the remaining part $R(t)$ called the *reinsured amount*. Under any *reinsurance treaty*, the risk is therefore shared between insurer and reinsurer and the specific form of $R(t)$ is of course crucial. Acting like that, the insurer himself is taking an insurance with a reinsurance company and hence himself becomes a client. The reinsurer can then again redivide the risk and go to another reinsurance company, hence building up a *reinsurance chain*.

In the following, we introduce some reinsurance treaties which are standard in the literature. For discussions on these treaties, we refer to Embrechts et al. [24], Rolski et al. [54], Teugels [62] and Mikosch [49]. It is convenient to distinguish between two different types. On the one hand, treaties of *random walk type* which are based on the total claim amount $S_{N(t)}$ or a modified version of it. On the other hand, treaties of *extreme value type* which aim at covering the largest claims in the

portfolio since they are based on some of the largest order statistics of the sequence $X_1, \ldots, X_{N(t)}$. Of course, there are possibilities for *combined reinsurance treaties* (see Chapter 4).

2.3.1 Treaties of Random Walk Type

Quota-share reinsurance. Simply a fraction $p \in (0,1]$ of the total portfolio is covered by the reinsurer. This means that the reinsured amount takes the special form:

$$R_p(t) := p\, S_{N(t)}$$

where p is called the *proportionality factor*.

The insurer keeps as deductible:

$$D_p(t) := (1-p)\, S_{N(t)}$$

in order that the sum of deductible and reinsured amount add up to $S_{N(t)}$.

This form of reinsurance is very popular in almost all insurance branches. One of its shortcomings is that all claims are partly reinsured, not only the largest claims. Moreover, the proportion $1-p$ of these largest claims is still carried by the insurer.

Surplus reinsurance. This form of reinsurance is somewhat akin to quota-share reinsurance. The reinsured amount is determined by the value of the insured object. Let V denote the value insured for each of the claim policies. As long as V lies below a positive *retention level* L, any claim X_i is carried by the insurer. If however $V \geqslant L$, the insurer deducts a proportion L/V and shifts the remaining part to the reinsurer. The reinsured amount over the entire portfolio is thus given by:

$$R_L(t) := \mathbb{1}_{\{N(t)=0\}}\, 0 + \mathbb{1}_{\{N(t)\geqslant 1\}} \sum_{i=1}^{N(t)} \mathbb{1}_{\{V>L\}} \left(1 - \frac{L}{V}\right) X_i$$

while the deductible is:

$$D_L(t) := \mathbb{1}_{\{N(t)=0\}}\, 0 + \mathbb{1}_{\{N(t)\geqslant 1\}} \sum_{i=1}^{N(t)} \left(\mathbb{1}_{\{V\leqslant L\}}\, X_i + \mathbb{1}_{\{V>L\}}\, \frac{L}{V}\, X_i \right).$$

Surplus reinsurance is very popular and even dominates quota-share reinsurance in the fire insurance branch. Note that the overall value of the insured amount is the key factor when choosing this form of reinsurance.

Excess-of-loss reinsurance. The reinsurance company covers the overshoot over some positive amount M stipulated by the ceding insurer for all individual claims whether or not they are considered to be large. This means that the reinsurer pays for:

$$R_M(t) := \mathbb{1}_{\{N(t)=0\}}\, 0 + \mathbb{1}_{\{N(t)\geqslant 1\}} \sum_{i=1}^{N(t)} \max(0, X_i - M).$$

Depending on the insurance branch, the amount M is called *deductible*, *priority* or *retention level*.

For the insurer, all claims that are passed on to the reinsurer still require the priority M to be paid out since the deductible is:

$$D_M(t) := \mathbb{1}_{\{N(t)=0\}}\, 0 + \mathbb{1}_{\{N(t)\geqslant 1\}} \sum_{i=1}^{N(t)} \min(X_i, M)\,.$$

It is clear that this form of reinsurance limits the liability of the insurer. It is used in particular in general liability and, to a lesser extent, in motor-liability and windstorm reinsurance. Moreover, excess-of-loss reinsurance is particularly interesting if a relatively small number of risks is insured and the individual claim size distribution is heavy-tailed.

In excess-of-loss reinsurance with *limit of coverage*, the reinsurer pays for all individual claims in excess of the priority M but only up to an amount L. The area between M and the *limit* $M + L$ is refereed to as the *layer*. The reinsured amount is therefore given by:

$$R_{M,L}(t) := \mathbb{1}_{\{N(t)=0\}}\, 0 + \mathbb{1}_{\{N(t)\geqslant 1\}} \sum_{i=1}^{N(t)} \min\{L, \max(0, X_i - M)\}$$

while the deductible is:

$$D_{M,L}(t) := \mathbb{1}_{\{N(t)=0\}}\, 0 + \mathbb{1}_{\{N(t)\geqslant 1\}} \sum_{i=1}^{N(t)} \{\min(X_i, M) + \max(0, X_i - (M + L))\}\,.$$

Stop-loss reinsurance. The reinsurer is liable for that portion of the insurer's total claim amount which is in excess of a predetermined positive *retention level* C. The reinsured amount is then equal to:

$$R_C(t) := \max\big(0, S_{N(t)} - C\big)$$

and the deductible for the insurer is simply given by:

$$D_C(t) := \min\big(S_{N(t)}, C\big)\,.$$

In stop-loss reinsurance, each claim is considered as an integral part of the entire portfolio. Hence, the small claims also show their influence on the reinsured amount. In particular, when the number of claims is very large, the total claim amount is highly dependent upon the small claims as well. This form of reinsurance seems a natural adaptation of excess-of-loss reinsurance but then to the portfolio as a whole. Stop-loss reinsurance is used in windstorm and hail reinsurance and, occasionally, in fire insurance. In general, one would not use stop-loss reinsurance unless the number of policies were large.

In stop-loss reinsurance with *limit of coverage*, the reinsurer will pay that part of the total claim amount in excess of the ceding insurer's retention C up to his limit of liability L. The reinsured amount is consequently given by:

$$R_{C,L}(t) := \min\big\{L, \max\big(0, S_{N(t)} - C\big)\big\}$$

and the insurer keeps as deductible:

$$D_{C,L}(t) := \min\big(S_{N(t)}, C\big) + \max\big(0, S_{N(t)} - (C + L)\big)\,.$$

2.3.2 Treaties of Extreme Value Type

Thinking especially about coverage against large claims, it seems natural to consider reinsurance treaties based on the largest claim sizes in the portfolio. Treaties of extreme value type then come into play because they are defined in terms of upper order statistics of the sequence $X_1, \dots, X_{N(t)}$ of successive claim sizes up to time t.

If $N(t) \geqslant 1$, we arrange the sequence $X_1, \dots, X_{N(t)}$ in nondecreasing order of size as:

$$X_{1:N(t)} \leqslant \cdots \leqslant X_{N(t):N(t)}$$

where $X_{N(t)-i+1:N(t)}$ is called the ith *largest order statistic* ($i \in \mathbb{N}$). Note that for each $n \in \mathbb{N}_0$, we use the conventions given in Remark 1.10 on the set where $N(t) = n$.

In spite of their conceptual appeal, treaties of extreme value type are extremely unpopular within actuarial circles. This is partly due to their mathematical intractability. However, extreme value theory techniques can overcome this problem. This will be illustrated in Chapters 3 and 5.

ECOMOR reinsurance (Excédent du coût moyen relatif). This treaty has been introduced by the French actuary Thépaut [63]. It can be considered as an excess-of-loss reinsurance with a *random priority* determined by the $(r+1)$th largest claim size for some fixed $r \in \mathbb{N}$. This means that the reinsured amount is defined by:

$$\begin{aligned} R_r(t) &:= \mathbb{1}_{\{N(t)\leqslant r\}}\, S_{N(t)} + \mathbb{1}_{\{N(t)\geqslant r+1\}} \sum_{i=1}^{N(t)} \max\bigl(0, X_i - X_{N(t)-r:N(t)}\bigr) \\ &= \mathbb{1}_{\{N(t)\leqslant r\}}\, S_{N(t)} + \mathbb{1}_{\{N(t)\geqslant r+1\}} \left(\sum_{i=1}^{r} X_{N(t)-i+1:N(t)} - r X_{N(t)-r:N(t)} \right). \end{aligned} \tag{2.4}$$

It is then clear that the ECOMOR-quantity $R_r(t)$ is a function of the $r+1$ upper order statistics $X_{N(t)-r:N(t)}, \dots, X_{N(t):N(t)}$ in the portfolio.

Of all the claims that are forwarded to the reinsurer, the insurer has to carry the responsibility for the random priority. His deductible is therefore given by:

$$\begin{aligned} D_r(t) &:= \mathbb{1}_{\{N(t)\leqslant r\}}\, 0 + \mathbb{1}_{\{N(t)\geqslant r+1\}} \sum_{i=1}^{N(t)} \min\bigl(X_i, X_{N(t)-r:N(t)}\bigr) \\ &= \mathbb{1}_{\{N(t)\leqslant r\}}\, 0 + \mathbb{1}_{\{N(t)\geqslant r+1\}} \left(\sum_{i=1}^{N(t)-r} X_{i:N(t)} + r X_{N(t)-r:N(t)} \right). \end{aligned}$$

Originally, the ECOMOR treaty was motivated by the deficiency of the excess-of-loss treaty from the reinsurer's view in case of strong increase of the claim sizes when keeping the priority fixed.

LC reinsurance (Largest claims). The reinsurer simply covers the r largest claim sizes but without a *priority* for some fixed $r \in \mathbb{N}$. The reinsured amount is then defined by:

$$L_r(t) := \mathbb{1}_{\{N(t)\leqslant r-1\}}\, S_{N(t)} + \mathbb{1}_{\{N(t)\geqslant r\}} \sum_{i=1}^{r} X_{N(t)-i+1:N(t)}. \tag{2.5}$$

A drawback of the LC treaty is that no part of the r largest claims is carried by the insurer whose deductible is then given by:

$$D_r(t) := \mathbb{1}_{\{N(t)\leqslant r\}}\, 0 + \mathbb{1}_{\{N(t)\geqslant r+1\}} \sum_{i=1}^{N(t)-r} X_{i:N(t)}.$$

DDXL reinsurance (Drop down excess-of-loss). This treaty is an excess-of-loss reinsurance with limit of coverage where the *layer* may vary in function of the order of the claim sizes. The general form of the reinsured amount over the entire portfolio is therefore given by:

$$R(t) := \mathbb{1}_{\{N(t)=0\}}\, 0 + \mathbb{1}_{\{N(t)\geqslant 1\}} \sum_{i=1}^{N(t)} \min\big\{L_i, \max\big(0, X_{i:N(t)} - M_i\big)\big\}\,.$$

Each ordered claim size $X_{i:N(t)}$ is then curtailed on both ends at some positive *priority* M_i and *limit* $M_i + L_i$ where the amount L_i reflects the liability of the reinsurer.

It follows that the deductible for the insurer is defined by:

$$D(t) := \mathbb{1}_{\{N(t)=0\}}\, 0 + \mathbb{1}_{\{N(t)\geqslant 1\}} \sum_{i=1}^{N(t)} \big\{\min\big(X_{i:N(t)}, M_i\big) + \max\big(0, X_{i:N(t)} - (M_i + L_i)\big)\big\}\,.$$

Although it is not only defined in terms of upper order statistics, the DDXL treaty is nevertheless classified as being of extreme value type. Indeed, as soon as the layers vary, it is based on the whole sequence of order statistics $X_{1:N(t)}, \ldots, X_{N(t):N(t)}$ and as such requires the use of extreme value theory. Also, these layers permit to cover mainly the largest claims but partly. Finally, note that DDXL reinsurance is the most common within the class of treaties of extreme value type.

QDXL reinsurance. Ladoucette and Teugels [46] have introduced this treaty which is defined as a quota-share reinsurance on top of an excess-of-loss reinsurance with limit of coverage on the ith largest claim size for some fixed $i \in \mathbb{N}$. More precisely, the reinsurer covers the amount:

$$\tilde{X}_{i,t} := \mathbb{1}_{\{N(t)\leqslant i-1\}}\, 0 + p\, \mathbb{1}_{\{N(t)\geqslant i\}} \min\big\{L_i(t), \max\big(0, X_{N(t)-i+1:N(t)} - M_i(t)\big)\big\} \tag{2.6}$$

where the *priority* $M_i(t) \in [0, \infty)$ and the *limit* $M_i(t)+L_i(t) \in (0, \infty]$ both depend on the time t and the order of the ith largest claim size $X_{N(t)-i+1:N(t)}$ and where $p \in (0, 1]$ is a *proportionality factor*.

Clearly, the reinsurer buys a *reinsurance layer* whose liability is limited to the amount $L_i(t)$. Note that this can be done directly or alternatively by obtaining reinsurance from another reinsurer.

2.4 Major Models for the Claim Number Process

Three major models are used for the claim number process $\{N(t);\, t \geqslant 0\}$ in the framework of collective risk theory: the renewal counting process (see Definition 1.14), the homogeneous Poisson process (see Definition 1.15) and the mixed Poisson process (see Definition 1.16). The last two processes are modifications of the homogeneous Poisson process which yield greater flexibility in one way or another.

The homogeneous Poisson process has very attractive mathematical properties and plays a central role in insurance mathematics where it has made its way from the very beginning. Although it is the most popular among all claim number processes in the actuarial literature, the homogeneous Poisson process is not always realistic for real-life claim arrival times. For example, it is unlikely that windstorm claims arrive according to a homogeneous Poisson process since large gaps between claim arrivals are usually observed. In such a case, it is more natural to assume an inter-arrival time distribution which describes these large time intervals in an adequate way. The renewal counting process then comes into play since it allows for more flexibility in choosing the distribution of the inter-arrival times. Contrary to the homogeneous Poisson process where the random variable $N(t)$ has a Poisson distribution for every $t > 0$, the distribution of $N(t)$ is in general unknown for a renewal counting process. Nevertheless, precise statements about $\mathbb{E}[N(t)]$ and $\mathbb{V}[N(t)]$ for large t can be made by appealing to renewal theory. However, note that by giving up the hypothesis of a homogeneous Poisson process, we lose most of the nice properties of this process which are closely related to the exponential distribution of the inter-arrival times.

A third prominent model for the claim number process is the mixed Poisson process. Introduced to actuaries by Dubourdieu [21], the mixed Poisson process has always been very popular among (re)insurance modelers (e.g., Panjer and Willmot [51] and Bening and Korolev [7]). Clearly, the underlying hypothesis of this model is that the claims come from a heterogeneous group of policyholders. In particular, it then allows for risk fluctuations as asked in automobile and fire insurance. For example, think of the claim number process of a portfolio of car insurance policies where each driver causes a $\mathrm{Poi}(\lambda t)$-distributed number of accidents in the time period $(0, t]$ with an intensity λ which varies from one driver to another. The over-dispersion property has been at the origin of the success of mixed Poisson processes in actuarial data fitting. Seal [56] points out that this phenomenon is often encountered in insurance data. Also, Seal [56, 57] stresses that, apart from the homogeneous Poisson process, the Pólya process is the main realistic model for the number of claims in insurance applications (see Section 1.5).

2.5 Standard Claim Size Distributions

Treaties of extreme value type (see Subsection 2.3.2) aim at covering the largest claim sizes in the portfolio whether they are actually large or not. For *large claims reinsurance*, it is then necessary to use claim size distributions which allow the largest claim sizes to be extremal. Heavy-tailed distributions (see Definition 1.1) naturally come into play and some of them are better suited to model large claims than others. In (re)insurance mathematics, subexponentiality is considered as a synonym for heavy-tailedness. The class $\mathcal{S}$ of subexponential distributions (see Definition 1.3) therefore provides adequate candidates for modeling large claims. The major drawback of $\mathcal{S}$ is that this class of distributions is defined by a nonparametric condition. However, practitioners usually advocate claim size distributions that are subexponential but that contain at the same time enough parameters to allow greater flexibility for data fitting. Several examples of such subexponential distributions are given in Table 2.1. The heavy-tailed class $\mathcal{S}$ is very flexible since it contains *moderately* heavy-tailed distributions that belong to the extremal class $\mathcal{C}_0$ (such as Benktander type I, Benktander type II, lognormal and Weibull with $\tau \in (0, 1)$ distributions) but also *very* heavy-tailed distributions that belong to an extremal class $\mathcal{C}_\gamma$ with $\gamma > 0$ (such as the Burr distribution $\mathrm{Burr}(\alpha, \kappa, \tau)$ that belongs to $\mathcal{C}_{1/(\alpha\tau)}$, the loggamma distribution $\mathrm{L}\Gamma(\alpha, \beta)$ that belongs to $\mathcal{C}_{1/\beta}$ and the Pareto distribution $\mathrm{Par}(\alpha, \kappa)$, truncated α-stable distribution T_α and truncated Student distribution $\mathrm{TStud}(\alpha)$ that all belong to $\mathcal{C}_{1/\alpha}$). The Pareto-type distributions

(see Definition 1.7) constitute another natural class of heavy-tailed claim size distributions from a theoretical but also from a practical point of view. A Pareto-type distribution has the specific advantage of remaining invariant under a fair set of mathematical operations. Moreover, recall that any Pareto-type distribution is subexponential (see Subsection 1.2.1) and that a distribution function F on $(0,\infty)$ is of Pareto-type with index $\alpha > 0$ if and only if F belongs to the extremal class $\mathcal{C}_{1/\alpha}$ (see Subsection 1.2.2). Finally, we mention that the class $\mathcal{L}$ of long-tailed distributions (see Definition 1.2) can also be considered in the context of large claims reinsurance since any subexponential distribution is long-tailed (see Subsection 1.2.1).

Distribution F Name	Tail $1-F$ or density f	Parameters
$\mathrm{Burr}(\alpha,\kappa,\tau)$ Burr	$1-F(x)=\left(\frac{\kappa}{\kappa+x^\tau}\right)^\alpha$	$\alpha>0$, $\kappa>0$, $\tau>0$
$\mathrm{L}\Gamma(\alpha,\beta)$ Loggamma	$f(x)=\frac{\beta^\alpha}{\Gamma(\alpha)}\,(\log x)^{\alpha-1}\,x^{-(\beta+1)}$	$\alpha>0$, $\beta>0$
$\mathrm{Par}(\alpha,\kappa)$ Pareto	$1-F(x)=\left(\frac{\kappa}{\kappa+x}\right)^\alpha$	$\alpha>0$, $\kappa>0$
T_α Truncated α-stable	$1-F(x)=\mathbb{P}[\lvert X\rvert>x]$ where X is an α-stable random variable	$\alpha\in(0,2)$
$\mathrm{TStud}(\alpha)$ Truncated Student	$f(x)=\frac{2\,\Gamma(\frac{\alpha+1}{2})}{\sqrt{\pi\alpha}\,\Gamma(\frac{\alpha}{2})}\left(1+\frac{x^2}{\alpha}\right)^{-\frac{\alpha+1}{2}}$	$\alpha>0$
$\mathrm{BenI}(a,b)$ Benktander type I	$1-F(x)=x^{-(a+1)}\,\mathrm{e}^{-b(\log x)^2}(1+2(b/a)\log x)$	$a>0$, $b>0$ with $a(a+1)\geqslant 2b$
$\mathrm{BenII}(a,b)$ Benktander type II	$1-F(x)=\mathrm{e}^{a/b}\,x^{-(1-b)}\,\mathrm{e}^{-(a/b)x^b}$	$a>0$, $b\in(0,1)$
$\mathrm{LN}(\mu,\sigma^2)$ Lognormal	$f(x)=\frac{1}{\sqrt{2\pi}\sigma x}\,\mathrm{e}^{-(\log x-\mu)^2/(2\sigma^2)}$	$\mu\in\mathbb{R}$, $\sigma>0$
$\mathrm{W}(\tau,c)$ Weibull	$1-F(x)=\mathrm{e}^{-cx^\tau}$	$\tau\in(0,1)$, $c>0$

Table 2.1: Claim size distributions for large claims. All distributions have support $(0,\infty)$ except for the Benktander and loggamma cases with $(1,\infty)$.

In contrast to heavy-tailed claim size distributions as plausible models for large claims, (re)insurance mathematics is also concerned with claim size distributions that adequately model *small claims*. Most of them are classical distributions in probability theory or statistics and we refer to Table 2.2 for some examples. Note that the exponential, gamma, truncated normal and Weibull with $\tau \geqslant 1$ distributions all belong to the extremal class $\mathcal{C}_0$. Moreover, if a distribution with bounded support in $(0, \infty)$ belongs to an extremal class $\mathcal{C}_\gamma$, then $\gamma < 0$ necessarily.

Distribution F Name	Tail $1 - F$ or density f	Parameters
$\mathrm{Exp}(\lambda)$ Exponential	$1 - F(x) = \mathrm{e}^{-\lambda x}$	$\lambda > 0$
$\Gamma(\alpha, \beta)$ Gamma	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}\, \mathrm{e}^{-\beta x}$	$\alpha > 0$, $\beta > 0$
$\mathrm{TN}(0, 1)$ Truncated normal	$f(x) = \sqrt{\frac{2}{\pi}}\, \mathrm{e}^{-x^2/2}$	
$\mathrm{W}(\tau, c)$ Weibull	$1 - F(x) = \mathrm{e}^{-cx^\tau}$	$\tau \geqslant 1$, $c > 0$
Any distribution with bounded support in $(0, \infty)$		

Table 2.2: Claim size distributions for small claims. All distributions have support $(0, \infty)$ except for the last case.

Chapter 3

On ECOMOR and LC Reinsurance

3.1 Introduction

Let $\{X_i;\, i \in \mathbb{N}\}$ be a sequence of a.s. positive i.i.d. random variables from the distribution function F of a generic random variable X. Denote by U the tail quantile function of F. Let $\{N(t);\, t \geqslant 0\}$ be a counting process independent of $\{X_i;\, i \in \mathbb{N}\}$. We interpret $\{X_i;\, i \in \mathbb{N}\}$ as the sequence of consecutive claim sizes arising from an insurance portfolio and $\{N(t);\, t \geqslant 0\}$ as the underlying claim number process, i.e. the random variable $N(t)$ counts the number of claims up to time t. If $N(t) \geqslant 1$, we therefore interpret $X_1, \ldots, X_{N(t)}$ as the sequence of successive claim sizes up to time t and we denote the corresponding sequence of nondescending order statistics by $X_{1:N(t)}, \ldots, X_{N(t):N(t)}$ as in Subsection 2.3.2. Also, recall from Section 2.2 that the total claim amount at time $t \geqslant 0$ is defined by $S_{N(t)} := \sum_{i=1}^{N(t)} X_i$ if $N(t) \geqslant 1$ while $S_{N(t)} := 0$ if $N(t) = 0$.

As mentioned in Section 2.1, one of the principal reasons for taking reinsurance is the possible appearance of *large claims*. It is therefore somewhat remarkable that classical reinsurance treaties such as quota-share, surplus, excess-of-loss and stop-loss (see Subsection 2.3.1) are not expressed in terms of the largest claim sizes. Although treaties of extreme value type (see Subsection 2.3.2) allow the coverage of large claims, they are playing a minor - even non-existent - role in practice. This is partly due to their mathematical intractability. In this chapter, we will show that extreme value theory is capable to overcome this problem. We will restrict attention to two specific treaties of extreme value type, namely ECOMOR reinsurance and LC reinsurance. Note that similar results can be obtained for other reinsurance treaties based on the largest claim sizes.

Let $r \in \mathbb{N}$ and $t \geqslant 0$ be fixed. Recall from (2.4) that the reinsured amount $R_r(t)$ in the ECOMOR treaty is defined by:

$$R_r(t) := \mathbb{1}_{\{N(t) \leqslant r\}}\, S_{N(t)} + \mathbb{1}_{\{N(t) \geqslant r+1\}} \sum_{i=1}^{r} \left(X_{N(t)-i+1:N(t)} - X_{N(t)-r:N(t)} \right) \tag{3.1}$$

so that the reinsurer covers that part of the r largest claim sizes $X_{N(t)-r+1:N(t)}, \ldots, X_{N(t):N(t)}$ that overshoots the random priority $X_{N(t)-r:N(t)}$ if $N(t) \geqslant r+1$ while the total claim amount $S_{N(t)}$ is covered if $N(t) \leqslant r$.

Also, recall from (2.5) that the reinsured amount $L_r(t)$ in the LC treaty is defined by:

$$L_r(t) := \mathbb{1}_{\{N(t)\leqslant r-1\}}\, S_{N(t)} + \mathbb{1}_{\{N(t)\geqslant r\}} \sum_{i=1}^{r} X_{N(t)-i+1:N(t)} \tag{3.2}$$

which means that the reinsurer covers the r largest claim sizes $X_{N(t)-r+1:N(t)}, \ldots, X_{N(t):N(t)}$ if $N(t) \geqslant r$ while the total claim amount $S_{N(t)}$ is covered if $N(t) \leqslant r-1$.

In this chapter, we derive asymptotic results related to the reinsured amounts $R_r(t)$ and $L_r(t)$. However, we will mainly focus on ECOMOR reinsurance. Not much is known about this treaty of extreme value type which has been largely neglected by the reinsurance world. Helbig [33] is one of the first actuaries to investigate some of the strengths and weaknesses of the ECOMOR treaty in a Poisson-Pareto setting. Within the framework of premium calculations in connection with ECOMOR reinsurance, Ammeter [3] develops an approximation for the net risk premium under the Poisson-Pareto assumption. Kremer [35, 36] gives general asymptotic premium formulas and crude upper bounds for the net premium under a Pareto claim size distribution. Kremer [38] allows the claim sizes to be not necessarily independent but only derives an upper premium bound. Otherwise, Ammeter [3] points out the influence of excluding one or more of the largest claims on the expected amount of the remaining total claim amount. He also makes the important observation that - even within a portfolio with Pareto distributed claim sizes - there is a preponderance of small claims. The first publication that treated of problems on the convergence in distribution for the ECOMOR-quantity $R_r(t)$ is Teugels [61] with proofs based on transform techniques. These results have been rederived later on by Embrechts et al. [24] with probabilistic methods. Finally, Beirlant and Teugels [6] give some asymptotic theory for $R_r(t)$ by assuming that r increases when $t \to \infty$ and that the claim size distribution belongs either to the domain of attraction of the Fréchet-Pareto distribution or to that of the Gumbel distribution.

Concerning LC reinsurance, the calculation of premiums quickly runs into mathematically intractable formulas. This has been recognized by Benktander [8] who deals with a relation between excess-of-loss and LC situations. For the calculation of the net premium, see Berglund [10] and the references therein. Kupper [39] compares the pure premium for the excess-of-loss cover with that for the LC cover when the claim size distribution is Pareto. Also, Kremer [36] gives crude upper bounds for the net premium under a Pareto claim size distribution. Otherwise, Berliner [11] considers a set of interesting problems connected with LC reinsurance. Under the Poisson-Pareto assumption, he derives the joint distribution of two large claims and then computes the covariance between them. He also calculates the covariance between the LC-quantity $L_r(t)$ and the total claim amount $S_{N(t)}$. Finally, the asymptotic efficiency of the LC treaty is discussed in Kremer [37].

Since we want to pay special attention to large claims, we need to allow the largest claim sizes (or upper order statistics) to be extremal. We then have to assume that the claim size distribution F belongs to a specific class of heavy-tailed distributions (see Definition 1.1). A first obvious candidate is the class $\mathcal{S}$ of subexponential distributions (see Definition 1.3) or even the more general class $\mathcal{L}$ of long-tailed distributions (see Definition 1.2). Alternatively, we can take the extremal classes $\mathcal{C}_\gamma$ with $\gamma \geqslant 0$ (see Definition 1.8). Indeed, recall from Section 2.5 that for *large claims reinsurance*, the relevant values of γ are either $\gamma > 0$ (in which case F is of Pareto-type with index $1/\gamma$ (see Definition 1.7) and as such is subexponential) or $\gamma = 0$ (in which case F belongs to the domain of attraction of the Gumbel distribution). Also, recall that the set of distributions in the extremal classes coincides with the set of distributions that belong to the domain of attraction

of an extreme value distribution, i.e. such that (1.5) holds. It is therefore natural that these classes of distributions appear when one deals with upper order statistics.

In Section 3.2, we state our results for ECOMOR reinsurance. In Subsection 3.2.1, we are interested in asymptotic relations between the (right) tail of the distribution function F of the generic claim size X and that of the ECOMOR-quantity $R_r(t)$ for any fixed $t > 0$. Note that since $R_r(0) = 0$ a.s., the case $t = 0$ is excluded to avoid trivialities. Depending on the tail character of F, that is whether subexponential or long-tailed, different kinds of results will show up. In particular, we will head for accurate *asymptotic equivalences* and *asymptotic bounds*. In the subexponential case, we will even derive a result showing the interplay between the total claim amount $S_{N(t)}$, the largest claim size $X_{N(t):N(t)}$ and the ECOMOR-quantity $R_r(t)$. A further question is to see what happens with the distribution of $R_r(t)$ when $t \to \infty$. As expected, we tackle the problem by assuming that $F \in \mathcal{C}_\gamma(a)$ for some $\gamma \in \mathbb{R}$. We then deal with *convergence in distribution* for $R_r(t)/a(t)$ as $t \to \infty$. Our results will be obtained under the additional assumption that the claim number process $\{N(t);\, t \geqslant 0\}$ is mixed Poisson (see Definition 1.16). Subsection 3.2.2 will be devoted to the case $r = 1$. An explicit expression for $\lim_{t\to\infty} \mathbb{P}[R_1(t) > a(t)s]$ is obtained. This is important because the complicated distribution of $R_1(t)$ can thus be replaced by a much simpler distribution. Since no such expression is available for a general integer $r \geqslant 2$, the limiting distribution of $R_r(t)/a(t)$ as $t \to \infty$ will be determined by its Laplace-Stieltjes transform. The result is given in Subsection 3.2.3. Finally, we turn in Subsection 3.2.4 to the study of the convergence of the ratio $R_r(t)/S_{N(t)}$ as $t \to \infty$. In particular, we will show that the ECOMOR-quantity $R_r(t)$ has a dominant influence on the total claim amount $S_{N(t)}$ when the claim size distribution F is super heavy-tailed (see Definition 1.6).

In Section 3.3, we give our results for LC reinsurance. Concerning the LC-quantity $L_r(t)$, we only focus on the same asymptotic distributional problems as for the ECOMOR case. Again assuming that $F \in \mathcal{C}_\gamma(a)$ for some $\gamma \in \mathbb{R}$ and that the claim number process $\{N(t);\, t \geqslant 0\}$ is mixed Poisson, we deal with *convergence in distribution* for $(L_r(t) - rU(t))/a(t)$ as $t \to \infty$. Subsection 3.3.1 concerns the case $r = 1$. Subsection 3.3.2 is devoted to the general case $r \in \mathbb{N}$. The limiting distribution of $(L_r(t) - rU(t))/a(t)$ as $t \to \infty$ will be determined by its characteristic function. We conclude with some remarks in Section 3.4. This chapter is mainly based on Ladoucette and Teugels [44]. Subsection 3.2.4 comes from Ladoucette and Teugels [45]. An overview of the main results of the chapter is also given in Ladoucette and Teugels [42].

3.2 Results for ECOMOR Reinsurance

In this section, we give asymptotic results related to ECOMOR reinsurance. The first part concerns asymptotic relations between the (right) tail of the distribution function F of the generic claim size X and that of the ECOMOR-quantity $R_r(t)$ for any fixed $t > 0$. More precisely, we derive asymptotic equivalences and bounds under the condition that the claim size distribution F belongs to either the subexponential class $\mathcal{S}$ or the long-tailed class $\mathcal{L}$. The second and third parts deal with convergence in distribution for $R_r(t)/a(t)$ as $t \to \infty$ under the conditions that $F \in \mathcal{C}_\gamma(a)$ for some $\gamma \in \mathbb{R}$ and that the claim number process $\{N(t);\, t \geqslant 0\}$ is mixed Poisson. The second part concerns $R_1(t)$ and the third part focuses on $R_r(t)$ for a general $r \in \mathbb{N}$. The last part deals with the convergence of the ratio $R_r(t)/S_{N(t)}$ as $t \to \infty$ when F is super heavy-tailed, i.e. $1 - F \in \mathrm{RV}_0$.

3.2.1 Asymptotic Equivalences and Bounds

In this subsection, the time $t > 0$ is fixed. Let us start with a general result that gives a necessary and sufficient condition for an asymptotic equivalence between the tail of the distribution of the largest claim size $X_{N(t):N(t)}$ and that of the generic claim size X.

Lemma 3.1. *Let $t > 0$ be fixed. Assume that the distribution function F satisfies $F(x) < 1$ for all $x > 0$. Then:*

$$\left(\mathbb{P}[X_{N(t):N(t)} > s] \sim \mathbb{E}[N(t)]\,\mathbb{P}[X > s] \;\; as \;\; s \to \infty\right) \iff \mathbb{E}[N(t)] < \infty.$$

Proof. Let $t > 0$ be fixed. By using a conditioning argument and the probability generating function of $N(t)$, we obtain for every $s \geqslant 0$:

$$\mathbb{P}[X_{N(t):N(t)} > s] = \sum_{n=0}^{\infty} \mathbb{P}[X_{n:n} > s]\, p_n(t) = \sum_{n=0}^{\infty} (1 - F^n(s))\, p_n(t) = 1 - Q_t(F(s))$$

so that:

$$\lim_{s\to\infty} \frac{\mathbb{P}[X_{N(t):N(t)} > s]}{\mathbb{P}[X > s]} = \lim_{s\to\infty} \frac{1 - Q_t(F(s))}{1 - F(s)} = \lim_{u\uparrow 1} \frac{1 - Q_t(u)}{1 - u} = Q_t^{(1)}(1-) = \mathbb{E}[N(t)] \leqslant \infty.$$

Now, the left-hand term of the claim holds if and only if the expectation of $N(t)$ is finite. □

Remark 3.1. It is easily seen that Lemma 3.1 remains true if the support of F is contained in $\mathbb{R}$.

A neat connection between factorial moments of the random variable $N(t)$ and derivatives of its probability generating function is given in the next auxiliary result.

Proposition 3.2. *Let $r \in \mathbb{N}$, $t \geqslant 0$ and $\alpha \geqslant 0$ be fixed. Then:*

$$\int_0^1 z^\alpha\, Q_t^{(r)}(z)\,\mathrm{d}z = \mathbb{E}\left[\frac{\mathbb{1}_{\{N(t)\geqslant r\}}\, N(t)!}{(N(t)-r)!\,(N(t)-r+\alpha+1)}\right].$$

Moreover, both sides are finite if $\mathbb{E}[N(t)^{r-1}] < \infty$.

Proof. Let $r \in \mathbb{N}$, $t \geqslant 0$ and $\alpha \geqslant 0$ be fixed. Using (1.12), we get by an application of Fubini's theorem:

$$\begin{aligned}
\int_0^1 z^\alpha\, Q_t^{(r)}(z)\,\mathrm{d}z &= \int_0^1 z^\alpha \sum_{n=r}^{\infty} \left\{\frac{n!}{(n-r)!}\, p_n(t)\, z^{n-r}\right\} \mathrm{d}z \\
&= \sum_{n=r}^{\infty} \left\{\frac{n!}{(n-r)!}\, p_n(t) \int_0^1 z^{n-r+\alpha}\,\mathrm{d}z\right\} \\
&= \sum_{n=r}^{\infty} \frac{n!}{(n-r)!\,(n-r+\alpha+1)}\, p_n(t) \\
&= \mathbb{E}\left[\frac{\mathbb{1}_{\{N(t)\geqslant r\}}\, N(t)!}{(N(t)-r)!\,(N(t)-r+\alpha+1)}\right].
\end{aligned}$$

Now, the random variable $\mathbb{1}_{\{N(t)\geqslant r\}}\, N(t)!\,\{(N(t)-r)!\,(N(t)-r+\alpha+1)\}^{-1}$ is a.s. bounded above by $N(t)^{r-1}$. Its expectation is therefore finite if we assume $\mathbb{E}[N(t)^{r-1}] < \infty$. □

Here is the first of our main results. We give an asymptotic upper bound for the ratio of the tail of the distribution of the ECOMOR-quantity $R_r(t)$ to that of the generic claim size X.

Theorem 3.3. *Let $r \in \mathbb{N}$ and $t > 0$ be fixed. Assume that $F \in \mathcal{S}$ and that $Q_t(z)$ is analytic at the point $z = 1$. Then:*

$$\limsup_{s\to\infty} \frac{\mathbb{P}[R_r(t) > s]}{\mathbb{P}[X > s]} \leqslant \mathbb{E}[N(t)].$$

Proof. Let $r \in \mathbb{N}$ and $t > 0$ be fixed. For each $n \in \mathbb{N}_0$, define the random variable:

$$R_r^n := \begin{cases} S_n & \text{if } n \leqslant r \\ \sum_{i=1}^r X_{n-i+1:n} - rX_{n-r:n} & \text{if } n \geqslant r+1. \end{cases}$$

For every $n \in \mathbb{N}_0$, since $R_r^n \leqslant S_n$ a.s., we obtain by using Lemma 1.1(iii) that:

$$\limsup_{s\to\infty} \frac{\mathbb{P}[R_r^n > s]}{\mathbb{P}[X > s]} \leqslant \limsup_{s\to\infty} \frac{\mathbb{P}[S_n > s]}{\mathbb{P}[X > s]} = n.$$

For each $\delta > 0$, there exists a constant $K = K(\delta) < \infty$ such that for all $s \geqslant 0$ and $n \in \mathbb{N}_0$:

$$\frac{\mathbb{P}[R_r^n > s]}{\mathbb{P}[X > s]} \leqslant \frac{\mathbb{P}[S_n > s]}{\mathbb{P}[X > s]} \leqslant K(1+\delta)^n$$

by Lemma 1.1(ii). The analyticity of $Q_t(z)$ at $z = 1$ yields that $\sum_{n=0}^\infty p_n(t)\,(1+\epsilon)^n < \infty$ for some $\epsilon > 0$. By choosing $\delta \in (0, \epsilon)$, it then follows that $\sum_{n=0}^\infty p_n(t)\,(1+\delta)^n \leqslant \sum_{n=0}^\infty p_n(t)\,(1+\epsilon)^n < \infty$.

A conditioning argument and Fatou's lemma then yield:

$$\begin{aligned} \limsup_{s\to\infty} \frac{\mathbb{P}[R_r(t) > s]}{\mathbb{P}[X > s]} &= \limsup_{s\to\infty} \sum_{n=0}^\infty \frac{\mathbb{P}[R_r^n > s]}{\mathbb{P}[X > s]}\, p_n(t) \\ &\leqslant \sum_{n=0}^\infty p_n(t) \limsup_{s\to\infty} \frac{\mathbb{P}[R_r^n > s]}{\mathbb{P}[X > s]} \\ &\leqslant \sum_{n=0}^\infty n p_n(t) = \mathbb{E}[N(t)] \end{aligned}$$

and the assertion is proved. □

Remark 3.2. The condition on $N(t)$ in Theorem 3.3 means in particular that the probability of a large number of claims in the time interval $[0, t]$ is exponentially small.

In the following result, we give an asymptotic lower bound for the ratio of the tail of the distribution of the ECOMOR-quantity $R_r(t)$ to that of the generic claim size X. Note that the result is established under weaker conditions on F and on $N(t)$ than in Theorem 3.3.

Theorem 3.4. *Let $r \in \mathbb{N}$ and $t > 0$ be fixed. Assume that $F \in \mathcal{L}$ and that $Q_t^{(r)}(1-) < \infty$. Then:*

$$\liminf_{s\to\infty} \frac{\mathbb{P}[R_r(t) > s]}{\mathbb{P}[X > s]} \geqslant \mathbb{E}[N(t)].$$

Proof. Let $r \in \mathbb{N}$, $t > 0$ and $y \geqslant 0$ be fixed. For each $s \geqslant 0$, we have:

$$\begin{aligned}\mathbb{P}[R_r(t) > s] &\geqslant \mathbb{P}[X_{N(t):N(t)} > s + y, R_r(t) > s] \\ &= \mathbb{P}[X_{N(t):N(t)} > s + y] - \mathbb{P}[X_{N(t):N(t)} > s + y, R_r(t) \leqslant s].\end{aligned}$$

Consider the second term on the right-hand side. Using a conditioning argument and (3.1) gives:

$$\begin{aligned}&\mathbb{P}[X_{N(t):N(t)} > s + y, R_r(t) \leqslant s] \\ &= \mathbb{E}\Bigg[N(t)(N(t)-1)\dots(N(t)-r)\int_{s+y}^{\infty}\int_{x_1-s}^{x_1}\int_{\frac{x_1+x_2-s}{2}}^{x_2}\cdots\int_{\frac{x_1+\dots+x_{r-1}-s}{r-1}}^{x_{r-1}}\int_{\frac{x_1+\dots+x_r-s}{r}}^{x_r} \\ &\qquad F^{N(t)-(r+1)}(x_{r+1})\,\mathrm{d}F(x_{r+1})\dots\mathrm{d}F(x_1)\Bigg] \\ &= \mathbb{E}\Bigg[N(t)(N(t)-1)\dots(N(t)-r+1)\int_{s+y}^{\infty}\int_{x_1-s}^{x_1}\int_{\frac{x_1+x_2-s}{2}}^{x_2}\cdots\int_{\frac{x_1+\dots+x_{r-1}-s}{r-1}}^{x_{r-1}} \\ &\qquad \left(F^{N(t)-r}(x_r) - F^{N(t)-r}\left((x_1+\dots+x_r-s)/r\right)\right)\mathrm{d}F(x_r)\dots\mathrm{d}F(x_1)\Bigg] \\ &\leqslant \mathbb{E}\Bigg[N(t)(N(t)-1)\dots(N(t)-r+1)\int_{s+y}^{\infty}\int_{0}^{x_1}\int_{0}^{x_2}\cdots\int_{0}^{x_{r-1}} \\ &\qquad \left(1 - F^{N(t)-r}\left((x_1+\dots+x_r-s)/r\right)\right)\mathrm{d}F(x_r)\dots\mathrm{d}F(x_1)\Bigg] \\ &\leqslant \mathbb{E}\Bigg[N(t)(N(t)-1)\dots(N(t)-r+1)\int_{s+y}^{\infty}\int_{0}^{x_1}\int_{0}^{x_2}\cdots\int_{0}^{x_{r-2}} \\ &\qquad \left(1 - F^{N(t)-r}\left((x_1+\dots+x_{r-1}-s)/r\right)\right)\mathrm{d}F(x_{r-1})\dots\mathrm{d}F(x_1)\Bigg] \\ &\vdots \\ &\leqslant \mathbb{E}\Bigg[N(t)(N(t)-1)\dots(N(t)-r+1)\int_{s+y}^{\infty}\left(1 - F^{N(t)-r}((x_1-s)/r)\right)\mathrm{d}F(x_1)\Bigg] \\ &\leqslant (1-F(s+y))\,\mathbb{E}\Big[N(t)(N(t)-1)\dots(N(t)-r+1)\left(1-F^{N(t)-r}(y/r)\right)\Big].\end{aligned}$$

Since $F \in \mathcal{L}$, we therefore obtain the following asymptotic inequality:

$$\begin{aligned}&\limsup_{s\to\infty}\frac{\mathbb{P}[X_{N(t):N(t)} > s + y, R_r(t) \leqslant s]}{\mathbb{E}[N(t)]\,\mathbb{P}[X > s]} \\ &\quad\leqslant \limsup_{s\to\infty}\frac{1-F(s+y)}{1-F(s)}\,\frac{\mathbb{E}\big[N(t)(N(t)-1)\dots(N(t)-r+1)\left(1-F^{N(t)-r}(y/r)\right)\big]}{\mathbb{E}[N(t)]} \\ &\quad= \frac{\mathbb{E}\big[N(t)(N(t)-1)\dots(N(t)-r+1)\left(1-F^{N(t)-r}(y/r)\right)\big]}{\mathbb{E}[N(t)]}.\end{aligned}$$

It then follows:

$$\begin{aligned}\liminf_{s\to\infty}\frac{\mathbb{P}[R_r(t) > s]}{\mathbb{E}[N(t)]\,\mathbb{P}[X > s]} &\geqslant \liminf_{s\to\infty}\left(\frac{\mathbb{P}[X_{N(t):N(t)} > s + y]}{\mathbb{E}[N(t)]\,\mathbb{P}[X > s]} - \frac{\mathbb{P}[X_{N(t):N(t)} > s + y, R_r(t) \leqslant s]}{\mathbb{E}[N(t)]\,\mathbb{P}[X > s]}\right) \\ &\geqslant \liminf_{s\to\infty}\frac{\mathbb{P}[X_{N(t):N(t)} > s + y]}{\mathbb{E}[N(t)]\,\mathbb{P}[X > s]} \\ &\quad + \liminf_{s\to\infty}\left(-\frac{\mathbb{P}[X_{N(t):N(t)} > s + y, R_r(t) \leqslant s]}{\mathbb{E}[N(t)]\,\mathbb{P}[X > s]}\right) \\ &= \liminf_{s\to\infty}\frac{\mathbb{P}[X_{N(t):N(t)} > s + y]}{\mathbb{E}[N(t)]\,\mathbb{P}[X > s]}\end{aligned}$$

$$
\begin{aligned}
& -\limsup_{s\to\infty} \frac{\mathbb{P}[X_{N(t):N(t)} > s+y, R_r(t) \leqslant s]}{\mathbb{E}[N(t)]\,\mathbb{P}[X>s]} \\
\geqslant\ & \liminf_{s\to\infty} \frac{\mathbb{P}[X_{N(t):N(t)} > s+y]}{\mathbb{E}[N(t)]\,\mathbb{P}[X>s]} \\
& -\frac{\mathbb{E}\big[N(t)(N(t)-1)\dots(N(t)-r+1)\left(1-F^{N(t)-r}(y/r)\right)\big]}{\mathbb{E}[N(t)]}.
\end{aligned}
$$

For the first term on the right-hand side, we have:

$$
\begin{aligned}
\liminf_{s\to\infty} \frac{\mathbb{P}[X_{N(t):N(t)} > s+y]}{\mathbb{E}[N(t)]\,\mathbb{P}[X>s]} &= \liminf_{s\to\infty} \frac{\mathbb{P}[X_{N(t):N(t)} > s+y]}{\mathbb{E}[N(t)]\,\mathbb{P}[X>s+y]} \frac{\mathbb{P}[X>s+y]}{\mathbb{P}[X>s]} \\
&= \lim_{s\to\infty} \frac{1-F(s+y)}{1-F(s)} = 1
\end{aligned}
$$

by Lemma 3.1 and the hypothesis $F \in \mathcal{L}$. It then follows:

$$
\liminf_{s\to\infty} \frac{\mathbb{P}[R_r(t) > s]}{\mathbb{E}[N(t)]\,\mathbb{P}[X>s]} \geqslant 1 - \frac{\mathbb{E}\big[N(t)(N(t)-1)\dots(N(t)-r+1)\left(1-F^{N(t)-r}(y/r)\right)\big]}{\mathbb{E}[N(t)]}.
$$

The proof of the theorem is finished if we take the limit as $y \to \infty$ on both sides. Indeed, let us show that the second term on the right-hand side goes to 0 when $y \to \infty$. The random variable $N(t)(N(t)-1)\dots(N(t)-r+1)\left(1-F^{N(t)-r}(y/r)\right)$ is a.s. bounded above by the random variable $N(t)(N(t)-1)\dots(N(t)-r+1)$ for all $y \geqslant 0$ and is a.s. monotone nonincreasing to 0 as $y \to \infty$. Moreover, $\mathbb{E}[N(t)(N(t)-1)\dots(N(t)-r+1)] = Q_t^{(r)}(1-) < \infty$ by assumption. Note that dividing by $\mathbb{E}[N(t)]$ is not a problem since $\mathbb{E}[N(t)] \leqslant Q_t^{(r)}(1-) + \sum_{n=1}^{r-1} n\,p_n(t) < \infty$ if $r \geqslant 2$ while $\mathbb{E}[N(t)] = Q_t^{(1)}(1-) < \infty$ if $r = 1$. A simple application of the monotone convergence theorem then proves the claim. □

Remark 3.3. Theorem 3.4 still holds if F is long-tailed on $\mathbb{R}$ (see Remark 1.1) rather than on $(0,\infty)$. Few changes in the proof are then required.

Now, we derive an asymptotic equivalence between the tail of the distribution of the ECOMOR-quantity $R_1(t)$ and that of the generic claim size X under the long-tailed assumption for the claim size distribution F.

Theorem 3.5. *Let $t > 0$ be fixed. Assume that $F \in \mathcal{L}$ and that $\mathbb{E}[N(t)] < \infty$. Then:*

$$
\mathbb{P}[R_1(t) > s] \sim \mathbb{E}[N(t)]\,\mathbb{P}[X>s] \quad \textit{as} \quad s \to \infty.
$$

Proof. Let $t > 0$ be fixed. By using (3.1) and a conditioning argument, we get for every $s \geqslant 0$:

$$
\begin{aligned}
\mathbb{P}[R_1(t) > s] &= \sum_{n=0}^{\infty} \mathbb{P}[R_1(t) > s \,|N(t) = n]\, p_n(t) \\
&= \sum_{n=0}^{1} \mathbb{P}[S_n > s]\, p_n(t) + \sum_{n=2}^{\infty} \mathbb{P}[X_{n:n} - X_{n-1:n} > s]\, p_n(t) \\
&= \mathbb{P}[X>s]\, p_1(t) + \sum_{n=2}^{\infty} \left\{ n(n-1)\, p_n(t) \int_0^{\infty} F^{n-2}(y) \int_{s+y}^{\infty} \mathrm{d}F(x)\, \mathrm{d}F(y) \right\} \\
&= (1-F(s))\, p_1(t) + \int_0^{\infty} (1-F(s+y)) \sum_{n=2}^{\infty} \left\{ n(n-1)\, p_n(t)\, F^{n-2}(y) \right\} \mathrm{d}F(y) \\
&= (1-F(s))\, p_1(t) + \int_0^{\infty} (1-F(s+y))\, Q_t^{(2)}(F(y))\, \mathrm{d}F(y) \qquad (3.3)
\end{aligned}
$$

so that:

$$\frac{\mathbb{P}[R_1(t) > s]}{\mathbb{P}[X > s]} = p_1(t) + \int_0^\infty \frac{1 - F(s+y)}{1 - F(s)} \, Q_t^{(2)}(F(y)) \, \mathrm{d}F(y).$$

For each $s \geqslant 0$, $\frac{1-F(s+y)}{1-F(s)} \, Q_t^{(2)}(F(y)) \leqslant Q_t^{(2)}(F(y))$ for all $y \geqslant 0$. Also, Proposition 3.2 with $r = 2$ and $\alpha = 0$ gives $\int_0^\infty Q_t^{(2)}(F(y)) \, \mathrm{d}F(y) = \int_0^1 Q_t^{(2)}(z) \, \mathrm{d}z = \mathbb{E}\big[\mathbb{1}_{\{N(t)\geqslant 2\}} N(t)\big] = \mathbb{E}[N(t)] - p_1(t) < \infty$ since $\mathbb{E}[N(t)] < \infty$. Finally, since $F \in \mathcal{L}$, we have $\lim_{s\to\infty} \frac{1-F(s+y)}{1-F(s)} \, Q_t^{(2)}(F(y)) = Q_t^{(2)}(F(y))$ for all $y \geqslant 0$. Applying Lebesgue's dominated convergence theorem then gives:

$$\lim_{s\to\infty} \frac{\mathbb{P}[R_1(t) > s]}{\mathbb{P}[X > s]} = p_1(t) + \int_0^\infty Q_t^{(2)}(F(y)) \, \mathrm{d}F(y) = \mathbb{E}[N(t)]$$

and the result follows since $\mathbb{E}[N(t)] < \infty$. □

As a consequence of the previous results, we obtain the following corollary pertaining to asymptotic equivalences under the subexponential assumption for the claim size distribution. The second assertion extends Theorem 3.5 to a general $r \in \mathbb{N}$ under stronger assumptions. Also, the last assertion shows the interplay between the ECOMOR-quantity $R_r(t)$, the largest claim size $X_{N(t):N(t)}$ and the total claim amount $S_{N(t)}$.

Corollary 3.6. *Let $t > 0$ be fixed. Assume that $F \in \mathcal{S}$.*

(*i*) *If $\mathbb{E}[N(t)] < \infty$, then:*

$$\mathbb{P}[R_1(t) > s] \sim \mathbb{E}[N(t)] \, \mathbb{P}[X > s] \quad \text{as} \quad s \to \infty.$$

(*ii*) *If $Q_t(z)$ is analytic at the point $z = 1$, then for every fixed $r \in \mathbb{N}$:*

$$\mathbb{P}[R_r(t) > s] \sim \mathbb{E}[N(t)] \, \mathbb{P}[X > s] \quad \text{as} \quad s \to \infty.$$

(*iii*) *If $Q_t(z)$ is analytic at the point $z = 1$, then for every fixed $r \in \mathbb{N}$:*

$$\mathbb{P}[R_r(t) > s] \sim \mathbb{P}[X_{N(t):N(t)} > s] \sim \mathbb{P}[S_{N(t)} > s] \sim \mathbb{E}[N(t)] \, \mathbb{P}[X > s] \quad \text{as} \quad s \to \infty.$$

Proof. Let $t > 0$ be fixed. Recall that $\mathcal{S}$ is a (proper) subset of $\mathcal{L}$.

(*i*) Since $\mathbb{E}[N(t)] < \infty$, Theorem 3.5 applies.

(*ii*) Let $r \in \mathbb{N}$ be fixed. The analyticity of $Q_t(z)$ at $z = 1$ implies that $Q_t^{(r)}(1-) < \infty$. If $r = 1$, the asymptotic equivalence in (*i*) holds since $\mathbb{E}[N(t)] = Q_t^{(1)}(1-) < \infty$. If $r \geqslant 2$, we combine Theorem 3.3 and Theorem 3.4 to obtain:

$$\mathbb{E}[N(t)] \leqslant \liminf_{s\to\infty} \frac{\mathbb{P}[R_r(t) > s]}{\mathbb{P}[X > s]} \leqslant \limsup_{s\to\infty} \frac{\mathbb{P}[R_r(t) > s]}{\mathbb{P}[X > s]} \leqslant \mathbb{E}[N(t)]$$

which proves the result.

(*iii*) Repeating the arguments of the proof of Theorem 3.3 gives $\mathbb{P}[S_{N(t)} > s] \sim \mathbb{E}[N(t)] \, \mathbb{P}[X > s]$ as $s \to \infty$. By using the latter together with Lemma 3.1 and (*i*) for the case $r = 1$ or (*ii*) for the case $r \geqslant 2$, the claim follows. □

3.2.2 Convergence in Distribution for $R_1(t)$

Given that the claim size distribution F belongs to an extremal class $\mathcal{C}_\gamma(a)$ for some $\gamma \in \mathbb{R}$, we derive in the following result the limiting distribution of $R_1(t)/a(t)$ as $t \to \infty$ under the additional condition that the claim number process $\{N(t);\, t \geqslant 0\}$ is mixed Poisson.

Theorem 3.7. *Assume that $F \in \mathcal{C}_\gamma(a)$ for some $\gamma \in \mathbb{R}$ and that $\{N(t);\, t \geqslant 0\}$ is a mixed Poisson process with mixing random variable Λ. Then:*

$$\frac{R_1(t)}{a(t)} \xrightarrow{\mathcal{D}} Y_{1,\gamma} \quad \textit{as} \quad t \to \infty$$

where the distribution function of the limiting random variable $Y_{1,\gamma}$ is given by:

$$\mathbb{P}[Y_{1,\gamma} > s] = \int_0^\infty w\, \eta_\gamma(sw^\gamma)\, q_2(w)\, \mathrm{d}w =: I_\gamma(s), \quad s \geqslant 0$$

provided $1 + \gamma s w^\gamma > 0$. Specifically:

(i) If $\gamma = 0$: $I_0(s) = \mathrm{e}^{-s}$;

(ii) If $\gamma > 0$: $I_\gamma(s) = \int_0^\infty w(1+\gamma s w^\gamma)^{-1/\gamma}\, q_2(w)\, \mathrm{d}w$;

(iii) If $\gamma < 0$: $I_\gamma(s) = \int_{(|\gamma| s)^{1/|\gamma|}}^\infty w(1+\gamma s w^\gamma)^{1/|\gamma|}\, q_2(w)\, \mathrm{d}w$.

Proof. Let $\gamma \in \mathbb{R}$ and $s \geqslant 0$ be fixed. By using (3.3) and the change of variable $w = t(1 - F(y))$, we obtain for every $t > 0$:

$$\begin{aligned}
\mathbb{P}[R_1(t) > a(t)s] &= (1 - F(a(t)s))\, p_1(t) + \int_0^\infty (1 - F(y + a(t)s))\, Q_t^{(2)}(F(y))\, \mathrm{d}F(y) \\
&= (1 - F(a(t)s))\, p_1(t) \\
&\quad + \int_0^\infty w \frac{t}{w} \left\{ 1 - F\left(U(t/w) + a(t/w) \frac{a(t)}{a(t/w)}\, s \right) \right\} \frac{Q_t^{(2)}(1 - w/t)}{t^2}\, \mathbb{1}_{(0,t]}(w)\, \mathrm{d}w \\
&= (1 - F(a(t)s))\, p_1(t) \\
&\quad + \int_0^\infty w \frac{t}{w} \left\{ 1 - F\left(U(t/w) + a(t/w) \frac{a(t)}{a(t/w)}\, s \right) \right\} q_2(w)\, \mathbb{1}_{(0,t]}(w)\, \mathrm{d}w
\end{aligned}$$

where the last equality follows from (1.15).

The first term on the right-hand side goes to 0 as $t \to \infty$ since $|(1 - F(a(t)s))\, p_1(t)| \leqslant p_1(t)$ for every $t > 0$ and $\lim_{t\to\infty} p_1(t) = 0$.

For every $t > 0$, define $f_t(x, w) := \frac{t}{w}\{1 - F(U(t/w) + a(t/w)x)\}\, \mathbb{1}_{(0,t]}(w)$ for $x \in \mathbb{R}$ and $w > 0$ and $g_t(w; s) := \frac{a(t)}{a(t/w)} s$ for $w > 0$. Since $a \in \mathrm{RV}_\gamma$, we get $\lim_{t\to\infty} g_t(w; s) = sw^\gamma =: g_\gamma(w; s)$ for all $w > 0$. For any fixed $w > 0$, it follows from Theorem 1.3(*iii*) that $\lim_{t\to\infty} f_t(x, w) = \eta_\gamma(x)$ for all $x \in \mathbb{R}$ such that $1 + \gamma x > 0$.

For each $t > 0$, $|w f_t(g_t(w; s), w)\, q_2(w)| \leqslant w\, q_2(w)$ for all $w > 0$. Moreover, $\int_0^\infty w\, q_2(w)\, \mathrm{d}w = 1 < \infty$ by (1.14). Now, it remains to show that $\lim_{t\to\infty} f_t(g_t(w; s), w) = \eta_\gamma(g_\gamma(w; s))$ for all $w > 0$ such that $1 + \gamma s w^\gamma > 0$. Fix $w > 0$ such that $1 + \gamma s w^\gamma > 0$ and write the following triangular inequality:

$$|f_t(g_t(w; s), w) - \eta_\gamma(g_\gamma(w; s))| \leqslant |f_t(g_t(w; s), w) - \eta_\gamma(g_t(w; s))| + |\eta_\gamma(g_t(w; s)) - \eta_\gamma(g_\gamma(w; s))|.$$

First, $\lim_{t\to\infty} |\eta_\gamma(g_t(w;s)) - \eta_\gamma(g_\gamma(w;s))| = 0$ since η_γ is continuous and $\lim_{t\to\infty} g_t(w;s) = g_\gamma(w;s)$. Now, there exist real constants a, b with $-1/\max(0,\gamma) < a < g_\gamma(w;s) < b < 1/\max(-\gamma, 0)$ such that $g_t(w;s) \in [a,b]$ for t large enough. We then have $\lim_{t\to\infty} |f_t(g_t(w;s), w) - \eta_\gamma(g_t(w;s))| = 0$ if and only if $\lim_{t\to\infty} \sup_{x\in[a,b]} |f_t(x,w) - \eta_\gamma(x)| = 0$ if and only if $\lim_{t\to\infty} \sup_{x\in[a,b]} |f_t(x,1) - \eta_\gamma(x)| = 0$. The latter assertion proves to be true since the convergence in Theorem 1.3(iii) holds locally uniformly in $(-1/\max(0,\gamma), 1/\max(-\gamma,0))$, see Remark 1.17.

Hence, using Lebesgue's dominated convergence theorem, we conclude that:

$$\lim_{t\to\infty} \mathbb{P}[R_1(t) > a(t)s] = \int_0^\infty w\, \eta_\gamma(sw^\gamma)\, q_2(w)\, \mathrm{d}w$$

provided $1 + \gamma s w^\gamma > 0$. Furthermore, we have:

(i) $\gamma = 0$: $\eta_0(s) = \mathrm{e}^{-s}$ and $\mathrm{e}^{-s} \int_0^\infty w\, q_2(w)\, \mathrm{d}w = \mathrm{e}^{-s}$ since $\int_0^\infty w\, q_2(w)\, \mathrm{d}w = 1$ by (1.14);

(ii) $\gamma > 0$: $\eta_\gamma(sw^\gamma) = (1+\gamma s w^\gamma)^{-1/\gamma}$ and $1 + \gamma s w^\gamma$ is positive for all $w > 0$;

(iii) $\gamma < 0$: $\eta_\gamma(sw^\gamma) = (1+\gamma s w^\gamma)^{1/|\gamma|}$ and $1 + \gamma s w^\gamma > 0 \iff w > (|\gamma| s)^{1/|\gamma|}$. □

The appearance of the exponential distribution in the case $\gamma = 0$ is pleasing. However, for the other values of γ, no simplification seems to be possible.

Remark 3.4. Theorem 3.7 is still valid if F has support in $\mathbb{R}$ without any changes in the proof.

3.2.3 Convergence in Distribution for $R_r(t)$

Under the same assumptions as in Theorem 3.7, we now determine the limiting distribution of $R_r(t)/a(t)$ as $t \to \infty$ by deriving the general form of its Laplace-Stieltjes transform. Note that using Laplace-Stieltjes transforms makes sense since the random variable $R_r(t)/a(t)$ is a.s. nonnegative. Also, the very definition of the ECOMOR-quantity $R_r(t)$ makes a further centering unnecessary.

Theorem 3.8. *Let $r \in \mathbb{N}$ be fixed. Assume that $F \in \mathcal{C}_\gamma(a)$ for some $\gamma \in \mathbb{R}$ and that $\{N(t);\, t \geqslant 0\}$ is a mixed Poisson process with mixing random variable Λ. Then:*

$$\frac{R_r(t)}{a(t)} \xrightarrow{\mathcal{D}} Y_{r,\gamma} \quad \textit{as} \quad t \to \infty$$

where the limiting random variable $Y_{r,\gamma}$ has the Laplace-Stieltjes transform:

$$\mathbb{E}\big[\mathrm{e}^{-\theta Y_{r,\gamma}}\big] = \frac{1}{r!} \int_0^\infty \mathrm{e}^{r\theta h_\gamma(1/w)} \left(\int_0^w \mathrm{e}^{-\theta h_\gamma(1/z)}\, \mathrm{d}z \right)^r q_{r+1}(w)\, \mathrm{d}w, \quad \theta \geqslant 0. \tag{3.4}$$

Proof. Let $r \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and $\theta \geqslant 0$ be fixed. By using (3.1), a conditioning argument, the change of variable $w = t(1 - F(y))$ and (1.15), we get for every $t > 0$:

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\theta\, \frac{R_r(t)}{a(t)}\right)\right] \\
&= \sum_{n=0}^{r} p_n(t) \left(\mathbb{E}\left[\mathrm{e}^{-\theta \frac{X}{a(t)}}\right]\right)^n + \frac{1}{r!} \int_0^\infty \left(\int_y^\infty \mathrm{e}^{-\theta\{x-y\}/a(t)}\, \mathrm{d}F(x) \right)^r Q_t^{(r+1)}(F(y))\, \mathrm{d}F(y)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{r} p_n(t) \left(\mathbb{E}\left[\mathrm{e}^{-\theta \frac{X}{a(t)}} \right] \right)^n \\
&\quad + \frac{1}{r!} \int_0^\infty \left(\int_0^w \mathrm{e}^{-\theta \{U(t/z) - U(t/w)\}/a(t)} \, \mathrm{d}z \right)^r \frac{Q_t^{(r+1)}(1 - w/t)}{t^{r+1}} \, \mathbb{1}_{(0,t]}(w) \, \mathrm{d}w \\
&= \sum_{n=0}^{r} p_n(t) \left(\mathbb{E}\left[\mathrm{e}^{-\theta \frac{X}{a(t)}} \right] \right)^n + \frac{1}{r!} \int_0^\infty \left(\int_0^w \mathrm{e}^{-\theta \{U(t/z) - U(t/w)\}/a(t)} \, \mathrm{d}z \right)^r q_{r+1}(w) \, \mathbb{1}_{(0,t]}(w) \, \mathrm{d}w \\
&= \sum_{n=0}^{r} p_n(t) \left(\mathbb{E}\left[\mathrm{e}^{-\theta \frac{X}{a(t)}} \right] \right)^n + \frac{1}{r!} \int_0^\infty f_t(w;\theta) \, \mathrm{d}w
\end{aligned}
$$

where $f_t(w;\theta) := \left(\int_0^w g_t(z,w;\theta) \, \mathrm{d}z \right)^r q_{r+1}(w) \, \mathbb{1}_{(0,t]}(w)$ for $w > 0$ with an integrand defined by $g_t(z,w;\theta) := \mathrm{e}^{-\theta \{U(t/z) - U(t/w)\}/a(t)}$ for $w > 0$ and $z \in (0,w]$.

Since $\left| \sum_{n=0}^{r} p_n(t) \left(\mathbb{E}\left[\mathrm{e}^{-\theta X/a(t)} \right] \right)^n \right| \leqslant \Pi_{r+1}(t) \to 0$ as $t \to \infty$, the first term on the right-hand side goes to 0 as $t \to \infty$.

Now, fix $w > 0$. First, $|g_t(z,w;\theta)| \leqslant 1$ for all $z \in (0,w]$ and $t > 0$. Also, $\int_0^w \mathrm{d}z = w < \infty$. Finally, $\lim_{t\to\infty} g_t(z,w;\theta) = \mathrm{e}^{-\theta\{h_\gamma(1/z) - h_\gamma(1/w)\}} =: g_\gamma(z,w;\theta)$ for all $z \in (0,w]$ since U satisfies (1.4) with the auxiliary function a. Hence, by virtue of the bounded convergence theorem, we have:

$$
\lim_{t\to\infty} \int_0^w g_t(z,w;\theta) \, \mathrm{d}z = \int_0^w g_\gamma(z,w;\theta) \, \mathrm{d}z
$$

so that:

$$
\lim_{t\to\infty} f_t(w;\theta) = \left(\int_0^w g_\gamma(z,w;\theta) \, \mathrm{d}z \right)^r q_{r+1}(w) =: f_\gamma(w;\theta).
$$

For each $t > 0$, $|f_t(w;\theta)| \leqslant w^r q_{r+1}(w)$ for all $w > 0$. Also, $\int_0^\infty w^r q_{r+1}(w) \, \mathrm{d}w = r! < \infty$ by (1.14). Lebesgue's dominated convergence theorem then gives $\lim_{t\to\infty} \int_0^\infty f_t(w;\theta) \, \mathrm{d}w = \int_0^\infty f_\gamma(w;\theta) \, \mathrm{d}w$, that is:

$$
\lim_{t\to\infty} \mathbb{E}\left[\exp\left(-\theta \, \frac{R_r(t)}{a(t)} \right) \right] = \frac{1}{r!} \int_0^\infty \left(\int_0^w \mathrm{e}^{-\theta\{h_\gamma(1/z) - h_\gamma(1/w)\}} \, \mathrm{d}z \right)^r q_{r+1}(w) \, \mathrm{d}w =: \varphi_{r,\gamma}(\theta).
$$

For every $\theta \geqslant 0$, define $k_\theta(z,w;\gamma) := \mathrm{e}^{-\theta\{h_\gamma(1/z) - h_\gamma(1/w)\}}$ for $w > 0$ and $z \in (0,w]$. Fix $w > 0$. We have $|k_\theta(z,w;\gamma)| \leqslant 1$ for all $z \in (0,w]$ and $\theta \geqslant 0$. Also, $\lim_{\theta \downarrow 0} k_\theta(z,w;\gamma) = 1$ for all $z \in (0,w]$. By using the bounded convergence theorem and Lebesgue's dominated convergence theorem, it is then easy to prove that $\varphi_{r,\gamma}(\theta) \to 1$ as $\theta \downarrow 0$.

By the continuity theorem for Laplace-Stieltjes transforms (e.g., Theorem XIII.1.2 of Feller [25]), we therefore conclude that $R_r(t)/a(t) \xrightarrow{\mathcal{D}} Y_{r,\gamma}$ as $t \to \infty$ where the random variable $Y_{r,\gamma}$ has the Laplace-Stieltjes transform $\varphi_{r,\gamma}$. □

Remark 3.5. Assuming in Theorem 3.8 that the support of F is contained in $\mathbb{R}$, we find that we can replace θ by $-\mathrm{i}\theta$ in (3.4) so that the limiting random variable $Y_{r,\gamma}$ has the characteristic function:

$$
\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\theta Y_{r,\gamma}} \right] = \frac{1}{r!} \int_0^\infty \mathrm{e}^{-\mathrm{i}r\theta h_\gamma(1/w)} \left(\int_0^w \mathrm{e}^{\mathrm{i}\theta h_\gamma(1/z)} \, \mathrm{d}z \right)^r q_{r+1}(w) \, \mathrm{d}w, \quad \theta \in \mathbb{R}.
$$

Remark 3.6. In the proof of Theorem 3.8, we have used the condition (1.4) rather than Theorem 1.3 as in the proof of Theorem 3.7. The distribution function of the limiting random variable $Y_{1,\gamma}$ given in Theorem 3.7 also follows from Theorem 3.8, but no inversion of the Laplace-Stieltjes transform (3.4) seems possible for a general integer $r \geqslant 2$.

The distribution function of the limiting random variable $Y_{r,\gamma}$ is very hard to recover from its Laplace-Stieltjes transform (3.4) except when F is in the domain of attraction of the Gumbel distribution ($\gamma = 0$). Also, an explicit result may be obtained with $\gamma = -1$ as shown below.

Special cases. Under the assumptions of Theorem 3.8, two values of γ give something special for the limiting distribution of $R_r(t)/a(t)$ as $t \to \infty$.

(i) When $\gamma = 0$, we easily obtain from (3.4):

$$\mathbb{E}\big[\mathrm{e}^{-\theta Y_{r,0}}\big] = \frac{1}{(1+\theta)^r}, \quad \theta \geqslant 0$$

showing that the limiting random variable $Y_{r,0}$ is $\Gamma(r,1)$-distributed. This can alternatively be written as:

$$\lim_{t\to\infty} \mathbb{P}[R_r(t) > a(t)s] = \frac{1}{(r-1)!}\int_s^\infty x^{r-1}\mathrm{e}^{-x}\,\mathrm{d}x, \quad s \geqslant 0.$$

Hence, the difficult distribution of $R_r(t)$ can be approximated for large t as:

$$\mathbb{P}[R_r(t) > y] \approx \frac{1}{(r-1)!}\int_{y/a(t)}^\infty x^{r-1}\mathrm{e}^{-x}\,\mathrm{d}x = \Gamma_r(y/a(t)), \quad y \geqslant 0.$$

The case $\gamma = 0$ covers a wide range of claim size distributions like the exponential, gamma, lognormal and Weibull distributions (see Section 2.5). In Table 3.1, we give the expression of an auxiliary function a associated with each of these distributions.

Distribution $F \in \mathcal{C}_0(a)$	Auxiliary function a
Exp(λ), $\lambda > 0$	$a(t) = \lambda^{-1}$, $t > 0$
$\Gamma(\alpha,\beta)$, $\alpha > 0$, $\beta > 0$	$a(t) = \beta^{-1}$, $t > 0$
LN(μ,σ^2), $\mu \in \mathbb{R}$, $\sigma > 0$	$a(t) = \sigma(2\log t)^{-1/2}\exp\{\mu+\sigma b(t)\}$, $t > \mathrm{e}$ where $b(t) = (2\log t)^{1/2} - \frac{\log\log t + \log 4\pi}{2(2\log t)^{1/2}}$
W(τ,c), $\tau > 0$, $c > 0$	$a(t) = \tau^{-1}c^{-1/\tau}(\log t)^{-1+1/\tau}$, $t > 1$

Table 3.1: Auxiliary functions a for distributions $F \in \mathcal{C}_0(a)$.

(ii) When $\gamma = -1$, a simple calculation yields:

$$\mathbb{E}\big[\mathrm{e}^{-\theta Y_{r,-1}}\big] = \mathbb{E}\left[\prod_{j=1}^{r}\frac{\Lambda}{\Lambda + j\theta}\right], \quad \theta \geqslant 0. \tag{3.5}$$

Proof. Let $\theta \geqslant 0$ be fixed. Let H denote the distribution function of the mixing random variable Λ. Setting $\gamma = -1$ in (3.4) gives:

$$\begin{aligned}
\mathbb{E}\big[\mathrm{e}^{-\theta Y_{r,-1}}\big] &= \frac{1}{r!}\int_0^\infty \mathrm{e}^{-r\theta w}\bigg(\int_0^w \mathrm{e}^{\theta z}\,\mathrm{d}z\bigg)^r q_{r+1}(w)\,\mathrm{d}w \\
&= \frac{1}{r!\,\theta^r}\int_0^\infty \big(1-\mathrm{e}^{-\theta w}\big)^r q_{r+1}(w)\,\mathrm{d}w \\
&= \frac{1}{r!\,\theta^r}\int_0^\infty \big(1-\mathrm{e}^{-\theta w}\big)^r \int_0^\infty \mathrm{e}^{-\lambda w}\lambda^{r+1}\,\mathrm{d}H(\lambda)\,\mathrm{d}w \\
&= \frac{1}{r!\,\theta^r}\int_0^\infty \lambda^{r+1}\int_0^\infty \mathrm{e}^{-\lambda w}\big(1-\mathrm{e}^{-\theta w}\big)^r\,\mathrm{d}w\,\mathrm{d}H(\lambda).
\end{aligned}$$

The inner integral reduces for every $\lambda > 0$ to:

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-\lambda w}\big(1-\mathrm{e}^{-\theta w}\big)^r\,\mathrm{d}w &= \frac{1}{\theta}\int_0^1 x^r(1-x)^{\lambda/\theta-1}\,\mathrm{d}x \\
&= \frac{1}{\theta}\,\mathrm{B}(r+1,\lambda/\theta) \\
&= \frac{1}{\theta}\,\frac{r\theta}{\lambda+r\theta}\,\mathrm{B}(r,\lambda/\theta) \\
&\;\;\vdots \\
&= \frac{1}{\theta}\,\frac{r\theta}{\lambda+r\theta}\,\frac{(r-1)\theta}{\lambda+(r-1)\theta}\cdots\frac{\theta}{\lambda+\theta}\,\frac{\theta}{\lambda} \\
&= \frac{r!\,\theta^r}{\lambda\prod_{j=1}^r(\lambda+j\theta)}.
\end{aligned}$$

Thus, the Laplace-Stieltjes transform of the limiting random variable $Y_{r,-1}$ is given by:

$$\mathbb{E}\big[\mathrm{e}^{-\theta Y_{r,-1}}\big] = \int_0^\infty \frac{\lambda^r}{\prod_{j=1}^r(\lambda+j\theta)}\,\mathrm{d}H(\lambda) = \mathbb{E}\left[\prod_{j=1}^r \frac{\Lambda}{\Lambda+j\theta}\right]$$

and the assertion is proved. □

Note that when the mixing random variable Λ is degenerate at the point $\lambda \in (0,\infty)$, it follows from (3.5) that $Y_{r,-1} \stackrel{\mathcal{D}}{=} E_1 + \cdots + E_r$ for independent random variables $E_1, \ldots, E_r$, where E_j is $\mathrm{Exp}(\lambda/j)$-distributed for all $j = 1, \ldots, r$.

Under the assumptions of Theorem 3.8, it is possible to derive the limiting expression of the βth moment of $R_r(t)/a(t)$ as $t \to \infty$ for all $\beta \in \mathbb{N}$. However, we need to impose the following conditions:

(*i*) If $\gamma = 0$: $\mathbb{E}[\Lambda^{-\beta\delta}] < \infty$ for some $\delta > 0$ and $\mathbb{E}[\Lambda^{\beta\delta}] < \infty$ for some $\delta \in (0, 1/\beta)$.

(*ii*) If $\gamma > 0$: $\mathbb{E}[\Lambda^{\beta\delta}] < \infty$ for some $\delta \in (\gamma, 1/\beta)$.

(*iii*) If $\gamma < 0$: $\mathbb{E}[\Lambda^{-\beta\delta}] < \infty$ for some $\delta > |\gamma|$.

In the case $\gamma \leqslant 0$, we also assume that the distribution function of the mixing random variable Λ is regularly varying at 0 with index $\alpha > -\beta\gamma$ (see Definition 1.10). Then, we find that the moments of $R_r(t)/a(t)$ converge as $t \to \infty$ to the corresponding moments of the limiting random variable $Y_{r,\gamma}$. For instance, we get for the mean when $\gamma < 1$:

$$\lim_{t\to\infty}\frac{\mathbb{E}[R_r(t)]}{a(t)} = \frac{\Gamma(r-\gamma+1)}{(r-1)!\,(1-\gamma)}\,\mathbb{E}[\Lambda^\gamma] = \mathbb{E}[Y_{r,\gamma}] \tag{3.6}$$

while we get for the moment of second order when $\gamma < 1/2$:

$$\lim_{t\to\infty} \frac{\mathbb{E}[R_r(t)^2]}{a^2(t)} = \frac{\{1+r(1-2\gamma)\}\,\Gamma(r-2\gamma+1)}{(r-1)!\,(1-\gamma)^2\,(1-2\gamma)}\,\mathbb{E}[\Lambda^{2\gamma}] = \mathbb{E}[Y_{r,\gamma}^2].$$

3.2.4 Asymptotic Comparison of $R_r(t)$ with $S_{N(t)}$

We turn to a comparison of the ECOMOR-quantity $R_r(t)$ with the total claim amount $S_{N(t)}$ by studying the convergence of the ratio $R_r(t)/S_{N(t)}$ as $t \to \infty$. We derive a result that shows the dominant influence of $R_r(t)$ on $S_{N(t)}$ in the case where the claim size distribution F is super heavy-tailed, i.e. $1 - F \in \mathrm{RV}_0$. To do so, we first establish a similar result for the ratio R_r^n/S_n where R_r^n is the random variable $R_r(t)$ when the claim number process $\{N(t);\, t \geqslant 0\}$ is non-random, i.e. for any fixed $r \in \mathbb{N}$:

$$R_r^n := \begin{cases} S_n, & n \leqslant r \\ \sum_{i=1}^r X_{n-i+1:n} - rX_{n-r:n}, & n \geqslant r+1. \end{cases}$$

Note that we put $R_r^0/S_0 := 0$.

Lemma 3.9. *Let $r \in \mathbb{N}$ be fixed. If $1 - F \in \mathrm{RV}_0$, then for any $s > 0$:*

$$\frac{R_r^n}{S_n} \xrightarrow{s\text{th}} 1 \quad \textit{as} \quad n \to \infty.$$

Proof. Let $r \in \mathbb{N}$ be fixed. First, we prove that $R_r^n/S_n \xrightarrow{p} 1$ as $n \to \infty$. For each integer $n \geqslant r+1$, rewrite the ratio R_r^n/S_n as:

$$\frac{R_r^n}{S_n} = \frac{X_{n:n}}{S_n}\left(1 + \frac{X_{n-1:n}}{X_{n:n}} + \cdots + \frac{X_{n-r+1:n}}{X_{n:n}} - r\frac{X_{n-r:n}}{X_{n:n}}\right).$$

Since $1 - F \in \mathrm{RV}_0$, we know from Darling [20] that $X_{n:n}/S_n \xrightarrow{p} 1$ as $n \to \infty$. Because we have:

$$0 \leqslant \frac{X_{n-r:n}}{X_{n:n}} \leqslant \cdots \leqslant \frac{X_{n-1:n}}{X_{n:n}} \quad \text{a.s.}$$

for all integers $n \geqslant r+1$, it then suffices to prove that $X_{n-1:n}/X_{n:n} \xrightarrow{p} 0$ as $n \to \infty$. Since $\mathbb{P}[X_{n-1:n} > \epsilon X_{n:n}] = 0$ for every $\epsilon \geqslant 1$, choose $\epsilon \in (0,1)$ arbitrarily. We have:

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}[X_{n-1:n} > \epsilon X_{n:n}] &= \lim_{n\to\infty} n(n-1)\int_0^\infty \int_{\epsilon x}^x F^{n-2}(y)\,\mathrm{d}F(y)\,\mathrm{d}F(x) \\
&= \lim_{n\to\infty} n\int_0^\infty \left(F^{n-1}(x) - F^{n-1}(\epsilon x)\right)\mathrm{d}F(x) \\
&= 1 - \lim_{n\to\infty} n\int_0^\infty F^{n-1}(\epsilon x)\,\mathrm{d}F(x) \\
&= 0
\end{aligned}$$

where the last equality follows from Lemma 1.3(b) of Maller and Resnick [48] since $1 - F \in \mathrm{RV}_0$. Hence, $X_{n-1:n}/X_{n:n} \xrightarrow{p} 0$ as $n \to \infty$ which entails $R_r^n/S_n \xrightarrow{p} 1$ as $n \to \infty$ by virtue of Slutsky's theorem (e.g., Corollary page 97 of Chung [19]).

Now, it is easily shown that the convergence even holds in sth mean for any $s > 0$. Indeed, since $|R_r^n/S_n| \leqslant 1$ a.s. for all $n \in \mathbb{N}_0$, an application of Theorem 4.1.4 of Chung [19] gives $R_r^n/S_n \xrightarrow{s\text{th}} 1$ as $n \to \infty$ for any $s > 0$. □

Now, armed with Lemma 3.9, we deal with the convergence of the ratio $R_r(t)/S_{N(t)}$ as $t \to \infty$.

Theorem 3.10. *Let $r \in \mathbb{N}$ be fixed. If $1 - F \in \mathrm{RV}_0$, then for any $s > 0$:*

$$\frac{R_r(t)}{S_{N(t)}} \xrightarrow{\text{sth}} 1 \quad as \quad t \to \infty.$$

Proof. Let $r \in \mathbb{N}$ and $s > 0$ be fixed. Choose $\epsilon > 0$ arbitrarily. It follows from Lemma 3.9 that there exists $n_0 := n_0(\epsilon) \in \mathbb{N}_0$ such that $\mathbb{E}\big[\big|R_r^n/S_n - 1\big|^s\big] \leqslant \epsilon$ for all integers $n > n_0$. Since $0 \leqslant R_r^n/S_n \leqslant 1$ a.s. for all $n \in \mathbb{N}_0$, we also have $\mathbb{E}\big[\big|R_r^n/S_n - 1\big|^s\big] \leqslant 1$ for all $n = 0, \ldots, n_0$ so that:

$$\begin{aligned}
\mathbb{E}\left[\left|\frac{R_r(t)}{S_{N(t)}} - 1\right|^s\right] &= \sum_{n=0}^{\infty} p_n(t)\, \mathbb{E}\left[\left|\frac{R_r^n}{S_n} - 1\right|^s\right] \\
&= \left(\sum_{n=0}^{n_0} + \sum_{n=n_0+1}^{\infty}\right) p_n(t)\, \mathbb{E}\left[\left|\frac{R_r^n}{S_n} - 1\right|^s\right] \\
&\leqslant \Pi_{n_0+1}(t) + \epsilon.
\end{aligned}$$

Since $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$, it follows that $\limsup_{t\to\infty} \mathbb{E}\big[\big|R_r(t)/S_{N(t)} - 1\big|^s\big] \leqslant \epsilon$ and the result is established upon letting $\epsilon \downarrow 0$. □

Under the assumption of Theorem 3.10, the total claim amount $S_{N(t)}$ is then completely dominated by the ECOMOR-quantity $R_r(t)$. It follows from Theorem 4.5.1 of Chung [19] that for any $s > 0$:

$$\lim_{t\to\infty} \mathbb{E}\left[\left(\frac{R_r(t)}{S_{N(t)}}\right)^s\right] = 1.$$

3.3 Results for LC Reinsurance

In this section, we give asymptotic results related to LC reinsurance. More precisely, we focus on convergence in distribution for $(L_r(t) - rU(t))/a(t)$ as $t \to \infty$ when the claim size distribution F belongs to an extremal class $\mathcal{C}_\gamma(a)$ for some $\gamma \in \mathbb{R}$ and the counting process $\{N(t);\, t \geqslant 0\}$ is mixed Poisson. The first part concerns $L_1(t)$ and the second part deals with $L_r(t)$ for a general $r \in \mathbb{N}$.

3.3.1 Convergence in Distribution for $L_1(t)$

The limiting distribution of the appropriately normalized LC-quantity $L_1(t)$ as $t \to \infty$ is derived in the following result. Contrary to the ECOMOR case, a centering is necessary here.

Theorem 3.11. *Assume that $F \in \mathcal{C}_\gamma(a)$ for some $\gamma \in \mathbb{R}$ and that $\{N(t);\, t \geqslant 0\}$ is a mixed Poisson process with mixing random variable Λ. Then:*

$$\frac{L_1(t) - U(t)}{a(t)} \xrightarrow{\mathcal{D}} Z_{1,\gamma} \quad as \quad t \to \infty$$

where the distribution function of the limiting random variable $Z_{1,\gamma}$ is given by:

$$\mathbb{P}[Z_{1,\gamma} \leqslant s] = \mathbb{E}\big[\mathrm{e}^{-\eta_\gamma(s)\,\Lambda}\big], \quad 1 + \gamma s > 0.$$

Proof. Let $\gamma \in \mathbb{R}$ be fixed. For every $t > 0$, define $\psi_t(z) := t\{1 - F(U(t) + a(t)z)\}$ for $z \in \mathbb{R}$. Fix $s \in \mathbb{R}$ such that $1 + \gamma s > 0$. By using (3.2), a conditioning argument, the change of variable $w = t(1 - F(y))$ and (1.15), we get for every $t > 0$:

$$\begin{aligned}
\mathbb{P}[L_1(t) - U(t) \leqslant a(t)s] &= \sum_{n=0}^{\infty} \mathbb{P}[L_1(t) \leqslant U(t) + a(t)s \,|N(t) = n]\, p_n(t) \\
&= \mathbb{P}[S_0 \leqslant U(t) + a(t)s]\, p_0(t) + \sum_{n=1}^{\infty} \mathbb{P}[X_{n:n} \leqslant U(t) + a(t)s]\, p_n(t) \\
&= p_0(t)\, \mathbb{1}_{[-U(t)/a(t),\infty)}(s) + \sum_{n=1}^{\infty} \left\{ n\, p_n(t) \int_0^{U(t)+a(t)s} F^{n-1}(y)\, \mathrm{d}F(y) \right\} \\
&= p_0(t)\, \mathbb{1}_{[-U(t)/a(t),\infty)}(s) + \int_0^{U(t)+a(t)s} \sum_{n=1}^{\infty} \left\{ n\, p_n(t)\, F^{n-1}(y) \right\} \mathrm{d}F(y) \\
&= p_0(t)\, \mathbb{1}_{[-U(t)/a(t),\infty)}(s) + \int_0^{U(t)+a(t)s} Q_t^{(1)}(F(y))\, \mathrm{d}F(y) \\
&= p_0(t)\, \mathbb{1}_{[-U(t)/a(t),\infty)}(s) + \int_0^{\infty} \frac{Q_t^{(1)}(1 - w/t)}{t}\, \mathbb{1}_{[\psi_t(s),t]}(w)\, \mathrm{d}w \\
&= p_0(t)\, \mathbb{1}_{[-U(t)/a(t),\infty)}(s) + \int_0^{\infty} q_1(w)\, \mathbb{1}_{[\psi_t(s),t]}(w)\, \mathrm{d}w.
\end{aligned}$$

The first term on the right-hand side goes to 0 as $t \to \infty$ since $\left|p_0(t)\, \mathbb{1}_{[-U(t)/a(t),\infty)}(s)\right| \leqslant p_0(t)$ for every $t > 0$ and $\lim_{t\to\infty} p_0(t) = 0$.

Now, for each $t > 0$, $|q_1(w)\, \mathbb{1}_{[\psi_t(s),t]}(w)| \leqslant q_1(w)$ for all $w \geqslant 0$, and $\int_0^{\infty} q_1(w)\, \mathrm{d}w = 1 < \infty$ by (1.14). Moreover, $\lim_{t\to\infty} q_1(w)\, \mathbb{1}_{[\psi_t(s),t]}(w) = q_1(w)\, \mathbb{1}_{[\eta_\gamma(s),\infty)}(w)$ for all $w \geqslant 0$ by Theorem 1.3(*iii*). Since $\lim_{t\to\infty} p_0(t) = 0$, Lebesgue's dominated convergence theorem thus gives:

$$\lim_{t\to\infty} \mathbb{P}[L_1(t) - U(t) \leqslant a(t)s] = \int_{\eta_\gamma(s)}^{\infty} q_1(w)\, \mathrm{d}w = -\int_{\eta_\gamma(s)}^{\infty} \frac{\partial q_0(w)}{\partial w}\, \mathrm{d}w = q_0(\eta_\gamma(s)) = \mathbb{E}\big[\mathrm{e}^{-\eta_\gamma(s)\,\Lambda}\big]$$

and the proof is finished. □

Replacing $a(t)/U(t)$ by its limit γ when $\gamma > 0$ and $a(t)/(U(\infty) - U(t))$ by its limit $|\gamma|$ when $\gamma < 0$ (see Proposition 1.2(*i*)-(*ii*)), we obtain after an affine transformation depending on the sign of γ:

$$\lim_{t\to\infty} \mathbb{P}[L_1(t) - c(t) \leqslant d(t)s] = \mathbb{E}\big[\mathrm{e}^{-\chi(s)\,\Lambda}\big]$$

where one of the three following cases emerges necessarily:

(*i*) If $\gamma = 0$: $c(t) = U(t)$, $d(t) = a(t)$ and $\chi(s) = \mathrm{e}^{-s}$, $s \in \mathbb{R}$;

(*ii*) If $\gamma > 0$: $c(t) = 0$, $d(t) = U(t)$ and $\chi(s) = s^{-1/\gamma}$, $s > 0$;

(*iii*) If $\gamma < 0$: $c(t) = U(\infty)$, $d(t) = U(\infty) - U(t)$ and $\chi(s) = |s|^{1/|\gamma|}$, $s \leqslant 0$.

Remark 3.7. If we assume in Theorem 3.11 that the mixing random variable Λ is degenerate at the point 1, the distribution function of the limiting random variable $Z_{1,\gamma}$ then reduces to the extreme value distribution G_γ defined in (1.6).

Remark 3.8. Theorem 3.11 still holds if F has support in $\mathbb{R}$ since the proof is then identical.

3.3.2 Convergence in Distribution for $L_r(t)$

Now, we determine the limiting distribution of the suitably normalized LC-quantity $L_r(t)$ as $t \to \infty$ by deriving the general form of its characteristic function. Note that characteristic functions are used instead of Laplace-Stieltjes transforms because the normalized random variable $L_r(t)$ may assume negative values.

Theorem 3.12. *Let $r \in \mathbb{N}$ be fixed. Assume that $F \in \mathcal{C}_\gamma(a)$ for some $\gamma \in \mathbb{R}$ and that $\{N(t);\, t \geqslant 0\}$ is a mixed Poisson process with mixing random variable Λ. Then:*

$$\frac{L_r(t) - rU(t)}{a(t)} \xrightarrow{\mathcal{D}} Z_{r,\gamma} \quad \text{as} \quad t \to \infty$$

where the limiting random variable $Z_{r,\gamma}$ has the characteristic function:

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}\theta Z_{r,\gamma}}\big] = \frac{1}{r!}\int_0^\infty \left(\int_0^w \mathrm{e}^{\mathrm{i}\theta h_\gamma(1/z)}\,\mathrm{d}z\right)^r q_{r+1}(w)\,\mathrm{d}w, \quad \theta \in \mathbb{R}. \tag{3.7}$$

Proof. Let $r \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and $\theta \in \mathbb{R}$ be fixed. By using (3.2), a conditioning argument, the change of variable $w = t(1 - F(y))$ and (1.15), we obtain for every $t > 0$:

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\mathrm{i}\theta\,\frac{L_r(t) - rU(t)}{a(t)}\right)\right] \\
&= \sum_{n=0}^r p_n(t)\,\mathbb{E}\Big[\mathrm{e}^{\mathrm{i}\theta\frac{S_n - rU(t)}{a(t)}}\Big] + \frac{1}{r!}\int_0^\infty \left(\int_y^\infty \mathrm{e}^{\mathrm{i}\theta\{x-U(t)\}/a(t)}\,\mathrm{d}F(x)\right)^r Q_t^{(r+1)}(F(y))\,\mathrm{d}F(y) \\
&= \sum_{n=0}^r p_n(t)\,\mathbb{E}\Big[\mathrm{e}^{\mathrm{i}\theta\frac{S_n - rU(t)}{a(t)}}\Big] \\
&\quad + \frac{1}{r!}\int_0^\infty \left(\int_0^w \mathrm{e}^{\mathrm{i}\theta\{U(t/z)-U(t)\}/a(t)}\,\mathrm{d}z\right)^r \frac{Q_t^{(r+1)}(1 - w/t)}{t^{r+1}}\,\mathbb{1}_{(0,t]}(w)\,\mathrm{d}w \\
&= \sum_{n=0}^r p_n(t)\,\mathbb{E}\Big[\mathrm{e}^{\mathrm{i}\theta\frac{S_n - rU(t)}{a(t)}}\Big] + \frac{1}{r!}\int_0^\infty \left(\int_0^w \mathrm{e}^{\mathrm{i}\theta\{U(t/z)-U(t)\}/a(t)}\,\mathrm{d}z\right)^r q_{r+1}(w)\,\mathbb{1}_{(0,t]}(w)\,\mathrm{d}w \\
&= \sum_{n=0}^r p_n(t)\,\mathbb{E}\Big[\mathrm{e}^{\mathrm{i}\theta\frac{S_n - rU(t)}{a(t)}}\Big] + \frac{1}{r!}\int_0^\infty f_t(w;\theta)\,\mathrm{d}w
\end{aligned}$$

where $f_t(w;\theta) := \big(\int_0^w g_t(z;\theta)\,\mathrm{d}z\big)^r q_{r+1}(w)\,\mathbb{1}_{(0,t]}(w)$ for $w > 0$ with $g_t(z;\theta) := \mathrm{e}^{\mathrm{i}\theta\{U(t/z)-U(t)\}/a(t)}$ for $z \in (0, w]$.

Since $\big|\sum_{n=0}^r p_n(t)\,\mathbb{E}\big[\mathrm{e}^{\mathrm{i}\theta\frac{S_n - rU(t)}{a(t)}}\big]\big| \leqslant \Pi_{r+1}(t) \to 0$ as $t \to \infty$, the first term on the right-hand side goes to 0 as $t \to \infty$.

Fix $w > 0$. First, $|g_t(z;\theta)| \leqslant 1$ for all $z \in (0, w]$ and $t > 0$. Also, $\int_0^w \mathrm{d}z = w < \infty$. Finally, $\lim_{t\to\infty} g_t(z;\theta) = \mathrm{e}^{\mathrm{i}\theta h_\gamma(1/z)} =: g_\gamma(z;\theta)$ for all $z \in (0, w]$ since U satisfies (1.4) with auxiliary function a. Hence, applying the bounded convergence theorem gives:

$$\lim_{t\to\infty}\int_0^w g_t(z;\theta)\,\mathrm{d}z = \int_0^w g_\gamma(z;\theta)\,\mathrm{d}z$$

so that:

$$\lim_{t\to\infty} f_t(w;\theta) = \left(\int_0^w g_\gamma(z;\theta)\,\mathrm{d}z\right)^r q_{r+1}(w) =: f_\gamma(w;\theta).$$

For each $t > 0$, $|f_t(w;\theta)| \leqslant w^r q_{r+1}(w)$ for all $w > 0$. Also, $\int_0^\infty w^r q_{r+1}(w)\,\mathrm{d}w = r! < \infty$ by (1.14). Lebesgue's dominated convergence theorem then gives $\lim_{t\to\infty} \int_0^\infty f_t(w;\theta)\,\mathrm{d}w = \int_0^\infty f_\gamma(w;\theta)\,\mathrm{d}w$, that is:

$$\lim_{t\to\infty} \mathbb{E}\left[\exp\left(\mathrm{i}\theta\,\frac{L_r(t) - rU(t)}{a(t)}\right)\right] = \frac{1}{r!}\int_0^\infty \left(\int_0^w \mathrm{e}^{\mathrm{i}\theta h_\gamma(1/z)}\,\mathrm{d}z\right)^r q_{r+1}(w)\,\mathrm{d}w =: \phi_{r,\gamma}(\theta).$$

Fix $w > 0$. For every $\theta \in \mathbb{R}$, define $k_\theta(z;\gamma) := \mathrm{e}^{\mathrm{i}\theta h_\gamma(1/z)}$ for $z \in (0, w]$. We have $|k_\theta(z;\gamma)| \leqslant 1$ for all $z \in (0, w]$ and $\theta \in \mathbb{R}$. Also, $\lim_{\theta\to 0} k_\theta(z;\gamma) = 1$ for all $z \in (0, w]$. By combining the bounded convergence theorem and Lebesgue's dominated convergence theorem, it is then easy to prove that $\phi_{r,\gamma}$ is continuous at 0.

By the continuity theorem for characteristic functions (e.g., Theorem XV.3.2 of Feller [25]), we therefore conclude that $(L_r(t) - rU(t))/a(t) \xrightarrow{\mathcal{D}} Z_{r,\gamma}$ as $t \to \infty$ where the random variable $Z_{r,\gamma}$ has the characteristic function $\phi_{r,\gamma}$. □

Remark 3.9. The distribution function of the limiting random variable $Z_{1,\gamma}$ given in Theorem 3.11 can be recovered from the characteristic function (3.7) with $r = 1$.

Remark 3.10. If the support of F is contained in $\mathbb{R}$, it turns out that Theorem 3.12 remains true without any changes in the arguments of the proof.

The case $\gamma = 0$ in Theorem 3.12 gives something special. Indeed, when F is in the domain of attraction of the Gumbel distribution, the inner integral of the characteristic function (3.7) reduces for every $w > 0$ to:

$$\int_0^w \mathrm{e}^{\mathrm{i}\theta h_0(1/z)}\,\mathrm{d}z = \int_0^w z^{-\mathrm{i}\theta}\,\mathrm{d}z = \frac{w^{1-\mathrm{i}\theta}}{1-\mathrm{i}\theta}$$

so that the limiting random variable $Z_{r,0}$ has the characteristic function:

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}\theta Z_{r,0}}\big] = \frac{\Gamma(r(1-\mathrm{i}\theta)+1)}{r!\,(1-\mathrm{i}\theta)^r}\,\mathbb{E}[\Lambda^{\mathrm{i}r\theta}], \quad \theta \in \mathbb{R}.$$

Under the assumptions of Theorem 3.12, we can use Laplace-Stieltjes transforms to determine the limiting distribution of $L_r(t)/U(t)$ as $t \to \infty$ when $\gamma > 0$. Replacing θ by $\mathrm{i}\theta$ in (3.7) and $a(t)/U(t)$ by its limit γ, we then obtain:

$$\frac{L_r(t)}{U(t)} \xrightarrow{\mathcal{D}} W_{r,\gamma} \quad \text{as} \quad t \to \infty$$

where the limiting random variable $W_{r,\gamma}$ has the Laplace-Stieltjes transform:

$$\mathbb{E}\big[\mathrm{e}^{-\theta W_{r,\gamma}}\big] = \frac{1}{r!}\int_0^\infty \left(\int_0^w \mathrm{e}^{-\theta z^{-\gamma}}\mathrm{d}z\right)^r q_{r+1}(w)\,\mathrm{d}w, \quad \theta \geqslant 0.$$

Also, we can derive the limiting expression of the βth moment of $L_r(t)/U(t)$ as $t \to \infty$ for any $\beta \in \mathbb{N}$ when $\gamma > 0$. The moment actually converges to the corresponding moment of the limiting random variable $W_{r,\gamma}$. For instance, we get for the mean when $\gamma \in (0, 1)$:

$$\lim_{t\to\infty} \frac{\mathbb{E}[L_r(t)]}{U(t)} = \frac{\Gamma(r-\gamma+1)}{(r-1)!\,(1-\gamma)}\,\mathbb{E}[\Lambda^\gamma] = \mathbb{E}[W_{r,\gamma}] \tag{3.8}$$

under the condition $\mathbb{E}[\Lambda^{\delta}] < \infty$ for some $\delta \in (\gamma, 1)$, while we get for the moment of second order when $\gamma \in (0, 1/2)$:

$$\lim_{t\to\infty} \frac{\mathbb{E}[L_r(t)^2]}{U^2(t)} = \frac{\{\gamma^2 + r(1-2\gamma)\}\,\Gamma(r-2\gamma+1)}{(r-1)!\,(1-\gamma)^2\,(1-2\gamma)}\,\mathbb{E}[\Lambda^{2\gamma}] = \mathbb{E}[W_{r,\gamma}^2]$$

under the condition $\mathbb{E}[\Lambda^{2\delta}] < \infty$ for some $\delta \in (\gamma, 1/2)$.

3.4 Concluding Remarks

In this chapter, we have been concerned with ECOMOR reinsurance and LC reinsurance, two treaties of extreme value type. Reinsurance mathematics is one important field of mathematical risk theory, and both treaties are typical examples of applications of extreme value theory to reinsurance. However, they are well-known not to be very popular among reinsurers, partly because of their technical complexity. We have derived asymptotic results for the ECOMOR-quantity $R_r(t)$ and the LC-quantity $L_r(t)$ that perhaps can reopen the discussion on the usefulness of including the largest claims in the decision making procedure.

Let us conclude with some observations and possible extensions.

(*i*) There is some need to get further information on the accuracy of the approximations given in Theorems 3.7 and 3.11 concerning the convergence in distribution of $R_1(t)$ and $L_1(t)$. It would be interesting to get second-order results when the claim size distribution F belongs to an extremal class $\mathcal{C}_\gamma$ with remainder (see Section 3.3 of Beirlant et al. [5]) and the claim number process $\{N(t);\, t \geqslant 0\}$ is mixed Poisson.

(*ii*) In Theorem 3.10, we have given a condition for the dominant influence of the ECOMOR-quantity $R_r(t)$ on the total claim amount $S_{N(t)}$. However, it would be interesting to find conditions for the significant (but not dominant) influence of $R_r(t)$ on $S_{N(t)}$, i.e. such that the limiting distribution of $R_r(t)/S_{N(t)}$ as $t \to \infty$ is nondegenerate. In particular, one may wonder what happens when the claim size distribution F is of Pareto-type.

(*iii*) Asymptotic results when $t \to \infty$ are not always relevant in catastrophic reinsurance. For example, earthquake claims or claims resulting from windstorms and hurricanes are commonly settled quickly. However, the number of claims is traditionally very high. It is worth noting that the condition $t \to \infty$ in our results on convergence in distribution can be replaced by a condition of the form $\mathbb{E}[N] \to \infty$. The changes in the arguments are then easily made.

In the next chapter, we will be interested in the analysis of common risk measures for reinsurance layers in a fixed time setting.

Chapter 4

Analysis of Risk Measures for Reinsurance Layers

4.1 Introduction

Let $\{Y_i;\, i \in \mathbb{N}\}$ be a sequence of a.s. positive i.i.d. random variables from the distribution function F_Y of a generic random variable Y. Let N be an a.s. nonnegative integer-valued random variable independent of $\{Y_i;\, i \in \mathbb{N}\}$. We interpret $\{Y_i;\, i \in \mathbb{N}\}$ as the sequence of consecutive claim sizes arising from an insurance portfolio and N as the number of claims incurred by the insurance company in a given time period. Note that there is no index referring to time since this parameter is unimportant here. We then use $Y_1, \ldots, Y_N$ as the sequence of successive claim sizes in the time period and we denote the corresponding sequence of nondescending order statistics by $Y_{1:N}, \ldots, Y_{N:N}$ (see Subsection 2.3.2). The total claim amount over the time period is described by the random variable S_Y defined by $S_Y := \sum_{i=1}^{N} Y_i$ if $N \geqslant 1$ and $S_Y := 0$ if $N = 0$. The above model is clearly the collective risk model introduced in Section 2.2 where the time has been fixed.

As mentioned in Section 2.1, reinsurance is considered as one possible way of risk sharing. The reinsurance forms all have in common the intention to diminish an excessive number of claims or the impact of the large claims. Of course, the taking of reinsurance diminishes the volatility in the portfolio as it is shared between the insurer and the reinsurer. The decision to involve other partners in the risk sharing depends on many factors.

An insurer will always try to safeguard his position by subscribing himself to a variety of insurance contracts with an equally varied set of (re)insurance companies. As such, the insurer himself becomes an insured client by paying a specific premium to a reinsurance company in exchange for a policy covering the reinsured quantity. For the insurer, it obviously does not make sense to sell the entire portfolio to a reinsurance company because he will then lose all premium income from that portfolio. Hence, the insurer has to ponder how he wants the claims to be split between himself and the reinsurer.

Remember from Section 2.3 that we distinguish between reinsurance treaties of *random walk type* and reinsurance treaties of *extreme value type*. Traditional treaties of random walk type are quota-share, surplus, excess-of-loss and stop-loss reinsurance (see Subsection 2.3.1). Note that when a

single risk comes into play, excess-of-loss and stop-loss covers are the same. Treaties of extreme value type such as ECOMOR, LC, DDXL and QDXL reinsurance aim at covering the largest claims in the portfolio since they are defined in terms of (upper) order statistics of the claim sizes (see Subsection 2.3.2). In Chapter 3, the treaties ECOMOR and LC have been studied. Chapter 5 will be devoted to QDXL reinsurance.

In accordance with current practice, quota-share and stop-loss treaties are somewhat popular if the number of claims is large while surplus and excess-of-loss treaties are more traditional if the claim sizes are large. Also, treaties of extreme value type are almost never used. This fact is rather surprising if reinsurance is meant to protect the insurer against large claims since, in such a case, it looks necessary to use a reinsurance treaty that is based on the largest claim sizes.

Combinations of different reinsurance treaties are easily constructed. Schmitter [55] combines quota-share and stop-loss treaties so that the reinsured amount is:

$$R_{p,C} := \max(0, p\, S_Y - C)\,.$$

Quota-share and excess-of-loss treaties are combined in Centeno [17]. The reinsured amount then has the following expression:

$$R_{p,M} := \mathbb{1}_{\{N=0\}}\, 0 + \mathbb{1}_{\{N\geqslant 1\}} \sum_{i=1}^{N} \max(0, p\, Y_i - M)\,.$$

Benktander and Ohlin [9] combine a surplus treaty with an excess-of-loss treaty. The reinsured amount is then given by:

$$R_{L,M} := \mathbb{1}_{\{N=0\}}\, 0 + \mathbb{1}_{\{N\geqslant 1\}} \sum_{i=1}^{N} \max\left\{0, \mathbb{1}_{\{V>L\}} \left(1 - \frac{L}{V}\right) Y_i - M\right\}$$

where V is the value insured for each of the claim policies.

Even more popular is the combination of a stop-loss treaty after an excess-of-loss treaty. In this case, one has for the reinsured amount:

$$R_{M,C} := \mathbb{1}_{\{N=0\}}\, 0 + \mathbb{1}_{\{N\geqslant 1\}} \max\left\{0, \sum_{i=1}^{N} \max(0, Y_i - M) - C\right\}.$$

Consider an a.s. nonnegative random variable X with distribution function F_X. Let U_X denote the tail quantile function of F_X. As examples, X may represent an individual claim size Y_i, the ith largest claim size $Y_{N-i+1:N}$ among $Y_1, \ldots, Y_N$ or even the total claim amount S_Y. Choose constants $u \in [0, \infty)$, $v \in (0, \infty]$ and $p \in (0, 1]$ and define the random variable $X_{p,u,v}$ by:

$$X_{p,u,v} := p\, \min\{v, \max(0, X - u)\}\,. \tag{4.1}$$

We can interpret p as a *proportionality factor*, u as a *lower retention*, *deductible* or *priority*, $u + v$ as an *upper retention* or *limit* and the interval $(u, u + v]$ as a *layer*.

It is readily seen that the random variable $X_{p,u,v}$ can be rewritten as:

$$X_{p,u,v} = \begin{cases} 0, & X \leqslant u \\ p\,(X - u), & u < X \leqslant u + v \\ p\,v, & X > u + v. \end{cases}$$

Clearly, we have $0 \leqslant X_{p,u,v} \leqslant p\,v$ a.s. so that $X_{p,u,v}$ is a.s. a nonnegative random variable which is bounded if $v \neq \infty$. Moreover, the distribution function of $X_{p,u,v}$ is simply given by:

$$F_{X_{p,u,v}}(x) := \mathbb{P}[X_{p,u,v} \leqslant x] = \begin{cases} 0, & x < 0 \\ F_X(u + x/p), & 0 \leqslant x < p\,v \\ 1, & x \geqslant p\,v. \end{cases} \tag{4.2}$$

If X stands for an individual claim size Y_i, then summing the form (4.1) over all $i = 1, \ldots, N$ yields the combination of a quota-share reinsurance after an excess-of-loss reinsurance with limit of coverage. The case $v = \infty$ leads to a quota-share cover if $u = 0$, an excess-of-loss cover if $p = 1$ or a combination of the two otherwise. Similarly, if X is used in (4.1) as the total claim amount S_Y, we obtain a quota-share reinsurance on top of a stop-loss reinsurance with limit of coverage. These forms of reinsurance make the classical treaties a bit more flexible by also allowing an upper retention. This is worthwhile in view of further applications. Indeed, the situations vary between different *reinsurance layers*. For the *first line reinsurer*, $u = 0$ while v may take any positive value (finite or infinite). If no part of the risk is shifted to a *second line reinsurer* then $v = \infty$. Otherwise, $v < \infty$ which then equals the lower retention of the second line reinsurer. The latter can also share his risk with a *third line reinsurer* and so on. By not specifying u and v, our results apply to any layer in a *reinsurance chain* of this type.

In the following, we are interested in the behavior of some *risk measures* which are frequently used in practical applications of reinsurance. Risk measures are useful in evaluating or estimating the risk associated with a random quantity. In the context of (re)insurance, such measures also permit to illustrate and quantify the effect of a premium scheme. The need for a variety of risk measures in insurance and even more in reinsurance has been advocated in Venter [64].

In Section 4.2, we analyze a couple of risk measures for the random variable $X_{p,u,v}$. We deal with the Value-at-Risk in Subsection 4.2.1, the variance in Subsection 4.2.2 and the coefficient of variation in Subsection 4.2.3. In Section 4.3, we assume a reinsurance chain for the fraction p of the insurer's total claim amount S_Y. Each layer corresponds to the sum reinsured when the cover (4.1) is applied to each of N individual claim sizes $Y_1, \ldots, Y_N$ but with different values of u and v. We then compare several risk measures among the different layers, in that way investigating properties of the risk sharing. We deal with the variance in Subsection 4.3.1, the dispersion in Subsection 4.3.2, the coefficient of variation in Subsection 4.3.3 and the reduction effect in Subsection 4.3.4. We refer to Mack [47] and Kaas et al. [34] for theoretical and practical results on the coefficient of variation within the context of (re)insurance. For asymptotic properties of the sample coefficient of variation and the sample dispersion, see Chapter 7 and the recent papers by Albrecher et al. [1] and Ladoucette and Teugels [43]. The conclusion is given in Section 4.4. This chapter is based on the work presented in Ladoucette and Teugels [41].

4.2 Risk Measures for $X_{p,u,v}$

In this section, we investigate some usual risk measures (Value-at-Risk, variance and coefficient of variation) for the random variable $X_{p,u,v}$.

We start with a couple of relations that will prove useful later on. Since $X_{p,u,v}$ is an a.s. nonnegative

random variable, we can define its moment of order $k > 0$ by:

$$\tilde{\mu}_k := \mathbb{E}\big[X_{p,u,v}^k\big] = k \int_0^\infty x^{k-1} \left(1 - F_{X_{p,u,v}}(x)\right) \mathrm{d}x.$$

It easily follows from (4.2) that:

$$\tilde{\mu}_k = k \int_0^{pv} x^{k-1} \left(1 - F_X(u + x/p)\right) \mathrm{d}x = kp^k \int_u^{u+v} (x-u)^{k-1} \left(1 - F_X(x)\right) \mathrm{d}x. \tag{4.3}$$

This quickly leads to the following lower and upper bounds:

$$(p\,v)^k \left(1 - F_X(u+v)\right) \leqslant \tilde{\mu}_k \leqslant (p\,v)^k \left(1 - F_X(u)\right).$$

Since $X_{p,u,v} \leqslant p\,v$ a.s., we also have $\mathbb{E}\big[X_{p,u,v}^k\big] \leqslant p\,v\,\mathbb{E}\big[X_{p,u,v}^{k-1}\big]$ for any $k > 1$, i.e.:

$$\tilde{\mu}_k \leqslant p\,v\,\tilde{\mu}_{k-1}. \tag{4.4}$$

Using (4.3), we calculate the partial derivative of $\tilde{\mu}_k$ with respect to v to be:

$$\frac{\partial \tilde{\mu}_k}{\partial v} = p\,k\,(p\,v)^{k-1} \left(1 - F_X(u+v)\right). \tag{4.5}$$

We also deduce the following relation between the partial derivatives of two successive moments:

$$\frac{\partial \tilde{\mu}_{k+1}}{\partial v} = \frac{k+1}{k}\,p\,v \frac{\partial \tilde{\mu}_k}{\partial v}. \tag{4.6}$$

4.2.1 Value-at-Risk

The *Value-at-Risk function* of an a.s. nonnegative random variable Z with distribution function F_Z is defined by:

$$\mathrm{VaR}_y(Z) := U_Z(y), \quad y > 1$$

where U_Z is the tail quantile function of F_Z. If Z represents a claim size, it is clear that when y is large, the latter is only defined for the reinsurer that carries the ultimate tail of the claim size.

For the quantity $X_{p,u,v}$, we are naturally led to the following left-continuous Value-at-Risk function:

$$\mathrm{VaR}_y(X_{p,u,v}) = \begin{cases} 0, & 1 < y \leqslant U_X^{\leftarrow}(u) \\ p\,(U_X(y) - u), & U_X^{\leftarrow}(u) < y \leqslant U_X^{\leftarrow}(u+v) \\ p\,v, & y > U_X^{\leftarrow}(u+v). \end{cases}$$

In addition, note that we have:

$$\mathrm{VaR}_y(X_{p,u,v}) \leqslant \mathrm{VaR}_y(p\,X) = p\,\mathrm{VaR}_y(X) \leqslant \mathrm{VaR}_y(X), \quad y > 1.$$

4.2.2 Variance

Here, we deal with the *variance* of the random variable $X_{p,u,v}$. From (4.3), it is immediate that:

$$\mathbb{V}[X_{p,u,v}] = 2\int_0^{pv} x\,(1 - F_X(u + x/p))\,\mathrm{d}x - \left(\int_0^{pv} (1 - F_X(u + x/p))\,\mathrm{d}x\right)^2. \tag{4.7}$$

By using (4.5), we therefore get:

$$\begin{aligned}
\frac{\partial \mathbb{V}[X_{p,u,v}]}{\partial v} &= \frac{\partial \tilde{\mu}_2}{\partial v} - 2\tilde{\mu}_1 \frac{\partial \tilde{\mu}_1}{\partial v} \\
&= 2p^2 v\,(1 - F_X(u + v)) - 2p\,(1 - F_X(u + v)) \int_0^{pv} (1 - F_X(u + x/p))\,\mathrm{d}x \\
&= 2p\,(1 - F_X(u + v)) \int_0^{pv} F_X(u + x/p)\,\mathrm{d}x \geqslant 0.
\end{aligned}$$

The partial derivative with respect to v being nonnegative, the variance is nondecreasing in v so that:

$$\begin{aligned}
\mathbb{V}[X_{p,u,v}] &\leqslant 2\int_0^{\infty} x\,(1 - F_X(u + x/p))\,\mathrm{d}x - \left(\int_0^{\infty} (1 - F_X(u + x/p))\,\mathrm{d}x\right)^2 \\
&= 2p^2 \int_u^{\infty} (y - u)\,(1 - F_X(y))\,\mathrm{d}y - p^2 \left(\int_u^{\infty} (1 - F_X(y))\,\mathrm{d}y\right)^2 =: g_p(u).
\end{aligned}$$

Now, we compute:

$$\begin{aligned}
\frac{\partial g_p(u)}{\partial u} &= -2p^2 \int_u^{\infty} (1 - F_X(y))\,\mathrm{d}y + 2p^2\,(1 - F_X(u)) \int_u^{\infty} (1 - F_X(y))\,\mathrm{d}y \\
&= -2p^2 F_X(u) \int_u^{\infty} (1 - F_X(y))\,\mathrm{d}y \leqslant 0.
\end{aligned}$$

Thus, the function $g_p(u)$ is nonincreasing in u and therefore smaller than the same expression where we put $u = 0$, that is $g_p(0)$. However, this quantity is equal to $p^2\mathbb{V}[X]$. Hence, we get the following result for the variance of $X_{p,u,v}$:

$$\mathbb{V}[X_{p,u,v}] \leqslant p^2 \mathbb{V}[X].$$

Since $p \in (0, 1]$, we also remark that $\mathbb{V}[X_{p,u,v}] \leqslant \mathbb{V}[X]$ as it should be.

4.2.3 Coefficient of Variation

A risk measure which is popular among actuaries is the *coefficient of variation*, a dimensionless relative measure of dispersion around the mean. It is defined for an a.s. nonnegative random variable Z satisfying $\mathbb{E}[Z] \neq 0$ by:

$$\mathrm{CoVar}(Z) := \frac{\sqrt{\mathbb{V}[Z]}}{\mathbb{E}[Z]}.$$

Note that $\mathrm{CoVar}(cZ) = \mathrm{CoVar}(Z)$ for any $c > 0$.

For the random variable $X_{p,u,v}$, we have:

$$\mathrm{CoVar}(X_{p,u,v}) = \sqrt{\frac{\tilde{\mu}_2}{\tilde{\mu}_1^2} - 1}$$

showing that the coefficient of variation depends monotonically on the ratio under the square root. Incorporating both retentions into the notation and since $\mathrm{CoVar}(X_{p,u,v}) = \mathrm{CoVar}(X_{1,u,v})$, we write $\mathrm{CoVar}(u,v) := \mathrm{CoVar}(X_{p,u,v})$ for arbitrary u and v.

Inequality (4.4) together with (4.6) leads to the following:

$$\begin{aligned}\frac{\partial}{\partial v}\frac{\tilde{\mu}_2}{\tilde{\mu}_1^2} &= (\tilde{\mu}_1)^{-3}\left(\tilde{\mu}_1\frac{\partial\tilde{\mu}_2}{\partial v} - 2\tilde{\mu}_2\frac{\partial\tilde{\mu}_1}{\partial v}\right)\\ &\geqslant (\tilde{\mu}_1)^{-2}\left(\frac{\partial\tilde{\mu}_2}{\partial v} - 2p\,v\frac{\partial\tilde{\mu}_1}{\partial v}\right) = 0.\end{aligned}$$

The above then shows that $\mathrm{CoVar}(u,v)$ is nondecreasing in v. Hence, we get:

$$\mathrm{CoVar}(u,v) \leqslant \mathrm{CoVar}(u,\infty). \tag{4.8}$$

The dependence on u is more intricate. Let us introduce the *retention distribution*:

$$G_{p,u,v}(x) := \begin{cases}\frac{F_X(u+x/p)-F_X(u)}{F_X(u+v)-F_X(u)}, & 0 \leqslant x < p\,v\\ 1, & x \geqslant p\,v\end{cases}$$

whose moment of order $k>0$ is defined by $\nu_k := k\int_0^\infty x^{k-1}\,(1-G_{p,u,v}(x))\,\mathrm{d}x$.

If we abbreviate $\Delta_X := F_X(u+v) - F_X(u)$, we then get:

$$\begin{aligned}\Delta_X\nu_k &= k\,(F_X(u+v)-F_X(u))\int_0^{pv} x^{k-1}\left(1-\frac{F_X(u+x/p)-F_X(u)}{F_X(u+v)-F_X(u)}\right)\mathrm{d}x\\ &= -k\int_0^{pv} x^{k-1}\,\{1-F_X(u+v)-(1-F_X(u+x/p))\}\,\mathrm{d}x\\ &= -(p\,v)^k\,(1-F_X(u+v)) + k\int_0^{pv} x^{k-1}\,(1-F_X(u+x/p))\,\mathrm{d}x\\ &= \tilde{\mu}_k - (p\,v)^k\,(1-F_X(u+v))\,.\end{aligned} \tag{4.9}$$

The latter relation is handy in rewriting the partial derivative of $\mathrm{CoVar}(u,v)$ with respect to u. Indeed, it follows that:

$$\frac{\partial\tilde{\mu}_1}{\partial u} = -\int_0^{p\,v}\frac{\partial}{\partial u}F_X(u+x/p)\,\mathrm{d}x = -p\,(F_X(u+v)-F_X(u)) = -p\,\Delta_X$$

and:

$$\begin{aligned}\frac{\partial\tilde{\mu}_2}{\partial u} &= -2p\int_0^{pv} x\,\mathrm{d}_x F_X(u+x/p)\\ &= -2p\,x\,F_X(u+x/p)\Big|_0^{pv} + 2p\int_0^{pv} F_X(u+x/p)\,\mathrm{d}x\\ &= -2p^2v\,F_X(u+v) + 2p\int_0^{pv} F_X(u+x/p)\,\mathrm{d}x\\ &= -2p\int_0^{pv}(1-F_X(u+x/p))\,\mathrm{d}x + 2p^2v - 2p^2v\,F_X(u+v)\\ &= 2p^2v\,(1-F_X(u+v)) - 2p\int_0^{pv}(1-F_X(u+x/p))\,\mathrm{d}x\\ &= -2p\,\Delta_X\,\nu_1.\end{aligned}$$

But then, we have:

$$\tilde{\mu}_1^3 \frac{\partial}{\partial u} \frac{\tilde{\mu}_2}{\tilde{\mu}_1^2} = \tilde{\mu}_1 \frac{\partial \tilde{\mu}_2}{\partial u} - 2\tilde{\mu}_2 \frac{\partial \tilde{\mu}_1}{\partial u} = 2p\,\Delta_X(\tilde{\mu}_2 - \tilde{\mu}_1\nu_1)\,.$$

Replacing in the last expression the moments $\tilde{\mu}_k$ by their analogues ν_k from (4.9), we get:

$$\begin{aligned} \tilde{\mu}_1^3 \frac{\partial}{\partial u} \frac{\tilde{\mu}_2}{\tilde{\mu}_1^2} &= 2p\,\Delta_X\big\{\Delta_X\nu_2 + (p\,v)^2\,(1 - F_X(u+v)) - \nu_1\,(\Delta_X\nu_1 + p\,v\,(1 - F_X(u+v)))\big\} \\ &= 2p\,\Delta_X^2\big(\nu_2 - \nu_1^2\big) + 2p^2\Delta_X v\,(p\,v - \nu_1)\,(1 - F_X(u+v)) \geqslant 0. \end{aligned}$$

Indeed, the quantity $\nu_2 - \nu_1^2$ is the variance of the distribution $G_{p,u,v}$ while by definition $\nu_1 \leqslant p\,v$. This then shows that the requested partial derivative is nonnegative and hence that $\mathrm{CoVar}(u,v)$ is also nondecreasing in u. Consequently, $\mathrm{CoVar}(u,v)$ is nondecreasing both in u and v. In particular, we find that:

$$\mathrm{CoVar}(u,v) \geqslant \mathrm{CoVar}(0,v). \tag{4.10}$$

Applying both inequalities (4.8) and (4.10), we get the following lower and upper bounds for the coefficient of variation of $X_{p,u,v}$:

$$\mathrm{CoVar}(0,v) \leqslant \mathrm{CoVar}(u,v) \leqslant \mathrm{CoVar}(u,\infty)$$

comparing the risk measure between different layers for any $p \in (0,1]$. Applying the inequality on the left for $v = \infty$ and that on the right for $u = 0$, we also get the following inequalities:

$$\mathrm{CoVar}(0,v) \leqslant \mathrm{CoVar}(X) \leqslant \mathrm{CoVar}(u,\infty)$$

since $\mathrm{CoVar}(0,\infty)$ is the coefficient of variation of the initial risk X.

4.3 Risk Measures for Reinsurance Layers

Recall that $Y_1, \ldots, Y_N$ is used as the sequence of successive claim sizes occurred in a given time period and that S_Y denotes the corresponding total claim amount. Combining a quota-share reinsurance after an excess-of-loss reinsurance with limit of coverage leads to the following reinsured amount over the entire portfolio:

$$R_{p,u,v} := \begin{cases} 0 & \text{if } N = 0 \\ \sum_{i=1}^{N} Y_{i;p,u,v} & \text{if } N \geqslant 1 \end{cases} \tag{4.11}$$

where $Y_{i;p,u,v} := p\,\min\{v, \max(0, Y_i - u)\}$ is used as the reinsured part of the original claim size Y_i under the cover (4.1). Write $Y_{p,u,v}$ for the analogous reinsured part of the generic claim size Y.

Using the formulas (2.2) and (2.3), we deduce the following expressions for the mean and the variance of the quantity $R_{p,u,v}$:

$$\begin{aligned} \mathbb{E}[R_{p,u,v}] &= \tilde{\mu}_1\,\mathbb{E}[N] \\ \mathbb{V}[R_{p,u,v}] &= \mathbb{V}[Y_{p,u,v}]\,\mathbb{E}[N] + \tilde{\mu}_1^2\,\mathbb{V}[N] \end{aligned}$$

where $\tilde{\mu}_1$ and $\mathbb{V}[Y_{p,u,v}]$ are respectively given by (4.3) and (4.7) when replacing F_X by F_Y.

By way of example and without loosing too much of generality, we assume from now on that there are exactly three partners involved in a reinsurance chain for the fraction p of the insurer's total claim amount S_Y. The first reinsurer covers the proportion p of that part of the claim sizes below some retention u. The second reinsurer is responsible for the proportion p of that part of the claim sizes above u but below some retention $u+v$. Finally, the third and last reinsurer deals with the proportion p of that part of the claim sizes above $u+v$. We need to assume that $F_Y(u)\,\Delta_Y(1-F_Y(u+v))>0$ in order that the three layers be nontrivial. To investigate the effect of reinsurance on the partners involved in the risk sharing, let us introduce some notation.

1. For the first reinsurer, the aggregate loss is $S_L := \sum_{i=1}^N L_i$ if $N \geqslant 1$ and $S_L := 0$ if $N = 0$ where L_i has the same distribution as the random variable:

$$L := p\,\mathbb{1}_{\{Y\leqslant u\}}Y + p\,u\,\mathbb{1}_{\{Y>u\}} = p\,\min(u,Y)\,.$$

   This corresponds to the choice $u=0$ and $v=u$ in the general cover (4.11). We use the letter L to point at the lower part of the ceded claims.

2. The aggregate loss of the second reinsurer is $S_M := \sum_{i=1}^N M_i$ if $N \geqslant 1$ and $S_M := 0$ if $N = 0$ where M_i has the same distribution as the random variable:

$$M := p\,\mathbb{1}_{\{u<Y\leqslant u+v\}}\,(Y-u) + p\,v\,\mathbb{1}_{\{Y>u+v\}} = p\,\min\{v,\max(0,Y-u)\}\,.$$

   This case corresponds to the general cover (4.11) with u and v. The letter M points to the middle part of the ceded claims.

3. The top layer is carried by the third reinsurer. His aggregate loss is $S_U := \sum_{i=1}^N U_i$ if $N \geqslant 1$ and $S_U := 0$ if $N = 0$ where U_i has the same distribution as the random variable:

$$U := p\,\mathbb{1}_{\{Y>u+v\}}\,(Y-(u+v)) = p\,\max(0,Y-(u+v))\,.$$

   The choice is $u = u+v$ and $v=\infty$ in the general cover (4.11). We use the letter U to indicate that the upper part of the ceded claims is now at stake.

Note that since $S_L + S_M + S_U = p\,S_Y = R_{p,0,\infty}$ a.s., it is clear that for $p=1$, the first reinsurer reduces to the insurer and so on.

As an additional risk measure, let us also look at the *covariance*. From the definitions of the three random variables L, M and U, it follows that $\mathbb{E}[LM] = p\,u\,\mathbb{E}[M]$, $\mathbb{E}[LU] = p\,u\,\mathbb{E}[U]$ and $\mathbb{E}[MU] = p\,v\,\mathbb{E}[U]$. Therefore, the three covariances $\mathrm{Cov}(L,M)$, $\mathrm{Cov}(L,U)$ and $\mathrm{Cov}(M,U)$ are all positive. To see the influence on the three separate aggregate losses, we calculate the covariances between the quantities S_L, S_M and S_U. An easy calculation learns us that:

$$\begin{aligned}\mathbb{E}[S_L S_M] &= \sum_{n=1}^{\infty}\mathbb{P}[N=n]\sum_{i=1}^{n}\mathbb{E}[L_i M_i] + \sum_{n=1}^{\infty}\mathbb{P}[N=n]\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mathbb{E}[L_i M_j]\\ &= \mathbb{E}[LM]\,\mathbb{E}[N] + \mathbb{E}[L]\,\mathbb{E}[M]\,(\mathbb{E}[N^2]-\mathbb{E}[N])\end{aligned}$$

and hence:

$$\mathrm{Cov}(S_L,S_M) = \mathrm{Cov}(L,M)\,\mathbb{E}[N] + \mathbb{E}[L]\,\mathbb{E}[M]\,\mathbb{V}[N]. \tag{4.12}$$

In (4.12), one can replace the pair (L, M) by (L, U) or (M, U) without any problem. Using the expressions for the covariances between the variables L, M and U, one finds the expressions:

$$\begin{aligned}
\mathrm{Cov}(S_L, S_M) &= \mathbb{E}[L]\,\mathbb{E}[M]\,(\mathbb{V}[N] - \mathbb{E}[N]) + p\,u\,\mathbb{E}[M]\,\mathbb{E}[N] \\
\mathrm{Cov}(S_L, S_U) &= \mathbb{E}[L]\,\mathbb{E}[U]\,(\mathbb{V}[N] - \mathbb{E}[N]) + p\,u\,\mathbb{E}[U]\,\mathbb{E}[N] \\
\mathrm{Cov}(S_M, S_U) &= \mathbb{E}[M]\,\mathbb{E}[U]\,(\mathbb{V}[N] - \mathbb{E}[N]) + p\,v\,\mathbb{E}[U]\,\mathbb{E}[N].
\end{aligned}$$

In the following, we compare some risk measures for the different partners in the chain, namely the variance, the dispersion, the coefficient of variation and the reduction effect. Let A and B be any two distinct letters from the set $\{L, M, U\}$.

4.3.1 Variance

Here, we deal with the *variance*. If one uses the variance as a risk measure then one should compare the variance of a combined risk with the sum of the individual variances. It is clear that:

$$\mathbb{V}[S_A + S_B] = \mathbb{V}[A + B]\,\mathbb{E}[N] + (\mathbb{E}[A] + \mathbb{E}[B])^2\,\mathbb{V}[N].$$

Since (4.12) learns us that $\mathrm{Cov}(S_A, S_B) > 0$, we get by a simple calculation:

$$\mathbb{V}[S_A + S_B] = \mathbb{V}[S_A] + \mathbb{V}[S_B] + 2\,\mathrm{Cov}(S_A, S_B) > \mathbb{V}[S_A] + \mathbb{V}[S_B].$$

A similar relation holds between the two other pairs. Hence, there is an increase in variance if one combines different layers in the reinsurance chain.

4.3.2 Dispersion

The situation is different for the *dispersion*. This risk measure is defined for an a.s. nonnegative random variable Z satisfying $\mathbb{E}[Z] \neq 0$ by:

$$\mathrm{D}(Z) := \frac{\mathbb{V}[Z]}{\mathbb{E}[Z]}.$$

Since the correlation coefficient between S_A and S_B is at most 1 in absolute value, it follows that $\mathrm{Cov}(S_A, S_B) < \sqrt{\mathbb{V}[S_A]\mathbb{V}[S_B]}$ and as a result:

$$\sqrt{\mathbb{V}[S_A + S_B]} < \sqrt{\mathbb{V}[S_A]} + \sqrt{\mathbb{V}[S_B]}. \tag{4.13}$$

Using (4.13), we calculate:

$$\begin{aligned}
\mathrm{D}(S_A + S_B) &= \frac{\mathbb{V}[S_A + S_B]}{\mathbb{E}[S_A + S_B]} \\
&< \frac{\mathbb{V}[S_A] + \mathbb{V}[S_B] + 2\sqrt{\mathbb{V}[S_A]\mathbb{V}[S_B]}}{\mathbb{E}[S_A + S_B]} \\
&= \mathrm{D}(S_A) + \mathrm{D}(S_B) - \frac{\mathbb{V}[S_A]\,\mathbb{E}[S_B]}{\mathbb{E}[S_A]\,\mathbb{E}[S_A + S_B]} - \frac{\mathbb{V}[S_B]\,\mathbb{E}[S_A]}{\mathbb{E}[S_B]\,\mathbb{E}[S_A + S_B]} + 2\frac{\sqrt{\mathbb{V}[S_A]\mathbb{V}[S_B]}}{\mathbb{E}[S_A + S_B]} \\
&= \mathrm{D}(S_A) + \mathrm{D}(S_B) - \frac{\left(\sqrt{\mathbb{V}[S_A]}\,\mathbb{E}[S_B] - \sqrt{\mathbb{V}[S_B]}\,\mathbb{E}[S_A]\right)^2}{\mathbb{E}[S_A]\,\mathbb{E}[S_B]\,\mathbb{E}[S_A + S_B]} \\
&\leqslant \mathrm{D}(S_A) + \mathrm{D}(S_B).
\end{aligned}$$

Therefore, the dispersion decreases by combining different layers.

4.3.3 Coefficient of Variation

Let us now look at the *coefficient of variation*. Using (4.13), we easily obtain:

$$\mathrm{CoVar}(S_A + S_B) = \frac{\sqrt{\mathbb{V}[S_A + S_B]}}{\mathbb{E}[S_A + S_B]} < \frac{\sqrt{\mathbb{V}[S_A]}}{\mathbb{E}[S_A]} + \frac{\sqrt{\mathbb{V}[S_B]}}{\mathbb{E}[S_B]} = \mathrm{CoVar}(S_A) + \mathrm{CoVar}(S_B).$$

Thus, the coefficient of variation decreases by combining different layers in the reinsurance chain.

In addition, a simple calculation gives:

$$\frac{\mathbb{V}[S_A]}{(\mathbb{E}[S_A])^2} = \frac{\mathbb{V}[A]\,\mathbb{E}[N] + (\mathbb{E}[A])^2\,\mathbb{V}[N]}{(\mathbb{E}[A]\,\mathbb{E}[N])^2} = \frac{\mathbb{V}[A]}{(\mathbb{E}[A])^2\,\mathbb{E}[N]} + \frac{\mathbb{V}[N]}{(\mathbb{E}[N])^2}$$

yielding the following formula for the coefficient of variation of S_A:

$$\mathrm{CoVar}(S_A) = \sqrt{\frac{\mathrm{CoVar}^2(A)}{\mathbb{E}[N]} + \mathrm{CoVar}^2(N)}.$$

The second term under the square root depends on the number of claims and is the same for all partners involved. Also the factor $(\mathbb{E}[N])^{-1}$ is the same. Therefore, the differences in the risk measure solely depend on the coefficient of variation for the retained risk. However, the comparison of this risk measure or the quantity $\tilde{\mu}_2/\tilde{\mu}_1^2$ for the different values of u and v is by no means simple.

4.3.4 Reduction Effect

As another risk measure, we finally look at the *reduction effect*. There are two ways of writing out a reduction effect depending on the partner one is looking at. For the first reinsurer, it does hardly matter how the parameter v of the second reinsurer is decided. Therefore, it seems natural to calculate his reduction effect by the quantity:

$$r_L := \frac{\mathbb{E}[S_L]}{\mathbb{E}[S_L + S_M + S_U]}.$$

However, this is easily calculated since:

$$r_L = \frac{\mathbb{E}[L]}{p\,\mathbb{E}[Y]} = \frac{1}{\mathbb{E}[Y]} \int_0^u (1 - F_Y(x))\,\mathrm{d}x.$$

The latter quantity is well known from renewal theory where it is called the *equilibrium distribution* of F_Y. Properties of this distribution can be found in standard treatises of renewal theory.

For the second reinsurer, the natural reduction effect is:

$$r_M := \frac{\mathbb{E}[S_M]}{\mathbb{E}[S_M + S_U]}$$

as only the reduction with respect to the last reinsurer needs to be taken into account. We obtain:

$$r_M = \frac{\mathbb{E}[M]}{\mathbb{E}[M] + \mathbb{E}[U]} = \frac{\int_0^v (1 - F_Y(u+x))\,\mathrm{d}x}{\int_0^\infty (1 - F_Y(u+x))\,\mathrm{d}x} =: H_u(v).$$

Note that the quantity on the right is a proper distribution which for every fixed value of u can be considered as an equilibrium distribution.

4.4 Conclusion

In this chapter, we have been concerned with the analysis of common risk measures for reinsurance layers. To be more precise, we have dealt with Value-at-Risk, variance, coefficient of variation, dispersion and reduction effect. In a first part, we have derived some risk measures for a general layer. In a second part, we have investigated the effect of reinsurance on the partners involved in a reinsurance chain by comparing several risk measures among the different layers.

The next chapter will be devoted to QDXL reinsurance (see Subsection 2.3.2). Recall from (2.6) that the reinsured amount in this treaty of extreme value type is closely related to the quantity defined in (4.1) because X is then used as the ith largest order statistic $Y_{N(t)-i+1:N(t)}$ in the sequence $Y_1, \ldots, Y_{N(t)}$ of successive claim sizes that occurred up to some fixed time t. Also, the parameters u and v are replaced by deterministic functions of the time t both depending on the rank i. Asymptotic and finite time results for the QDXL-quantity will be derived.

Chapter 5

On QDXL Reinsurance

5.1 Introduction

Let $\{X_i;\, i \in \mathbb{N}\}$ be a sequence of a.s. positive i.i.d. random variables with distribution function F. Denote by U the tail quantile function of F. Let $\{N(t);\, t \geqslant 0\}$ be a counting process independent of $\{X_i;\, i \in \mathbb{N}\}$. We interpret $\{X_i;\, i \in \mathbb{N}\}$ as the sequence of consecutive claim sizes arising from an insurance portfolio and $\{N(t);\, t \geqslant 0\}$ as the underlying claim number process, i.e. the random variable $N(t)$ counts the number of claims up to time t. If $N(t) \geqslant 1$, we then use $X_1, \ldots, X_{N(t)}$ as the sequence of successive claim sizes up to time t and we denote the corresponding sequence of nondescending order statistics by $X_{1:N(t)}, \ldots, X_{N(t):N(t)}$ as in Subsection 2.3.2.

In this chapter, we derive *asymptotic* and *finite time results* for the reinsured amount in the QDXL treaty. Recall from Subsection 2.3.2 that QDXL reinsurance is a treaty of extreme value type defined as a quota-share cover on top of an excess-of-loss reinsurance with limit of coverage on an ordered claim size. In Chapter 3, we have focused on two other treaties of extreme value type, namely ECOMOR reinsurance and LC reinsurance. Note that the present chapter can naturally be seen as a sequel to Chapter 4.

Let $i \in \mathbb{N}$ and $t \geqslant 0$ be fixed. Recall from (2.6) that the reinsured amount $\tilde{X}_{i,t}$ in the QDXL treaty is mathematically defined by:

$$\tilde{X}_{i,t} := \mathbb{1}_{\{N(t) \leqslant i-1\}}\, 0 + p\, \mathbb{1}_{\{N(t) \geqslant i\}} \min\big\{L_i(t), \max\big(0, X_{N(t)-i+1:N(t)} - M_i(t)\big)\big\} \tag{5.1}$$

where $p \in (0,1]$, $M_i(t) \in [0,\infty)$, $L_i(t) \in (0,\infty]$ and $X_{N(t)-i+1:N(t)}$ is the ith largest order statistic in the sequence $X_1, \ldots, X_{N(t)}$. Because $\tilde{X}_{i,t} = 0$ a.s. for $t = 0$, we assume from now on that $t > 0$ to avoid trivialities.

From the reinsurance point of view, we see that the QDXL treaty is related to the single loss amount $X_{N(t)-i+1:N(t)}$. If $N(t) \geqslant i$, the reinsurer pays to the ceding insurer the proportion p of that part of the ith largest claim size $X_{N(t)-i+1:N(t)}$ that exceeds the *lower retention* $M_i(t)$ subject to a maximum of $L_i(t)$. The reinsurer's liability is then limited to the amount $p\, L_i(t)$ so that the payment is never more even if the loss exceeds the *upper retention* $M_i(t) + L_i(t)$. The part of the

loss borne by the reinsurer can then be rewritten on the set $\{N(t) \geqslant i\}$ as:

$$\tilde{X}_{i,t} = \begin{cases} 0, & X_{N(t)-i+1:N(t)} \leqslant M_i(t) \\ p\,(X_{N(t)-i+1:N(t)} - M_i(t)), & M_i(t) < X_{N(t)-i+1:N(t)} \leqslant M_i(t) + L_i(t) \\ p\,L_i(t), & X_{N(t)-i+1:N(t)} > M_i(t) + L_i(t). \end{cases} \tag{5.2}$$

It is then clear that $0 \leqslant \tilde{X}_{i,t} \leqslant p\,L_i(t)$ a.s., meaning that $\tilde{X}_{i,t}$ is a.s. a nonnegative random variable which is bounded provided $L_i(t) \neq \infty$.

With a QDXL treaty, one has the choice between a pure quota-share treaty ($M_i(t) = 0$, $L_i(t) = \infty$), a pure excess-of-loss treaty ($p = 1$) with or without limit of coverage ($L_i(t) \leqslant \infty$) on the ith largest claim size or a combination of the two. Note that since only one loss amount is involved here, one may speak about stop-loss instead of excess-of-loss reinsurance. Furthermore, the quantity $\tilde{S}_{N(t)}$ defined by $\tilde{S}_{N(t)} := \sum_{i=1}^{N(t)} \tilde{X}_{i,t}$ if $N(t) \geqslant 1$ and $\tilde{S}_{N(t)} := 0$ if $N(t) = 0$ is the reinsured amount in the combination of a quota-share treaty after a DDXL treaty with time dependent retentions (see Subsection 2.3.2).

The cover granted for an insurance portfolio can be split into several segments known as *layers*. By not specifying $M_i(t)$ and $L_i(t)$, the form (5.1) defines a generic layer in a QDXL *reinsurance chain* for the fraction p of the ith largest claim size $X_{N(t)-i+1:N(t)}$. Indeed, for the *first line reinsurer*, $M_i(t) = 0$ while $L_i(t) \leqslant \infty$. If no part of the risk is shifted to a *second line reinsurer* then $L_i(t) = \infty$. Otherwise, $L_i(t) < \infty$ which then matches the lower retention of the second line reinsurer. The latter can also share his risk with a *third line reinsurer* whose lower retention is then the upper retention of the second line reinsurer and so on. The results of the chapter will therefore apply to any of these layers.

Since we have the coverage of *large claims* in mind, we need to allow the claim size distribution F to be heavy-tailed (see Definition 1.1) so that the largest claim sizes (or upper order statistics) can really be extremal. Natural heavy-tailed distributions are incorporated within the extremal classes (see Definition 1.8). Indeed, any Pareto-type distribution (see Definition 1.7) belongs to an extremal class $\mathcal{C}_\gamma$ for some $\gamma > 0$. Also, the heavy-tailed class $\mathcal{S}$ of subexponential distributions (see Definition 1.3) contains distributions that belong to the extremal class $\mathcal{C}_0$ (see Section 2.5). Furthermore, recall that the set of distributions in the extremal classes corresponds to the set of distributions that belong to a maximum domain of attraction. Hence, these specific classes suggest themselves for applications of extreme value analysis.

Recall from Section 1.1 that the tail quantile function U of F is defined by $U(y) := F^{\leftarrow}(1 - 1/y)$ for $y > 1$ and then that $x_\ell := U(1+) \geqslant 0$ and $U(\infty) := \lim_{y\to\infty} U(y) \leqslant \infty$ are respectively the left and right endpoints of the support of F. For convenience, we set $U(y) := 0$ for $y \in [0, 1]$. For our purposes, we express both retentions in (5.1) in terms of U by putting for some $0 \leqslant \ell_i < r_i \leqslant \infty$:

$$\begin{aligned} M_i(t) &:= U(\ell_i t), \quad t > 0 \\ M_i(t) + L_i(t) &:= U(r_i t), \quad t > 0. \end{aligned}$$

In addition, we assume that F is continuous and increasing on its support and satisfies $F(x_\ell) = 0$. This guarantees that the retentions are determined by unique quantities ℓ_i and r_i as long as $\ell_i > 1/t$ because $x_\ell < U(\ell_i t) < U(r_i t)$ in this case. Note that the functions M_i and L_i take values

in $\{0\} \cup (x_\ell, U(\infty))$ and $\{0\} \cup (x_\ell, U(\infty)]$ respectively. For any $t > 0$, it is clear that $L_i(t) \neq 0$ whenever $r_i > 1/t$ since then $0 \leqslant U(\ell_i t) < U(r_i t)$. Also, we define the functions:

$$\ell_i(t) := p\,\{U(t) - U(\ell_i t)\}, \quad t > 0$$
$$r_i(t) := p\,\{U(r_i t) - U(t)\}, \quad t > 0.$$

Finally, we use the conventions $1/0 = \infty$, $1/\infty = 0$, $\log 0 = -\infty$, $\log \infty = \infty$. Accordingly, the limiting function h_γ appearing in the extremal class condition (1.4) satisfies for any $\gamma \in \mathbb{R}$:

$$h_\gamma(0) = -\frac{1}{\max(0,\gamma)}, \quad h_\gamma(\infty) = \frac{1}{\max(0,-\gamma)}.$$

In our asymptotic results, we will always assume that the claim size distribution F belongs to an extremal class $\mathcal{C}_\gamma(a)$ for some $\gamma \in \mathbb{R}$ and that the claim number process $\{N(t);\, t \geqslant 0\}$ is mixed Poisson (see Definition 1.16). This will be referred as the *mixed Poisson-extremal class assumption*.

In Section 5.2, we start by giving the distribution function of the QDXL-quantity $\tilde{X}_{i,t}$. Then, we derive the limit distribution of the random variable $(\tilde{X}_{i,t} - \ell_i(t))/a(t)$ as $t \to \infty$ under the mixed Poisson-extremal class assumption. In Section 5.3, we give the moments of all positive orders of $\tilde{X}_{i,t}$ and, under the mixed Poisson-extremal class assumption, we determine the asymptotic behavior of these moments as $t \to \infty$. Finally, again under the mixed Poisson-extremal class assumption, we derive the limiting expression of the βth moment of $(\tilde{X}_{i,t} - \ell_i(t))/a(t)$ as $t \to \infty$ for any $\beta \in \mathbb{N}_0$.

In Section 5.4, we look at some *risk measures* for the reinsured amount $\tilde{X}_{i,t}$ in a fixed time setting. We successively deal with the Value-at-Risk in Subsection 5.4.1, the variance in Subsection 5.4.2 and the coefficient of variation in Subsection 5.4.3. For the coefficient of variation within the context of (re)insurance, see Mack [47] and Kaas et al. [34]. For the asymptotic behavior of the sample coefficient of variation, see Chapter 7 and the recent papers by Albrecher et al. [1] and Ladoucette and Teugels [43]. We conclude in Section 5.5. Note that to avoid overloading Section 5.3, the long proofs of Theorems 5.4 and 5.5 have been deferred to Section 5.6. This chapter is based on Ladoucette and Teugels [46].

5.2 Distributional Results

We first specify the distribution function of the ith largest claim size $X_{N(t)-i+1:N(t)}$ before deriving that of the QDXL-quantity $\tilde{X}_{i,t}$.

Fix $i \in \mathbb{N}$ and $t > 0$. The ith largest claim size $X_{N(t)-i+1:N(t)}$ has a distribution function given by:

$$F_{i,t}(x) := \mathbb{P}[X_{N(t)-i+1:N(t)} \leqslant x] = \begin{cases} 0, & x < 0 \\ \Pi_i(t) + \frac{1}{(i-1)!} \int_{t\{1-F(x)\}}^{t} w^{i-1}\, \frac{Q_t^{(i)}(1-w/t)}{t^i}\, \mathrm{d}w, & x \geqslant 0. \end{cases} \tag{5.3}$$

Notice that if $x_\ell > 0$, we have $F_{i,t}(x) = \Pi_i(t)$ for $x \in [0, x_\ell)$ since there is a jump of size $\Pi_i(t)$ at the origin (i.e., $\mathbb{P}[X_{N(t)-i+1:N(t)} = 0] = \Pi_i(t)$). Moreover, in the case $U(\infty) < \infty$, we have $F_{i,t}(x) = F(x) = 1$ for $x \geqslant U(\infty)$.

Now, we give the distribution function of the QDXL-quantity $\tilde{X}_{i,t}$.

Proposition 5.1. *Let $i \in \mathbb{N}$ and $t > 0$ be fixed. Then the distribution function of $\tilde{X}_{i,t}$ is given by:*

$$\tilde{F}_{i,t}(x) := \mathbb{P}[\tilde{X}_{i,t} \leqslant x] = \begin{cases} 0, & x < 0 \\ F_{i,t}(U(\ell_i t) + x/p), & 0 \leqslant x < p\,\{U(r_i t) - U(\ell_i t)\} \\ 1, & x \geqslant p\,\{U(r_i t) - U(\ell_i t)\} \end{cases} \tag{5.4}$$

where $F_{i,t}$ is the distribution function defined in (5.3).

Proof. Let $i \in \mathbb{N}$ and $t > 0$ be fixed. Using (5.2), we easily compute:

- $\star$ $x < 0$: $\mathbb{P}[\tilde{X}_{i,t} \leqslant x] = 0$;
- $\star$ $0 \leqslant x < p\,\{U(r_i t) - U(\ell_i t)\}$:

$$\begin{aligned} \mathbb{P}[\tilde{X}_{i,t} \leqslant x] &= \mathbb{P}[X_{N(t)-i+1:N(t)} \leqslant U(\ell_i t)] + \mathbb{P}[U(\ell_i t) < X_{N(t)-i+1:N(t)} \leqslant U(\ell_i t) + x/p] \\ &= \mathbb{P}[X_{N(t)-i+1:N(t)} \leqslant U(\ell_i t) + x/p]; \end{aligned}$$

- $\star$ $x \geqslant p\,\{U(r_i t) - U(\ell_i t)\}$:

$$\begin{aligned} \mathbb{P}[\tilde{X}_{i,t} \leqslant x] &= \mathbb{P}[X_{N(t)-i+1:N(t)} \leqslant U(\ell_i t)] + \mathbb{P}[U(\ell_i t) < X_{N(t)-i+1:N(t)} \leqslant U(r_i t)] \\ &\quad + \mathbb{P}[X_{N(t)-i+1:N(t)} > U(r_i t)] \\ &= 1 \end{aligned}$$

and the proof is finished. □

Clearly, the scaled QDXL-quantity $\tilde{X}_{i,t}/p$ has a distribution which is similar to that of the ith largest claim size $X_{N(t)-i+1:N(t)}$ but truncated at the retentions. Also, note that $\tilde{X}_{i,t} = 0$ a.s. when $r_i \in (0, 1/t]$ because then $U(\ell_i t) = U(r_i t) = 0$.

The case $U(\ell_i t) < x_\ell$ is equivalent to $U(\ell_i t) = 0$ and $x_\ell > 0$. In this situation, the distribution function $\tilde{F}_{i,t}$ of $\tilde{X}_{i,t}$ becomes:

$$\tilde{F}_{i,t}(x) = \begin{cases} 0, & x < 0 \\ \Pi_i(t), & 0 \leqslant x < p\,x_\ell \\ F_{i,t}(x/p), & p\,x_\ell \leqslant x < p\,U(r_i t) \\ 1, & x \geqslant p\,U(r_i t). \end{cases} \tag{5.5}$$

In the following result, we derive the limiting distribution of $(\tilde{X}_{i,t} - \ell_i(t))/a(t)$ as $t \to \infty$ under the mixed Poisson-extremal class assumption.

Theorem 5.2. *Let $i \in \mathbb{N}$ be fixed. Assume that $F \in \mathcal{C}_\gamma(a)$ for some $\gamma \in \mathbb{R}$ and that $\{N(t);\, t \geqslant 0\}$ is a mixed Poisson process with mixing random variable Λ. Then:*

$$\frac{\tilde{X}_{i,t} - \ell_i(t)}{a(t)} \xrightarrow{\mathcal{D}} \tilde{Y}_{i,\gamma} \quad \text{as} \quad t \to \infty$$

where the distribution function of the limiting random variable $\tilde{Y}_{i,\gamma}$ is given by:

$$\tilde{H}_{i,\gamma}(x) := \mathbb{P}[\tilde{Y}_{i,\gamma} \leqslant x] = \begin{cases} 0, & x < p\,h_\gamma(\ell_i) \\ \frac{1}{(i-1)!} \int_{\eta_\gamma(x/p)}^{\infty} w^{i-1} q_i(w)\,\mathrm{d}w, & p\,h_\gamma(\ell_i) \leqslant x < p\,h_\gamma(r_i) \\ 1, & x \geqslant p\,h_\gamma(r_i). \end{cases}$$

Proof. Let $i \in \mathbb{N}$ be fixed. Define $t_0 := r_i^{-1}$ if $\ell_i = 0$ (with $r_i^{-1} = 0$ if $r_i = \infty$) or $t_0 := \ell_i^{-1}$ if $\ell_i \neq 0$.

1. The analysis below holds when $\ell_i \neq 0$ or when $\ell_i = x_\ell = 0$. It follows from (5.4) that the distribution function of $(\tilde{X}_{i,t} - \ell_i(t))/a(t)$ is given for every $t > t_0$ by:

$$\tilde{H}_{i,t}(x) := \mathbb{P}\left[\frac{\tilde{X}_{i,t} - \ell_i(t)}{a(t)} \leqslant x\right] = \begin{cases} 0, & x < -\frac{\ell_i(t)}{a(t)} \\ F_{i,t}(U(t) + a(t)x/p), & -\frac{\ell_i(t)}{a(t)} \leqslant x < \frac{r_i(t)}{a(t)} \\ 1, & x \geqslant \frac{r_i(t)}{a(t)}. \end{cases} \tag{5.6}$$

Combining (5.3) and (1.15), it then follows that for every $t > t_0$:

$$\tilde{H}_{i,t}(x) = \Pi_i(t) + \frac{1}{(i-1)!}\int_0^\infty w^{i-1} q_i(w) \mathbb{1}_{[\psi_t(x/p),t)}(w)\,\mathrm{d}w, \quad x \in \left[-\frac{\ell_i(t)}{a(t)}, \frac{r_i(t)}{a(t)}\right)$$

where we define for every $t > 0$, $\psi_t(z) := t\{1 - F(U(t) + a(t)z)\}$ for $z \in \mathbb{R}$. For each $w > 0$, we have $w^{i-1} q_i(w) \mathbb{1}_{[\psi_t(x/p),t)}(w) \leqslant w^{i-1} q_i(w)$ for all $t > t_0$. Moreover, $\frac{1}{(i-1)!}\int_0^\infty w^{i-1} q_i(w)\,\mathrm{d}w = 1 < \infty$ by (1.14). Finally, we get $\lim_{t\to\infty} w^{i-1} q_i(w) \mathbb{1}_{[\psi_t(x/p),t)}(w) = w^{i-1} q_i(w) \mathbb{1}_{[\eta_\gamma(x/p),\infty)}(w)$ for each $w > 0$ by Theorem 1.3(*iii*). Applying Lebesgue's dominated convergence theorem and because $\lim_{t\to\infty} -\frac{\ell_i(t)}{a(t)} = p\, h_\gamma(\ell_i)$ and $\lim_{t\to\infty} \frac{r_i(t)}{a(t)} = p\, h_\gamma(r_i)$, we then obtain:

$$\lim_{t\to\infty} \tilde{H}_{i,t}(x) = \frac{1}{(i-1)!}\int_{\eta_\gamma(x/p)}^\infty w^{i-1} q_i(w)\,\mathrm{d}w, \quad x \in [p\, h_\gamma(\ell_i), p\, h_\gamma(r_i))$$

since $\lim_{t\to\infty} \Pi_i(t) = 0$ for any fixed $i \in \mathbb{N}$.

2. When $\ell_i = 0$ and $x_\ell > 0$, it follows from (5.5) that for every $t > t_0$:

$$\tilde{H}_{i,t}(x) = \mathbb{P}\left[\frac{\tilde{X}_{i,t} - p\,U(t)}{a(t)} \leqslant x\right] = \begin{cases} 0, & x < \frac{-p\,U(t)}{a(t)} \\ \Pi_i(t), & \frac{-p\,U(t)}{a(t)} \leqslant x < \frac{p(x_\ell - U(t))}{a(t)} \\ F_{i,t}(U(t) + a(t)x/p), & \frac{p(x_\ell - U(t))}{a(t)} \leqslant x < \frac{r_i(t)}{a(t)} \\ 1, & x \geqslant \frac{r_i(t)}{a(t)}. \end{cases} \tag{5.7}$$

It is easily shown that $\lim_{t\to\infty} \frac{-p\,U(t)}{a(t)} = \lim_{t\to\infty} \frac{p(x_\ell - U(t))}{a(t)} = p\, h_\gamma(0)$ for any $\gamma \in \mathbb{R}$. Now, the rest of the proof is similar to the one above. □

The result of Theorem 5.2 easily compares with that from the classical theory of order statistics which yields under the same assumptions:

$$\frac{X_{N(t)-i+1:N(t)} - U(t)}{a(t)} \xrightarrow{\mathcal{D}} Y_{i,\gamma} \quad \text{as} \quad t \to \infty$$

where the distribution function of the limiting random variable $Y_{i,\gamma}$ is given by:

$$H_{i,\gamma}(x) := \mathbb{P}[Y_{i,\gamma} \leqslant x] = \frac{1}{(i-1)!}\int_{\eta_\gamma(x)}^\infty w^{i-1} q_i(w)\,\mathrm{d}w, \quad 1 + \gamma x > 0. \tag{5.8}$$

In Theorem 5.2, it is then easy to see that $\tilde{H}_{i,\gamma}$ may be expressed with the help of $H_{i,\gamma}$. Moreover, $\tilde{H}_{i,\gamma} = H_{i,\gamma}$ when $\ell_i = 0$, $r_i = \infty$ and $p = 1$ since $\tilde{X}_{i,t}$ reduces to $X_{N(t)-i+1:N(t)}$ in this case.

5.3 Results on the Moments

We start by deriving the expression of the moment of order $\beta > 0$ of the QDXL-quantity $\tilde{X}_{i,t}$. In this fixed time result, we avoid triviality by assuming $t > 0$ and $r_i \in (1/t, \infty]$.

Proposition 5.3. *Let $i \in \mathbb{N}$ and $t > 0$ be fixed. Assume that $r_i \in (1/t, \infty]$. Then for any $\beta > 0$:*

$$\begin{aligned}\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] = \frac{p^{\beta}}{(i-1)!} \int_{r_i^{-1}}^{t\{1-F(U(\ell_i t))\}} \{U(t/w) - U(\ell_i t)\}^{\beta}\, w^{i-1}\, \frac{Q_t^{(i)}(1-w/t)}{t^i}\, \mathrm{d}w \\ + \frac{(p\,\{U(r_i t) - U(\ell_i t)\})^{\beta}}{(i-1)!} \int_0^{r_i^{-1}} w^{i-1}\, \frac{Q_t^{(i)}(1-w/t)}{t^i}\, \mathrm{d}w.\end{aligned} \tag{5.9}$$

When $r_i = \infty$, the second integral disappears while in the first the integration starts at 0. Note that $U(\ell_i t) = 0$ and $t\{1 - F(U(\ell_i t))\} = t$ when $\ell_i \in [0, 1/t]$ while $t\{1 - F(U(\ell_i t))\} = \ell_i^{-1}$ when $\ell_i > 1/t$.

Proof. Let $i \in \mathbb{N}$, $t > 0$ and $\beta > 0$ be fixed. To prove the assertion, we use the fact that $\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] = \beta \int_0^{\infty} x^{\beta-1}(1 - \tilde{F}_{i,t}(x))\,\mathrm{d}x$ since the random variable $\tilde{X}_{i,t}$ is a.s. nonnegative.

1. Case $\ell_i \in (1/t, \infty)$. Note that $t\{1 - F(U(\ell_i t))\} = \ell_i^{-1}$.

a. First, assume that $r_i \neq \infty$. Since $1/t < \ell_i < r_i < \infty$, we have $0 \leqslant x_\ell < U(\ell_i t) < U(r_i t) < U(\infty)$ and $t\{1 - F(U(r_i t))\} = r_i^{-1} \in (0, \infty)$. By using (5.4), we then obtain:

$$\begin{aligned}\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] &= \beta \int_0^{p\,\{U(r_i t) - U(\ell_i t)\}} x^{\beta-1}\,(1 - F_{i,t}(U(\ell_i t) + x/p))\,\mathrm{d}x \\ &= \frac{\beta}{(i-1)!} \int_0^{p\,\{U(r_i t) - U(\ell_i t)\}} x^{\beta-1} \int_0^{t\{1-F(U(\ell_i t)+x/p)\}} w^{i-1}\, \frac{Q_t^{(i)}(1-w/t)}{t^i}\, \mathrm{d}w\,\mathrm{d}x \\ &= \frac{p^{\beta}}{(i-1)!} \int_{r_i^{-1}}^{\ell_i^{-1}} \{U(t/w) - U(\ell_i t)\}^{\beta}\, w^{i-1}\, \frac{Q_t^{(i)}(1-w/t)}{t^i}\, \mathrm{d}w \\ &\quad + \frac{(p\,\{U(r_i t) - U(\ell_i t)\})^{\beta}}{(i-1)!} \int_0^{r_i^{-1}} w^{i-1}\, \frac{Q_t^{(i)}(1-w/t)}{t^i}\, \mathrm{d}w\end{aligned}$$

where the last step follows from an interchange of the order of integration.

b. Since $t\{1 - F(U(\infty))\} = 0$, the case $r_i = \infty$ similarly leads to:

$$\begin{aligned}\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] &= \beta \int_0^{p\,\{U(\infty) - U(\ell_i t)\}} x^{\beta-1}\,(1 - F_{i,t}(U(\ell_i t) + x/p))\,\mathrm{d}x \\ &= \frac{\beta}{(i-1)!} \int_0^{p\,\{U(\infty) - U(\ell_i t)\}} x^{\beta-1} \int_0^{t\{1-F(U(\ell_i t)+x/p)\}} w^{i-1}\, \frac{Q_t^{(i)}(1-w/t)}{t^i}\, \mathrm{d}w\,\mathrm{d}x \\ &= \frac{p^{\beta}}{(i-1)!} \int_0^{\ell_i^{-1}} \{U(t/w) - U(\ell_i t)\}^{\beta}\, w^{i-1}\, \frac{Q_t^{(i)}(1-w/t)}{t^i}\, \mathrm{d}w.\end{aligned}$$

2. Case $\ell_i \in [0, 1/t]$. Note that $U(\ell_i t) = 0$ and $t\{1 - F(U(\ell_i t))\} = t$.

a. Assume that $r_i \neq \infty$. The case $r_i \in (0, 1/t]$ with $r_i > \ell_i$ obviously leads to $\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] = 0$. When $r_i \in (1/t, \infty)$, we have $U(\ell_i t) = 0 \leqslant x_\ell < U(r_i t) < U(\infty)$ and $t\{1 - F(U(r_i t))\} = r_i^{-1} \in (0, \infty)$. By using (5.5), we then obtain:

$$\begin{aligned}
\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] &= (p\,x_\ell)^\beta\,(1-\Pi_i(t)) + \beta\int_{p\,x_\ell}^{p\,U(r_i t)} x^{\beta-1}\,(1-F_{i,t}(x/p))\,\mathrm{d}x \\
&= (p\,x_\ell)^\beta\,(1-\Pi_i(t)) + \frac{\beta}{(i-1)!}\int_{p\,x_\ell}^{p\,U(r_i t)} x^{\beta-1}\int_0^{t\{1-F(x/p)\}} w^{i-1}\frac{Q_t^{(i)}(1-w/t)}{t^i}\,\mathrm{d}w\,\mathrm{d}x \\
&= (p\,x_\ell)^\beta\left\{1-\Pi_i(t) - \frac{1}{(i-1)!}\int_0^t w^{i-1}\frac{Q_t^{(i)}(1-w/t)}{t^i}\,\mathrm{d}w\right\} \\
&\quad + \frac{p^\beta}{(i-1)!}\int_{r_i^{-1}}^t U^\beta(t/w)\,w^{i-1}\frac{Q_t^{(i)}(1-w/t)}{t^i}\,\mathrm{d}w + \frac{(p\,U(r_i t))^\beta}{(i-1)!}\int_0^{r_i^{-1}} w^{i-1}\frac{Q_t^{(i)}(1-w/t)}{t^i}\,\mathrm{d}w \\
&= \frac{p^\beta}{(i-1)!}\int_{r_i^{-1}}^t U^\beta(t/w)\,w^{i-1}\frac{Q_t^{(i)}(1-w/t)}{t^i}\,\mathrm{d}w + \frac{(p\,U(r_i t))^\beta}{(i-1)!}\int_0^{r_i^{-1}} w^{i-1}\frac{Q_t^{(i)}(1-w/t)}{t^i}\,\mathrm{d}w
\end{aligned}$$

since $\frac{1}{(i-1)!}\int_0^t w^{i-1}\frac{Q_t^{(i)}(1-w/t)}{t^i}\,\mathrm{d}w = 1 - \Pi_i(t)$.

b. Since $t\{1 - F(U(\infty))\} = 0$, the case $r_i = \infty$ similarly gives:

$$\begin{aligned}
\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] &= (p\,x_\ell)^\beta\,(1-\Pi_i(t)) + \beta\int_{p\,x_\ell}^{p\,U(\infty)} x^{\beta-1}\,(1-F_{i,t}(x/p))\,\mathrm{d}x \\
&= (p\,x_\ell)^\beta\,(1-\Pi_i(t)) + \frac{\beta}{(i-1)!}\int_{p\,x_\ell}^{p\,U(\infty)} x^{\beta-1}\int_0^{t\{1-F(x/p)\}} w^{i-1}\frac{Q_t^{(i)}(1-w/t)}{t^i}\,\mathrm{d}w\,\mathrm{d}x \\
&= (p\,x_\ell)^\beta\,\{1-\Pi_i(t) - (1-\Pi_i(t))\} + \frac{p^\beta}{(i-1)!}\int_0^t U^\beta(t/w)\,w^{i-1}\frac{Q_t^{(i)}(1-w/t)}{t^i}\,\mathrm{d}w \\
&= \frac{p^\beta}{(i-1)!}\int_0^t U^\beta(t/w)\,w^{i-1}\frac{Q_t^{(i)}(1-w/t)}{t^i}\,\mathrm{d}w.
\end{aligned}$$

This concludes the proof. □

With $\ell_i = 0$, $r_i = \infty$ and $p = 1$ in Proposition 5.3, we obtain the following expression for the moment of order $\beta > 0$ of the non-truncated ith largest claim size $X_{N(t)-i+1:N(t)}$:

$$\mathbb{E}\big[X_{N(t)-i+1:N(t)}^{\beta}\big] = \frac{1}{(i-1)!}\int_0^t U^\beta(t/w)\,w^{i-1}\frac{Q_t^{(i)}(1-w/t)}{t^i}\,\mathrm{d}w.$$

Note that we readily obtain from (5.9) the following lower and upper bounds for the βth moment of $\tilde{X}_{i,t}$ with $\beta > 0$:

$$(p\,\{U(r_i t) - U(\ell_i t)\})^\beta\,(1-F_{i,t}(U(r_i t))) \leqslant \mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] \leqslant (p\,\{U(r_i t) - U(\ell_i t)\})^\beta\,(1-F_{i,t}(U(\ell_i t)))\,.$$

Since $\tilde{X}_{i,t} \leqslant p\,\{U(r_i t) - U(\ell_i t)\}$ a.s., we also have for any $\beta > 1$:

$$\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] \leqslant p\,\{U(r_i t) - U(\ell_i t)\}\,\mathbb{E}\big[\tilde{X}_{i,t}^{\beta-1}\big].$$

Now, consider that the claim number process $\{N(t);\, t \geqslant 0\}$ is mixed Poisson with mixing random variable Λ. The integrand on the right-hand side of the distribution function $F_{i,t}$ of $X_{N(t)-i+1:N(t)}$ given in (5.3) becomes independent of t because of (1.15), i.e. we obtain:

$$F_{i,t}(x) = \begin{cases} 0, & x < 0 \\ \Pi_i(t) + \frac{1}{(i-1)!} \int_{t\{1-F(x)\}}^{t} w^{i-1} q_i(w)\,\mathrm{d}w, & x \geqslant 0. \end{cases} \tag{5.10}$$

Likewise, the integrands on the right-hand side of (5.9) can be simplified by using (1.15). With the additional assumption that the claim size distribution F belongs to an extremal class $\mathcal{C}_\gamma(a)$ for some $\gamma \in \mathbb{R}$, it is then possible to determine the asymptotic behavior of $\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big]$ as $t \to \infty$ for $\beta > 0$.

Theorem 5.4. *Let $i \in \mathbb{N}$ and $\beta > 0$ be fixed. Assume that $F \in \mathcal{C}_\gamma(a)$ for some $\gamma \in \mathbb{R}$ and that $\{N(t);\, t \geqslant 0\}$ is a mixed Poisson process with mixing random variable Λ. Assume further the following conditions:*

(i) If $\gamma > 0$: $\mathbb{E}\big[\Lambda^{\beta(\gamma+\epsilon)}\big] < \infty$ for some $\epsilon \in (0, i/\beta - \gamma)$ when $r_i = \infty$.

(ii) If $\gamma < 0$: $\ell_i \neq 0$.

(iii) If $\gamma = 0$: $\ell_i \neq 0$ and when $r_i = \infty$, $\mathbb{E}\big[\Lambda^{\beta\epsilon}\big] < \infty$ for some $\epsilon \in (0, i/\beta)$.

Then $\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] \in \mathrm{RV}_{\beta\gamma}$ and:

$$\begin{aligned} \mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] \sim \frac{(p\,a(t))^\beta}{(i-1)!} \Bigg\{ & \int_{r_i^{-1}}^{\ell_i^{-1}} \big(h_\gamma(1/w) - h_\gamma(\ell_i)\big)^\beta\, w^{i-1} q_i(w)\,\mathrm{d}w \\ & + \big(h_\gamma(r_i) - h_\gamma(\ell_i)\big)^\beta \int_0^{r_i^{-1}} w^{i-1} q_i(w)\,\mathrm{d}w \Bigg\} \quad \text{as} \quad t \to \infty. \end{aligned} \tag{5.11}$$

When $r_i = \infty$, the second integral disappears while in the first the integration starts at 0.

Proof. Consult Subsection 5.6.1. □

Remark 5.1. Knowing that $\ell_i \neq 0$ if $\gamma \leqslant 0$, the right-hand side of (5.11) may be simplified using that $h_\gamma(1/w) - h_\gamma(\ell_i) = (w^{-\gamma} - \ell_i^\gamma)/\gamma$ if $\gamma \neq 0$ while $h_0(1/w) - h_0(\ell_i) = |\log(\ell_i w)|$ for $w \in (r_i^{-1}, \ell_i^{-1})$ and, when $r_i \neq \infty$, that $h_\gamma(r_i) - h_\gamma(\ell_i) = (r_i^\gamma - \ell_i^\gamma)/\gamma$ if $\gamma \neq 0$ while $h_0(r_i) - h_0(\ell_i) = \log(r_i/\ell_i)$.

By setting $\ell_i = 0$, $r_i = \infty$ and $p = 1$ in Theorem 5.4, we deduce the asymptotic behavior of the moments of the ith largest claim size $X_{N(t)-i+1:N(t)}$ in the case $\gamma > 0$. To be more precise, since $\lim_{t\to\infty} a(t)/U(t) = \gamma$ by Proposition 1.2(i), we obtain by using (1.13) that for any $\gamma \in (0, i/\beta)$:

$$\mathbb{E}\big[X_{N(t)-i+1:N(t)}^{\beta}\big] \sim \frac{U^\beta(t)}{(i-1)!} \int_0^\infty w^{i-\beta\gamma-1} q_i(w)\,\mathrm{d}w = \frac{\Gamma(i-\beta\gamma)}{(i-1)!}\, \mathbb{E}\big[\Lambda^{\beta\gamma}\big]\, U^\beta(t) \quad \text{as} \quad t \to \infty$$

under the existence of some $\epsilon \in (0, i/\beta - \gamma)$ such that $\mathbb{E}\big[\Lambda^{\beta(\gamma+\epsilon)}\big] < \infty$.

Recalling the definition of the LC-quantity $L_r(t)$ given in (3.2), it is clear that the above asymptotic relation can be used in particular to derive the expression for the limit of $\mathbb{E}\big[L_r(t)\big]/U(t)$ as $t \to \infty$ for any $\gamma \in (0, 1)$. Of course, this expression coincides with that given in (3.8).

The following result concerns the convergence of the βth moment of $(\tilde{X}_{i,t} - \ell_i(t))/a(t)$ as $t \to \infty$ under the mixed Poisson-extremal class assumption. Because the QDXL-quantity $\tilde{X}_{i,t}$ normalized in that way may assume negative values, we consider $\beta \in \mathbb{N}_0$ only.

Theorem 5.5. *Let $i \in \mathbb{N}$ and $\beta \in \mathbb{N}_0$ be fixed. Assume that $F \in \mathcal{C}_\gamma(a)$ for some $\gamma \in \mathbb{R}$ and that $\{N(t);\, t \geqslant 0\}$ is a mixed Poisson process with mixing random variable Λ. In the case $\gamma \leqslant 0$, assume that the distribution function H of Λ satisfies $H(\lambda) \sim \lambda^\alpha \ell(1/\lambda)$ as $\lambda \downarrow 0$ for some $\ell \in \mathrm{RV}_0$ and $\alpha > -\beta\gamma$. Assume further the following conditions:*

(i) If $\gamma > 0$: $\mathbb{E}\big[\Lambda^{\beta(\gamma+\epsilon)}\big] < \infty$ for some $\epsilon \in (0, i/\beta - \gamma)$ when $r_i = \infty$.

(ii) If $\gamma < 0$: $\mathbb{E}\big[\Lambda^{-\beta(\epsilon-\gamma)}\big] < \infty$ for some $\epsilon > 0$ when $\ell_i = 0$.

(iii) If $\gamma = 0$: $\mathbb{E}\big[\Lambda^{-\beta\epsilon}\big] < \infty$ for some $\epsilon > 0$ when $\ell_i = 0$, $\mathbb{E}\big[\Lambda^{\beta\epsilon}\big] < \infty$ for some $\epsilon \in (0, i/\beta)$ when $r_i = \infty$.

Then:

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}\left[\left(\frac{\tilde{X}_{i,t} - \ell_i(t)}{a(t)}\right)^\beta\right] &= \left(p\, h_\gamma(\ell_i)\right)^\beta H_{i,\gamma}(h_\gamma(\ell_i)) + \frac{p^\beta}{(i-1)!}\int_{r_i^{-1}}^{\ell_i^{-1}} h_\gamma^\beta(1/w)\, w^{i-1} q_i(w)\,\mathrm{d}w \\ &\quad + \left(p\, h_\gamma(r_i)\right)^\beta \left(1 - H_{i,\gamma}(h_\gamma(r_i))\right) \\ &=: I_1 + I_2 + I_3\end{aligned}$$

where $H_{i,\gamma}$ is the distribution function defined in (5.8)*. When $\ell_i = 0$, $I_1 = 0$ while the integration in I_2 ends at ∞. When $r_i = \infty$, $I_3 = 0$ while the integration in I_2 starts at* 0*.*

Proof. Consult Subsection 5.6.2. □

Remark 5.2. The assumption on H in Theorem 5.5 means that H is regularly varying at 0 with index α (see Definition 1.10). Recall that this is equivalent to saying $H(1/\lambda) \in \mathrm{RV}_{-\alpha}$.

As a particular case, setting $\ell_i = 0$, $r_i = \infty$ and $p = 1$ in Theorem 5.5 gives the following limit for the moment of order $\beta > 0$ of the normalized ith largest claim size $(X_{N(t)-i+1:N(t)} - U(t))/a(t)$:

$$\lim_{t\to\infty} \mathbb{E}\left[\left(\frac{X_{N(t)-i+1:N(t)} - U(t)}{a(t)}\right)^\beta\right] = \frac{1}{(i-1)!}\int_0^\infty h_\gamma^\beta(1/w)\, w^{i-1} q_i(w)\,\mathrm{d}w$$

under the conditions (i)-(iii) of the theorem.

It is clear from the definition of the ECOMOR-quantity $R_r(t)$ given in (3.1) that the above relation can be used in particular to obtain the expression for the limit of $\mathbb{E}\big[R_r(t)\big]/a(t)$ as $t \to \infty$ if $\gamma < 1$. Of course, we retrieve the expression given in (3.6).

5.4 Risk Measures

In this section, we look at some risk measures for the QDXL-quantity $\tilde{X}_{i,t}$ when the time t is fixed. We successively deal with the Value-at-Risk, the variance and the coefficient of variation. These risk measures are frequently used in practical applications.

Let $i \in \mathbb{N}$ and $t > 0$ be fixed. For what follows, we assume that $r_i \in (1/t, \infty]$. No specific assumption is made neither on the number of claims $N(t)$ nor on the claim size distribution F.

For any $\beta > 0$, we write the βth moment of $\tilde{X}_{i,t}$ as:

$$\begin{aligned}\tilde{\mu}_{i,\beta}^t := \mathbb{E}\big[\tilde{X}_{i,t}^\beta\big] &= \beta \int_0^{p\,\{U(r_it)-U(\ell_it)\}} x^{\beta-1}\,(1-F_{i,t}(U(\ell_it)+x/p))\,\mathrm{d}x \\ &= \beta\, p^\beta \int_{U(\ell_it)}^{U(r_it)} (x-U(\ell_it))^{\beta-1}\,(1-F_{i,t}(x))\,\mathrm{d}x \end{aligned} \tag{5.12}$$

where $F_{i,t}$ is the distribution function of the ith largest claim size $X_{N(t)-i+1:N(t)}$ given in (5.3).

In Section 5.3, we have obtained the following relation for any $\beta > 0$:

$$(p\,\{U(r_it)-U(\ell_it)\})^\beta\,(1-F_{i,t}(U(r_it))) \leqslant \tilde{\mu}_{i,\beta}^t \leqslant (p\,\{U(r_it)-U(\ell_it)\})^\beta\,(1-F_{i,t}(U(\ell_it)))$$

which can be easily retrieved from (5.12). Moreover, we have also obtained for any $\beta > 1$:

$$\tilde{\mu}_{i,\beta}^t \leqslant p\,\{U(r_it)-U(\ell_it)\}\,\tilde{\mu}_{i,\beta-1}^t. \tag{5.13}$$

The next two relations will prove to be useful later on. The partial derivative of $\tilde{\mu}_{i,\beta}^t$ with respect to $U(r_it)$ is calculated to be:

$$\frac{\partial \tilde{\mu}_{i,\beta}^t}{\partial U(r_it)} = \beta\, p^\beta \{U(r_it)-U(\ell_it)\}^{\beta-1}\,(1-F_{i,t}(U(r_it))) \tag{5.14}$$

leading to the following relation between the partial derivatives of two successive moments:

$$\frac{\partial \tilde{\mu}_{i,\beta+1}^t}{\partial U(r_it)} = \frac{\beta+1}{\beta}\,p\,\{U(r_it)-U(\ell_it)\}\frac{\partial \tilde{\mu}_{i,\beta}^t}{\partial U(r_it)}. \tag{5.15}$$

5.4.1 Value-at-Risk

The *Value-at-Risk function* of an a.s. nonnegative random variable Y with distribution function F_Y is defined by:

$$\mathrm{VaR}_y(Y) := U_Y(y), \quad y > 1$$

where U_Y is the tail quantile function of F_Y.

For the QDXL-quantity $\tilde{X}_{i,t}$, we obtain the following left-continuous Value-at-Risk function:

$$\mathrm{VaR}_y(\tilde{X}_{i,t}) = \begin{cases} 0, & 1 < y \leqslant \mathrm{VaR}^{\leftarrow}_{U(\ell_it)}(X_{N(t)-i+1:N(t)}) \\ p\,\{\mathrm{VaR}_y(X_{N(t)-i+1:N(t)}) - U(\ell_it)\}, & \mathrm{VaR}^{\leftarrow}_{U(\ell_it)}(X_{N(t)-i+1:N(t)}) \\ & < y \leqslant \mathrm{VaR}^{\leftarrow}_{U(r_it)}(X_{N(t)-i+1:N(t)}) \\ p\,\{U(r_it)-U(\ell_it)\}, & y > \mathrm{VaR}^{\leftarrow}_{U(r_it)}(X_{N(t)-i+1:N(t)}) \end{cases}$$

where $\mathrm{VaR}^{\leftarrow}_{\cdot}(X_{N(t)-i+1:N(t)})$ is the left-continuous inverse of $\mathrm{VaR}_{\cdot}(X_{N(t)-i+1:N(t)})$ given by:

$$\mathrm{VaR}^{\leftarrow}_x(X_{N(t)-i+1:N(t)}) := \begin{cases} \frac{1}{1-\Pi_i(t)}, & x \in [0, x_\ell] \\ \inf\{y > 1/(1-\Pi_i(t))\colon \mathrm{VaR}_y(X_{N(t)-i+1:N(t)}) \geqslant x\}, & x \in (x_\ell, U(\infty)]. \end{cases}$$

In addition, note that we have:

$$\mathrm{VaR}_y(\tilde{X}_{i,t}) \leqslant p\,\mathrm{VaR}_y(X_{N(t)-i+1:N(t)}) \leqslant \mathrm{VaR}_y(X_{N(t)-i+1:N(t)}), \quad y > 1.$$

5.4.2 Variance

Now, we deal with the *variance* of the reinsured amount $\tilde{X}_{i,t}$. It follows from (5.12) that:

$$\mathbb{V}\big[\tilde{X}_{i,t}\big] = 2p^2 \int_{U(\ell_i t)}^{U(r_i t)} (x - U(\ell_i t))\,(1 - F_{i,t}(x))\,\mathrm{d}x - p^2 \left(\int_{U(\ell_i t)}^{U(r_i t)} (1 - F_{i,t}(x))\,\mathrm{d}x \right)^2. \tag{5.16}$$

By using (5.14), it is immediate that:

$$\frac{\partial \mathbb{V}\big[\tilde{X}_{i,t}\big]}{\partial U(r_i t)} = \frac{\partial \tilde{\mu}_{i,2}^t}{\partial U(r_i t)} - 2\tilde{\mu}_{i,1}^t \frac{\partial \tilde{\mu}_{i,1}^t}{\partial U(r_i t)} = 2p^2\,(1 - F_{i,t}(U(r_i t))) \int_{U(\ell_i t)}^{U(r_i t)} F_{i,t}(x)\,\mathrm{d}x \geqslant 0$$

proving that the variance of $\tilde{X}_{i,t}$ is nondecreasing in $U(r_i t)$. By setting $r_i = \infty$ in the term on the right-hand side of (5.16), we then obtain:

$$\mathbb{V}\big[\tilde{X}_{i,t}\big] \leqslant 2p^2 \int_{U(\ell_i t)}^{U(\infty)} (x - U(\ell_i t))\,(1 - F_{i,t}(x))\,\mathrm{d}x - p^2 \left(\int_{U(\ell_i t)}^{U(\infty)} (1 - F_{i,t}(x))\,\mathrm{d}x \right)^2 =: g_p(U(\ell_i t)).$$

Now, we compute:

$$\frac{\partial g_p(U(\ell_i t))}{\partial U(\ell_i t)} = -2p^2 F_{i,t}(U(\ell_i t)) \int_{U(\ell_i t)}^{U(\infty)} (1 - F_{i,t}(x))\,\mathrm{d}x \leqslant 0.$$

The latter shows that the function $g_p(U(\ell_i t))$ is nonincreasing in $U(\ell_i t)$ and therefore smaller than the same expression where we put $U(\ell_i t) = 0$ by choosing $\ell_i \in [0, 1/t]$. Since $g_p(0) = p^2 \mathbb{V}\big[X_{N(t)-i+1:N(t)}\big]$ and $p \in (0, 1]$, we therefore obtain the following result:

$$\mathbb{V}\big[\tilde{X}_{i,t}\big] \leqslant p^2 \mathbb{V}\big[X_{N(t)-i+1:N(t)}\big] \leqslant \mathbb{V}\big[X_{N(t)-i+1:N(t)}\big].$$

5.4.3 Coefficient of Variation

The *coefficient of variation* of an a.s. nonnegative random variable Y satisfying $\mathbb{E}[Y] \neq 0$ is defined by:

$$\mathrm{CoVar}(Y) := \frac{\sqrt{\mathbb{V}[Y]}}{\mathbb{E}[Y]}.$$

This risk measure is rather popular among actuaries.

Incorporating the retentions of the QDXL-quantity $\tilde{X}_{i,t}$ into the notation and since $\mathrm{CoVar}(cY) = \mathrm{CoVar}(Y)$ for any $c > 0$, we write:

$$\mathrm{CoVar}(U(\ell_i t), U(r_i t)) := \mathrm{CoVar}(\tilde{X}_{i,t}) = \sqrt{\frac{\tilde{\mu}_{i,2}^t}{\big(\tilde{\mu}_{i,1}^t\big)^2} - 1}.$$

Inequality (5.13) together with (5.15) leads to the following:

$$\begin{aligned} \frac{\partial}{\partial U(r_i t)} \frac{\tilde{\mu}_{i,2}^t}{\big(\tilde{\mu}_{i,1}^t\big)^2} &= \big(\tilde{\mu}_{i,1}^t\big)^{-3} \left(\tilde{\mu}_{i,1}^t \frac{\partial \tilde{\mu}_{i,2}^t}{\partial U(r_i t)} - 2\tilde{\mu}_{i,2}^t \frac{\partial \tilde{\mu}_{i,1}^t}{\partial U(r_i t)} \right) \\ &\geqslant \big(\tilde{\mu}_{i,1}^t\big)^{-2} \left(\frac{\partial \tilde{\mu}_{i,2}^t}{\partial U(r_i t)} - 2p\,\{U(r_i t) - U(\ell_i t)\} \frac{\partial \tilde{\mu}_{i,1}^t}{\partial U(r_i t)} \right) = 0. \end{aligned}$$

It follows that $\mathrm{CoVar}(U(\ell_i t), U(r_i t))$ is nondecreasing in $U(r_i t)$ because it depends monotonically on the ratio $\tilde{\mu}^t_{i,2}/(\tilde{\mu}^t_{i,1})^2$. We therefore obtain the following upper bound for the coefficient of variation of $\tilde{X}_{i,t}$:

$$\mathrm{CoVar}(U(\ell_i t), U(r_i t)) \leqslant \mathrm{CoVar}(U(\ell_i t), U(\infty)).$$

Note that applying the latter inequality for any $\ell_i \in [0, 1/t]$ (or equivalently for $U(\ell_i t) = 0$) yields:

$$\mathrm{CoVar}(0, U(r_i t)) \leqslant \mathrm{CoVar}(X_{N(t)-i+1:N(t)})$$

since $\mathrm{CoVar}(0, U(\infty))$ is the coefficient of variation of the ith largest claim size $X_{N(t)-i+1:N(t)}$.

5.5 Conclusion

In this chapter, we have been concerned with QDXL reinsurance which is a treaty of extreme value type. We have established asymptotic and finite time results for the QDXL-quantity $\tilde{X}_{i,t}$. In a fixed time setting, we have derived the distribution of $\tilde{X}_{i,t}$ as well as the expression of its moments. Next, we have given the asymptotic behavior of these moments as $t \to \infty$. Also, we have derived the limiting distribution and the limit of the moments of the suitably normalized reinsured amount $\tilde{X}_{i,t}$ as $t \to \infty$. All our asymptotic results have been established under the assumptions that the claim size distribution F belongs to an extremal class $\mathcal{C}_\gamma$ for some $\gamma \in \mathbb{R}$ and that the claim number process $\{N(t);\, t \geqslant 0\}$ is mixed Poisson. Finally, we have analyzed some popular risk measures for $\tilde{X}_{i,t}$ when the time t is fixed (Value-at-Risk, variance and coefficient of variation). We hope that our results will give some insight into the structure of QDXL reinsurance in view of future applications. Note that since $\tilde{X}_{i,t} = X_{N(t)-i+1:N(t)}$ a.s. when $\ell_i = 0$, $r_i = \infty$ and $p = 1$, the results of this chapter are easily adapted to the ith largest claim size $X_{N(t)-i+1:N(t)}$.

5.6 Deferred Proofs

5.6.1 Proof of Theorem 5.4

Let $i \in \mathbb{N}$ and $\beta > 0$ be fixed.

I. We first examine the case $r_i \neq \infty$. We take $t > r_i^{-1}$ for $t\{1 - F(U(r_i t))\} = r_i^{-1}$ to hold. It follows from Proposition 5.3 and (1.15) that we have for every $t > r_i^{-1}$:

$$\begin{aligned}\mathbb{E}\big[\tilde{X}^\beta_{i,t}\big] = \frac{p^\beta}{(i-1)!} \int_{r_i^{-1}}^{t\{1-F(U(\ell_i t))\}} \{U(t/w) - U(\ell_i t)\}^\beta \, w^{i-1} q_i(w) \, \mathrm{d}w \\ + \frac{(p\,\{U(r_i t) - U(\ell_i t)\})^\beta}{(i-1)!} \int_0^{r_i^{-1}} w^{i-1} q_i(w) \, \mathrm{d}w.\end{aligned}$$

1. First assume that $\ell_i = 0$, leading for each $t > r_i^{-1}$ to:

$$\mathbb{E}\big[\tilde{X}^\beta_{i,t}\big] = \frac{p^\beta}{(i-1)!} \int_{r_i^{-1}}^{t} U^\beta(t/w) \, w^{i-1} q_i(w) \, \mathrm{d}w + \frac{(p\,U(r_i t))^\beta}{(i-1)!} \int_0^{r_i^{-1}} w^{i-1} q_i(w) \, \mathrm{d}w.$$

a. Case $\gamma \leqslant 0$. We prove that $\mathbb{E}\big[\tilde{X}^\beta_{i,t}\big]/a^\beta(t) \to \infty$ as $t \to \infty$. First of all, $U \in \mathrm{RV}_0$ when $\gamma \leqslant 0$. Indeed, $U \in \mathrm{RV}_0$ when $\gamma = 0$ since $U \in \Pi_a$ (see Definition 1.9). When $\gamma < 0$, we obviously have

$U \in \mathrm{RV}_0$ since $U(\infty) \neq 0$. Now, choose b such that $0 < r_i^{-1} < b < \infty$. Then, given $\epsilon \in (0,1)$, there exists $t_0 > \max(1,b)$ such that for $t \geqslant t_0$ and $w \in [r_i^{-1}, b]$:

$$1 - \epsilon \leqslant \frac{U(t/w)}{U(t)} \leqslant 1 + \epsilon$$

by virtue of the uniform convergence theorem for slowly varying functions (see Theorem 1.6(i)), yielding:

$$\frac{\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big]}{a^{\beta}(t)} \geqslant \frac{p^{\beta}}{(i-1)!}\int_{r_i^{-1}}^{b}\left(\frac{U(t/w)}{a(t)}\right)^{\beta} w^{i-1}q_i(w)\,\mathrm{d}w \geqslant \left(\frac{U(t)}{a(t)}\right)^{\beta}\frac{(p\,(1-\epsilon))^{\beta}}{(i-1)!}\int_{r_i^{-1}}^{b} w^{i-1}q_i(w)\,\mathrm{d}w.$$

The result follows since $\int_{r_i^{-1}}^{b} w^{i-1}q_i(w)\,\mathrm{d}w \leqslant (i-1)! < \infty$ by (1.14) and $U(t)/a(t) \to \infty$ as $t \to \infty$ by Proposition 1.2(iii) if $\gamma = 0$ and Remark 1.16 if $\gamma < 0$ because $U(\infty) \neq 0$.

b. Case $\gamma > 0$. Define $f_t(w) := \left(\frac{U(t/w)}{a(t)}\right)^{\beta} w^{i-1}q_i(w)\mathbb{1}_{\left[r_i^{-1},\,t\right)}(w)$ for $w \geqslant r_i^{-1}$. For each $w \geqslant r_i^{-1}$, we have:

$$0 \leqslant f_t(w) \leqslant \left(\frac{U(r_i t)}{a(t)}\right)^{\beta} w^{i-1}q_i(w) := g_t(w) \quad \text{for all } t > r_i^{-1}.$$

For each $w \geqslant r_i^{-1}$, $\lim_{t\to\infty} f_t(w) = \left(\frac{w^{-\gamma}}{\gamma}\right)^{\beta} w^{i-1}q_i(w)\mathbb{1}_{\left[r_i^{-1},\,\infty\right)}(w) =: f(w)$ and $\lim_{t\to\infty} g_t(w) = \left(\frac{r_i^{\gamma}}{\gamma}\right)^{\beta} w^{i-1}q_i(w) =: g(w)$ by using Proposition 1.2(i). Moreover:

$$\lim_{t\to\infty}\int_{r_i^{-1}}^{\infty} g_t(w)\,\mathrm{d}w = \left(\frac{r_i^{\gamma}}{\gamma}\right)^{\beta}\int_{r_i^{-1}}^{\infty} w^{i-1}q_i(w)\,\mathrm{d}w = \int_{r_i^{-1}}^{\infty} g(w)\,\mathrm{d}w \leqslant (i-1)!\left(\frac{r_i^{\gamma}}{\gamma}\right)^{\beta} < \infty$$

where (1.14) is used. Therefore, we apply Pratt's theorem (see Pratt [53]) to get:

$$\lim_{t\to\infty}\int_{r_i^{-1}}^{\infty} f_t(w)\,\mathrm{d}w = \int_{r_i^{-1}}^{\infty} f(w)\,\mathrm{d}w = \frac{1}{\gamma^{\beta}}\int_{r_i^{-1}}^{\infty} w^{i-\beta\gamma-1}q_i(w)\,\mathrm{d}w < \infty$$

and consequently:

$$\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] \sim \left(\frac{p\,a(t)}{\gamma}\right)^{\beta}\frac{1}{(i-1)!}\left\{\int_{r_i^{-1}}^{\infty} w^{i-\beta\gamma-1}q_i(w)\,\mathrm{d}w + r_i^{\beta\gamma}\int_{0}^{r_i^{-1}} w^{i-1}q_i(w)\,\mathrm{d}w\right\} \quad \text{as} \quad t \to \infty.$$

2. Now, assume that $\ell_i \neq 0$. This permits to consider $t > \ell_i^{-1} > r_i^{-1}$ so that $t\{1-F(U(r_i t))\} = r_i^{-1}$ and $t\{1-F(U(\ell_i t))\} = \ell_i^{-1}$ hold. We then get for each $t > \ell_i^{-1}$:

$$\begin{aligned}\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] = {} & \frac{p^{\beta}}{(i-1)!}\int_{r_i^{-1}}^{\ell_i^{-1}}\{U(t/w) - U(\ell_i t)\}^{\beta} w^{i-1}q_i(w)\,\mathrm{d}w \\ & + \frac{(p\,\{U(r_i t) - U(\ell_i t)\})^{\beta}}{(i-1)!}\int_{0}^{r_i^{-1}} w^{i-1}q_i(w)\,\mathrm{d}w.\end{aligned}$$

Fix any $\gamma \in \mathbb{R}$. We have $0 < \ell_i < r_i < \infty$ and $-\infty < h_\gamma(\ell_i) < h_\gamma(r_i) < \infty$. It follows from Theorem 1.4 that $\lim_{t\to\infty}\left(\frac{U(t/w)-U(\ell_i t)}{a(t)}\right)^{\beta} w^{i-1}q_i(w) = (h_\gamma(1/w) - h_\gamma(\ell_i))^{\beta}\, w^{i-1}q_i(w)$ uniformly

in $w \in [r_i^{-1}, \ell_i^{-1}]$. Also, $\int_0^{r_i^{-1}} w^{i-1} q_i(w)\,\mathrm{d}w \leqslant (i-1)! < \infty$ by (1.14) and $\lim_{t\to\infty} \frac{U(r_i t)-U(\ell_i t)}{a(t)} = h_\gamma(r_i) - h_\gamma(\ell_i) \in (0,\infty)$. We thus obtain:

$$\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] \sim \frac{(p\,a(t))^\beta}{(i-1)!} \left\{ \int_{r_i^{-1}}^{\ell_i^{-1}} \left(h_\gamma(1/w) - h_\gamma(\ell_i)\right)^\beta w^{i-1} q_i(w)\,\mathrm{d}w \right. \\ \left. + \left(h_\gamma(r_i) - h_\gamma(\ell_i)\right)^\beta \int_0^{r_i^{-1}} w^{i-1} q_i(w)\,\mathrm{d}w \right\} \quad \text{as} \quad t \to \infty.$$

II. Now, we pass to the case $r_i = \infty$. It follows from Proposition 5.3 and (1.15) that we have for every $t > 0$:

$$\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] = \frac{p^\beta}{(i-1)!} \int_0^{t\{1-F(U(\ell_i t))\}} \{U(t/w) - U(\ell_i t)\}^\beta\, w^{i-1} q_i(w)\,\mathrm{d}w.$$

1. First assume that $\ell_i = 0$, leading for each $t > 0$ to:

$$\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] = \frac{p^\beta}{(i-1)!} \int_0^t U^\beta(t/w)\, w^{i-1} q_i(w)\,\mathrm{d}w.$$

a. Case $\gamma \leqslant 0$. Here, $\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big]/a^\beta(t) \to \infty$ as $t \to \infty$. Indeed, for all $t \geqslant 1$, we have $U(t/w) \geqslant U(t)$ for each $w \in (0,1]$ and therefore:

$$\frac{\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big]}{a^\beta(t)} \geqslant \frac{p^\beta}{(i-1)!} \int_0^1 \left(\frac{U(t/w)}{a(t)}\right)^\beta w^{i-1} q_i(w)\,\mathrm{d}w \geqslant \left(\frac{U(t)}{a(t)}\right)^\beta \frac{p^\beta}{(i-1)!} \int_0^1 w^{i-1} q_i(w)\,\mathrm{d}w.$$

The result follows since $\int_0^1 w^{i-1} q_i(w)\,\mathrm{d}w \leqslant (i-1)! < \infty$ by (1.14) and $U(t)/a(t) \to \infty$ as $t \to \infty$ by Proposition 1.2(iii) if $\gamma = 0$ and Remark 1.16 if $\gamma < 0$ because $U(\infty) \neq 0$.

b. Case $\gamma > 0$. Define $f_t(w) := \left(\frac{U(t/w)}{a(t)}\right)^\beta w^{i-1} q_i(w) \mathbb{1}_{[0,\,t)}(w)$ for $w > 0$. Using Proposition 1.2(i), we obtain $\lim_{t\to\infty} f_t(w) = \left(\frac{w^{-\gamma}}{\gamma}\right)^\beta w^{i-1} q_i(w) \mathbb{1}_{[0,\,\infty)}(w) =: f(w)$ for every $w > 0$.

(i) When $w \geqslant 1$, we have $t/w \leqslant t$. Consequently, for each $w \geqslant 1$:

$$0 \leqslant f_t(w) \leqslant \left(\frac{U(t)}{a(t)}\right)^\beta w^{i-1} q_i(w) =: g_t(w) \quad \text{for all } t > 0$$

and $\lim_{t\to\infty} g_t(w) = \left(\frac{1}{\gamma}\right)^\beta w^{i-1} q_i(w) =: g(w)$. Moreover:

$$\lim_{t\to\infty} \int_1^\infty g_t(w)\,\mathrm{d}w = \frac{1}{\gamma^\beta} \int_1^\infty w^{i-1} q_i(w)\,\mathrm{d}w = \int_1^\infty g(w)\,\mathrm{d}w \leqslant \frac{(i-1)!}{\gamma^\beta} < \infty$$

where (1.14) is used. Applying Pratt's theorem, we thus obtain:

$$\lim_{t\to\infty} \int_1^\infty f_t(w)\,\mathrm{d}w = \int_1^\infty f(w)\,\mathrm{d}w = \frac{1}{\gamma^\beta} \int_1^\infty w^{i-\beta\gamma-1} q_i(w) \mathbb{1}_{[0,\,\infty)}(w)\,\mathrm{d}w < \infty. \tag{5.17}$$

(ii) When $w \in (0,1)$, we have $t/w > t$. Proposition 1.2(i) learns us that $a(t)/U(t) \to \gamma$ as $t \to \infty$. Hence, there exist $C > 1$ and $t_1 > 0$ such that for $t \geqslant t_1$:

$$0 \leqslant \frac{U(t)}{a(t)} \leqslant \frac{C}{\gamma}.$$

Assume that $\mathbb{E}\big[\Lambda^{\beta(\gamma+\epsilon)}\big] < \infty$ for some $\epsilon \in (0, i/\beta - \gamma)$. For this ϵ, there exists $t_2 > 1$ such that for $t \geqslant t_2$ and $w \in (0,1)$:

$$1 \leqslant \frac{U(t/w)}{U(t)} \leqslant (1+\epsilon)\, w^{-(\gamma+\epsilon)}$$

by virtue of Potter's theorem (see Theorem 1.7(iii)) since $U \in \mathrm{RV}_\gamma$. Hence, for $t \geqslant t_0 := \max(t_1, t_2)$ and $w \in (0,1)$:

$$0 \leqslant \frac{U(t/w)}{a(t)} = \frac{U(t/w)}{U(t)}\frac{U(t)}{a(t)} \leqslant \frac{C(1+\epsilon)}{\gamma}\, w^{-(\gamma+\epsilon)}.$$

Therefore, for each $w \in (0,1)$, $|f_t(w)| \leqslant \left(\frac{C(1+\epsilon)}{\gamma}\right)^\beta w^{i-\beta(\gamma+\epsilon)-1} q_i(w) =: g(w)$ for all $t \geqslant t_0$. Moreover:

$$\int_0^1 g(w)\,\mathrm{d}w \leqslant \int_0^\infty g(w)\,\mathrm{d}w = \left(\frac{C(1+\epsilon)}{\gamma}\right)^\beta \Gamma\left(i - \beta(\gamma+\epsilon)\right) \mathbb{E}\big[\Lambda^{\beta(\gamma+\epsilon)}\big] < \infty$$

by using (1.13) and the assumption on ϵ. Applying Lebesgue's dominated convergence theorem then gives:

$$\lim_{t\to\infty} \int_0^1 f_t(w)\,\mathrm{d}w = \int_0^1 f(w)\,\mathrm{d}w = \frac{1}{\gamma^\beta}\int_0^1 w^{i-\beta\gamma-1} q_i(w) \mathbb{1}_{[0,\,\infty)}(w)\,\mathrm{d}w < \infty. \tag{5.18}$$

Finally, combining (5.17) together with (5.18) yields:

$$\mathbb{E}\big[\tilde{X}_{i,t}^\beta\big] \sim \left(\frac{p\,a(t)}{\gamma}\right)^\beta \frac{1}{(i-1)!}\int_0^\infty w^{i-\beta\gamma-1} q_i(w)\,\mathrm{d}w \quad \text{as} \quad t \to \infty.$$

2. Now, assume that $\ell_i \neq 0$. This permits us to consider $t > \ell_i^{-1}$ so that $t\{1 - F(U(\ell_i t))\} = \ell_i^{-1}$ holds, leading for each $t > \ell_i^{-1}$ to:

$$\mathbb{E}\big[\tilde{X}_{i,t}^\beta\big] = \frac{p^\beta}{(i-1)!}\int_0^{\ell_i^{-1}} \{U(t/w) - U(\ell_i t)\}^\beta\, w^{i-1} q_i(w)\,\mathrm{d}w.$$

Define $f_t(w) := \left(\frac{U(t/w)-U(\ell_i t)}{a(t)}\right)^\beta w^{i-1} q_i(w)$ for $w \in (0, \ell_i^{-1}]$. By using (1.4), we get $\lim_{t\to\infty} f_t(w) = \left(h_\gamma(1/w) - h_\gamma(\ell_i)\right)^\beta w^{i-1} q_i(w) =: f(w)$ for every $w \in (0, \ell_i^{-1}]$.

a. Case $\gamma < 0$. We have $\ell_i \in (0,\infty)$ and $h_\gamma(\ell_i) \in (-\infty, -1/\gamma)$. It follows from Theorem 1.4(iii) that $f_t(w) \to f(w)$ as $t \to \infty$ uniformly in $w \in (0, \ell_i^{-1}]$. We therefore conclude:

$$\mathbb{E}\big[\tilde{X}_{i,t}^\beta\big] \sim \left(\frac{p\,a(t)}{\gamma}\right)^\beta \frac{1}{(i-1)!}\int_0^{\ell_i^{-1}} \left(w^{-\gamma} - \ell_i^\gamma\right)^\beta w^{i-1} q_i(w)\,\mathrm{d}w \quad \text{as} \quad t \to \infty$$

since $h_\gamma(1/w) - h_\gamma(\ell_i) = (w^{-\gamma} - \ell_i^\gamma)/\gamma$ for $w \in (0, \ell_i^{-1}]$.

b. Case $\gamma \geqslant 0$. Since $\ell_i t > 1$ and $\ell_i w \leqslant 1$, we have $t/w \geqslant \ell_i t > 1$. Hence, for each $w \in (0, \ell_i^{-1}]$:

$$0 \leqslant \frac{U(t/w) - U(\ell_i t)}{a(t)} = \frac{U(\ell_i t/(\ell_i w)) - U(\ell_i t)}{a(\ell_i t)}\frac{a(\ell_i t)}{a(t)} \quad \text{for all } t > \ell_i^{-1}.$$

(i) Let $\gamma > 0$. By Proposition 1.2(i), we know that $a \in \mathrm{RV}_\gamma$ and that $a(t)/U(t) \to \gamma$ as $t \to \infty$. Hence, on the one hand, there exist $C > 1$ and $t_1 > 0$ such that for $t \geqslant t_1$:

$$0 \leqslant \frac{a(\ell_i t)}{a(t)} \leqslant C\,\ell_i^\gamma$$

and, on the other hand, there exist $C' > 1$ and $t_2 > 0$ such that for $t \geqslant t_2$:

$$0 \leqslant \frac{U(t)}{a(t)} \leqslant \frac{C'}{\gamma}.$$

Assume that $\mathbb{E}\big[\Lambda^{\beta(\gamma+\epsilon)}\big] < \infty$ for some $\epsilon \in (0, i/\beta - \gamma)$. For this ϵ, there exists $t_3 > \ell_i^{-1}$ such that for $t \geqslant t_3$ and $w \in (0, \ell_i^{-1}]$:

$$0 \leqslant \frac{U(\ell_i t/(\ell_i w))}{U(\ell_i t)} - 1 \leqslant (1+\epsilon)(\ell_i w)^{-(\gamma+\epsilon)} - 1 \leqslant \frac{1+\epsilon}{\ell_i^{\gamma+\epsilon}}\, w^{-(\gamma+\epsilon)}$$

by virtue of Potter's theorem (see Theorem 1.7(iii)) since $U \in \mathrm{RV}_\gamma$. It then follows that for $t \geqslant t_1' := \max(t_2, t_3)$ and $w \in (0, \ell_i^{-1}]$:

$$0 \leqslant \frac{U(\ell_i t/(\ell_i w)) - U(\ell_i t)}{a(\ell_i t)} = \left(\frac{U(\ell_i t/(\ell_i w))}{U(\ell_i t)} - 1\right)\frac{U(\ell_i t)}{a(\ell_i t)} \leqslant \frac{C'(1+\epsilon)}{\gamma\,\ell_i^{\gamma+\epsilon}}\, w^{-(\gamma+\epsilon)}.$$

Hence, for $t \geqslant t_0 := \max(t_1, t_1')$ and $w \in (0, \ell_i^{-1}]$:

$$0 \leqslant \frac{U(t/w) - U(\ell_i t)}{a(t)} \leqslant \frac{CC'(1+\epsilon)}{\gamma\,\ell_i^{\epsilon}}\, w^{-(\gamma+\epsilon)}.$$

Therefore, for each $w \in (0, \ell_i^{-1}]$, $|f_t(w)| \leqslant \left(\frac{CC'(1+\epsilon)}{\gamma\,\ell_i^{\epsilon}}\right)^\beta w^{i-\beta(\gamma+\epsilon)-1} q_i(w) =: g(w)$ for all $t \geqslant t_0$. Moreover:

$$\int_0^{\ell_i^{-1}} g(w)\,\mathrm{d}w \leqslant \int_0^{\infty} g(w)\,\mathrm{d}w = \left(\frac{CC'(1+\epsilon)}{\gamma\,\ell_i^{\epsilon}}\right)^\beta \Gamma(i - \beta(\gamma+\epsilon))\,\mathbb{E}\big[\Lambda^{\beta(\gamma+\epsilon)}\big] < \infty$$

by using (1.13) and the assumption on ϵ. Applying Lebesgue's dominated convergence theorem then gives:

$$\mathbb{E}\big[\tilde{X}_{i,t}^\beta\big] \sim \left(\frac{p\,a(t)}{\gamma}\right)^\beta \frac{1}{(i-1)!}\int_0^{\ell_i^{-1}} \big(w^{-\gamma} - \ell_i^\gamma\big)^\beta\, w^{i-1} q_i(w)\,\mathrm{d}w \quad \text{as} \quad t \to \infty$$

since $h_\gamma(1/w) - h_\gamma(\ell_i) = (w^{-\gamma} - \ell_i^\gamma)/\gamma$ for $w \in (0, \ell_i^{-1}]$.

(ii) Let $\gamma = 0$. Since $a \in \mathrm{RV}_0$, there exist $C > 1$ and $t_1 > 0$ such that for $t \geqslant t_1$:

$$0 \leqslant \frac{a(\ell_i t)}{a(t)} \leqslant C.$$

Assume that $\mathbb{E}\big[\Lambda^{\beta\epsilon}\big] < \infty$ for some $\epsilon \in (0, i/\beta)$. For this ϵ, there exist $C' > 1$ and $t_2 > \ell_i^{-1}$ such that for $t \geqslant t_2$ and $w \in (0, \ell_i^{-1}]$:

$$0 \leqslant \frac{U(\ell_i t/(\ell_i w)) - U(\ell_i t)}{a(\ell_i t)} \leqslant \frac{C'}{\ell_i^{\epsilon}}\, w^{-\epsilon}$$

by virtue of Corollary 1.15 of Geluk and de Haan [27] since $U \in \Pi_a$. Hence, for $t \geqslant t_0 := \max(t_1, t_2)$ and $w \in (0, \ell_i^{-1}]$:

$$0 \leqslant \frac{U(t/w) - U(\ell_i t)}{a(t)} \leqslant \frac{CC'}{\ell_i^{\epsilon}}\, w^{-\epsilon}.$$

Therefore, for each $w \in (0, \ell_i^{-1}]$, $|f_t(w)| \leqslant \left(\frac{CC'}{\ell_i^{\epsilon}}\right)^{\beta} w^{i-\beta\epsilon-1} q_i(w) =: g(w)$ for all $t \geqslant t_0$. Moreover:

$$\int_0^{\ell_i^{-1}} g(w)\,\mathrm{d}w \leqslant \int_0^{\infty} g(w)\,\mathrm{d}w = \left(\frac{CC'}{\ell_i^{\epsilon}}\right)^{\beta} \Gamma(i-\beta\epsilon)\,\mathbb{E}\big[\Lambda^{\beta\epsilon}\big] < \infty$$

by using (1.13) and the assumption on ϵ. Applying Lebesgue's dominated convergence theorem, we thus get:

$$\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] \sim \frac{(p\,a(t))^{\beta}}{(i-1)!} \int_0^{\ell_i^{-1}} |\log(\ell_i w)|^{\beta} w^{i-1} q_i(w)\,\mathrm{d}w \quad \text{as} \quad t \to \infty$$

since $h_0(1/w) - h_0(\ell_i) = |\log(\ell_i w)|$ for $w \in (0, \ell_i^{-1}]$.

Finally, since $a \in \mathrm{RV}_{\gamma}$ and $\lim_{t\to\infty} \mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big]/a^{\beta}(t) \in (0,\infty)$ under the conditions (i)-(iii) of the theorem, it follows from Proposition 1.9(i) that $\mathbb{E}\big[\tilde{X}_{i,t}^{\beta}\big] \in \mathrm{RV}_{\beta\gamma}$.

5.6.2 Proof of Theorem 5.5

Let $i \in \mathbb{N}$ and $\beta \in \mathbb{N}_0$ be fixed.

A. The analysis below holds when $\ell_i \neq 0$ or when $\ell_i = x_\ell = 0$. Recall that for every $t > 0$, $\psi_t(z) := t\{1 - F(U(t) + a(t)z)\}$ for $z \in \mathbb{R}$. By using (5.6) and (5.10), we obtain for every $t > 0$:

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{\tilde{X}_{i,t} - \ell_i(t)}{a(t)}\right)^{\beta}\right] &= \int_{-\infty}^{\infty} x^{\beta}\,\mathrm{d}\tilde{H}_{i,t}(x) \\
&= \left(\frac{-\ell_i(t)}{a(t)}\right)^{\beta} F_{i,t}(U(\ell_i t)) + \int_{-\frac{\ell_i(t)}{a(t)}}^{\frac{r_i(t)}{a(t)}} x^{\beta}\,\mathrm{d}_x F_{i,t}(U(t) + a(t)x/p) + \left(\frac{r_i(t)}{a(t)}\right)^{\beta} (1 - F_{i,t}(U(r_i t))) \\
&= \left(\frac{-\ell_i(t)}{a(t)}\right)^{\beta} \left\{\Pi_i(t) + \frac{1}{(i-1)!} \int_{t\{1-F(U(\ell_i t))\}}^{t} w^{i-1} q_i(w)\,\mathrm{d}w\right\} \\
&\quad - \frac{1}{(i-1)!} \int_{-\frac{\ell_i(t)}{a(t)}}^{\frac{r_i(t)}{a(t)}} x^{\beta} \{\psi_t(x/p)\}^{i-1}\, q_i(\psi_t(x/p))\,\mathrm{d}_x\psi_t(x/p) \\
&\quad + \left(\frac{r_i(t)}{a(t)}\right)^{\beta} \frac{1}{(i-1)!} \int_0^{t\{1-F(U(r_i t))\}} w^{i-1} q_i(w)\,\mathrm{d}w \\
&=: I_1(t) + I_2(t) + I_3(t).
\end{aligned}$$

I. We start by dealing with $I_1(t)$.

1. When $\ell_i = 0$, we have $\ell_i(t) = p\,U(t)$ and $t\{1 - F(U(\ell_i t))\} = t$. First, the second term in the sum defining $I_1(t)$ obviously equals 0. Now, let us show that $\lim_{t\to\infty} \left(\frac{-p\,U(t)}{a(t)}\right)^{\beta} \Pi_i(t) = 0$ for any

$\gamma \in \mathbb{R}$. The case $\gamma > 0$ is readily proved since $\Pi_i(t) \to 0$ and $a(t)/U(t) \to \gamma$ as $t \to \infty$. The case $\gamma \leqslant 0$ is a bit more intricate and we need the following result.

Lemma 5.6. *Let $r \in \mathbb{N}$, $t \geqslant 0$ and $\alpha \geqslant 0$ be fixed. Then:*

$$\int_0^1 (1-z)^\alpha \, Q_t^{(r)}(z) \, \mathrm{d}z = \Gamma(\alpha+1) \, \mathbb{E}\left[\frac{\mathbb{1}_{\{N(t) \geqslant r\}} \, N(t)!}{(N(t)-r+\alpha+1)!} \right].$$

Proof. Let $r \in \mathbb{N}$, $t \geqslant 0$ and $\alpha \geqslant 0$ be fixed. Using (1.12), we get by an application of Fubini's theorem:

$$\begin{aligned}
\int_0^1 (1-z)^\alpha \, Q_t^{(r)}(z) \, \mathrm{d}z &= \int_0^1 (1-z)^\alpha \sum_{n=r}^{\infty} \left\{ \frac{n!}{(n-r)!} \, p_n(t) \, z^{n-r} \right\} \mathrm{d}z \\
&= \sum_{n=r}^{\infty} \left\{ \frac{n!}{(n-r)!} \, p_n(t) \int_0^1 z^{n-r} \, (1-z)^\alpha \, \mathrm{d}z \right\} \\
&= \sum_{n=r}^{\infty} \frac{n! \, \mathrm{B}(n-r+1, \alpha+1)}{(n-r)!} \, p_n(t) \\
&= \Gamma(\alpha+1) \sum_{n=r}^{\infty} \frac{n!}{(n-r+\alpha+1)!} \, p_n(t) \\
&= \Gamma(\alpha+1) \, \mathbb{E}\left[\frac{\mathbb{1}_{\{N(t) \geqslant r\}} \, N(t)!}{(N(t)-r+\alpha+1)!} \right]
\end{aligned}$$

and the proof is finished. □

Setting $r = i$ and $\alpha = i - 1$ in Lemma 5.6 and since $\Pi_i(t) = 1 - \mathbb{P}[N(t) \geqslant i]$, it follows from the change of variable $w = t(1-z)$ and the relations (1.15) and (1.14) that for each $t > 0$:

$$\begin{aligned}
\Pi_i(t) &= 1 - \frac{1}{(i-1)!} \int_0^1 (1-z)^{i-1} \, Q_t^{(i)}(z) \, \mathrm{d}z \\
&= 1 - \frac{1}{(i-1)!} \int_0^t w^{i-1} \frac{Q_t^{(i)}(1-w/t)}{t^i} \, \mathrm{d}w \\
&= 1 - \frac{1}{(i-1)!} \int_0^t w^{i-1} q_i(w) \, \mathrm{d}w \\
&= \frac{1}{(i-1)!} \int_t^\infty w^{i-1} q_i(w) \, \mathrm{d}w \\
&= \frac{1}{(i-1)!} \int_0^\infty \int_{\lambda t}^\infty v^{i-1} \, \mathrm{e}^{-v} \, \mathrm{d}v \, \mathrm{d}H(\lambda) \\
&= \frac{t^i}{(i-1)!} \int_0^\infty c(\lambda) \, \mathrm{e}^{-\lambda t} \, \mathrm{d}\lambda
\end{aligned}$$

with $c(\lambda) := \lambda^{i-1} H(\lambda)$ for $\lambda > 0$. It follows from the assumption on H that $c(\lambda) \sim \lambda^{\alpha+i-1} \ell(1/\lambda)$ as $\lambda \downarrow 0$. Now, because c is a nondecreasing function and $\alpha + i > 0$, we have by Karamata's Tauberian theorem (e.g., Theorem 8.2 of Grandell [29]):

$$c(\lambda) \sim \lambda^{\alpha+i-1} \ell(1/\lambda) \;\text{ as }\; \lambda \downarrow 0 \iff \int_0^\infty c(\lambda) \, \mathrm{e}^{-\lambda t} \, \mathrm{d}\lambda \sim \Gamma(\alpha+i) \, t^{-(\alpha+i)} \ell(t) \;\text{ as }\; t \to \infty.$$

Hence, $\Pi_i(t) \sim \frac{\Gamma(\alpha+i)}{(i-1)!}\, t^{-\alpha}\ell(t)$ as $t \to \infty$. Since $a/U \in \mathrm{RV}_\gamma$ if $\gamma \leqslant 0$ (recall that $U(\infty) \neq 0$ if $\gamma < 0$) by Remark 1.16, we deduce that $\left(\frac{U(t)}{a(t)}\right)^\beta t^{-\alpha}\ell(t) \in \mathrm{RV}_{-(\beta\gamma+\alpha)}$ where $\beta\gamma + \alpha > 0$ by assumption. Therefore, we have:

$$\left(\frac{U(t)}{a(t)}\right)^\beta \Pi_i(t) \sim \frac{\Gamma(\alpha+i)}{(i-1)!}\left(\frac{U(t)}{a(t)}\right)^\beta t^{-\alpha}\ell(t) \to 0 \quad \text{as} \quad t \to \infty$$

so that the claim follows. As a consequence, we conclude that $I_1 := \lim_{t\to\infty} I_1(t) = 0$ when $\ell_i = 0$.

2. When $\ell_i \neq 0$, take $t > \ell_i^{-1}$ for $t\{1 - F(U(\ell_i t))\} = \ell_i^{-1} \in (0,\infty)$ to hold. Since $\lim_{t\to\infty} \Pi_i(t) = 0$ and $\lim_{t\to\infty} \frac{-\ell_i(t)}{a(t)} = p\, h_\gamma(\ell_i) \neq \pm\infty$, we easily obtain by using Lebesgue's dominated convergence theorem:

$$\lim_{t\to\infty} I_1(t) = \frac{p^\beta h_\gamma^\beta(\ell_i)}{(i-1)!}\int_{\ell_i^{-1}}^\infty w^{i-1} q_i(w)\,\mathrm{d}w = (p\, h_\gamma(\ell_i))^\beta\, H_{i,\gamma}(h_\gamma(\ell_i)) =: I_1.$$

II. Now, we deal with $I_3(t)$.

1. When $r_i = \infty$, we obviously have $I_3(t) = 0$ for any $t > 0$ so that $I_3 := \lim_{t\to\infty} I_3(t) = 0$.

2. When $r_i \neq \infty$, we consider $t > r_i^{-1}$ so that $t\{1 - F(U(r_i t))\} = r_i^{-1} \in (0,\infty)$ holds. Because $\lim_{t\to\infty} \frac{r_i(t)}{a(t)} = p\, h_\gamma(r_i) \neq \pm\infty$, we simply deduce:

$$\lim_{t\to\infty} I_3(t) = \frac{p^\beta h_\gamma^\beta(r_i)}{(i-1)!}\int_0^{r_i^{-1}} w^{i-1} q_i(w)\,\mathrm{d}w = (p\, h_\gamma(r_i))^\beta\,(1 - H_{i,\gamma}(h_\gamma(r_i))) =: I_3.$$

III. Finally, we deal with $I_2(t)$. Since $r_i \in (0,\infty]$, we take $t > r_i^{-1}$ so that $t\{1 - F(U(r_i t))\} = r_i^{-1}$ holds with $r_i^{-1} = 0$ in the case $r_i = \infty$. The change of variable $w = \psi_t(x/p)$, or equivalently $x = p\{U(t/w) - U(t)\}/a(t)$, leads for each $t > r_i^{-1}$ to:

$$\begin{aligned} I_2(t) &= -\frac{1}{(i-1)!}\int_{-\frac{\ell_i(t)}{a(t)}}^{\frac{r_i(t)}{a(t)}} x^\beta \{\psi_t(x/p)\}^{i-1}\, q_i(\psi_t(x/p))\,\mathrm{d}_x\psi_t(x/p) \\ &= \frac{p^\beta}{(i-1)!}\int_{t\{1-F(U(r_i t))\}}^{t\{1-F(U(\ell_i t))\}} \left(\frac{U(t/w) - U(t)}{a(t)}\right)^\beta w^{i-1} q_i(w)\,\mathrm{d}w \\ &= \frac{p^\beta}{(i-1)!}\int_0^\infty \left(\frac{U(t/w) - U(t)}{a(t)}\right)^\beta w^{i-1} q_i(w)\,\mathbb{1}_{[r_i^{-1},\, t\{1-F(U(\ell_i t))\})}(w)\,\mathrm{d}w. \end{aligned}$$

Define $f_t(w) := \left(\frac{U(t/w)-U(t)}{a(t)}\right)^\beta w^{i-1} q_i(w)\mathbb{1}_{[r_i^{-1},\, t\{1-F(U(\ell_i t))\})}(w)$ for $w > 0$ and write:

$$I_2(t) = \frac{p^\beta}{(i-1)!}\left\{\int_0^1 f_t(w)\,\mathrm{d}w + \int_1^\infty f_t(w)\,\mathrm{d}w\right\} =: \frac{p^\beta}{(i-1)!}\left(I_2^1(t) + I_2^2(t)\right).$$

1. In a first step, we deal with $I_2^1(t)$. In this situation, we have $w \in (0,1]$ and then $t/w \geqslant t$. Moreover, we assume $r_i \in (1,\infty]$ in order to have $I_2^1(t) \neq 0$.

When $\ell_i = 0$, we have for each $t > 1 > r_i^{-1}$:

$$I_2^1(t) = \int_{r_i^{-1}}^1 \left(\frac{U(t/w) - U(t)}{a(t)} \right)^\beta w^{i-1} q_i(w) \, \mathrm{d}w$$

and when $\ell_i \in (0, \infty)$, we have for each $t > \max(1, \ell_i^{-1}) > r_i^{-1}$:

$$I_2^1(t) = \int_{r_i^{-1}}^{\min(1, \ell_i^{-1})} \left(\frac{U(t/w) - U(t)}{a(t)} \right)^\beta w^{i-1} q_i(w) \, \mathrm{d}w$$

with $r_i^{-1} = 0$ if $r_i = \infty$.

a. Case $\gamma > 0$.

(i) Assume that $r_i \in (1, \infty)$. Since by Theorem 1.4(ii) we have $\lim_{t\to\infty} \left(\frac{U(t/w)-U(t)}{a(t)} \right)^\beta w^{i-1} q_i(w) = h_\gamma^\beta(1/w)\, w^{i-1} q_i(w)$ uniformly in $w \in [r_i^{-1}, \min(1, \ell_i^{-1})]$, where $\ell_i^{-1} = \infty$ if $\ell_i = 0$, we deduce:

$$\lim_{t\to\infty} I_2^1(t) = \int_{r_i^{-1}}^{\min(1, \ell_i^{-1})} h_\gamma^\beta(1/w)\, w^{i-1} q_i(w) \, \mathrm{d}w =: I_2^1.$$

(ii) Assume that $r_i = \infty$. The integration in $I_2^1(t)$ then starts at 0. By Proposition 1.2(i), we know that $a(t)/U(t) \to \gamma$ as $t \to \infty$. Hence, there exist $C > 1$ and $t_1 > 0$ such that for $t \geqslant t_1$:

$$0 \leqslant \frac{U(t)}{a(t)} \leqslant \frac{C}{\gamma}.$$

Assume that $\mathbb{E}\big[\Lambda^{\beta(\gamma+\epsilon)}\big] < \infty$ for some $\epsilon \in (0, i/\beta - \gamma)$. For this ϵ, there exists $t_2 > 1$ if $\ell_i = 0$ or $t_2 > \max(1, \ell_i^{-1})$ if $\ell_i \in (0, \infty)$ such that for $t \geqslant t_2$ and $w \in (0, 1]$:

$$0 \leqslant \frac{U(t/w)}{U(t)} - 1 \leqslant (1+\epsilon)\, w^{-(\gamma+\epsilon)} - 1 \leqslant (1+\epsilon)\, w^{-(\gamma+\epsilon)}$$

by virtue of Potter's theorem (see Theorem 1.7(iii)) since $U \in \mathrm{RV}_\gamma$. It then follows that for $t \geqslant t_0 := \max(t_1, t_2)$ and $w \in (0, 1]$:

$$0 \leqslant \frac{U(t/w) - U(t)}{a(t)} = \left(\frac{U(t/w)}{U(t)} - 1 \right) \frac{U(t)}{a(t)} \leqslant \frac{C(1+\epsilon)}{\gamma}\, w^{-(\gamma+\epsilon)}.$$

Therefore, for each $w \in (0, 1]$, $|f_t(w)| \leqslant \left(\frac{C(1+\epsilon)}{\gamma} \right)^\beta w^{i-\beta(\gamma+\epsilon)-1} q_i(w) =: g(w)$ for all $t \geqslant t_0$. Moreover:

$$\int_0^1 g(w) \, \mathrm{d}w \leqslant \int_0^\infty g(w) \, \mathrm{d}w = \left(\frac{C(1+\epsilon)}{\gamma} \right)^\beta \Gamma(i - \beta(\gamma+\epsilon))\, \mathbb{E}\big[\Lambda^{\beta(\gamma+\epsilon)}\big] < \infty$$

by using (1.13) and the assumption on ϵ. Finally, $\lim_{t\to\infty} f_t(w) = h_\gamma^\beta(1/w)\, w^{i-1} q_i(w) \mathbb{1}_{[0,\, \ell_i^{-1})}(w)$ for every $w \in (0, 1]$, with $\ell_i^{-1} = \infty$ if $\ell_i = 0$. Applying Lebesgue's dominated convergence theorem, we thus get:

$$\lim_{t\to\infty} I_2^1(t) = \int_0^1 h_\gamma^\beta(1/w)\, w^{i-1} q_i(w) \mathbb{1}_{[0,\, \ell_i^{-1})}(w) \, \mathrm{d}w =: I_2^1.$$

b. Case $\gamma < 0$. It follows from Theorem 1.4(iii) that we have $\lim_{t\to\infty} \left(\frac{U(t/w)-U(t)}{a(t)}\right)^\beta w^{i-1}q_i(w) = h_\gamma^\beta(1/w)\, w^{i-1}q_i(w)$ uniformly in $w \in (r_i^{-1},\, \min(1, \ell_i^{-1})]$, where $r_i^{-1} = 0$ if $r_i = \infty$ and $\ell_i^{-1} = \infty$ if $\ell_i = 0$. Hence, we deduce:

$$\lim_{t\to\infty} I_2^1(t) = \int_{r_i^{-1}}^{\min(1,\ell_i^{-1})} h_\gamma^\beta(1/w)\, w^{i-1}q_i(w)\,\mathrm{d}w =: I_2^1.$$

c. Case $\gamma = 0$. Since $a \in \mathrm{RV}_0$, there exists $T > 0$ such that a is bounded away from 0 on each compact subset of $[T, \infty)$ (e.g., Lemma 1.3.2 of Bingham et al. [13]). Defining the function a_1 by $a_1(t) := a(t)$ on $[T, \infty)$ and $a_1(t) := a(T) \neq 0$ on $(0, T)$, we obtain that $a_1 \in \mathrm{RV}_0$ and that both a_1 and $1/a_1$ are bounded on each compact subset of $(0, \infty)$. Since $a(t) \sim a_1(t)$ as $t \to \infty$, on the one hand, there exist $A > 1$ and $t_0 > 0$ such that for $t \geqslant t_0$:

$$\left|\frac{U(t/w)-U(t)}{a(t)}\right| = \frac{a_1(t)}{a(t)}\left|\frac{U(t/w)-U(t)}{a_1(t)}\right| \leqslant A\left|\frac{U(t/w)-U(t)}{a_1(t)}\right|$$

and, on the other hand, $U \in \Pi_{a_1}$ since $U \in \Pi_a$. Fix $\epsilon > 0$ arbitrarily. There exists $A' > 1$ such that for $t > 0$ and $w > 0$:

$$\left|\frac{U(t/w)-U(t)}{a_1(t)}\right| \leqslant A' \max(w^\epsilon, w^{-\epsilon})$$

by virtue of Theorem 3.8.6(b) of Bingham et al. [13]. Consequently, for $t \geqslant t_0$ and $w > 0$:

$$\left|\frac{U(t/w)-U(t)}{a(t)}\right| \leqslant AA' \max(w^\epsilon, w^{-\epsilon}). \tag{5.19}$$

(i) Assume that $r_i \in (1, \infty)$. Since by Theorem 1.4(i) we have $\lim_{t\to\infty} \left(\frac{U(t/w)-U(t)}{a(t)}\right)^\beta w^{i-1}q_i(w) = (-\log w)^\beta\, w^{i-1}q_i(w)$ uniformly in $w \in [r_i^{-1},\, \min(1, \ell_i^{-1})]$, where $\ell_i^{-1} = \infty$ if $\ell_i = 0$, we deduce:

$$\lim_{t\to\infty} I_2^1(t) = \int_{r_i^{-1}}^{\min(1,\ell_i^{-1})} (-\log w)^\beta\, w^{i-1}q_i(w)\,\mathrm{d}w =: I_2^1.$$

(ii) Assume that $r_i = \infty$. Assume that $\mathbb{E}\big[\Lambda^{\beta\epsilon}\big] < \infty$ for some $\epsilon \in (0, i/\beta)$. For this ϵ and thanks to (5.19), there exist $C > 1$ and $t_0 > 1$ if $\ell_i = 0$ or $t_0 > \max(1, \ell_i^{-1})$ if $\ell_i \in (0, \infty)$ such that for $t \geqslant t_0$ and $w \in (0, 1]$:

$$0 \leqslant \frac{U(t/w)-U(t)}{a(t)} \leqslant C\, w^{-\epsilon}.$$

Therefore, for each $w \in (0, 1]$, $|f_t(w)| \leqslant C^\beta w^{i-\beta\epsilon-1}q_i(w) =: g(w)$ for all $t \geqslant t_0$. Moreover:

$$\int_0^1 g(w)\,\mathrm{d}w \leqslant \int_0^\infty g(w)\,\mathrm{d}w = C^\beta\,\Gamma(i-\beta\epsilon)\,\mathbb{E}\big[\Lambda^{\beta\epsilon}\big] < \infty$$

by using (1.13) and the assumption on ϵ. Finally, $\lim_{t\to\infty} f_t(w) = (-\log w)^\beta\, w^{i-1}q_i(w)\mathbb{1}_{[0,\,\ell_i^{-1})}(w)$ for every $w \in (0, 1]$, with $\ell_i^{-1} = \infty$ if $\ell_i = 0$. Applying Lebesgue's dominated convergence theorem, we thus get:

$$\lim_{t\to\infty} I_2^1(t) = \int_0^1 (-\log w)^\beta\, w^{i-1}q_i(w)\mathbb{1}_{[0,\,\ell_i^{-1})}(w)\,\mathrm{d}w =: I_2^1.$$

2. In a second step, we deal with $I_2^2(t)$. In this situation, we have $w \geqslant 1$ and then $t/w \leqslant t$. Moreover, we assume $\ell_i \in [0,1)$ in order to have $I_2^2(t) \neq 0$.

When $\ell_i = 0$, we have for each $t > \max(1, r_i^{-1})$:

$$I_2^2(t) = \int_{\max(1,r_i^{-1})}^{t} \left(\frac{U(t/w) - U(t)}{a(t)}\right)^{\beta} w^{i-1} q_i(w)\,\mathrm{d}w$$

and when $\ell_i \in (0,1)$, we have for each $t > \ell_i^{-1} > \max(1, r_i^{-1})$:

$$I_2^2(t) = \int_{\max(1,r_i^{-1})}^{\ell_i^{-1}} \left(\frac{U(t/w) - U(t)}{a(t)}\right)^{\beta} w^{i-1} q_i(w)\,\mathrm{d}w$$

with $r_i^{-1} = 0$ if $r_i = \infty$.

a. Assume that $\ell_i \in (0,1)$. It follows from Theorem 1.4 that $\lim_{t\to\infty} \left(\frac{U(t/w)-U(t)}{a(t)}\right)^{\beta} w^{i-1} q_i(w) = h_\gamma^\beta(1/w)\, w^{i-1} q_i(w)$ uniformly in $w \in [\max(1, r_i^{-1}),\, \ell_i^{-1}]$ for any $\gamma \in \mathbb{R}$, where $r_i^{-1} = 0$ if $r_i = \infty$, leading to:

$$\lim_{t\to\infty} I_2^2(t) = \int_{\max(1,r_i^{-1})}^{\ell_i^{-1}} h_\gamma^\beta(1/w)\, w^{i-1} q_i(w)\,\mathrm{d}w =: I_2^2.$$

b. Assume that $\ell_i = 0$.

(i) Case $\gamma > 0$. For each $w \geqslant 1$, we have:

$$|f_t(w)| \leqslant \left(\frac{U(t)}{a(t)}\right)^{\beta} w^{i-1} q_i(w) =: g_t(w) \quad \text{for all } t > \max(1, r_i^{-1})$$

and $\lim_{t\to\infty} g_t(w) = \left(\frac{1}{\gamma}\right)^{\beta} w^{i-1} q_i(w) =: g(w)$. Moreover:

$$\lim_{t\to\infty} \int_1^\infty g_t(w)\,\mathrm{d}w = \frac{1}{\gamma^\beta} \int_1^\infty w^{i-1} q_i(w)\,\mathrm{d}w = \int_1^\infty g(w)\,\mathrm{d}w \leqslant \frac{(i-1)!}{\gamma^\beta} < \infty$$

where (1.14) is used. Finally, $\lim_{t\to\infty} f_t(w) = h_\gamma^\beta(1/w)\, w^{i-1} q_i(w) \mathbb{1}_{[r_i^{-1},\,\infty)}(w)$ for every $w \geqslant 1$, with $r_i^{-1} = 0$ if $r_i = \infty$. Applying Pratt's theorem, we thus obtain:

$$\lim_{t\to\infty} I_2^2(t) = \int_1^\infty h_\gamma^\beta(1/w)\, w^{i-1} q_i(w) \mathbb{1}_{[r_i^{-1},\,\infty)}(w)\,\mathrm{d}w =: I_2^2.$$

(ii) Case $\gamma < 0$. Assume that $\mathbb{E}\big[\Lambda^{-\beta(\epsilon-\gamma)}\big] < \infty$ for some $\epsilon > 0$. For this ϵ, there exist $C > 1$ and $t_1 > \max(1, r_i^{-1})$ such that for $t \geqslant t_1$ and $w \geqslant 1$:

$$0 \leqslant \frac{U(\infty) - U(t/w)}{U(\infty) - U(t)} - 1 \leqslant C\, w^{\epsilon-\gamma} - 1 \leqslant C\, w^{\epsilon-\gamma}$$

by virtue of Proposition 1.7.7 of Geluk and de Haan [27] since $U(\infty) - U \in \mathrm{RV}_\gamma$ is bounded on finite intervals of $(0,\infty)$. Moreover, since $a(t)/(U(\infty) - U(t)) \to |\gamma|$ as $t \to \infty$ by Proposition 1.2(ii), there exist $C' > 1$ and $t_2 > 0$ such that for $t \geqslant t_2$:

$$0 \leqslant \frac{U(\infty) - U(t)}{a(t)} \leqslant \frac{C'}{|\gamma|}.$$

Consequently, for $t \geqslant t_0 := \max(t_1, t_2)$ and $w \geqslant 1$:

$$\left|\frac{U(t/w) - U(t)}{a(t)}\right| = \left(\frac{U(\infty) - U(t/w)}{U(\infty) - U(t)} - 1\right) \frac{U(\infty) - U(t)}{a(t)} \leqslant \frac{CC'}{|\gamma|}\, w^{\epsilon-\gamma}.$$

Therefore, for each $w \geqslant 1$, $|f_t(w)| \leqslant \left(\frac{CC'}{|\gamma|}\right)^{\beta} w^{i+\beta(\epsilon-\gamma)-1} q_i(w) =: g(w)$ for all $t \geqslant t_0$. Moreover:

$$\int_1^{\infty} g(w)\,\mathrm{d}w \leqslant \int_0^{\infty} g(w)\,\mathrm{d}w = \left(\frac{CC'}{|\gamma|}\right)^{\beta} \Gamma(i + \beta(\epsilon - \gamma))\, \mathbb{E}\big[\Lambda^{-\beta(\epsilon-\gamma)}\big] < \infty$$

by using (1.13) and the assumption on ϵ. Finally, $\lim_{t\to\infty} f_t(w) = h_\gamma^\beta(1/w)\, w^{i-1} q_i(w) \mathbb{1}_{[r_i^{-1}, \infty)}(w)$ for every $w \geqslant 1$, with $r_i^{-1} = 0$ if $r_i = \infty$. Applying Lebesgue's dominated convergence theorem, we thus get:

$$\lim_{t\to\infty} I_2^2(t) = \int_1^{\infty} h_\gamma^\beta(1/w)\, w^{i-1} q_i(w) \mathbb{1}_{[r_i^{-1}, \infty)}(w)\,\mathrm{d}w =: I_2^2.$$

(iii) Case $\gamma = 0$. Assume that $\mathbb{E}\big[\Lambda^{-\beta\epsilon}\big] < \infty$ for some $\epsilon > 0$. For this ϵ and thanks to (5.19), there exist $C > 1$ and $t_0 > \max(1, r_i^{-1})$ such that for $t \geqslant t_0$ and $w \geqslant 1$:

$$\left|\frac{U(t/w) - U(t)}{a(t)}\right| \leqslant C\, w^{\epsilon}.$$

Therefore, for each $w \geqslant 1$, $|f_t(w)| \leqslant C^\beta w^{i+\beta\epsilon-1} q_i(w) =: g(w)$ for all $t \geqslant t_0$. Moreover:

$$\int_1^{\infty} g(w)\,\mathrm{d}w \leqslant \int_0^{\infty} g(w)\,\mathrm{d}w = C^\beta\, \Gamma(i + \beta\epsilon)\, \mathbb{E}\big[\Lambda^{-\beta\epsilon}\big] < \infty$$

by using (1.13) and the assumption on ϵ. Finally, $\lim_{t\to\infty} f_t(w) = (-\log w)^\beta\, w^{i-1} q_i(w) \mathbb{1}_{[r_i^{-1}, \infty)}(w)$ for every $w \geqslant 1$, with $r_i^{-1} = 0$ if $r_i = \infty$. Applying Lebesgue's dominated convergence theorem, we thus get:

$$\lim_{t\to\infty} I_2^2(t) = \int_1^{\infty} (-\log w)^\beta\, w^{i-1} q_i(w) \mathbb{1}_{[r_i^{-1}, \infty)}(w)\,\mathrm{d}w =: I_2^2.$$

B. Now, consider the case $\ell_i = 0$ and $x_\ell > 0$. Because $r_i \in (0, \infty]$, we take $t > r_i^{-1}$ so that $t\{1 - F(U(r_i t))\} = r_i^{-1}$ holds with $r_i^{-1} = 0$ if $r_i = \infty$. By using (5.7) and (5.10), we readily obtain for every $t > r_i^{-1}$:

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{\tilde{X}_{i,t} - p\,U(t)}{a(t)}\right)^{\beta}\right] &= \int_{-\infty}^{\infty} x^\beta\,\mathrm{d}\tilde{H}_{i,t}(x) \\
&= \left(\frac{-p\,U(t)}{a(t)}\right)^{\beta} \Pi_i(t) + \int_{\frac{p(x_\ell - U(t))}{a(t)}}^{\frac{r_i(t)}{a(t)}} x^\beta\,\mathrm{d}_x F_{i,t}(U(t) + a(t)x/p) + \left(\frac{r_i(t)}{a(t)}\right)^{\beta} (1 - F_{i,t}(U(r_i t))) \\
&= \left(\frac{-p\,U(t)}{a(t)}\right)^{\beta} \Pi_i(t) + \frac{p^\beta}{(i-1)!} \int_{r_i^{-1}}^{t} \left(\frac{U(t/w) - U(t)}{a(t)}\right)^{\beta} w^{i-1} q_i(w)\,\mathrm{d}w \\
&\quad + \left(\frac{r_i(t)}{a(t)}\right)^{\beta} \frac{1}{(i-1)!} \int_0^{r_i^{-1}} w^{i-1} q_i(w)\,\mathrm{d}w.
\end{aligned}$$

Thus, the analysis of this case comes down to the one above when $\ell_i = 0$.

Part II

Asymptotic Analysis of the Ratio of the Random Sum of Squares to the Square of the Random Sum

Chapter 6

Asymptotic Behavior of the Moments of the Ratio

6.1 Introduction

Let $\{X_i;\, i \in \mathbb{N}\}$ be a sequence of a.s. positive i.i.d. random variables with distribution function F. Let $\{N(t);\, t \geqslant 0\}$ be a counting process independent of $\{X_i;\, i \in \mathbb{N}\}$. For any fixed $t \geqslant 0$, define the random variable $T_{N(t)}$ by:

$$T_{N(t)} := \frac{X_1^2 + X_2^2 + \cdots + X_{N(t)}^2}{\big(X_1 + X_2 + \cdots + X_{N(t)}\big)^2}$$

if $N(t) \geqslant 1$ and $T_{N(t)} := 0$ if $N(t) = 0$.

Denote by T_n the random variable $T_{N(t)}$ when the counting process $\{N(t);\, t \geqslant 0\}$ is non-random, i.e. $T_n := (\sum_{i=1}^n X_i^2)/(\sum_{i=1}^n X_i)^2$ if $n \in \mathbb{N}$ and $T_0 := 0$. An asymptotic analysis of T_n is provided by Albrecher et al. [1] and Albrecher and Teugels [2] under the condition that the distribution function F of X_1 is of Pareto-type with index $\alpha > 0$ (see Definition 1.7). To be more precise, Albrecher and Teugels [2] study the asymptotic behavior of arbitrary moments of T_n as $n \to \infty$, generalizing earlier results pertaining to $\mathbb{E}[T_n]$ by Fuchs et al. [26], and Albrecher et al. [1] derive limit distributions for the suitably normalized quantity T_n as $n \to \infty$. This chapter deals with *moment convergence*. We derive the asymptotic expression of any moment of order $k \in \mathbb{N}$ of the ratio $T_{N(t)}$ as $t \to \infty$ under the conditions that the distribution function F of X_1 is of Pareto-type with index $\alpha > 0$ and that the counting process $\{N(t);\, t \geqslant 0\}$ is mixed Poisson (see Definition 1.16).

The general method for proving our results is based on an integral representation of $\mathbb{E}\big[T_{N(t)}^k\big]$ (in terms of the probability generating function of $N(t)$ and the Laplace-Stieltjes transform of X_1) and relies on the theory of functions of regular variation (see Section 1.3).

Recall from Definition 1.7 that a distribution function F on $(0, \infty)$ is of Pareto-type with index $\alpha > 0$ if $1 - F \in \mathrm{RV}_{-\alpha}$, i.e. there exists $\ell \in \mathrm{RV}_0$ such that:

$$1 - F(x) \sim x^{-\alpha} \ell(x) \quad \text{as} \quad x \to \infty. \tag{6.1}$$

In the context of extreme value theory, the tail condition (6.1) is necessary and sufficient for F to belong to the domain of attraction of the Fréchet-Pareto distribution (see Subsection 1.2.2). When $\alpha \in (0,2)$, the tail condition (6.1) is also necessary and sufficient for F to belong to the domain of attraction of a non-normal stable distribution with exponent α (see Remark 1.20).

Let μ_β denote the moment of order $\beta > 0$ of X_1, i.e.:

$$\mu_\beta := \mathbb{E}\big[X_1^\beta\big] = \beta \int_0^\infty x^{\beta-1}\,(1-F(x))\,\mathrm{d}x \leqslant \infty.$$

Clearly, both the numerator and the denominator defining $T_{N(t)}$ exhibit an erratic behavior if $\mu_1 = \infty$, whereas this is the case only for the numerator if $\mu_1 < \infty$ and $\mu_2 = \infty$. When X_1 has a Pareto-type distribution function F with index $\alpha > 0$, it turns out that μ_β is finite if $\beta < \alpha$ but infinite whenever $\beta > \alpha$ (see Remark 1.9). In particular, $\mu_1 < \infty$ if $\alpha > 1$ while $\mu_2 < \infty$ as soon as $\alpha > 2$. The asymptotic behavior of $\mathbb{E}\big[T_{N(t)}^k\big]$ as $t \to \infty$ will thus heavily depend on the range of α. This is expressed in our main results in Section 6.2.

The common Laplace-Stieltjes transform of the X_i's is denoted by:

$$\varphi(s) := \mathbb{E}\big[\mathrm{e}^{-sX_1}\big] = \int_0^\infty \mathrm{e}^{-sx}\,\mathrm{d}F(x), \quad s \geqslant 0.$$

Note that $\varphi(0) = \varphi(0+) = 1$. Also, $\lim_{s\to\infty}\varphi(s) = 0$ and $\varphi(s) < 1$ for all $s > 0$ (since $X_1 > 0$ a.s.). For every $n \in \mathbb{N}$, the nth derivative of $\varphi(s)$ with respect to s is given by:

$$\varphi^{(n)}(s) := \frac{\partial^n \varphi(s)}{\partial s^n} = (-1)^n \int_0^\infty \mathrm{e}^{-sx}\,x^n\,\mathrm{d}F(x) = (-1)^n\,\mathbb{E}\big[\mathrm{e}^{-sX_1}X_1^n\big], \quad s > 0.$$

It follows in particular that for any $n \in \mathbb{N}$, μ_n is finite if and only if a finite limit $\varphi^{(n)}(0+)$ exists.

The asymptotic behavior of $\varphi^{(n)}$ at the origin when the distribution function F of X_1 is of Pareto-type with index $\alpha > 0$ is given in the following result which is a combination of Lemma 3.1 of Albrecher and Teugels [2] and Bingham-Doney's lemma (e.g., Theorem 8.1.6 of Bingham et al. [13]).

Lemma 6.1. *Assume that the distribution function F of X_1 satisfies $1 - F(x) \sim x^{-\alpha}\ell(x)$ as $x \to \infty$ for some $\alpha > 0$ and $\ell \in \mathrm{RV}_0$. Then for any $n \in \mathbb{N}$, we have as $s \downarrow 0$:*

$$(-1)^n\,\varphi^{(n)}(s) \sim \begin{cases} \alpha\,\Gamma(n-\alpha)\,s^{\alpha-n}\,\ell\big(\frac{1}{s}\big) & \text{if } n > \alpha \\ \alpha\,\tilde{\ell}\big(\frac{1}{s}\big) & \text{if } n = \alpha \text{ and } \mu_n = \infty \\ \mu_n & \text{if } n < \alpha \text{ or if } n = \alpha \text{ and } \mu_n < \infty \end{cases}$$

where $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,\mathrm{d}u \in \mathrm{RV}_0$.

The conclusion is given in Section 6.3. The content of this chapter is based upon Ladoucette [40].

6.2 Moment Convergence for $T_{N(t)}$

We start by deriving an integral representation for the kth moment of $T_{N(t)}$ for fixed $t \geqslant 0$ and $k \in \mathbb{N}$. Note that we make no specific assumption neither on the distribution function F of X_1 nor on the counting process $\{N(t);\, t \geqslant 0\}$.

Lemma 6.2. *Let $t \geqslant 0$ and $k \in \mathbb{N}$ be fixed. The kth moment of $T_{N(t)}$ is then given by:*

$$\mathbb{E}\big[T_{N(t)}^k\big] = \sum_{r=1}^{k} \sum_{\substack{(k_1,\dots,k_r)\in\mathbb{N}^r \\ k_1+\cdots+k_r=k}} \frac{k!}{\prod_{i=1}^r k_i!} \frac{B_t(k_1,\dots,k_r)}{(2k-1)!\, r!} \tag{6.2}$$

with:

$$B_t(k_1,\dots,k_r) := \int_0^\infty s^{2k-1} \prod_{i=1}^r \varphi^{(2k_i)}(s)\, Q_t^{(r)}(\varphi(s))\,\mathrm{d}s. \tag{6.3}$$

Proof. Let $t \geqslant 0$ and $k \in \mathbb{N}$ be fixed. For each $n \in \mathbb{N}$, we interpret $\binom{n}{r} := 0$ whenever $r \geqslant n+1$. Using Lemma 2.1 of Albrecher and Teugels [2], we easily derive:

$$\begin{aligned}
\mathbb{E}\big[T_{N(t)}^k\big] &= \sum_{n=0}^\infty p_n(t)\, \mathbb{E}\big[T_{N(t)}^k \big| N(t)=n\big] = \sum_{n=1}^\infty p_n(t)\, \mathbb{E}\big[T_n^k\big] \\
&= \sum_{n=1}^\infty p_n(t) \sum_{r=1}^{k} \sum_{\substack{(k_1,\dots,k_r)\in\mathbb{N}^r \\ k_1+\cdots+k_r=k}} \frac{k!}{\prod_{i=1}^r k_i!} \frac{\binom{n}{r}}{(2k-1)!} \int_0^\infty s^{2k-1} \prod_{i=1}^r \varphi^{(2k_i)}(s)\, \varphi^{n-r}(s)\,\mathrm{d}s \\
&= \sum_{r=1}^{k} \sum_{\substack{(k_1,\dots,k_r)\in\mathbb{N}^r \\ k_1+\cdots+k_r=k}} \frac{k!}{\prod_{i=1}^r k_i!} \frac{1}{(2k-1)!} \int_0^\infty s^{2k-1} \prod_{i=1}^r \varphi^{(2k_i)}(s) \sum_{n=r}^\infty \binom{n}{r} p_n(t)\, \varphi^{n-r}(s)\,\mathrm{d}s \\
&= \sum_{r=1}^{k} \sum_{\substack{(k_1,\dots,k_r)\in\mathbb{N}^r \\ k_1+\cdots+k_r=k}} \frac{k!}{\prod_{i=1}^r k_i!} \frac{1}{(2k-1)!\, r!} \int_0^\infty s^{2k-1} \prod_{i=1}^r \varphi^{(2k_i)}(s)\, Q_t^{(r)}(\varphi(s))\,\mathrm{d}s
\end{aligned}$$

where the last equality follows from (1.12). □

From now on, the counting process $\{N(t);\, t \geqslant 0\}$ is assumed to be mixed Poisson with mixing random variable Λ. We rewrite the term $B_t(k_1,\dots,k_r)$ defined in (6.3) in a more convenient form which will be the basis of our analysis. Defining $\psi(s) := \varphi^{-1}(1-s)$ for $s \in [0,1)$, substituting $s = \psi\big(\frac{w}{t}\big)$ and using (1.15) thanks to the mixed Poisson assumption leads for every $t > 0$ to:

$$\begin{aligned}
B_t(k_1,\dots,k_r) &= \int_0^\infty s^{2k-1} \prod_{i=1}^r \varphi^{(2k_i)}(s)\, Q_t^{(r)}(\varphi(s))\,\mathrm{d}s \\
&= \int_0^t \psi^{2k-1}\Big(\frac{w}{t}\Big) \prod_{i=1}^r \varphi^{(2k_i)}\Big(\psi\Big(\frac{w}{t}\Big)\Big)\, Q_t^{(r)}\Big(1-\frac{w}{t}\Big)\,\mathrm{d}_w\psi\Big(\frac{w}{t}\Big) \\
&= -t^{r-1} \frac{\psi^{2k-1}\big(\frac{1}{t}\big)}{\varphi^{(1)}\big(\psi\big(\frac{1}{t}\big)\big)} \prod_{i=1}^r \varphi^{(2k_i)}\Big(\psi\Big(\frac{1}{t}\Big)\Big) \\
&\quad \times \int_0^t \left(\frac{\psi\big(\frac{w}{t}\big)}{\psi\big(\frac{1}{t}\big)}\right)^{2k-1} \prod_{i=1}^r \frac{\varphi^{(2k_i)}\big(\psi\big(\frac{w}{t}\big)\big)}{\varphi^{(2k_i)}\big(\psi\big(\frac{1}{t}\big)\big)} \frac{\varphi^{(1)}\big(\psi\big(\frac{1}{t}\big)\big)}{\varphi^{(1)}\big(\psi\big(\frac{w}{t}\big)\big)} \frac{Q_t^{(r)}\big(1-\frac{w}{t}\big)}{t^r}\,\mathrm{d}w \\
&= -t^{r-1} \frac{\psi^{2k-1}\big(\frac{1}{t}\big)}{\varphi^{(1)}\big(\psi\big(\frac{1}{t}\big)\big)} \prod_{i=1}^r \varphi^{(2k_i)}\Big(\psi\Big(\frac{1}{t}\Big)\Big) \\
&\quad \times \int_0^\infty \left(\frac{\psi\big(\frac{w}{t}\big)}{\psi\big(\frac{1}{t}\big)}\right)^{2k-1} \prod_{i=1}^r \frac{\varphi^{(2k_i)}\big(\psi\big(\frac{w}{t}\big)\big)}{\varphi^{(2k_i)}\big(\psi\big(\frac{1}{t}\big)\big)} \frac{\varphi^{(1)}\big(\psi\big(\frac{1}{t}\big)\big)}{\varphi^{(1)}\big(\psi\big(\frac{w}{t}\big)\big)}\, q_r(w)\, \mathbb{1}_{(0,t)}(w)\,\mathrm{d}w
\end{aligned}$$

$$= \quad f_t(k_1, \dots, k_r) \int_0^\infty g_t(w; k_1, \dots, k_r)\, \mathrm{d}w$$

where:

$$f_t(k_1, \dots, k_r) := -t^{r-1} \frac{\psi^{2k-1}\left(\frac{1}{t}\right)}{\varphi^{(1)}\left(\psi\left(\frac{1}{t}\right)\right)} \prod_{i=1}^{r} \varphi^{(2k_i)}\left(\psi\left(\frac{1}{t}\right)\right)$$

and:

$$g_t(w; k_1, \dots, k_r) := \left(\frac{\psi\left(\frac{w}{t}\right)}{\psi\left(\frac{1}{t}\right)}\right)^{2k-1} \prod_{i=1}^{r} \frac{\varphi^{(2k_i)}\left(\psi\left(\frac{w}{t}\right)\right)}{\varphi^{(2k_i)}\left(\psi\left(\frac{1}{t}\right)\right)} \frac{\varphi^{(1)}\left(\psi\left(\frac{1}{t}\right)\right)}{\varphi^{(1)}\left(\psi\left(\frac{w}{t}\right)\right)}\, q_r(w)\, \mathbb{1}_{(0,t)}(w), \quad w \geqslant 0.$$

Now, we give an easy but useful result which holds in particular for the mixing random variable Λ.

Lemma 6.3. *Let Λ be an a.s. positive random variable. Then, $\mathbb{E}[\Lambda^s] < \infty \implies \mathbb{E}[\Lambda^r] < \infty$ and $\mathbb{E}[\Lambda^{-s}] < \infty \implies \mathbb{E}[\Lambda^{-r}] < \infty$ for all $0 < r \leqslant s$.*

Proof. Let H denote the distribution function of Λ. Fix $0 < r \leqslant s$. If $\mathbb{E}[\Lambda^s] < \infty$, we have:

$$\mathbb{E}[\Lambda^r] = \int_0^1 \lambda^r \,\mathrm{d}H(\lambda) + \int_1^\infty \lambda^r \,\mathrm{d}H(\lambda) \leqslant \mathbb{P}[\Lambda \leqslant 1] + \mathbb{E}[\Lambda^s] < \infty.$$

Similarly, if $\mathbb{E}[\Lambda^{-s}] < \infty$, we have:

$$\mathbb{E}[\Lambda^{-r}] = \int_0^1 \lambda^{-r} \,\mathrm{d}H(\lambda) + \int_1^\infty \lambda^{-r} \,\mathrm{d}H(\lambda) \leqslant \mathbb{E}[\Lambda^{-s}] + \mathbb{P}[\Lambda > 1] < \infty$$

and the claim is proved. □

Additionally to the mixed Poisson assumption for the counting process $\{N(t);\, t \geqslant 0\}$, we also assume in what follows that the distribution function F of X_1 is of Pareto-type with index $\alpha > 0$, i.e. satisfies (6.1) for some $\ell \in \mathrm{RV}_0$. We then determine the asymptotic behavior of the kth moment of $T_{N(t)}$ as $t \to \infty$ for any $k \in \mathbb{N}$. Different cases are considered according to the range of α.

6.2.1 Case $\alpha \in (0, 1)$

Here is the first of our main results pertaining to moment convergence for $T_{N(t)}$. It concerns the case $\alpha \in (0, 1)$.

Theorem 6.4. *Assume that X_1 has a Pareto-type distribution function F with index $\alpha \in (0, 1)$ and that $\{N(t);\, t \geqslant 0\}$ is a mixed Poisson process with mixing random variable Λ. If $\mathbb{E}[\Lambda^\epsilon] < \infty$ and $\mathbb{E}[\Lambda^{-\epsilon}] < \infty$ for some $\epsilon > 0$, then for any fixed $k \in \mathbb{N}$:*

$$\lim_{t\to\infty} \mathbb{E}\big[T_{N(t)}^k\big] = \frac{k!}{(2k-1)!} \sum_{r=1}^{k} \frac{\alpha^{r-1}}{r\,\Gamma^r(1-\alpha)}\, G(r, k)$$

where $G(r, k)$ is the coefficient of x^k in the polynomial $\left(\sum_{i=1}^{k-r+1} \frac{\Gamma(2i-\alpha)}{i!}\, x^i\right)^r$.

Proof. Let $k \in \mathbb{N}$ and $\alpha \in (0,1)$ be fixed. Since $1 - F(x) \sim x^{-\alpha}\ell(x)$ as $x \to \infty$ for some $\ell \in \mathrm{RV}_0$, it follows from Corollary 8.1.7 of Bingham et al. [13] that $1 - \varphi(s) \sim \Gamma(1-\alpha)\, s^{\alpha}\ell(1/s)$ as $s \downarrow 0$. Since $\lim_{s\downarrow 0} \psi(s) = 0$, we then get $s = 1 - \varphi(\psi(s)) \sim \Gamma(1-\alpha)\, \psi^{\alpha}(s)\, \ell(1/\psi(s))$ as $s \downarrow 0$ so that:

$$\lim_{s\downarrow 0} s^{-1}\, \psi^{\alpha}(s)\, \ell\biggl(\frac{1}{\psi(s)}\biggr) = \frac{1}{\Gamma(1-\alpha)}. \tag{6.4}$$

First, we determine the asymptotic behavior of $B_t(k_1, \ldots, k_r)$ as $t \to \infty$ for any integer $1 \leqslant r \leqslant k$. It follows from (6.4) that $\psi \in \mathrm{RV}^0_{1/\alpha}$. For $i = 1, \ldots, r$, we then have $\varphi^{(2k_i)} \circ \psi \in \mathrm{RV}^0_{(\alpha-2k_i)/\alpha}$ since $\varphi^{(2k_i)} \in \mathrm{RV}^0_{\alpha-2k_i}$ by Lemma 6.1. Moreover, $-\varphi^{(1)} \circ \psi \in \mathrm{RV}^0_{(\alpha-1)/\alpha}$ since $-\varphi^{(1)} \in \mathrm{RV}^0_{\alpha-1}$ by Lemma 6.1. Note that the last two items follow from the translation of Proposition 1.9(ii) from regular variation at infinity to regular variation at 0. Set $\delta_r := \frac{\zeta}{2k+r}$ with $\zeta = \epsilon$ if $\epsilon \in (0,1)$ or $\zeta \in (0,1)$ otherwise. Thus, for this chosen $\delta_r > 0$, there exists $C_r := C_r(\delta_r) > 1$ such that for $t > 0$ and $w > 0$:

$$\begin{aligned} g_t(w; k_1, \ldots, k_r) &\leqslant C_r\, w^{\frac{2k-1}{\alpha}} \bigl(\max\bigl\{w^{\delta_r}, w^{-\delta_r}\bigr\}\bigr)^{2k-1}\, w^{\frac{r\alpha-2k}{\alpha}} \bigl(\max\bigl\{w^{\delta_r}, w^{-\delta_r}\bigr\}\bigr)^{r+1}\, w^{\frac{1-\alpha}{\alpha}}\, q_r(w) \\ &= C_r\, w^{r-1} \max\bigl\{w^{\zeta}, w^{-\zeta}\bigr\}\, q_r(w) =: h(w) \end{aligned}$$

by virtue of Potter's theorem (see Theorem 1.7(ii)).

Now, $\int_0^{\infty} h(w)\,\mathrm{d}w < \infty$ if and only if $\int_0^1 w^{r-1-\zeta}\, q_r(w)\,\mathrm{d}w < \infty$ and $\int_1^{\infty} w^{r-1+\zeta}\, q_r(w)\,\mathrm{d}w < \infty$. Since $\zeta \in (0,1)$ and $\zeta \leqslant \epsilon$, we use (1.13) together with Lemma 6.3 to get:

$$\int_0^1 w^{r-1-\zeta}\, q_r(w)\,\mathrm{d}w \leqslant \int_0^{\infty} w^{r-1-\zeta}\, q_r(w)\,\mathrm{d}w = \Gamma(r-\zeta)\, \mathbb{E}[\Lambda^{\zeta}] < \infty$$

and:

$$\int_1^{\infty} w^{r-1+\zeta}\, q_r(w)\,\mathrm{d}w \leqslant \int_0^{\infty} w^{r-1+\zeta}\, q_r(w)\,\mathrm{d}w = \Gamma(r+\zeta)\, \mathbb{E}[\Lambda^{-\zeta}] < \infty.$$

Hence, the function h is integrable.

Finally, $\lim_{t\to\infty} g_t(w; k_1, \ldots, k_r) = w^{\frac{2k-1}{\alpha}} w^{r-\frac{2k}{\alpha}} w^{\frac{1}{\alpha}-1} q_r(w) = w^{r-1} q_r(w)$ for every $w > 0$. Thus, applying Lebesgue's dominated convergence theorem and using (1.14), we deduce:

$$\lim_{t\to\infty} \int_0^{\infty} g_t(w; k_1, \ldots, k_r)\,\mathrm{d}w = \int_0^{\infty} w^{r-1}\, q_r(w)\,\mathrm{d}w = (r-1)!.$$

Using Lemma 6.1 leads to:

$$\begin{aligned} f_t(k_1, \ldots, k_r) &\underset{t\to\infty}{\sim} \frac{-\alpha^r \prod_{i=1}^r \Gamma(2k_i - \alpha)\, t^{r-1}\, \psi^{r\alpha-1}\bigl(\frac{1}{t}\bigr)\, \ell^r\bigl(1/\psi\bigl(\frac{1}{t}\bigr)\bigr)}{-\alpha\, \Gamma(1-\alpha)\, \psi^{\alpha-1}\bigl(\frac{1}{t}\bigr)\, \ell\bigl(1/\psi\bigl(\frac{1}{t}\bigr)\bigr)} \\ &= \alpha^{r-1} \frac{\prod_{i=1}^r \Gamma(2k_i - \alpha)}{\Gamma(1-\alpha)}\, t^{r-1}\, \psi^{\alpha(r-1)}\biggl(\frac{1}{t}\biggr)\, \ell^{r-1}\biggl(\frac{1}{\psi\bigl(\frac{1}{t}\bigr)}\biggr) \end{aligned}$$

so that we get by using the relation (6.4):

$$\lim_{t\to\infty} f_t(k_1, \ldots, k_r) = \frac{\alpha^{r-1} \prod_{i=1}^r \Gamma(2k_i - \alpha)}{\Gamma^r(1-\alpha)}.$$

Therefore, we obtain:

$$\lim_{t\to\infty} B_t(k_1,\dots,k_r) = \frac{(r-1)!\,\alpha^{r-1}\prod_{i=1}^r \Gamma(2k_i-\alpha)}{\Gamma^r(1-\alpha)}.$$

Summing over all $r=1,\dots,k$ in (6.2), we finally arrive at:

$$\lim_{t\to\infty} \mathbb{E}\big[T_{N(t)}^k\big] = \frac{k!}{(2k-1)!}\sum_{r=1}^k \frac{\alpha^{r-1}}{r\,\Gamma^r(1-\alpha)} \sum_{\substack{(k_1,\dots,k_r)\in\mathbb{N}^r\\ k_1+\cdots+k_r=k}} \prod_{i=1}^r \frac{\Gamma(2k_i-\alpha)}{k_i!}.$$

The assertion of the theorem follows since Albrecher and Teugels [2] have observed that:

$$G(r,k) := \sum_{\substack{(k_1,\dots,k_r)\in\mathbb{N}^r\\ k_1+\cdots+k_r=k}} \prod_{i=1}^r \frac{\Gamma(2k_i-\alpha)}{k_i!}$$

can be read off as the coefficient of x^k in the r-fold product $\Big(\sum_{i=1}^{k-r+1} \frac{\Gamma(2i-\alpha)}{i!}\,x^i\Big)^r$. □

6.2.2 Case $\alpha = 1$, $\mu_1 = \infty$

Our next result deals with the case $\alpha = 1$ if $\mu_1 = \infty$.

Theorem 6.5. *Assume that X_1 has a Pareto-type distribution function F with index $\alpha = 1$ satisfying $\mu_1 = \infty$. Assume that $\{N(t);\, t \geqslant 0\}$ is a mixed Poisson process with mixing random variable Λ. If $\mathbb{E}[\Lambda^\epsilon] < \infty$ and $\mathbb{E}[\Lambda^{-\epsilon}] < \infty$ for some $\epsilon > 0$, then for any fixed $k \in \mathbb{N}$:*

$$\mathbb{E}\big[T_{N(t)}^k\big] \sim \frac{1}{2k-1}\frac{\ell(a_t)}{\tilde{\ell}(a_t)} \quad \text{as} \quad t\to\infty$$

where $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,\mathrm{d}u \in \mathrm{RV}_0$ and $(a_t)_{t>0}$ is a sequence defined by $\lim_{t\to\infty} t\,a_t^{-1}\,\tilde{\ell}(a_t) = 1$.

Proof. Let $k \in \mathbb{N}$ be fixed. Since $1-F(x) \sim x^{-1}\ell(x)$ as $x\to\infty$ for some $\ell \in \mathrm{RV}_0$ such that $\mu_1 = \infty$, it follows from Lemma 6.1 that $\varphi^{(1)}(s) \sim -\tilde{\ell}(1/s)$ as $s \downarrow 0$ and then that $1-\varphi(s) \sim s\,\tilde{\ell}(1/s)$ as $s\downarrow 0$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,\mathrm{d}u \in \mathrm{RV}_0$. Since $\lim_{s\downarrow 0}\psi(s) = 0$, we then get $s = 1-\varphi(\psi(s)) \sim \psi(s)\,\tilde{\ell}(1/\psi(s))$ as $s\downarrow 0$ so that:

$$\lim_{s\downarrow 0} s^{-1}\,\psi(s)\,\tilde{\ell}\bigg(\frac{1}{\psi(s)}\bigg) = 1. \tag{6.5}$$

First, we determine the asymptotic behavior of $B_t(k_1,\dots,k_r)$ as $t\to\infty$ for any integer $1 \leqslant r \leqslant k$. It follows from (6.5) that $\psi \in \mathrm{RV}_1^0$. For $i=1,\dots,r$, we then have $\varphi^{(2k_i)} \circ \psi \in \mathrm{RV}_{1-2k_i}^0$ since $\varphi^{(2k_i)} \in \mathrm{RV}_{1-2k_i}^0$ by Lemma 6.1. Moreover, $-\varphi^{(1)} \circ \psi \in \mathrm{RV}_0^0$ since $-\varphi^{(1)} \in \mathrm{RV}_0^0$. Set $\delta_r := \frac{\zeta}{2k+r}$ with $\zeta = \epsilon$ if $\epsilon \in (0,1)$ or $\zeta \in (0,1)$ otherwise. Thus, for this chosen $\delta_r > 0$, there exists $C_r := C_r(\delta_r) > 1$ such that for $t > 0$ and $w > 0$:

$$\begin{aligned} g_t(w;k_1,\dots,k_r) &\leqslant C_r\,w^{2k-1}\big(\max\{w^{\delta_r},w^{-\delta_r}\}\big)^{2k-1}\,w^{r-2k}\big(\max\{w^{\delta_r},w^{-\delta_r}\}\big)^{r+1}\,q_r(w) \\ &= C_r\,w^{r-1}\max\{w^{\zeta},w^{-\zeta}\}\,q_r(w) =: h(w) \end{aligned}$$

by virtue of Potter's theorem (see Theorem 1.7(*ii*)).

Now, $\int_0^\infty h(w)\,\mathrm{d}w < \infty$ if and only if $\int_0^1 w^{r-1-\zeta}\,q_r(w)\,\mathrm{d}w < \infty$ and $\int_1^\infty w^{r-1+\zeta}\,q_r(w)\,\mathrm{d}w < \infty$. Since $\zeta \in (0,1)$ and $\zeta \leqslant \epsilon$, we use (1.13) together with Lemma 6.3 to get:

$$\int_0^1 w^{r-1-\zeta}\,q_r(w)\,\mathrm{d}w \leqslant \int_0^\infty w^{r-1-\zeta}\,q_r(w)\,\mathrm{d}w = \Gamma(r-\zeta)\,\mathbb{E}[\Lambda^\zeta] < \infty$$

and:

$$\int_1^\infty w^{r-1+\zeta}\,q_r(w)\,\mathrm{d}w \leqslant \int_0^\infty w^{r-1+\zeta}\,q_r(w)\,\mathrm{d}w = \Gamma(r+\zeta)\,\mathbb{E}[\Lambda^{-\zeta}] < \infty.$$

Hence, the function h is integrable.

Finally, $\lim_{t\to\infty} g_t(w;k_1,\ldots,k_r) = w^{2k-1}w^{r-2k}q_r(w) = w^{r-1}q_r(w)$ for every $w>0$. Thus, applying Lebesgue's dominated convergence theorem and using (1.14), we deduce:

$$\lim_{t\to\infty}\int_0^\infty g_t(w;k_1,\ldots,k_r)\,\mathrm{d}w = \int_0^\infty w^{r-1}\,q_r(w)\,\mathrm{d}w = (r-1)!.$$

Using Lemma 6.1, relation (6.5) and $\varphi^{(1)}(\psi(s)) \sim -\tilde{\ell}(1/\psi(s))$ as $s \downarrow 0$, we get as $t\to\infty$:

$$f_t(k_1,\ldots,k_r) \sim \prod_{i=1}^r \Gamma(2k_i-1)\,t^{r-1}\,\psi^{r-1}\!\left(\frac{1}{t}\right)\frac{\ell^r\big(1/\psi\big(\frac{1}{t}\big)\big)}{\tilde{\ell}\big(1/\psi\big(\frac{1}{t}\big)\big)} \sim \prod_{i=1}^r \Gamma(2k_i-1)\left(\frac{\ell\big(1/\psi\big(\frac{1}{t}\big)\big)}{\tilde{\ell}\big(1/\psi\big(\frac{1}{t}\big)\big)}\right)^r.$$

Therefore, we obtain:

$$B_t(k_1,\ldots,k_r) \sim (r-1)!\prod_{i=1}^r \Gamma(2k_i-1)\left(\frac{\ell\big(1/\psi\big(\frac{1}{t}\big)\big)}{\tilde{\ell}\big(1/\psi\big(\frac{1}{t}\big)\big)}\right)^r \quad \text{as} \quad t\to\infty.$$

Since $\lim_{t\to\infty}\frac{\ell\big(1/\psi\big(\frac{1}{t}\big)\big)}{\tilde{\ell}\big(1/\psi\big(\frac{1}{t}\big)\big)} = 0$ (e.g., Proposition 1.5.9a of Bingham et al. [13]), only the summand with $r=1$ contributes to the dominating asymptotic term of (6.2). Hence, we get:

$$\mathbb{E}\big[T_{N(t)}^k\big] \sim \frac{1}{2k-1}\frac{\ell\big(1/\psi\big(\frac{1}{t}\big)\big)}{\tilde{\ell}\big(1/\psi\big(\frac{1}{t}\big)\big)} \quad \text{as} \quad t\to\infty.$$

One easily notes that the limit relation (6.5) is equivalent to $\lim_{t\to\infty} t\,\psi\big(\frac{1}{t}\big)\,\tilde{\ell}\big(1/\psi\big(\frac{1}{t}\big)\big) = 1$. Defining a sequence $(a_t)_{t>0}$ by $\lim_{t\to\infty} t\,a_t^{-1}\tilde{\ell}(a_t) = 1$ implies that $\lim_{t\to\infty} a_t\,\psi\big(\frac{1}{t}\big) = 1$. By virtue of the uniform convergence theorem for slowly varying functions (see Theorem 1.6(i)), we thus get $\ell\big(1/\psi\big(\frac{1}{t}\big)\big) \sim \ell(a_t)$ and $\tilde{\ell}\big(1/\psi\big(\frac{1}{t}\big)\big) \sim \tilde{\ell}(a_t)$ as $t\to\infty$. Consequently, we finally arrive at:

$$\mathbb{E}\big[T_{N(t)}^k\big] \sim \frac{1}{2k-1}\frac{\ell(a_t)}{\tilde{\ell}(a_t)} \quad \text{as} \quad t\to\infty$$

and the proof is finished. □

6.2.3 Case $\alpha \in (1,2)$ or $\alpha = 1$, $\mu_1 < \infty$

In the following result, the case $\alpha \in (1,2)$ (including $\alpha = 1$ if $\mu_1 < \infty$) is treated.

Theorem 6.6. *Assume that X_1 has a Pareto-type distribution function F with index $\alpha \in (1,2)$ (including $\alpha = 1$ if $\mu_1 < \infty$) and that $\{N(t);\, t \geqslant 0\}$ is a mixed Poisson process with mixing random variable Λ. For any fixed $k \in \mathbb{N}$, if $\mathbb{E}[\Lambda^{\epsilon}] < \infty$ and $\mathbb{E}[\Lambda^{-(k(\alpha-1)+\epsilon)}] < \infty$ for some $\epsilon > 0$, then:*

$$\mathbb{E}\big[T_{N(t)}^k\big] \sim \frac{\alpha}{\mu_1^{\alpha}}\, \mathrm{B}(2k-\alpha,\alpha)\, \mathbb{E}[\Lambda^{-(\alpha-1)}]\, t^{-(\alpha-1)}\ell(t) \quad \textit{as} \quad t \to \infty.$$

Proof. Let $k \in \mathbb{N}$ be fixed. Let $1 - F(x) \sim x^{-\alpha}\ell(x)$ as $x \to \infty$ for some $\ell \in \mathrm{RV}_0$ and $\alpha \in (1,2)$ or $\alpha = 1$ if $\mu_1 < \infty$.

First, we determine the asymptotic behavior of $B_t(k_1,\dots,k_r)$ as $t \to \infty$ for any integer $1 \leqslant r \leqslant k$. Since $\mu_1 < \infty$, we have $\varphi^{(1)}(0+) = -\mu_1$ so that $1 - \varphi(s) \sim \mu_1 s$ as $s \downarrow 0$. Since $\lim_{s\downarrow 0}\psi(s) = 0$, we then get $s = 1 - \varphi(\psi(s)) \sim \mu_1\psi(s)$ as $s \downarrow 0$ so that $\psi(s) \underset{s\downarrow 0}{\sim} s/\mu_1 \in \mathrm{RV}_1^0$. For $i = 1,\dots,r$, we then have $\varphi^{(2k_i)} \circ \psi \in \mathrm{RV}^0_{\alpha-2k_i}$ since $\varphi^{(2k_i)} \in \mathrm{RV}^0_{\alpha-2k_i}$ by Lemma 6.1. Moreover, $-\varphi^{(1)} \circ \psi \in \mathrm{RV}_0^0$ since $-\varphi^{(1)}(s) \underset{s\downarrow 0}{\sim} \mu_1 \in \mathrm{RV}_0^0$. Set $\delta_r := \frac{\zeta}{2k+r}$ with $\zeta = \epsilon$ if $\epsilon \in (0,1)$ or $\zeta \in (0,1)$ otherwise. Thus, for this chosen $\delta_r > 0$, there exists $C_r := C_r(\delta_r) > 1$ such that for $t > 0$ and $w > 0$:

$$\begin{aligned} g_t(w;k_1,\dots,k_r) &\leqslant C_r\, w^{2k-1}\big(\max\{w^{\delta_r}, w^{-\delta_r}\}\big)^{2k-1}\, w^{r\alpha-2k}\big(\max\{w^{\delta_r}, w^{-\delta_r}\}\big)^{r+1}\, q_r(w) \\ &= C_r\, w^{r\alpha-1}\max\{w^{\zeta}, w^{-\zeta}\}\, q_r(w) =: h(w) \end{aligned}$$

by virtue of Potter's theorem (see Theorem 1.7(ii)).

Now, $\int_0^{\infty} h(w)\,\mathrm{d}w < \infty$ if and only if $\int_0^1 w^{r\alpha-1-\zeta}\, q_r(w)\,\mathrm{d}w < \infty$ and $\int_1^{\infty} w^{r\alpha-1+\zeta}\, q_r(w)\,\mathrm{d}w < \infty$. Since $\zeta \in (0,1)$ and $\zeta \leqslant \epsilon$, we use (1.13) together with Lemma 6.3 to get:

$$\int_0^1 w^{r\alpha-1-\zeta}\, q_r(w)\,\mathrm{d}w \leqslant \int_0^1 w^{r-1-\zeta}\, q_r(w)\,\mathrm{d}w \leqslant \int_0^{\infty} w^{r-1-\zeta}\, q_r(w)\,\mathrm{d}w = \Gamma(r-\zeta)\,\mathbb{E}[\Lambda^{\zeta}] < \infty$$

and since $-(k(\alpha-1)+\epsilon) \leqslant -(r(\alpha-1)+\zeta) < 0$:

$$\int_1^{\infty} w^{r\alpha-1+\zeta}\, q_r(w)\,\mathrm{d}w \leqslant \int_0^{\infty} w^{r\alpha-1+\zeta}\, q_r(w)\,\mathrm{d}w = \Gamma(r\alpha+\zeta)\,\mathbb{E}[\Lambda^{-(r(\alpha-1)+\zeta)}] < \infty.$$

Hence, the function h is integrable.

Finally, $\lim_{t\to\infty} g_t(w;k_1,\dots,k_r) = w^{2k-1}w^{r\alpha-2k}q_r(w) = w^{r\alpha-1}q_r(w)$ for every $w > 0$. Thus, applying Lebesgue's dominated convergence theorem and using (1.13), we deduce:

$$\lim_{t\to\infty}\int_0^{\infty} g_t(w;k_1,\dots,k_r)\,\mathrm{d}w = \int_0^{\infty} w^{r\alpha-1}\, q_r(w)\,\mathrm{d}w = \Gamma(r\alpha)\,\mathbb{E}[\Lambda^{-r(\alpha-1)}].$$

Since $\ell(1/s) \in \mathrm{RV}_0^0$, the uniform convergence theorem for slowly varying functions (see Theorem 1.6(i)) states that $\ell(x/s) \sim \ell(1/s)$ as $s \downarrow 0$ uniformly on each compact x-set in $(0,\infty)$. Since $\lim_{s\downarrow 0}\frac{s}{\psi(s)} = \mu_1 \in (0,\infty)$, we consequently get $\ell(1/\psi(s)) = \ell\big(\frac{s}{\psi(s)}\frac{1}{s}\big) \sim \ell(1/s)$ as $s \downarrow 0$. This together with Lemma 6.1 and $\lim_{s\downarrow 0}\varphi^{(1)}(\psi(s)) = -\mu_1$ yields as $t \to \infty$:

$$f_t(k_1,\dots,k_r) \sim \frac{\alpha^r}{\mu_1}\prod_{i=1}^{r}\Gamma(2k_i-\alpha)\, t^{r-1}\,\psi^{r\alpha-1}\!\left(\frac{1}{t}\right)\ell^r\!\left(\frac{1}{\psi\big(\frac{1}{t}\big)}\right) \sim \frac{\alpha^r}{\mu_1^{r\alpha}}\prod_{i=1}^{r}\Gamma(2k_i-\alpha)\, t^{-r(\alpha-1)}\ell^r(t).$$

Therefore, we obtain:

$$B_t(k_1,\dots,k_r) \sim \left(\frac{\alpha}{\mu_1^\alpha}\right)^r \Gamma(r\alpha)\prod_{i=1}^r \Gamma(2k_i-\alpha)\,\mathbb{E}\big[\Lambda^{-r(\alpha-1)}\big]\,\Big(t^{-(\alpha-1)}\ell(t)\Big)^r \quad \text{as} \quad t\to\infty.$$

Note that $\ell(x)=o(1)$ as $x\to\infty$ when $\alpha=1$ because $\mu_1<\infty$. Since $\lim_{t\to\infty} t^{-(\alpha-1)}\ell(t)=0$, the first-order asymptotic behavior of (6.2) is solely determined by the term with $r=1$ and we obtain:

$$\mathbb{E}\big[T_{N(t)}^k\big] \sim \frac{\alpha}{\mu_1^\alpha}\frac{\Gamma(2k-\alpha)\,\Gamma(\alpha)}{(2k-1)!}\,\mathbb{E}\big[\Lambda^{-(\alpha-1)}\big]\,t^{-(\alpha-1)}\ell(t) \quad \text{as} \quad t\to\infty.$$

Finally, $\mathbb{E}[\Lambda^{-(\alpha-1)}]<\infty$ since $-(k(\alpha-1)+\epsilon)<-(\alpha-1)<0$ if $\alpha\neq 1$ and $\alpha-1=0$ if $\alpha=1$. □

6.2.4 Case $\alpha>2$

We pass to the case $\alpha>2$.

Theorem 6.7. *Assume that X_1 has a distribution function F of Pareto-type with index $\alpha>2$ and that $\{N(t);\,t\geqslant 0\}$ is a mixed Poisson process with mixing random variable Λ. Let $k\in\mathbb{N}$ be fixed. If $k=1$, assume further that $\mathbb{E}[\Lambda^{-(1+\epsilon)}]<\infty$ for some $\epsilon>0$. If $k\geqslant 2$, assume further that $\mathbb{E}[\Lambda^{k-2+\epsilon}]<\infty$ and $\mathbb{E}[\Lambda^{-(2k-1+\epsilon)}]<\infty$ for some $\epsilon>0$. Then for $k<\alpha-1$:*

$$\mathbb{E}\big[T_{N(t)}^k\big] \sim \left(\frac{\mu_2}{\mu_1^2}\right)^k \mathbb{E}[\Lambda^{-k}]\,t^{-k} \quad \textit{as} \quad t\to\infty \tag{6.6}$$

and for $k>\alpha-1$:

$$\mathbb{E}\big[T_{N(t)}^k\big] \sim \frac{\alpha}{\mu_1^\alpha}\,\mathrm{B}(2k-\alpha,\alpha)\,\mathbb{E}[\Lambda^{-(\alpha-1)}]\,t^{-(\alpha-1)}\ell(t) \quad \textit{as} \quad t\to\infty. \tag{6.7}$$

If $k=\alpha-1$, then:

(i) (6.6) *holds if $\ell(x)=o(1)$ as $x\to\infty$;*

(ii) $\mathbb{E}\big[T_{N(t)}^k\big] \sim \left(\left(\frac{\mu_2}{\mu_1^2}\right)^k + C\frac{(k+1)\,\mathrm{B}(k-1,k+1)}{\mu_1^{k+1}}\right)\mathbb{E}[\Lambda^{-k}]\,t^{-k}$ *as $t\to\infty$ holds if $\lim_{x\to\infty}\ell(x)=C$ for a positive constant C;*

(iii) (6.7) *holds otherwise.*

Proof. Let $k\in\mathbb{N}$ be fixed. Assume that $1-F(x)\sim x^{-\alpha}\ell(x)$ as $x\to\infty$ for some $\ell\in\mathrm{RV}_0$ and $\alpha>2$. Since $\mu_1<\infty$, we have $\varphi^{(1)}(0+)=-\mu_1$ so that $1-\varphi(s)\sim\mu_1 s$ as $s\downarrow 0$. Since $\lim_{s\downarrow 0}\psi(s)=0$, we then get $s=1-\varphi(\psi(s))\sim\mu_1\psi(s)$ as $s\downarrow 0$ so that $\psi(s)\underset{s\downarrow 0}{\sim} s/\mu_1\in\mathrm{RV}_1^0$. For any $n\in\mathbb{N}$, we then have $\varphi^{(2n)}\circ\psi\in\mathrm{RV}_{\alpha-2n}^0$ if $2n>\alpha$ and $\varphi^{(2n)}\circ\psi\in\mathrm{RV}_0^0$ if $2n\leqslant\alpha$ since $\varphi^{(2n)}\in\mathrm{RV}_{\alpha-2n}^0$ if $2n>\alpha$ and $\varphi^{(2n)}\in\mathrm{RV}_0^0$ if $2n\leqslant\alpha$ by Lemma 6.1. Moreover, $-\varphi^{(1)}\circ\psi\in\mathrm{RV}_0^0$ since $-\varphi^{(1)}(s)\underset{s\downarrow 0}{\sim}\mu_1\in\mathrm{RV}_0^0$.

For simplicity, let us first assume that $\alpha\notin\mathbb{N}$. We start by determining the asymptotic behavior of $B_t(k_1,\dots,k_r)$ as $t\to\infty$ for any integer $1\leqslant r\leqslant k$. Set $\delta_r:=\frac{\zeta}{2k+r}$ with $\zeta=\epsilon$ if $\epsilon\in(0,1)$ or $\zeta\in(0,1)$ otherwise. If follows from Potter's theorem (see Theorem 1.7(*ii*)) that for this chosen $\delta_r>0$, there exists $C_r:=C_r(\delta_r)>1$ such that for $t>0$ and $w>0$:

$$g_t(w;k_1,\dots,k_r) \quad\leqslant\quad C_r\,w^{2k-1}\big(\max\{w^{\delta_r},w^{-\delta_r}\}\big)^{2k-1}\,w^{r_1\alpha-2u_1}\big(\max\{w^{\delta_r},w^{-\delta_r}\}\big)^{r+1}\,q_r(w)$$

$$= \quad C_r\, w^{2k-1+r_1\alpha-2u_1} \max\{w^{\zeta}, w^{-\zeta}\}\, q_r(w) =: h(w)$$

where r_1 denotes the number of integers among $k_1, \dots, k_r$ that are greater than $\alpha/2$ and u_1 is the sum of these. By convention, $\sum_{i\in\varnothing} a_i = 0$ for any natural number a_i and $\operatorname{card}\varnothing = 0$ where $\operatorname{card} A$ denotes the cardinality of a set $A \subseteq \mathbb{N}$.

Now, it is readily seen that $\int_0^\infty h(w)\,\mathrm{d}w < \infty$ if and only if $\int_0^1 w^{2k-1+r_1\alpha-2u_1-\zeta}\, q_r(w)\,\mathrm{d}w < \infty$ and $\int_1^\infty w^{2k-1+r_1\alpha-2u_1+\zeta}\, q_r(w)\,\mathrm{d}w < \infty$. We have $2-2k \leqslant r_1\alpha - 2u_1 \leqslant 0$. Indeed, if $r_1 = 0$ then obviously $r_1\alpha - 2u_1 = 0$. Now, if $r_1 \neq 0$ (i.e. $r_1 \geqslant 1$) then $r_1\alpha - 2u_1 < 0$ on the one hand, and $r_1\alpha - 2u_1 \geqslant \alpha - 2k$ on the other hand. Consequently, $1 \leqslant 2k - 1 + r_1\alpha - 2u_1 \leqslant 2k-1$. Since $\zeta \in (0,1)$ and $\zeta \leqslant \epsilon$, we get $0 < r - 2 + \zeta \leqslant k - 2 + \epsilon$ if $r \geqslant 2$ and $-(2k-1+\epsilon) < -(1-\zeta) < 0$ if $r = 1$. Moreover, $-(2k-1+\epsilon) \leqslant -(2k-r+\zeta) < 0$. Therefore, using (1.13) together with Lemma 6.3 leads to:

$$\int_0^1 w^{2k-1+r_1\alpha-2u_1-\zeta}\, q_r(w)\,\mathrm{d}w \leqslant \int_0^1 w^{1-\zeta}\, q_r(w)\,\mathrm{d}w \leqslant \Gamma(2-\zeta)\,\mathbb{E}[\Lambda^{r-2+\zeta}] < \infty$$

and:

$$\int_1^\infty w^{2k-1+r_1\alpha-2u_1+\zeta}\, q_r(w)\,\mathrm{d}w \leqslant \int_1^\infty w^{2k-1+\zeta}\, q_r(w)\,\mathrm{d}w \leqslant \Gamma(2k+\zeta)\,\mathbb{E}[\Lambda^{-(2k-r+\zeta)}] < \infty.$$

When $k = 1$, obviously $r = 1$ and we get $-(1+\epsilon) \leqslant -(1+\zeta) < -(1-\zeta) < 0$. The condition $\mathbb{E}[\Lambda^{-(1+\epsilon)}] < \infty$ is thus sufficient for $\mathbb{E}[\Lambda^{-(1-\zeta)}] < \infty$ and $\mathbb{E}[\Lambda^{-(1+\zeta)}] < \infty$ to hold. Hence, the function h is integrable.

Finally, $\lim_{t\to\infty} g_t(w; k_1, \dots, k_r) = w^{2k-1} w^{r_1\alpha-2u_1} q_r(w) = w^{2(k-u_1)+r_1\alpha-1} q_r(w)$ for every $w > 0$. Thus, applying Lebesgue's dominated convergence theorem and using (1.13), we deduce:

$$\begin{aligned}\lim_{t\to\infty} \int_0^\infty g_t(w; k_1, \dots, k_r)\,\mathrm{d}w &= \int_0^\infty w^{2(k-u_1)+r_1\alpha-1} q_r(w)\,\mathrm{d}w \\ &= \Gamma(2(k-u_1)+r_1\alpha)\,\mathbb{E}[\Lambda^{r-2(k-u_1)-r_1\alpha}].\end{aligned}$$

By virtue of the uniform convergence theorem for slowly varying functions (see Theorem 1.6(i)), we get $\ell(1/\psi(s)) \sim \ell(1/s)$ as $s \downarrow 0$. This together with Lemma 6.1 and $\lim_{s\downarrow 0} \varphi^{(1)}(\psi(s)) = -\mu_1$ yields as $t \to \infty$:

$$\begin{aligned} f_t(k_1, \dots, k_r) &\sim \frac{\alpha^{r_1} K_1 K_2}{\mu_1}\, t^{r-1}\, \psi^{2(k-u_1)+r_1\alpha-1}\!\left(\frac{1}{t}\right) \ell^{r_1}\!\left(\frac{1}{\psi\left(\frac{1}{t}\right)}\right) \\ &\sim \frac{\alpha^{r_1} K_1 K_2}{\mu_1^{2(k-u_1)+r_1\alpha}}\, t^{r-2(k-u_1)-r_1\alpha} \ell^{r_1}(t) \end{aligned} \tag{6.8}$$

where $K_1 := \prod_{i\in I_1} \Gamma(2k_i - \alpha)$ with $I_1 := \{i \in \{1, \dots, r\} \colon 2k_i > \alpha\}$ and $K_2 := \prod_{i\in I_2} \mu_{2k_i}$ with $I_2 := \{i \in \{1, \dots, r\} \colon 2k_i < \alpha\}$. By convention, $\prod_{i\in\varnothing} b_i = 1$ for any $b_i > 0$. Since $\operatorname{card} I_1 = r_1$ and $\alpha \notin \mathbb{N}$, we have $\operatorname{card} I_2 = r - r_1$. The asymptotic behavior of $B_t(k_1, \dots, k_r)$ as $t \to \infty$ is therefore given by:

$$B_t(k_1, \dots, k_r) \sim \frac{\alpha^{r_1} K_1 K_2}{\mu_1^{2(k-u_1)+r_1\alpha}}\, \Gamma(2(k-u_1)+r_1\alpha)\,\mathbb{E}[\Lambda^{r-2(k-u_1)-r_1\alpha}]\, t^{r-2(k-u_1)-r_1\alpha} \ell^{r_1}(t).$$

It remains to determine the dominating asymptotic term among all possible $B_t(k_1, \dots, k_r)$. For $r_1 > 0$, the largest exponent is obtained with $r_1 = 1$, $u_1 = k$ and therefore $r = 1$, so that the

asymptotic order is $t^{-(\alpha-1)}\ell(t)$. Note that $r_1 > 0$ is possible for $2k > \alpha$ only. For $r_1 = 0$, obviously $r = k$ (which implies $k_1 = \cdots = k_r = 1$) yields the largest exponent, leading to the asymptotic order t^{-k}. Hence, the asymptotically dominating power among all $B_t(k_1, \dots, k_r)$ is given by $\max\{-(\alpha-1), -k\}$. From this, we see that when $k < \alpha - 1$, the term with $r = k$ dominates and we obtain from (6.2):

$$\mathbb{E}\big[T_{N(t)}^k\big] \sim \frac{k!\,\mu_2^k\,\Gamma(2k)\,\mathbb{E}[\Lambda^{-k}]}{(2k-1)!\,k!\,\mu_1^{2k}}\,t^{-k} = \left(\frac{\mu_2}{\mu_1^2}\right)^k \mathbb{E}[\Lambda^{-k}]\,t^{-k} \quad \text{as} \quad t \to \infty.$$

Alternatively, when $k > \alpha - 1$, the term with $r = 1$ dominates and we find as $t \to \infty$:

$$\mathbb{E}\big[T_{N(t)}^k\big] \sim \frac{k!\,\alpha\,\Gamma(2k-\alpha)\,\Gamma(\alpha)\,\mathbb{E}[\Lambda^{-(\alpha-1)}]}{k!\,(2k-1)!\,\mu_1^{\alpha}}\,t^{-(\alpha-1)}\ell(t) = \frac{\alpha}{\mu_1^{\alpha}}\,\mathrm{B}(2k-\alpha,\alpha)\,\mathbb{E}[\Lambda^{-(\alpha-1)}]\,t^{-(\alpha-1)}\ell(t)$$

which is the same expression as the one obtained in Theorem 6.6 for $\alpha \in (1,2)$ or $\alpha = 1$ if $\mu_1 < \infty$.

The above conclusions also hold for $\alpha \in \mathbb{N}$ as long as $k \neq \alpha - 1$. Nevertheless, just note that instead of (6.8), we have the following by virtue of Lemma 6.1:

$$f_t(k_1, \dots, k_r) \sim \frac{\alpha^{r_1+r_2} K_1 K_3}{\mu_1^{2(k-u_1)+r_1\alpha}}\,t^{r-2(k-u_1)-r_1\alpha}\ell^{r_1}(t)\,\tilde{\ell}^{r_2}(t) \quad \text{as} \quad t \to \infty$$

where $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,\mathrm{d}u \in \mathrm{RV}_0$, $K_3 := \prod_{i \in I_2 \cup I_3} \mu_{2k_i}$, $I_3 := \{i \in \{1,\dots,r\}\colon 2k_i = \alpha,\, \mu_{2k_i} < \infty\}$ and $r_2 := \operatorname{card}\{i \in \{1,\dots,r\}\colon 2k_i = \alpha,\, \mu_{2k_i} = \infty\}$. Since $\operatorname{card}(I_2 \cup I_3) = r - r_1 - r_2$, note that $\operatorname{card} I_2 = r - r_1 - r_2 - \operatorname{card} I_3$. When $k = \alpha - 1$, the slowly varying function ℓ determines which of the two terms $t^{-(\alpha-1)}\ell(t)$ (corresponding to $r = 1$) and t^{-k} (corresponding to $r = k$) dominates the asymptotic behavior. If $\ell(x) = o(1)$ as $x \to \infty$, which is in particular fulfilled if $\mu_{k+1} < \infty$, the second term dominates. If $\lim_{x\to\infty} \ell(x) = C$ for a positive constant C, then both terms matter. Otherwise, the first term dominates.

To end the proof, it remains to check that $\mathbb{E}[\Lambda^{-k}] < \infty$ for $k \leqslant \alpha - 1$ and that $\mathbb{E}[\Lambda^{-(\alpha-1)}] < \infty$ for $k > \alpha - 1$. When $k > \alpha - 1$, we have $-(2k-1+\epsilon) < -k < -(\alpha-1) < 0$. When $k \leqslant \alpha - 1$, we have $-(2k-1+\epsilon) < -k < 0$. Thus, we conclude by using Lemma 6.3. □

6.2.5 Case $\alpha = 2$

Finally, we deal with the remaining case $\alpha = 2$.

Theorem 6.8. *Assume that X_1 has a distribution function F of Pareto-type with index $\alpha = 2$ and that $\{N(t);\, t \geqslant 0\}$ is a mixed Poisson process with mixing random variable Λ.*

(i) If $\mathbb{E}[\Lambda^{-(1+\epsilon)}] < \infty$ for some $\epsilon > 0$, then:

$$\mathbb{E}[T_{N(t)}] \sim \begin{cases} \frac{\mu_2\,\mathbb{E}[\Lambda^{-1}]}{\mu_1^2}\,\frac{1}{t} & \text{if} \quad \mu_2 < \infty \\ \frac{2\,\mathbb{E}[\Lambda^{-1}]}{\mu_1^2}\,\frac{\tilde{\ell}(t)}{t} & \text{if} \quad \mu_2 = \infty \end{cases} \quad \text{as} \quad t \to \infty$$

where $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,\mathrm{d}u \in \mathrm{RV}_0$.

(ii) For any fixed integer $k \geqslant 2$, if $\mathbb{E}[\Lambda^{k-2+\epsilon}] < \infty$ and $\mathbb{E}[\Lambda^{-(2k-1+\epsilon)}] < \infty$ for some $\epsilon > 0$, then:

$$\mathbb{E}\big[T_{N(t)}^k\big] \sim \frac{\mathbb{E}[\Lambda^{-1}]}{\mu_1^2\,(k-1)\,(2k-1)}\,\frac{\ell(t)}{t} \quad \text{as} \quad t \to \infty.$$

Proof. Let $k \in \mathbb{N}$ be fixed. Let $1 - F(x) \sim x^{-2}\ell(x)$ as $x \to \infty$ for some $\ell \in \mathrm{RV}_0$. One can easily verify that Theorem 6.7 remains true for $\alpha = 2$, except when $k = 1$ if $\mu_2 = \infty$. In the latter case, obviously $r = 1$ and $B_t(k_1, \ldots, k_r)$ then becomes:

$$\begin{aligned} B_t(1) &= \int_0^\infty s\,\varphi^{(2)}(s)\,Q_t^{(1)}(\varphi(s))\,\mathrm{d}s \\ &= \underbrace{\frac{-\psi\left(\frac{1}{t}\right)}{\varphi^{(1)}\left(\psi\left(\frac{1}{t}\right)\right)}\,\varphi^{(2)}\left(\psi\left(\frac{1}{t}\right)\right)}_{=:f_t(1)} \int_0^\infty \underbrace{\frac{\psi\left(\frac{w}{t}\right)}{\psi\left(\frac{1}{t}\right)}\,\frac{\varphi^{(2)}\left(\psi\left(\frac{w}{t}\right)\right)}{\varphi^{(2)}\left(\psi\left(\frac{1}{t}\right)\right)}\,\frac{\varphi^{(1)}\left(\psi\left(\frac{1}{t}\right)\right)}{\varphi^{(1)}\left(\psi\left(\frac{w}{t}\right)\right)}\,q_1(w)\,\mathbb{1}_{(0,t)}(w)}_{=:g_t(w;1)}\,\mathrm{d}w. \end{aligned}$$

Since $\mu_1 < \infty$, we have $\varphi^{(1)}(0+) = -\mu_1$ so that $1 - \varphi(s) \sim \mu_1 s$ as $s \downarrow 0$. Since $\lim_{s\downarrow 0}\psi(s) = 0$, we then get $s = 1-\varphi(\psi(s)) \sim \mu_1\psi(s)$ as $s \downarrow 0$ so that $\psi(s) \underset{s\downarrow 0}{\sim} s/\mu_1 \in \mathrm{RV}_1^0$. We then have $\varphi^{(2)}\circ\psi \in \mathrm{RV}_0^0$ since $\varphi^{(2)} \in \mathrm{RV}_0^0$ by Lemma 6.1. Moreover, $-\varphi^{(1)} \circ \psi \in \mathrm{RV}_0^0$ since $-\varphi^{(1)}(s) \underset{s\downarrow 0}{\sim} \mu_1 \in \mathrm{RV}_0^0$. Set $\delta := \frac{\zeta}{3}$ with $\zeta = \epsilon$ if $\epsilon \in (0,1)$ or $\zeta \in (0,1)$ otherwise. Therefore, for this chosen $\delta > 0$, there exists $C := C(\delta) > 1$ such that for $t > 0$ and $w > 0$:

$$g_t(w;1) \leqslant C\,w\left(\max\left\{w^\delta, w^{-\delta}\right\}\right)^3 q_1(w) = C\,w\,\max\left\{w^\zeta, w^{-\zeta}\right\} q_1(w) =: h(w)$$

by virtue of Potter's theorem (see Theorem 1.7(ii)).

Now, $\int_0^\infty h(w)\,\mathrm{d}w < \infty$ if and only if $\int_0^1 w^{1-\zeta}\,q_1(w)\,\mathrm{d}w < \infty$ and $\int_1^\infty w^{1+\zeta}\,q_1(w)\,\mathrm{d}w < \infty$. Since $\zeta \in (0,1)$ and $\zeta \leqslant \epsilon$, we get $-(1+\epsilon) \leqslant -(1+\zeta) < -(1-\zeta) < 0$. Therefore, using (1.13) together with Lemma 6.3 leads to:

$$\int_0^1 w^{1-\zeta}\,q_1(w)\,\mathrm{d}w \leqslant \int_0^\infty w^{1-\zeta}\,q_1(w)\,\mathrm{d}w = \Gamma(2-\zeta)\,\mathbb{E}[\Lambda^{-(1-\zeta)}] < \infty$$

and:

$$\int_1^\infty w^{1+\zeta}\,q_1(w)\,\mathrm{d}w \leqslant \int_0^\infty w^{1+\zeta}\,q_1(w)\,\mathrm{d}w = \Gamma(2+\zeta)\,\mathbb{E}[\Lambda^{-(1+\zeta)}] < \infty.$$

Hence, the function h is integrable.

Finally, $\lim_{t\to\infty} g_t(w;1) = w\,q_1(w)$ for every $w > 0$. Applying Lebesgue's dominated convergence theorem and using (1.13), we thus deduce:

$$\lim_{t\to\infty}\int_0^\infty g_t(w;1)\,\mathrm{d}w = \int_0^\infty w\,q_1(w)\,\mathrm{d}w = \mathbb{E}[\Lambda^{-1}].$$

By virtue of the uniform convergence theorem for slowly varying functions (see Theorem 1.6(i)), we get $\tilde{\ell}(1/\psi(s)) \sim \tilde{\ell}(1/s)$ as $s \downarrow 0$. This together with $\varphi^{(2)}(\psi(s)) \sim 2\,\tilde{\ell}(1/\psi(s))$ as $s \downarrow 0$ where $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,\mathrm{d}u \in \mathrm{RV}_0$ and $\lim_{s\downarrow 0}\varphi^{(1)}(\psi(s)) = -\mu_1$ yields:

$$f_t(1) \sim \frac{2}{\mu_1^2}\,\frac{\tilde{\ell}(t)}{t} \quad \text{as} \quad t \to \infty.$$

Consequently, we obtain:

$$\mathbb{E}[T_{N(t)}] = B_t(1) \sim \frac{2\,\mathbb{E}[\Lambda^{-1}]}{\mu_1^2}\,\frac{\tilde{\ell}(t)}{t} \quad \text{as} \quad t \to \infty.$$

Finally, note that $\mathbb{E}[\Lambda^{-1}] < \infty$ since $-(1+\epsilon) < -1$. □

We end by a remark.

Remark 6.1. As in Albrecher and Teugels [2], the integral representation approach that we have used does not permit to get a general asymptotic result for $\mathbb{E}\big[T_{N(t)}^k\big]$ when F belongs to the domain of attraction of a normal distribution, i.e. when F has a slowly varying truncated second moment function (see Theorem 1.10(i)). Note however that F belongs to the domain of attraction of a normal distribution when it is defined as in Theorem 6.8.

6.3 Conclusion

In this chapter, we have derived the asymptotic behavior of arbitrary moments of the ratio $T_{N(t)}$ as $t \to \infty$ under the assumptions that the distribution function F of X_1 is of Pareto-type with index $\alpha > 0$ and that the counting process $\{N(t);\, t \geqslant 0\}$ is mixed Poisson. Different results have shown up according to the range of α.

In the particular case where the mixing random variable Λ is degenerate at the point 1, i.e. when $\{N(t);\, t \geqslant 0\}$ is a standard homogeneous Poisson process (see Definition 1.15), our results are similar to those derived by Albrecher and Teugels [2] where the counting process is non-random. This is basically explained by the fact that for every $r \in \mathbb{N}$, $q_r(w) = \mathrm{e}^{-w}$ for both counting processes.

The next chapter will deal with *convergence in distribution* for the suitably normalized ratio $T_{N(t)}$ as $t \to \infty$ when F is of Pareto-type with index $\alpha > 0$ and the counting process $\{N(t);\, t \geqslant 0\}$ either $\mathcal{D}$-averages in time or p-averages in time (see Definitions 1.17 and 1.18) according to the range of α. Also, it will be shown how $T_{N(t)}$ is naturally connected to a well-known risk measure, namely the *sample coefficient of variation*. Armed with our asymptotic distributional results on $T_{N(t)}$, we will thus derive asymptotic properties of the sample coefficient of variation. Finally, we will adapt the methodology to derive asymptotic properties of another risk measure, namely the *sample dispersion*.

Chapter 7

Limit Distributions for the Ratio and Applications to Risk Measures

7.1 Introduction

Let $\{X_i;\, i \in \mathbb{N}\}$ be a sequence of a.s. positive i.i.d. random variables with distribution function F. Let $\{N(t);\, t \geqslant 0\}$ be a counting process independent of $\{X_i;\, i \in \mathbb{N}\}$. For any fixed $t \geqslant 0$, define the random variable $T_{N(t)}$ by:

$$T_{N(t)} := \frac{X_1^2 + X_2^2 + \cdots + X_{N(t)}^2}{\big(X_1 + X_2 + \cdots + X_{N(t)}\big)^2}$$

if $N(t) \geqslant 1$ and $T_{N(t)} := 0$ if $N(t) = 0$.

The asymptotic behavior of arbitrary moments of $T_{N(t)}$ as $t \to \infty$ has been studied in Chapter 6 (see also Ladoucette [40]) under the conditions that the distribution function F of X_1 is of Pareto-type with index $\alpha > 0$ (see Definition 1.7) and that the counting process $\{N(t);\, t \geqslant 0\}$ is mixed Poisson (see Definition 1.16). In this chapter, we focus on *weak convergence* by deriving limit distributions for the appropriately normalized ratio $T_{N(t)}$ as $t \to \infty$. We still assume that the distribution function F of X_1 is of Pareto-type with index $\alpha > 0$ except for one result where we assume that the fourth moment of X_1 exists. Our results are established under the additional condition that the counting process $\{N(t);\, t \geqslant 0\}$ either $\mathcal{D}$-averages in time (see Definition 1.18) or p-averages in time (see Definition 1.17) according to the range of α. Denote by T_n the random variable $T_{N(t)}$ when the counting process $\{N(t);\, t \geqslant 0\}$ is non-random, i.e. $T_n := (\sum_{i=1}^n X_i^2)/(\sum_{i=1}^n X_i)^2$ if $n \in \mathbb{N}$ and $T_0 := 0$. Albrecher et al. [1] derive limit distributions for the appropriately normalized random variable T_n as $n \to \infty$ under the same assumption on the distribution function F of X_1 as here. Our results therefore generalize those established by Albrecher et al. [1].

Recall from Definition 1.7 that a distribution function F on $(0, \infty)$ is of Pareto-type with index $\alpha > 0$ if $1 - F \in \mathrm{RV}_{-\alpha}$, i.e. there exists $\ell \in \mathrm{RV}_0$ such that:

$$1 - F(x) \sim x^{-\alpha}\ell(x) \quad \text{as} \quad x \to \infty. \tag{7.1}$$

It follows that our results rely on the theory of functions of regular variation (see Section 1.3). The tail condition (7.1) has an important role in applied probability. Indeed, the condition is necessary

and sufficient for F to belong to the domain of attraction of the Fréchet-Pareto distribution (see Subsection 1.2.2) and, when $\alpha \in (0,2)$, to belong to the domain of attraction of a non-normal stable distribution with exponent α (see Remark 1.20). Also from Remark 1.20, recall that a stable random variable Y is a.s. positive if and only if $Y \overset{\mathcal{D}}{=} c\,U_\gamma$ for some $c > 0$ and $\gamma \in (0,1)$ where the random variable U_γ has the Laplace-Stieltjes transform:

$$\mathbb{E}\big[\mathrm{e}^{-\theta U_\gamma}\big] = \mathrm{e}^{-\theta^\gamma \Gamma(1-\gamma)}, \quad \theta \geqslant 0. \tag{7.2}$$

Any random variable U_γ with Laplace-Stieltjes transform (7.2) for $\gamma \in (0,1)$ is then a.s. positive and stable with exponent γ.

Let μ_β denote the moment of order $\beta > 0$ of X_1, i.e.:

$$\mu_\beta := \mathbb{E}\big[X_1^\beta\big] = \beta \int_0^\infty x^{\beta-1}\,(1 - F(x))\,\mathrm{d}x \leqslant \infty.$$

Clearly, both the numerator and the denominator defining $T_{N(t)}$ exhibit an erratic behavior if $\mu_1 = \infty$, whereas this is the case only for the numerator if $\mu_1 < \infty$ and $\mu_2 = \infty$. When F is of Pareto-type with index $\alpha > 0$, we know that μ_β is finite if $\beta < \alpha$ but infinite whenever $\beta > \alpha$ (see Remark 1.9). In particular, $\mu_1 < \infty$ if $\alpha > 1$ while $\mu_2 < \infty$ as soon as $\alpha > 2$. Since the asymptotic behavior of $T_{N(t)}$ as $t \to \infty$ is influenced by the (non)finiteness of μ_1 and/or μ_2, different limiting distributions will consequently show up according to the range of α. This is expressed in our main results given in Section 7.2. Armed with these results, we study in Section 7.3 the asymptotic behavior of two popular risk measures, namely the *sample coefficient of variation* and the *sample dispersion*, through their limiting distributions. For numerous examples of fields of applications as well as empirical illustrations connected to these risk measures, we refer to Albrecher et al. [1].

Let X be an a.s. positive random variable with distribution function F and $\{N(t);\, t \geqslant 0\}$ be a counting process independent of X. Let $X_1, \ldots, X_{N(t)}$ be a random sample from F of random size $N(t)$.

The *coefficient of variation* of the a.s. positive random variable X is defined by:

$$\mathrm{CoVar}(X) := \frac{\sqrt{\mathbb{V}[X]}}{\mathbb{E}[X]}.$$

This risk measure is frequently used in practice. For instance, it is of particular interest to actuaries (e.g., Mack [47] and Kaas et al. [34]). From the random sample $X_1, \ldots, X_{N(t)}$, the coefficient of variation $\mathrm{CoVar}(X)$ is naturally estimated by the *sample coefficient of variation* defined by:

$$\widehat{\mathrm{CoVar}(X)} := \frac{S}{\overline{X}} \tag{7.3}$$

where $\overline{X} := \frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i$ is the sample mean and $S^2 := \frac{1}{N(t)} \sum_{i=1}^{N(t)} \big(X_i - \overline{X}\big)^2$ is the sample variance. The properties of $\widehat{\mathrm{CoVar}(X)}$ are usually studied under the tacite assumption of the finiteness of sufficiently many moments of X. However, the existence of moments of X is not always guaranteed in practical applications. It is therefore useful to investigate the asymptotic behavior of $\widehat{\mathrm{CoVar}(X)}$ also when such a moment condition is not satisfied. It turns out that this can be achieved by using results on $T_{N(t)}$ due to the following relation:

$$\widehat{\mathrm{CoVar}(X)} = \sqrt{N(t)\,T_{N(t)} - 1}. \tag{7.4}$$

In Subsection 7.3.1, we thus take advantage of (7.4) to derive limit distributions for $\widehat{\mathrm{CoVar}(X)}$ as $t \to \infty$ under the same assumptions on the distribution function F of X and on the counting process $\{N(t);\, t \geqslant 0\}$ as in Section 7.2. Note that this is done even when the first or second moment of X does not exist.

Remark 7.1. There is a slight abuse in the expression of the sample variance S^2 since it is usually defined with a normalizing factor of $N(t) - 1$ instead of $N(t)$. However, this does not matter asymptotically since $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$.

Another popular risk measure of the a.s. positive random variable X is the *dispersion* defined by:

$$\mathrm{D}(X) := \frac{\mathbb{V}[X]}{\mathbb{E}[X]}.$$

From the random sample $X_1, \ldots, X_{N(t)}$, the dispersion $\mathrm{D}(X)$ is typically estimated by the *sample dispersion* defined by:

$$\widehat{\mathrm{D}(X)} := \frac{S^2}{\overline{X}}. \tag{7.5}$$

For any fixed $t \geqslant 0$, define the random variable $C_{N(t)}$ by:

$$C_{N(t)} := \frac{X_1^2 + X_2^2 + \cdots + X_{N(t)}^2}{X_1 + X_2 + \cdots + X_{N(t)}} \tag{7.6}$$

if $N(t) \geqslant 1$ and $C_{N(t)} := 0$ otherwise. The following relation is then readily proved:

$$\widehat{\mathrm{D}(X)} = C_{N(t)} - \overline{X}. \tag{7.7}$$

It turns out that results from Section 7.2 can be used to derive limit distributions for $\widehat{\mathrm{D}(X)}$ as $t \to \infty$ from those of $C_{N(t)}$ by means of (7.7). The results are given in Subsection 7.3.2 under the same assumptions on the distribution function F of X and on the counting process $\{N(t);\, t \geqslant 0\}$ as in Section 7.2. As for $\widehat{\mathrm{CoVar}(X)}$, cases where the first moments of X do not exist are also considered. We conclude in Section 7.4. This chapter is mainly based upon Ladoucette and Teugels [43] and also contains parts of Albrecher et al. [1].

7.2 Convergence in Distribution for $T_{N(t)}$

We derive limit distributions for the suitably normalized ratio $T_{N(t)}$ as $t \to \infty$ when the distribution function F of X_1 is of Pareto-type with index $\alpha > 0$, i.e. satisfies (7.1) for some $\ell \in \mathrm{RV}_0$. Different cases are considered according to the range of α. The last result (Theorem 7.8) is established under the general condition that $\mu_4 < \infty$ and consequently holds in the cases $\alpha = 4$ if $\mu_4 < \infty$ and $\alpha > 4$. The counting process $\{N(t);\, t \geqslant 0\}$ is assumed to $\mathcal{D}$-average in time except for two results (Theorems 7.6 and 7.7) where we impose the stronger assumption that it p-averages in time.

Let us start with a general result that will be useful.

Lemma 7.1. *Let $\{Y_n;\, n \in \mathbb{N}\}$ be a sequence of random variables and $\{M(t);\, t \geqslant 0\}$ be a process of a.s. nonnegative integer-valued random variables. Assume that $\{Y_n;\, n \in \mathbb{N}\}$ and $\{M(t);\, t \geqslant 0\}$ are independent and that $M(t) \xrightarrow{p} \infty$ as $t \to \infty$. If $Y_n \xrightarrow{\mathcal{D}} Y$ as $n \to \infty$ then $Y_{M(t)} \xrightarrow{\mathcal{D}} Y$ as $t \to \infty$.*

Proof. We prove the assertion by showing that $\lim_{t\to\infty}\mathbb{P}[Y_{M(t)} \leqslant y] = F_Y(y)$ at all continuity points y of the distribution function F_Y of the random variable Y.

Let y be a point of continuity of F_Y. Choose $\epsilon \in (0,1)$ arbitrarily. Since $Y_n \xrightarrow{\mathcal{D}} Y$ as $n \to \infty$, there exists $n_0 := n_0(\epsilon, y) \in \mathbb{N}_0$ such that $|\mathbb{P}[Y_n \leqslant y] - F_Y(y)| \leqslant \epsilon$ for all integers $n > n_0$. The independence of $\{Y_n;\, n \in \mathbb{N}\}$ and $\{M(t);\, t \geqslant 0\}$ gives $\mathbb{P}[Y_{M(t)} \leqslant y] = \sum_{n=0}^{\infty}\mathbb{P}[Y_n \leqslant y]\,\mathbb{P}[M(t) = n]$ so that:

$$\begin{aligned}\big|\mathbb{P}[Y_{M(t)} \leqslant y] - F_Y(y)\big| &= \left|\left(\sum_{n=0}^{n_0} + \sum_{n=n_0+1}^{\infty}\right)\{\mathbb{P}[Y_n \leqslant y] - F_Y(y)\}\,\mathbb{P}[M(t)=n]\right| \\ &\leqslant \sum_{n=0}^{n_0}|\mathbb{P}[Y_n \leqslant y] - F_Y(y)|\,\mathbb{P}[M(t)=n] \\ &\quad + \sum_{n=n_0+1}^{\infty}|\mathbb{P}[Y_n \leqslant y] - F_Y(y)|\,\mathbb{P}[M(t)=n] \\ &\leqslant \mathbb{P}[M(t) \leqslant n_0] + \epsilon\,\mathbb{P}[M(t) > n_0].\end{aligned}$$

Since $M(t) \xrightarrow{p} \infty$ as $t \to \infty$, it follows that $\limsup_{t\to\infty}\big|\mathbb{P}[Y_{M(t)} \leqslant y] - F_Y(y)\big| \leqslant \epsilon$. The claim is proved upon letting $\epsilon \downarrow 0$. □

Before giving our main results, recall that the *bivariate Laplace-Stieltjes transform* of a random vector $(Z_1, Z_2)'$ with distribution function G on $(0,\infty)\times(0,\infty)$ is given by:

$$\mathbb{E}\big[\mathrm{e}^{-rZ_1 - sZ_2}\big] = \int_0^{\infty}\int_0^{\infty}\mathrm{e}^{-rz_1 - sz_2}\,\mathrm{d}G(z_1, z_2), \quad r \geqslant 0,\ s \geqslant 0.$$

7.2.1 Case $\alpha \in (0,1)$

We start with the case $\alpha \in (0,1)$ in the following theorem.

Theorem 7.2. *Assume that X_1 has a distribution function F of Pareto-type with index $\alpha \in (0,1)$ and that the counting process $\{N(t);\, t \geqslant 0\}$ $\mathcal{D}$-averages in time to the random variable Λ. Then:*

$$T_{N(t)} \xrightarrow{\mathcal{D}} \frac{U_{\alpha/2}}{U_\alpha^2} \quad \text{as} \quad t \to \infty$$

where the random vector $(U_{\alpha/2}, U_\alpha)'$ has the bivariate Laplace-Stieltjes transform:

$$\mathbb{E}\big[\mathrm{e}^{-rU_{\alpha/2} - sU_\alpha}\big] = \exp\left\{-\int_0^{\infty}\mathrm{e}^{-ru^2 - su}\,(2ru + s)\,u^{-\alpha}\,\mathrm{d}u\right\}, \quad r \geqslant 0,\ s \geqslant 0. \tag{7.8}$$

In particular, the random variables $U_{\alpha/2}$ and U_α are a.s. positive and stable with respective exponent $\alpha/2$ and α and have the Laplace-Stieltjes transform (7.2) with $\gamma = \alpha/2$ and $\gamma = \alpha$ respectively.

Proof. Let $1 - F(x) \sim x^{-\alpha}\ell(x)$ as $x \to \infty$ for some $\ell \in \mathrm{RV}_0$ and $\alpha \in (0,1)$. Define a sequence $(a_t)_{t>0}$ by $1 - F(a_t) \sim 1/t$ as $t \to \infty$, i.e. $\lim_{t\to\infty} t\,a_t^{-\alpha}\ell(a_t) = 1$. Notice that $a_t \in \mathrm{RV}_{1/\alpha}$.

Fix $r \geqslant 0$ and $s \geqslant 0$. By using conditioning and independence arguments, we obtain for $t > 0$:

$$\mathbb{E}\Big[\mathrm{e}^{-r\frac{1}{a_t^2}\sum_{i=1}^{N(t)}X_i^2 - s\frac{1}{a_t}\sum_{i=1}^{N(t)}X_i}\Big] = \sum_{n=0}^{\infty}p_n(t)\,\mathbb{E}\Big[\mathrm{e}^{-r\frac{1}{a_t^2}\sum_{i=1}^{n}X_i^2 - s\frac{1}{a_t}\sum_{i=1}^{n}X_i}\Big]$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} p_n(t) \left(\mathbb{E}\Big[\mathrm{e}^{-r\frac{1}{a_t^2}X_1^2 - s\frac{1}{a_t}X_1} \Big] \right)^n \\
&= \sum_{n=0}^{\infty} p_n(t) \left(\int_0^{\infty} \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{a_t}} \, \mathrm{d}F(x) \right)^n \\
&= Q_t\left(\int_0^{\infty} \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{a_t}} \, \mathrm{d}F(x) \right) \\
&= Q_t\left(\mathrm{e}^{-\frac{\delta_{\alpha,t}(r,s)}{t}} \right)
\end{aligned}$$

with $\delta_{\alpha,t}(r,s) := -t \log \int_0^{\infty} \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{a_t}} \, \mathrm{d}F(x) \in [0,\infty)$.

It is clear that $\lim_{t\to\infty} \int_0^{\infty} \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{a_t}} \, \mathrm{d}F(x) = 1$ by virtue of Lebesgue's dominated convergence theorem. It then follows:

$$\begin{aligned}
\delta_{\alpha,t}(r,s) &\underset{t\to\infty}{\sim} -t \left(\int_0^{\infty} \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{a_t}} \, \mathrm{d}F(x) - 1 \right) \\
&= t \int_0^{\infty} \left(1 - \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{a_t}} \right) \mathrm{d}F(x) \\
&= t \int_0^{\infty} (1 - F(x)) \, \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{a_t}} \left(\frac{2r}{a_t^2} \, x + \frac{s}{a_t} \right) \mathrm{d}x \\
&= \int_0^{\infty} \mathrm{e}^{-ru^2 - su} \, (2ru + s) \, t \, (1 - F(a_t u)) \, \mathrm{d}u.
\end{aligned}$$

Applying Potter's theorem (see Theorem 1.7) and Lebesgue's dominated convergence theorem then leads to:

$$\lim_{t\to\infty} \delta_{\alpha,t}(r,s) = \int_0^{\infty} \mathrm{e}^{-ru^2 - su} \, (2ru + s) \, u^{-\alpha} \, \mathrm{d}u =: \delta_{\alpha}(r,s) \in [0,\infty).$$

For every $t > 0$, define $\varphi_t(\theta) := Q_t(\mathrm{e}^{-\theta/t}) = \mathbb{E}\big[\mathrm{e}^{-\theta\frac{N(t)}{t}}\big]$ for $\theta \geqslant 0$ so that $\lim_{t\to\infty} \varphi_t(\theta) = \mathbb{E}[\mathrm{e}^{-\theta\Lambda}] =: \varphi(\theta)$ by (1.11). Write the following triangular inequality:

$$|\varphi_t(\delta_{\alpha,t}(r,s)) - \varphi(\delta_{\alpha}(r,s))| \leqslant |\varphi_t(\delta_{\alpha,t}(r,s)) - \varphi(\delta_{\alpha,t}(r,s))| + |\varphi(\delta_{\alpha,t}(r,s)) - \varphi(\delta_{\alpha}(r,s))| \, .$$

On the one hand, $\lim_{t\to\infty} |\varphi(\delta_{\alpha,t}(r,s)) - \varphi(\delta_{\alpha}(r,s))| = 0$ by continuity of φ. On the other hand, for t large enough, there exist reals a, b with $0 \leqslant a \leqslant \delta_{\alpha}(r,s) < b$ such that $\delta_{\alpha,t}(r,s) \in [a,b]$. Then, $\lim_{t\to\infty} |\varphi_t(\delta_{\alpha,t}(r,s)) - \varphi(\delta_{\alpha,t}(r,s))| = 0$ if and only if $\lim_{t\to\infty} \sup_{\theta\in[a,b]} |\varphi_t(\theta) - \varphi(\theta)| = 0$. The latter is true since the convergence in (1.11) holds uniformly in every finite θ-interval (see Remark 1.22). As a consequence, we have $\lim_{t\to\infty} \varphi_t(\delta_{\alpha,t}(r,s)) = \varphi(\delta_{\alpha}(r,s))$, that is:

$$\lim_{t\to\infty} \mathbb{E}\Big[\mathrm{e}^{-r\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2 - s\frac{1}{a_t}\sum_{i=1}^{N(t)} X_i} \Big] = \mathbb{E}\big[\mathrm{e}^{-\delta_{\alpha}(r,s)\,\Lambda}\big] =: \psi_{\alpha,\Lambda}(r,s), \quad r \geqslant 0, \, s \geqslant 0.$$

By taking a non-random counting process $\{N(t);\, t \geqslant 0\}$, it follows that $\Lambda = 1$ a.s. and, since $\psi_{\alpha,1}$ is continuous on the right at $(0,0)'$ with value 1 there, the above limiting relation then yields:

$$\left(\frac{1}{a_n^2} \sum_{i=1}^{n} X_i^2, \, \frac{1}{a_n} \sum_{i=1}^{n} X_i \right)' \xrightarrow{\mathcal{D}} (U_{\alpha/2}, U_{\alpha})' \quad \text{as} \quad n \to \infty$$

where the random vector $(U_{\alpha/2}, U_{\alpha})'$ has the bivariate Laplace-Stieltjes transform $\psi_{\alpha,1}$. It follows in particular that the random variables $U_{\alpha/2}$ and U_{α} each have the Laplace-Stieltjes transform (7.2)

with $\gamma = \alpha/2$ for the former and $\gamma = \alpha$ for the latter, meaning that $U_{\alpha/2}$ is a.s. positive and $\alpha/2$-stable and that U_α is a.s. positive and α-stable.

It is obvious that Λ is independent of $U_{\alpha/2}$ and U_α. Using conditioning arguments, we then readily compute:

$$\mathbb{E}\big[\mathrm{e}^{-rU_{\alpha/2}\Lambda^{2/\alpha}-sU_\alpha\Lambda^{1/\alpha}}\big] = \psi_{\alpha,\Lambda}(r,s), \quad r \geqslant 0,\ s \geqslant 0.$$

Hence, since $\psi_{\alpha,\Lambda}$ is continuous on the right at $(0,0)'$ with value 1 there, we have proved that:

$$\left(\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2,\ \frac{1}{a_t}\sum_{i=1}^{N(t)} X_i\right)' \xrightarrow{\mathcal{D}} (U_{\alpha/2}\,\Lambda^{2/\alpha}, U_\alpha\,\Lambda^{1/\alpha})' \quad \text{as} \quad t \to \infty \tag{7.9}$$

where the random vector $(U_{\alpha/2}\,\Lambda^{2/\alpha}, U_\alpha\,\Lambda^{1/\alpha})'$ has the bivariate Laplace-Stieltjes transform $\psi_{\alpha,\Lambda}$.

The continuous mapping theorem (e.g., Corollary 5.1 of Billingsley [12]) finally gives:

$$T_{N(t)} = \frac{\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2}{\left(\frac{1}{a_t}\sum_{i=1}^{N(t)} X_i\right)^2} \xrightarrow{\mathcal{D}} \frac{U_{\alpha/2}}{U_\alpha^2} \quad \text{as} \quad t \to \infty$$

where, as specified above, the random vector $(U_{\alpha/2}, U_\alpha)'$ has the bivariate Laplace-Stieltjes transform $\psi_{\alpha,1}$. This concludes the proof. □

7.2.2 Case $\alpha = 1$, $\mu_1 = \infty$

Our next result deals with the case $\alpha = 1$ if $\mu_1 = \infty$.

Theorem 7.3. *Assume that X_1 has a distribution function F of Pareto-type with index $\alpha = 1$ satisfying $\mu_1 = \infty$. Assume that the counting process $\{N(t);\, t \geqslant 0\}$ $\mathcal{D}$-averages in time to the random variable Λ. Then:*

$$\left(\frac{a_t'}{a_t}\right)^2 T_{N(t)} \xrightarrow{\mathcal{D}} U_{1/2} \quad \textit{as} \quad t \to \infty$$

where $U_{1/2}$ is an a.s. positive and $1/2$-stable random variable with Laplace-Stieltjes transform (7.2) for $\gamma = \alpha/2$ and where the sequences $(a_t)_{t>0}$ and $(a_t')_{t>0}$ are defined by $\lim_{t\to\infty} t\, a_t^{-1}\ell(a_t) = 1$ and $\lim_{t\to\infty} t\, a_t'^{-1}\tilde{\ell}(a_t') = 1$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,\mathrm{d}u \in \mathrm{RV}_0$ respectively.

Proof. Let $1 - F(x) \sim x^{-1}\ell(x)$ as $x \to \infty$ for some $\ell \in \mathrm{RV}_0$ such that $\mu_1 = \infty$. Define a sequence $(a_t)_{t>0}$ by $1 - F(a_t) \sim 1/t$ as $t \to \infty$, i.e. $\lim_{t\to\infty} t\, a_t^{-1}\ell(a_t) = 1$, and a sequence $(a_t')_{t>0}$ by $\lim_{t\to\infty} t\, a_t'^{-1}\tilde{\ell}(a_t') = 1$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,\mathrm{d}u$. Note that $\tilde{\ell} \in \mathrm{RV}_0$ and that $\lim_{x\to\infty} \tilde{\ell}(x)/\ell(x) = \infty$ (e.g., Proposition 1.5.9a of Bingham et al. [13]).

Let $r \geqslant 0$ and $s \geqslant 0$ be fixed. We readily compute for $t > 0$:

$$\mathbb{E}\Big[\mathrm{e}^{-r\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2 - s\frac{1}{a_t}\sum_{i=1}^{N(t)} X_i}\Big] = Q_t\Big(\mathrm{e}^{-\frac{\delta_t(r,s)}{t}}\Big)$$

with $\delta_t(r,s) := -t \log \int_0^\infty \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{a'_t}}\,\mathrm{d}F(x) \in [0,\infty)$.

Lebesgue's dominated convergence theorem gives $\lim_{t\to\infty} \int_0^\infty \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{a'_t}}\,\mathrm{d}F(x) = 1$, so that:

$$\begin{aligned}
\delta_t(r,s) &\underset{t\to\infty}{\sim} -t\left(\int_0^\infty \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{a'_t}}\,\mathrm{d}F(x) - 1\right)\\
&= t\int_0^\infty \left(1 - \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{a'_t}}\right)\mathrm{d}F(x)\\
&= t\int_0^\infty (1-F(x))\,\mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{a'_t}}\left(\frac{2r}{a_t^2}\,x + \frac{s}{a'_t}\right)\mathrm{d}x\\
&= 2r\int_0^\infty y\,\mathrm{e}^{-ry^2 - sy\frac{a_t}{a'_t}}\,t\,(1-F(a_t y))\,\mathrm{d}y + \frac{s^2 t}{a'_t}\int_0^\infty \mathrm{e}^{-sy}\int_0^{a'_t y}(1-F(x))\,\mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2}\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Since $\lim_{x\to\infty} \ell(x)/\tilde{\ell}(x) = 0$, we obtain by using de Bruijn conjugate arguments (see Theorem 1.8) that $\lim_{t\to\infty} a_t/a'_t = 0$. Note however that $a_t/a'_t \in \mathrm{RV}_0$ since $a_t \in \mathrm{RV}_1$ and $a'_t \in \mathrm{RV}_1$.

Potter's theorem (see Theorem 1.7) and Lebesgue's dominated convergence theorem then lead to:

$$\lim_{t\to\infty} 2r\int_0^\infty y\,\mathrm{e}^{-ry^2 - sy\frac{a_t}{a'_t}}\,t\,(1-F(a_t y))\,\mathrm{d}y = 2r\int_0^\infty \mathrm{e}^{-ry^2}\mathrm{d}y = \sqrt{r\pi}.$$

Since $\mu_1 = \infty$, we have $\tilde{\ell}(x) \sim \int_0^x (1-F(u))\,\mathrm{d}u$ as $x\to\infty$. For any $y > 0$, we then obtain:

$$\int_0^{a'_t y}(1-F(x))\,\mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2}\mathrm{d}x \sim \int_0^{a'_t y}(1-F(x))\,\mathrm{d}x \sim \tilde{\ell}(a'_t y) \sim \tilde{\ell}(a'_t) \sim \frac{a'_t}{t} \quad \text{as} \quad t\to\infty$$

so that Lebesgue's dominated convergence theorem leads to:

$$\lim_{t\to\infty} \frac{s^2 t}{a'_t}\int_0^\infty \mathrm{e}^{-sy}\int_0^{a'_t y}(1-F(x))\,\mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2}\mathrm{d}x\,\mathrm{d}y = s.$$

It follows that $\lim_{t\to\infty} \delta_t(r,s) = \sqrt{r\pi} + s$. A similar argument as in the proof of Theorem 7.2 applied to the convergence of $Q_t\big(\mathrm{e}^{-\delta_t(r,s)/t}\big)$ then yields:

$$\lim_{t\to\infty} \mathbb{E}\Big[\mathrm{e}^{-r\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2 - s\frac{1}{a'_t}\sum_{i=1}^{N(t)} X_i}\Big] = \mathbb{E}\big[\mathrm{e}^{-\sqrt{r\pi}\,\Lambda - s\Lambda}\big] =: \psi_\Lambda(r,s), \quad r\geqslant 0,\ s\geqslant 0.$$

Repeating the proof of Theorem 7.2 with $s=0$ gives $\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2 \xrightarrow{\mathcal{D}} U_{1/2}\,\Lambda^2$ as $t\to\infty$ where $U_{1/2}$ is an a.s. positive and 1/2-stable random variable independent of Λ with Laplace-Stieltjes transform (7.2) for $\gamma = 1/2$. From this independence, we get by using conditioning arguments:

$$\mathbb{E}\big[\mathrm{e}^{-rU_{1/2}\,\Lambda^2 - s\Lambda}\big] = \psi_\Lambda(r,s), \quad r\geqslant 0,\ s\geqslant 0.$$

Since ψ_Λ is continuous on the right at $(0,0)'$ with value 1 there, it then follows that:

$$\left(\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2,\ \frac{1}{a'_t}\sum_{i=1}^{N(t)} X_i\right)' \xrightarrow{\mathcal{D}} (U_{1/2}\,\Lambda^2, \Lambda)' \quad \text{as} \quad t\to\infty \tag{7.10}$$

where the random vector $(U_{1/2}\,\Lambda^2, \Lambda)'$ has the bivariate Laplace-Stieltjes transform ψ_Λ.

The continuous mapping theorem finally gives:

$$\left(\frac{a'_t}{a_t}\right)^2 T_{N(t)} = \frac{\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2}{\left(\frac{1}{a'_t}\sum_{i=1}^{N(t)} X_i\right)^2} \xrightarrow{\mathcal{D}} U_{1/2} \quad \text{as} \quad t \to \infty$$

and the proof is complete. □

Remark 7.2. In Theorem 7.3, we have $(a'_t/a_t)^2 \in \mathrm{RV}_0$ with $a'_t/a_t \to \infty$ as $t \to \infty$.

7.2.3 Case $\alpha \in (1,2)$ or $\alpha = 1$, $\mu_1 < \infty$

In the following theorem, the case $\alpha \in (1,2)$ (including $\alpha = 1$ if $\mu_1 < \infty$) is treated. Two different results are proved according to the choice for the normalization of $T_{N(t)}$.

Theorem 7.4. *Assume that X_1 has a distribution function F of Pareto-type with index $\alpha \in (1,2)$ (including $\alpha = 1$ if $\mu_1 < \infty$) and that the counting process $\{N(t);\, t \geqslant 0\}$ $\mathcal{D}$-averages in time to the random variable Λ.*

(*a*) *Then:*

$$\left(\frac{N(t)}{a_t}\right)^2 T_{N(t)} \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2}\, U_{\alpha/2}\, \Lambda^{2/\alpha} \quad \textit{as} \quad t \to \infty.$$

(*b*) *Then:*

$$\left(\frac{t}{a_t}\right)^2 T_{N(t)} \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2}\, \frac{U_{\alpha/2}}{\Lambda^{2-2/\alpha}} \quad \textit{as} \quad t \to \infty.$$

In (*a*) *and* (*b*), *$U_{\alpha/2}$ is an a.s. positive and $\alpha/2$-stable random variable independent of Λ with Laplace-Stieltjes transform* (7.2) *for $\gamma = \alpha/2$. Moreover, the sequence $(a_t)_{t>0}$ is defined by* $\lim_{t\to\infty} t\, a_t^{-\alpha}\ell(a_t) = 1$.

Proof. Let $1 - F(x) \sim x^{-\alpha}\ell(x)$ as $x \to \infty$ for some $\ell \in \mathrm{RV}_0$ and $\alpha \in (1,2)$ or $\alpha = 1$ if $\mu_1 < \infty$. Define a sequence $(a_t)_{t>0}$ by $1 - F(a_t) \sim 1/t$ as $t \to \infty$, i.e. $\lim_{t\to\infty} t\, a_t^{-\alpha}\ell(a_t) = 1$. Notice that $a_t \in \mathrm{RV}_{1/\alpha}$.

(*a*) Since $\mu_1 < \infty$ and $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$, we get $\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i \xrightarrow{p} \mu_1$ as $t \to \infty$ by the law of large numbers and Lemma 7.1. Repeating the proof of Theorem 7.2 with $s = 0$ shows that $\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2 \xrightarrow{\mathcal{D}} U_{\alpha/2}\,\Lambda^{2/\alpha}$ as $t \to \infty$ where $U_{\alpha/2}$ is an a.s. positive and $\alpha/2$-stable random variable independent of Λ with Laplace-Stieltjes transform (7.2) for $\gamma = \alpha/2$. Slutsky's theorem (e.g., Corollary page 97 of Chung [19]) and the continuous mapping theorem then yield:

$$\left(\frac{N(t)}{a_t}\right)^2 T_{N(t)} = \frac{\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2}{\left(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i\right)^2} \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2}\, U_{\alpha/2}\, \Lambda^{2/\alpha} \quad \text{as} \quad t \to \infty.$$

(*b*) Let $r \geqslant 0$ and $s \geqslant 0$ be fixed. We readily compute for $t > 0$:

$$\mathbb{E}\Big[\mathrm{e}^{-r\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2 - s\frac{1}{t}\sum_{i=1}^{N(t)} X_i}\Big] = Q_t\Big(\mathrm{e}^{-\frac{\delta_{\alpha,t}(r,s)}{t}}\Big)$$

with $\delta_{\alpha,t}(r,s) := -t \log \int_0^\infty \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{t}} \,\mathrm{d}F(x) \in [0,\infty)$.

It is clear that $\lim_{t\to\infty} \int_0^\infty \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{t}} \,\mathrm{d}F(x) = 1$ by virtue of Lebesgue's dominated convergence theorem. It then follows:

$$\begin{aligned}
\delta_{\alpha,t}(r,s) &\underset{t\to\infty}{\sim} -t\left(\int_0^\infty \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{t}} \,\mathrm{d}F(x) - 1\right) \\
&= t\int_0^\infty \left(1 - \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{t}}\right) \mathrm{d}F(x) \\
&= t\int_0^\infty (1-F(x))\, \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{t}} \left(\frac{2r}{a_t^2}\, x + \frac{s}{t}\right) \mathrm{d}x \\
&= 2r\int_0^\infty y\, \mathrm{e}^{-ry^2 - sy\frac{a_t}{t}}\, t\,(1-F(a_t y))\,\mathrm{d}y + s\int_0^\infty (1-F(x))\, \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{t}} \,\mathrm{d}x.
\end{aligned}$$

Let us prove that $\lim_{t\to\infty} a_t/t = 0$. If $\alpha = 1$, we have $\ell(x) = o(1)$ as $x \to \infty$ since $\mu_1 < \infty$ so that $a_t/t \sim \ell(a_t) \to 0$ as $t \to \infty$. If $\alpha \in (1,2)$, we have $a_t/t \sim a_t^{1-\alpha}\ell(a_t) \to 0$ as $t \to \infty$ since $1-\alpha \in (-1,0)$. Now, applying Potter's theorem (see Theorem 1.7) and Lebesgue's dominated convergence theorem then leads to:

$$\lim_{t\to\infty} 2r\int_0^\infty y\, \mathrm{e}^{-ry^2 - sy\frac{a_t}{t}}\, t\,(1-F(a_t y))\,\mathrm{d}y = 2r\int_0^\infty y^{1-\alpha}\, \mathrm{e}^{-ry^2}\mathrm{d}y = r^{\alpha/2}\Gamma(1-\alpha/2).$$

Since $\mu_1 < \infty$, an application of Lebesgue's dominated convergence theorem gives:

$$\lim_{t\to\infty} s\int_0^\infty (1-F(x))\, \mathrm{e}^{-r\left(\frac{x}{a_t}\right)^2 - s\frac{x}{t}} \,\mathrm{d}x = s\mu_1.$$

It follows that $\lim_{t\to\infty} \delta_{\alpha,t}(r,s) = r^{\alpha/2}\Gamma(1-\alpha/2) + s\mu_1$. A similar argument as in the proof of Theorem 7.2 applied to the convergence of $Q_t\big(\mathrm{e}^{-\delta_{\alpha,t}(r,s)/t}\big)$ then yields:

$$\lim_{t\to\infty} \mathbb{E}\Big[\mathrm{e}^{-r\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2 - s\frac{1}{t}\sum_{i=1}^{N(t)} X_i}\Big] = \mathbb{E}\big[\mathrm{e}^{-r^{\alpha/2}\Gamma(1-\alpha/2)\,\Lambda - s\mu_1\Lambda}\big] =: \psi_{\alpha,\Lambda}(r,s), \quad r \geqslant 0,\, s \geqslant 0.$$

We know from (a) that $\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2 \xrightarrow{\mathcal{D}} U_{\alpha/2}\,\Lambda^{2/\alpha}$ as $t \to \infty$ where $U_{\alpha/2}$ is an a.s. positive and $\alpha/2$-stable random variable independent of Λ with Laplace-Stieltjes transform (7.2) for $\gamma = \alpha/2$. Due to this independence, we get by using conditioning arguments:

$$\mathbb{E}\big[\mathrm{e}^{-rU_{\alpha/2}\,\Lambda^{2/\alpha} - s\mu_1\Lambda}\big] = \psi_{\alpha,\Lambda}(r,s), \quad r \geqslant 0,\, s \geqslant 0.$$

Since $\psi_{\alpha,\Lambda}$ is continuous on the right at $(0,0)'$ with value 1 there, it then follows that:

$$\left(\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2,\, \frac{1}{t}\sum_{i=1}^{N(t)} X_i\right)' \xrightarrow{\mathcal{D}} (U_{\alpha/2}\,\Lambda^{2/\alpha}, \mu_1\Lambda)' \quad \text{as} \quad t \to \infty \tag{7.11}$$

where the random vector $(U_{\alpha/2}\,\Lambda^{2/\alpha}, \mu_1\Lambda)'$ has the bivariate Laplace-Stieltjes transform $\psi_{\alpha,\Lambda}$.

The continuous mapping theorem finally gives:

$$\left(\frac{t}{a_t}\right)^2 T_{N(t)} = \frac{\frac{1}{a_t^2}\sum_{i=1}^{N(t)} X_i^2}{\left(\frac{1}{t}\sum_{i=1}^{N(t)} X_i\right)^2} \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2}\,\frac{U_{\alpha/2}}{\Lambda^{2-2/\alpha}} \quad \text{as} \quad t \to \infty$$

and the proof is finished. □

Remark 7.3. In Theorem 7.4(*b*), we have $(t/a_t)^2 \in \mathrm{RV}_{2-2/\alpha}$. If $\alpha \in (1,2)$ then $t/a_t \to \infty$ as $t \to \infty$ since $2 - 2/\alpha \in (0,1)$. If $\alpha = 1$ and $\mu_1 < \infty$ then $(t/a_t)^2 \in \mathrm{RV}_0$ but $t/a_t \to \infty$ as $t \to \infty$ since $\ell(x) = o(1)$ as $x \to \infty$.

7.2.4 Case $\alpha = 2$, $\mu_2 = \infty$

We pass to the case $\alpha = 2$ with $\mu_2 = \infty$. As in the preceding theorem, two different results are derived according to the way in which $T_{N(t)}$ is normalized.

Theorem 7.5. *Assume that X_1 has a distribution function F of Pareto-type with index $\alpha = 2$ satisfying $\mu_2 = \infty$. Assume that the counting process $\{N(t);\, t \geqslant 0\}$ $\mathcal{D}$-averages in time to the random variable Λ.*

(*a*) *Then:*

$$\left(\frac{N(t)}{a'_t}\right)^2 T_{N(t)} \xrightarrow{\mathcal{D}} \frac{2}{\mu_1^2}\,\Lambda \quad as \quad t \to \infty.$$

(*b*) *Then:*

$$\left(\frac{t}{a'_t}\right)^2 T_{N(t)} \xrightarrow{\mathcal{D}} \frac{2}{\mu_1^2}\,\frac{1}{\Lambda} \quad as \quad t \to \infty.$$

In (*a*) *and* (*b*), *the sequence* $(a'_t)_{t>0}$ *is defined by* $\lim_{t\to\infty} t\, a_t'^{-2}\tilde{\ell}(a'_t) = 1$ *with* $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,\mathrm{d}u \in \mathrm{RV}_0$.

Proof. Let $1 - F(x) \sim x^{-2}\ell(x)$ as $x \to \infty$ for some $\ell \in \mathrm{RV}_0$ such that $\mu_2 = \infty$. Define a sequence $(a'_t)_{t>0}$ by $\lim_{t\to\infty} t\, a_t'^{-2}\tilde{\ell}(a'_t) = 1$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,\mathrm{d}u \in \mathrm{RV}_0$. Note that $a'_t \in \mathrm{RV}_{1/2}$.

(*b*) Let $r \geqslant 0$ and $s \geqslant 0$ be fixed. We readily compute for $t > 0$:

$$\mathbb{E}\Big[\mathrm{e}^{-r\frac{1}{a_t'^2}\sum_{i=1}^{N(t)} X_i^2 - s\frac{1}{t}\sum_{i=1}^{N(t)} X_i}\Big] = Q_t\Big(\mathrm{e}^{-\frac{\delta_t(r,s)}{t}}\Big)$$

with $\delta_t(r,s) := -t \log \int_0^\infty \mathrm{e}^{-r\left(\frac{x}{a'_t}\right)^2 - s\frac{x}{t}}\,\mathrm{d}F(x) \in [0,\infty)$.

Lebesgue's dominated convergence theorem gives $\lim_{t\to\infty} \int_0^\infty \mathrm{e}^{-r\left(\frac{x}{a'_t}\right)^2 - s\frac{x}{t}}\,\mathrm{d}F(x) = 1$, so that:

$$\begin{aligned}
\delta_t(r,s) &\underset{t\to\infty}{\sim} -t\left(\int_0^\infty \mathrm{e}^{-r\left(\frac{x}{a'_t}\right)^2 - s\frac{x}{t}}\,\mathrm{d}F(x) - 1\right)\\
&= t\int_0^\infty \left(1 - \mathrm{e}^{-r\left(\frac{x}{a'_t}\right)^2 - s\frac{x}{t}}\right)\mathrm{d}F(x)\\
&= t\int_0^\infty (1 - F(x))\,\mathrm{e}^{-r\left(\frac{x}{a'_t}\right)^2 - s\frac{x}{t}}\left(\frac{2r}{a_t'^2}\,x + \frac{s}{t}\right)\mathrm{d}x\\
&= \frac{2r^2 t}{a_t'^2}\int_0^\infty x\,(1 - F(x))\,\mathrm{e}^{-s\frac{x}{t}}\int_{\left(\frac{x}{a'_t}\right)^2}^\infty \mathrm{e}^{-ry}\,\mathrm{d}y\,\mathrm{d}x + s\int_0^\infty (1 - F(x))\,\mathrm{e}^{-r\left(\frac{x}{a'_t}\right)^2 - s\frac{x}{t}}\,\mathrm{d}x\\
&= \frac{2r^2 t}{a_t'^2}\int_0^\infty \mathrm{e}^{-ry}\int_0^{a'_t\sqrt{y}} x\,(1 - F(x))\,\mathrm{e}^{-s\frac{x}{t}}\,\mathrm{d}x\,\mathrm{d}y + s\int_0^\infty (1 - F(x))\,\mathrm{e}^{-r\left(\frac{x}{a'_t}\right)^2 - s\frac{x}{t}}\,\mathrm{d}x.
\end{aligned}$$

Since $\mu_2 = \infty$, we have $\tilde{\ell}(x) \sim \int_0^x u\,(1 - F(u))\,\mathrm{d}u$ as $x \to \infty$. For any $y > 0$, we then obtain:

$$\int_0^{a_t'\sqrt{y}} x\,(1 - F(x))\,\mathrm{e}^{-s\frac{x}{t}}\,\mathrm{d}x \sim \int_0^{a_t'\sqrt{y}} x\,(1 - F(x))\,\mathrm{d}x \sim \tilde{\ell}(a_t'\sqrt{y}) \sim \tilde{\ell}(a_t') \sim \frac{a_t'^2}{t} \quad \text{as} \quad t \to \infty$$

so that Lebesgue's dominated convergence theorem leads to:

$$\lim_{t\to\infty} \frac{2r^2 t}{a_t'^2} \int_0^\infty \mathrm{e}^{-ry} \int_0^{a_t'\sqrt{y}} x\,(1 - F(x))\,\mathrm{e}^{-s\frac{x}{t}}\,\mathrm{d}x\,\mathrm{d}y = 2r.$$

Since $\mu_1 < \infty$, we have by virtue of Lebesgue's dominated convergence theorem:

$$\lim_{t\to\infty} s \int_0^\infty (1 - F(x))\,\mathrm{e}^{-r\left(\frac{x}{a_t'}\right)^2 - s\frac{x}{t}}\,\mathrm{d}x = s\mu_1.$$

It follows that $\lim_{t\to\infty} \delta_t(r,s) = 2r + s\mu_1$. A similar argument as in the proof of Theorem 7.2 applied to the convergence of $Q_t\big(\mathrm{e}^{-\delta_t(r,s)/t}\big)$ then yields:

$$\lim_{t\to\infty} \mathbb{E}\Big[\mathrm{e}^{-r\frac{1}{a_t'^2}\sum_{i=1}^{N(t)} X_i^2 - s\frac{1}{t}\sum_{i=1}^{N(t)} X_i}\Big] = \mathbb{E}\big[\mathrm{e}^{-2r\Lambda - s\mu_1\Lambda}\big] := \psi_\Lambda(r,s), \quad r \geqslant 0,\ s \geqslant 0.$$

Since ψ_Λ is continuous on the right at $(0,0)'$ with value 1 there, the latter limiting relation is equivalent to:

$$\left(\frac{1}{a_t'^2}\sum_{i=1}^{N(t)} X_i^2,\ \frac{1}{t}\sum_{i=1}^{N(t)} X_i\right)' \xrightarrow{\mathcal{D}} (2\Lambda, \mu_1\Lambda)' \quad \text{as} \quad t \to \infty. \tag{7.12}$$

The continuous mapping theorem finally gives:

$$\left(\frac{t}{a_t'}\right)^2 T_{N(t)} = \frac{\frac{1}{a_t'^2}\sum_{i=1}^{N(t)} X_i^2}{\left(\frac{1}{t}\sum_{i=1}^{N(t)} X_i\right)^2} \xrightarrow{\mathcal{D}} \frac{2}{\mu_1^2}\,\frac{1}{\Lambda} \quad \text{as} \quad t \to \infty.$$

(a) Since $\mu_1 < \infty$ and $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$, we get $\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i \xrightarrow{p} \mu_1$ as $t \to \infty$ by the law of large numbers and Lemma 7.1. Moreover, it follows from (7.12) that $\frac{1}{a_t'^2}\sum_{i=1}^{N(t)} X_i^2 \xrightarrow{\mathcal{D}} 2\Lambda$ as $t \to \infty$. Slutsky's theorem and the continuous mapping theorem then give:

$$\left(\frac{N(t)}{a_t'}\right)^2 T_{N(t)} = \frac{\frac{1}{a_t'^2}\sum_{i=1}^{N(t)} X_i^2}{\left(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i\right)^2} \xrightarrow{\mathcal{D}} \frac{2}{\mu_1^2}\,\Lambda \quad \text{as} \quad t \to \infty$$

and this ends the proof. □

Remark 7.4. In Theorem 7.5(b), we have $(t/a_t')^2 \in \mathrm{RV}_1$ so that $t/a_t' \to \infty$ as $t \to \infty$.

Remark 7.5. Note from Theorems 7.3, 7.4(b) and 7.5(b) that for all $\alpha \in [1,2]$, the sequences of normalizing constants that ensure nonzero limit distributions for $T_{N(t)}$ are regularly varying with index $2 - 2/\alpha$ which increases with α.

When the distribution function of X_1 is of Pareto-type with index $\alpha > 2$, we have $\mu_2 < \infty$ so that $N(t)\,T_{N(t)} \xrightarrow{p} \mu_2/\mu_1^2$ as $t \to \infty$ by the law of large numbers and Lemma 7.1 since $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$. In the sequel, we then derive second-order weak convergence results.

7.2.5 Case $\alpha \in (2,4)$ or $\alpha = 2$, $\mu_2 < \infty$

The following theorem concerns the case $\alpha \in (2,4)$ (including $\alpha = 2$ if $\mu_2 < \infty$) where it is assumed that the counting process p-averages in time.

Theorem 7.6. *Assume that X_1 has a distribution function F of Pareto-type with index $\alpha \in (2,4)$ (including $\alpha = 2$ if $\mu_2 < \infty$) and that the counting process $\{N(t);\, t \geqslant 0\}$ p-averages in time to the random variable Λ. Then:*

$$\frac{t^{1-2/\alpha}}{\ell_1^{\#}(t^{2/\alpha})}\left(N(t)\,T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2}\,\frac{W_{\alpha/2}}{\Lambda^{1-2/\alpha}} \quad \text{as} \quad t \to \infty$$

where $W_{\alpha/2}$ is an $\alpha/2$-stable random variable independent of Λ and where $\ell_1^{\#} \in \mathrm{RV}_0$ is a de Bruijn conjugate of $\ell_1(x) := \ell^{-2/\alpha}(\sqrt{x}) \in \mathrm{RV}_0$.

Proof. Let $1 - F(x) \sim x^{-\alpha}\ell(x)$ as $x \to \infty$ for some $\ell \in \mathrm{RV}_0$ and $\alpha \in (2,4)$ or $\alpha = 2$ if $\mu_2 < \infty$. For a sequence $(b_t)_{t>0}$ to be defined, consider the following identity:

$$b_t\left(N(t)\,T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) = \underbrace{b_t\left(\frac{N(t)}{X_1 + \cdots + X_{N(t)}}\right)^2\left(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i^2 - \mu_2\right)}_{=:A_t} + \underbrace{b_t\mu_2\left(\frac{N(t)}{X_1 + \cdots + X_{N(t)}}\right)^2 - \frac{b_t\mu_2}{\mu_1^2}}_{=:B_t}.$$

Since $\mathbb{P}[X_1^2 > x] = \mathbb{P}[X_1 > \sqrt{x}] \sim x^{-\alpha/2}\ell(\sqrt{x})$ as $x \to \infty$, the tail of the distribution function of X_1^2 is regularly varying with index $-\alpha/2$. Hence, we get $c_n^{-1}\big(\sum_{i=1}^n X_i^2 - n\mu_2\big) \xrightarrow{\mathcal{D}} W_{\alpha/2}$ as $n \to \infty$ for a stable random variable $W_{\alpha/2}$ with exponent $\alpha/2$ and a sequence of normalizing constants $(c_n)_{n\in\mathbb{N}}$ defined by $1 - F(\sqrt{c_n}) \sim 1/n$ as $n \to \infty$. Putting $\ell_1(x) := \ell^{-2/\alpha}(\sqrt{x}) \in \mathrm{RV}_0$, the latter is equivalent to $c_n\ell_1(c_n) \sim n^{2/\alpha}$ as $n \to \infty$ yielding $c_n \sim n^{2/\alpha}\ell_1^{\#}(n^{2/\alpha})$ as $n \to \infty$ where $\ell_1^{\#} \in \mathrm{RV}_0$ is a de Bruijn conjugate of ℓ_1 (see Theorem 1.8). Since $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$, it follows from Lemma 7.1 that:

$$\frac{N(t)^{1-2/\alpha}}{\ell_1^{\#}(N(t)^{2/\alpha})}\left(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i^2 - \mu_2\right) \xrightarrow{\mathcal{D}} W_{\alpha/2} \quad \text{as} \quad t \to \infty. \tag{7.13}$$

Similarly, since the central limit theorem gives $n^{-1/2}(\sum_{i=1}^n X_i - n\mu_1) \xrightarrow{\mathcal{D}} \mathrm{N}(0,\sigma^2)$ as $n \to \infty$ for a normal random variable $\mathrm{N}(0,\sigma^2)$ with mean 0 and variance $\sigma^2 := \mathbb{V}[X_1] < \infty$, we also get:

$$\sqrt{N(t)}\left(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i - \mu_1\right) \xrightarrow{\mathcal{D}} \mathrm{N}(0,\sigma^2) \quad \text{as} \quad t \to \infty. \tag{7.14}$$

Define the sequence $(b_t)_{t>0}$ by $b_t := \frac{t^{1-2/\alpha}}{\ell_1^{\#}(t^{2/\alpha})}$. This leads for any $t > 0$ to:

$$A_t = \frac{\ell_1^{\#}(N(t)^{2/\alpha})}{\ell_1^{\#}(t^{2/\alpha})}\left(\frac{t}{N(t)}\right)^{1-2/\alpha}\left(\frac{N(t)}{X_1 + \cdots + X_{N(t)}}\right)^2 \frac{N(t)^{1-2/\alpha}}{\ell_1^{\#}(N(t)^{2/\alpha})}\left(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i^2 - \mu_2\right).$$

Since $\ell_1^{\#} \in \mathrm{RV}_0$ and $\frac{N(t)}{t} \xrightarrow{p} \Lambda$ as $t \to \infty$ with $\Lambda > 0$ a.s., we get $\frac{\ell_1^{\#}(N(t)^{2/\alpha})}{\ell_1^{\#}(t^{2/\alpha})} \xrightarrow{p} 1$ as $t \to \infty$ by virtue of the uniform convergence theorem for slowly varying functions (see Theorem 1.6(i)),

the continuous mapping theorem and the subsequence principle. Since $\mu_1 < \infty$ and $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$, it follows from the law of large numbers and Lemma 7.1 that $\frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i \xrightarrow{p} \mu_1$ as $t \to \infty$. Recalling (7.13), Slutsky's theorem and the continuous mapping theorem therefore give:

$$A_t \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2} \frac{W_{\alpha/2}}{\Lambda^{1-2/\alpha}} \quad \text{as} \quad t \to \infty$$

due to the independence of $W_{\alpha/2}$ and Λ that is proved as follows. Successively, condition on $N(t)$, use that $\frac{N(t)}{t}$ is $\sigma(N(t))$-measurable, use the independence of $\{N(t);\, t \geqslant 0\}$ and $\{X_i;\, i \in \mathbb{N}\}$ and finally apply (7.13) to get:

$$\begin{aligned}
Y_t &:= \mathbb{P}\left[\frac{N(t)}{t} \leqslant x, \frac{N(t)^{1-2/\alpha}}{\ell_1^{\#}(N(t)^{2/\alpha})} \left(\frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i^2 - \mu_2 \right) \leqslant y \middle| N(t) \right] \\
&= \mathbb{E}\left[\mathbb{1}_{\left\{ \frac{N(t)}{t} \leqslant x \right\}} \mathbb{1}_{\left\{ \frac{N(t)^{1-2/\alpha}}{\ell_1^{\#}(N(t)^{2/\alpha})} \left(\frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i^2 - \mu_2 \right) \leqslant y \right\}} \middle| N(t) \right] \\
&= \mathbb{1}_{\left\{ \frac{N(t)}{t} \leqslant x \right\}} \mathbb{P}\left[\frac{N(t)^{1-2/\alpha}}{\ell_1^{\#}(N(t)^{2/\alpha})} \left(\frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i^2 - \mu_2 \right) \leqslant y \middle| N(t) \right] \\
&= \mathbb{1}_{\left\{ \frac{N(t)}{t} \leqslant x \right\}} \mathbb{P}\left[\frac{N(t)^{1-2/\alpha}}{\ell_1^{\#}(N(t)^{2/\alpha})} \left(\frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i^2 - \mu_2 \right) \leqslant y \right] \\
&\xrightarrow{\mathcal{D}} \mathbb{1}_{\{\Lambda \leqslant x\}} \mathbb{P}\big[W_{\alpha/2} \leqslant y\big] \quad \text{as} \quad t \to \infty
\end{aligned}$$

at any continuity points x of the distribution function of Λ and y of that of $W_{\alpha/2}$. The sequence of random variables $\{Y_t;\, t > 0\}$ being uniformly integrable, we apply Theorem 5.4 of Billingsley [12] to obtain:

$$\lim_{t \to \infty} \mathbb{P}\left[\frac{N(t)}{t} \leqslant x, \frac{N(t)^{1-2/\alpha}}{\ell_1^{\#}(N(t)^{2/\alpha})} \left(\frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i^2 - \mu_2 \right) \leqslant y \right] = \mathbb{P}[\Lambda \leqslant x]\, \mathbb{P}[W_{\alpha/2} \leqslant y]$$

from which the independence of $W_{\alpha/2}$ and Λ follows.

At the same time, we also have for any $t > 0$:

$$B_t = -\frac{\mu_2}{\mu_1} \left(\frac{N(t)}{X_1 + \cdots + X_{N(t)}} \right)^2 \sqrt{\frac{t}{N(t)}} \frac{t^{1/2-2/\alpha}}{\ell_1^{\#}(t^{2/\alpha})} \sqrt{N(t)} \left(\frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i - \mu_1 \right) \left(1 + \frac{1}{\mu_1} \frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i \right).$$

Since (7.14) holds and $\lim_{t \to \infty} \frac{t^{1/2-2/\alpha}}{\ell_1^{\#}(t^{2/\alpha})} = 0$, Slutsky's theorem, the continuous mapping theorem and the independence of $\mathrm{N}(0, \sigma^2)$ and Λ yield that $B_t \xrightarrow{p} 0$ as $t \to \infty$. Note that the independence of $\mathrm{N}(0, \sigma^2)$ and Λ is easily proved using the same kind of arguments as above for the independence of $W_{\alpha/2}$ and Λ.

Hence, another application of Slutsky's theorem leads to:

$$\frac{t^{1-2/\alpha}}{\ell_1^{\#}(t^{2/\alpha})} \left(N(t)\, T_{N(t)} - \frac{\mu_2}{\mu_1^2} \right) = A_t + B_t \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2} \frac{W_{\alpha/2}}{\Lambda^{1-2/\alpha}} \quad \text{as} \quad t \to \infty$$

and the proof is complete. □

Remark 7.6. In Theorem 7.6, we clearly have $b_t := \frac{t^{1-2/\alpha}}{\ell_1^{\#}(t^{2/\alpha})} \in \mathrm{RV}_{1-2/\alpha}$. If $\alpha \in (2,4)$ then $b_t \to \infty$ as $t \to \infty$ since $1 - 2/\alpha \in (0, 1/2)$. If $\alpha = 2$ and $\mu_2 < \infty$ then $b_t \in \mathrm{RV}_0$ but $b_t = 1/\ell_1^{\#}(t) \to \infty$ as $t \to \infty$ since $\ell_1^{\#}(x) = o(1)$ as $x \to \infty$ because of $\ell(x) = o(1)$ as $x \to \infty$.

7.2.6 Case $\alpha = 4$, $\mu_4 = \infty$

We now treat the case $\alpha = 4$ with $\mu_4 = \infty$, again assuming that the counting process p-averages in time. The result is proved using a procedure akin to that of Theorem 7.6.

Theorem 7.7. *Assume that X_1 has a distribution function F of Pareto-type with index $\alpha = 4$ satisfying $\mu_4 = \infty$. Assume that the counting process $\{N(t);\, t \geqslant 0\}$ p-averages in time to the random variable Λ. Then:*

$$\frac{\sqrt{t}}{\ell_2^{\#}\big(\sqrt{t}\big)} \left(N(t)\, T_{N(t)} - \frac{\mu_2}{\mu_1^2} \right) \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2} \frac{\mathrm{N}(0,1)}{\sqrt{\Lambda}} \quad \textit{as} \quad t \to \infty$$

where $\mathrm{N}(0,1)$ is a standard normal random variable independent of Λ and where $\ell_2^{\#} \in \mathrm{RV}_0$ is a de Bruijn conjugate of $\ell_2(x) := \frac{1}{2\sqrt{\tilde{\ell}(\sqrt{x})}} \in \mathrm{RV}_0$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\, \mathrm{d}u \in \mathrm{RV}_0$.

Proof. Let $1 - F(x) \sim x^{-4}\ell(x)$ as $x \to \infty$ for some $\ell \in \mathrm{RV}_0$ such that $\mu_4 = \infty$. For a sequence $(d_t)_{t>0}$ to be defined, consider the following identity:

$$d_t \left(N(t)\, T_{N(t)} - \frac{\mu_2}{\mu_1^2} \right) =: A_t + B_t$$

where the random variables A_t and B_t are defined as in the proof of Theorem 7.6 but with b_t replaced by d_t.

Let $V(x) := \int_0^x y^2\, \mathrm{d}\mathbb{P}[X_1^2 \leqslant y]$ for $x > 0$ denote the truncated second moment function of X_1^2. Obviously, $\lim_{x\to\infty} V(x) = \infty$ since $\mu_4 = \infty$. Moreover, $V \in \mathrm{RV}_0$ since $V(x) \sim 4\,\tilde{\ell}(\sqrt{x})$ as $x \to \infty$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\, \mathrm{d}u \in \mathrm{RV}_0$. Indeed, integrating by parts leads to:

$$\begin{aligned} V(x) &= -\int_0^x y^2\, \mathrm{d}\mathbb{P}[X_1^2 > y] \\ &= -x^2\, \mathbb{P}[X_1 > \sqrt{x}] + 2\int_0^x y\, \mathbb{P}[X_1 > \sqrt{y}]\, \mathrm{d}y \\ &= -x^2\, \mathbb{P}[X_1 > \sqrt{x}] + 4\int_0^{\sqrt{x}} u^3\, \mathbb{P}[X_1 > u]\, \mathrm{d}u \\ &\sim 4\,\tilde{\ell}\big(\sqrt{x}\big) \quad \text{as} \quad x \to \infty \end{aligned}$$

since $x^2\, \mathbb{P}[X_1 > \sqrt{x}] \sim \ell(\sqrt{x}) = o(\tilde{\ell}(\sqrt{x}))$ as $x \to \infty$ (e.g., Proposition 1.5.9a of Bingham et al. [13]) and $\int_0^{\sqrt{x}} u^3\, \mathbb{P}[X_1 > u]\, \mathrm{d}u \sim \tilde{\ell}(\sqrt{x})$ as $x \to \infty$ due to $u^3\, \mathbb{P}[X_1 > u] \sim \ell(u)/u$ as $u \to \infty$ and $\mu_4 = \infty$.

Since $V \in \mathrm{RV}_0$, we apply Theorem XVII.5.2 of Feller [25] to get $c_n'^{-1}\big(\sum_{i=1}^n X_i^2 - n\mu_2\big) \xrightarrow{\mathcal{D}} \mathrm{N}(0,1)$ as $n \to \infty$ for a standard normal random variable $\mathrm{N}(0,1)$ and a sequence of normalizing constants $(c_n')_{n\in\mathbb{N}}$ defined by $c_n'^{-2} V(c_n') \sim 1/n$ as $n \to \infty$. Putting $\ell_2(x) := \frac{1}{2\sqrt{\tilde{\ell}(\sqrt{x})}} \in \mathrm{RV}_0$, the latter is equivalent to $c_n' \ell_2(c_n') \sim \sqrt{n}$ as $n \to \infty$ yielding $c_n' \sim \sqrt{n}\, \ell_2^{\#}(\sqrt{n})$ as $n \to \infty$ where $\ell_2^{\#} \in \mathrm{RV}_0$

is a de Bruijn conjugate of ℓ_2 (see Theorem 1.8). Since $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$, it follows from Lemma 7.1 that:

$$\frac{\sqrt{N(t)}}{\ell_2^{\#}\big(\sqrt{N(t)}\big)} \left(\frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i^2 - \mu_2 \right) \xrightarrow{\mathcal{D}} \mathrm{N}(0,1) \quad \text{as} \quad t \to \infty. \tag{7.15}$$

Define the sequence $(d_t)_{t>0}$ by $d_t := \frac{\sqrt{t}}{\ell_2^{\#}(\sqrt{t})}$. This leads for any $t > 0$ to:

$$A_t = \frac{\ell_2^{\#}\big(\sqrt{N(t)}\big)}{\ell_2^{\#}\big(\sqrt{t}\big)} \sqrt{\frac{t}{N(t)}} \left(\frac{N(t)}{X_1 + \cdots + X_{N(t)}} \right)^2 \frac{\sqrt{N(t)}}{\ell_2^{\#}\big(\sqrt{N(t)}\big)} \left(\frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i^2 - \mu_2 \right).$$

Since $\ell_2^{\#} \in \mathrm{RV}_0$ and $\frac{N(t)}{t} \xrightarrow{p} \Lambda$ as $t \to \infty$ with $\Lambda > 0$ a.s., we have $\frac{\ell_2^{\#}(\sqrt{N(t)})}{\ell_2^{\#}(\sqrt{t})} \xrightarrow{p} 1$ as $t \to \infty$. Since $\mu_1 < \infty$ and $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$, it follows from the law of large numbers and Lemma 7.1 that $\frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i \xrightarrow{p} \mu_1$ as $t \to \infty$. Using (7.15), Slutsky's theorem and the continuous mapping theorem then give:

$$A_t \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2} \frac{\mathrm{N}(0,1)}{\sqrt{\Lambda}} \quad \text{as} \quad t \to \infty$$

thanks to the independence of $\mathrm{N}(0,1)$ and Λ. The latter claim is easily proved in the same way as for the independence of $W_{\alpha/2}$ and Λ in the proof of Theorem 7.6.

At the same time, we also have for any $t > 0$:

$$B_t = -\frac{\mu_2}{\mu_1} \left(\frac{N(t)}{X_1 + \cdots + X_{N(t)}} \right)^2 \sqrt{\frac{t}{N(t)}} \, \frac{1}{\ell_2^{\#}\big(\sqrt{t}\big)} \sqrt{N(t)} \left(\frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i - \mu_1 \right) \left(1 + \frac{1}{\mu_1} \frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i \right).$$

Since $\lim_{x\to\infty} V(x) = \infty$, we have $\lim_{x\to\infty} \tilde{\ell}(x) = \infty$ so that $\lim_{x\to\infty} \ell_2(x) = 0$. It then follows that $\lim_{x\to\infty} \ell_2^{\#}(x) = \infty$. Using (7.14) and the independence of $\mathrm{N}(0,\sigma^2)$ and Λ, Slutsky's theorem and the continuous mapping theorem then yield that $B_t \xrightarrow{p} 0$ as $t \to \infty$.

Finally, another application of Slutsky's theorem gives:

$$\frac{\sqrt{t}}{\ell_2^{\#}\big(\sqrt{t}\big)} \left(N(t)\, T_{N(t)} - \frac{\mu_2}{\mu_1^2} \right) = A_t + B_t \xrightarrow{\mathcal{D}} \frac{1}{\mu_1^2} \frac{\mathrm{N}(0,1)}{\sqrt{\Lambda}} \quad \text{as} \quad t \to \infty$$

and the proof is finished. □

Remark 7.7. In Theorem 7.7, we have $\frac{\sqrt{t}}{\ell_2^{\#}(\sqrt{t})} \to \infty$ as $t \to \infty$ since $\frac{\sqrt{t}}{\ell_2^{\#}(\sqrt{t})} \in \mathrm{RV}_{1/2}$. Moreover, the slowly varying function $\ell_2^{\#}$ satisfies $\ell_2^{\#}(x) \to \infty$ as $x \to \infty$.

7.2.7 Case $\mu_4 < \infty$

Finally, the case $\mu_4 < \infty$ is under consideration, covering the remaining α-cases. Indeed, the result holds in particular if $\alpha = 4$ with $\mu_4 < \infty$ or if $\alpha > 4$ (since $\mu_4 < \infty$ in both cases). It turns out that $N(t)\, T_{N(t)}$ is asymptotically normal (written AN).

Theorem 7.8. *Assume that X_1 has a distribution function F satisfying $\mu_4 < \infty$ and that the counting process $\{N(t);\, t \geqslant 0\}$ $\mathcal{D}$-averages in time to the random variable Λ. Then:*

$$\sqrt{t}\left(N(t)\, T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) \xrightarrow{\mathcal{D}} \frac{\mathrm{N}(0, \sigma_*^2)}{\sqrt{\Lambda}} \quad as \quad t \to \infty$$

where $\mathrm{N}(0, \sigma_*^2)$ is a normal random variable independent of Λ with mean 0 and variance σ_*^2 given by:

$$\sigma_*^2 := \frac{\mu_4}{\mu_1^4} - \left(\frac{\mu_2}{\mu_1^2}\right)^2 + 4\left(\frac{\mu_2}{\mu_1^2}\right)^3 - \frac{4\mu_2\mu_3}{\mu_1^5}. \tag{7.16}$$

Proof. Let F be such that $\mu_4 < \infty$. From the bivariate Lindeberg-Lévy central limit theorem (e.g., Theorem 1.9.1B of Serfling [59]), one deduces that:

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{Y_n} - \boldsymbol{\mu}\right) \xrightarrow{\mathcal{D}} \mathrm{N}(\mathbf{0}, \Sigma) \quad \text{as} \quad n \to \infty \tag{7.17}$$

where $\boldsymbol{Y_n} := \left(X_i, X_i^2\right)'$, $\boldsymbol{\mu} := (\mu_1, \mu_2)'$ and $\mathrm{N}(\mathbf{0}, \Sigma)$ is a bivariate normal random vector with mean $\mathbf{0} := (0, 0)'$ and covariance matrix Σ defined by:

$$\Sigma := \begin{pmatrix} \mu_2 - \mu_1^2 & \mu_3 - \mu_1\mu_2 \\ \mu_3 - \mu_1\mu_2 & \mu_4 - \mu_2^2 \end{pmatrix}.$$

Following the notation in Serfling [59], we write (7.17) as $\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{Y_n}$ is $\mathrm{AN}\big(\boldsymbol{\mu}, n^{-1}\Sigma\big)$.

By virtue of the multivariate delta method, the asymptotic normality carries over to the random variable $g\big(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{Y_n}\big) = g\big(\frac{1}{n}\sum_{i=1}^{n} X_i, \frac{1}{n}\sum_{i=1}^{n} X_i^2\big)$ for any function $g\colon (0,\infty)\times(0,\infty) \to \mathbb{R}$ that is continuously differentiable in a neighborhood of $\boldsymbol{\mu}$, so that $g\big(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{Y_n}\big)$ is $\mathrm{AN}\big(g(\boldsymbol{\mu}), n^{-1} D\Sigma D'\big)$ with $D := \left(\frac{\partial g}{\partial x}(\boldsymbol{\mu}), \frac{\partial g}{\partial y}(\boldsymbol{\mu})\right)$.

With the choice $g(x, y) = y/x^2$, we find that $n\, T_n$ is $\mathrm{AN}\left(\frac{\mu_2}{\mu_1^2}, \frac{\sigma_*^2}{n}\right)$ with σ_*^2 given by (7.16). Since $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$, it consequently follows from Lemma 7.1 that:

$$\sqrt{N(t)}\left(N(t)\, T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) \xrightarrow{\mathcal{D}} \mathrm{N}(0, \sigma_*^2) \quad \text{as} \quad t \to \infty$$

where $\mathrm{N}(0, \sigma_*^2)$ is a normal random variable with mean 0 and variance σ_*^2.

The continuous mapping theorem together with the independence of $\mathrm{N}(0, \sigma_*^2)$ and Λ (proved using the same arguments as for the independence of $W_{\alpha/2}$ and Λ in the proof of Theorem 7.6) finally gives:

$$\sqrt{t}\left(N(t)\, T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) = \sqrt{\frac{t}{N(t)}}\, \sqrt{N(t)}\left(N(t)\, T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right) \xrightarrow{\mathcal{D}} \frac{\mathrm{N}(0, \sigma_*^2)}{\sqrt{\Lambda}} \quad \text{as} \quad t \to \infty.$$

This completes the proof. □

7.3 Applications to Risk Measures

Let X be an a.s. positive random variable with distribution function F and $\{N(t);\, t \geqslant 0\}$ be a counting process independent of X. Let $X_1, \ldots, X_{N(t)}$ be a random sample from F of random size $N(t)$. Armed with the distributional results derived in Section 7.2, we use the relations (7.4) and (7.7) to study the asymptotic behavior of two popular risk measures through their limiting distributions. Subsection 7.3.1 deals with the sample coefficient of variation $\widehat{\mathrm{CoVar}(X)}$ defined in (7.3) and Subsection 7.3.2 concerns the sample dispersion $\widehat{\mathrm{D}(X)}$ defined in (7.5). Our results are established under the same assumptions on the distribution function F of X and on the counting process $\{N(t);\, t \geqslant 0\}$ as in Section 7.2.

7.3.1 Sample Coefficient of Variation

We derive limit distributions for the appropriately normalized random variable $\widehat{\mathrm{CoVar}(X)}$ by using the distributional results derived in Section 7.2 for $T_{N(t)}$ and by means of (7.4). Consequently, different cases will arise according to the range of α and on the (non)finiteness of the first few moments. In Cases 1-6, we assume that the distribution function F of X is of Pareto-type with index $\alpha > 0$, i.e. satisfies (7.1) for some $\ell \in \mathrm{RV}_0$. In Case 7, we assume that the distribution function F of X satisfies $\mu_4 < \infty$. Moreover, the counting process $\{N(t);\, t \geqslant 0\}$ is supposed to $\mathcal{D}$-average in time to the random variable Λ except in Cases 5-6 where it p-averages in time to Λ.

Case 1: $\alpha \in (0,1)$**.** Since $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$, it follows from Theorem 7.2, Slutsky's theorem and the continuous mapping theorem that:

$$\frac{\widehat{\mathrm{CoVar}(X)}}{\sqrt{N(t)}} = \sqrt{T_{N(t)} - \frac{1}{N(t)}} \xrightarrow{\mathcal{D}} \frac{\sqrt{U_{\alpha/2}}}{U_\alpha} \quad \text{as} \quad t \to \infty$$

where the distribution of the random vector $(U_{\alpha/2}, U_\alpha)'$ is determined by (7.8).

Case 2: $\alpha = 1$**,** $\mu_1 = \infty$**.** Define $(a_t)_{t>0}$ and $(a'_t)_{t>0}$ respectively by $\lim_{t\to\infty} t\, a_t^{-1} \ell(a_t) = 1$ and $\lim_{t\to\infty} t\, a_t'^{-1} \tilde{\ell}(a'_t) = 1$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\, \mathrm{d}u \in \mathrm{RV}_0$. Since $a'_t/a_t \sim \frac{(1/\tilde{\ell})^{\#}(t)}{(1/\ell)^{\#}(t)}$ as $t \to \infty$, where $(1/\tilde{\ell})^{\#} \in \mathrm{RV}_0$ and $(1/\ell)^{\#} \in \mathrm{RV}_0$ are de Bruijn conjugates of $1/\tilde{\ell} \in \mathrm{RV}_0$ and $1/\ell \in \mathrm{RV}_0$ respectively, it follows that $a'_t/a_t \in \mathrm{RV}_0$ and then $\lim_{t\to\infty} \frac{1}{t}\left(\frac{a'_t}{a_t}\right)^2 = 0$. Moreover, $\frac{N(t)}{t} \xrightarrow{\mathcal{D}} \Lambda$ as $t \to \infty$. Hence, Theorem 7.3 together with Slutsky's theorem and the continuous mapping theorem gives:

$$\frac{a'_t}{a_t} \frac{\widehat{\mathrm{CoVar}(X)}}{\sqrt{N(t)}} = \sqrt{\left(\frac{a'_t}{a_t}\right)^2 T_{N(t)} - \frac{1}{t}\left(\frac{a'_t}{a_t}\right)^2 \frac{t}{N(t)}} \xrightarrow{\mathcal{D}} \sqrt{U_{1/2}} \quad \text{as} \quad t \to \infty$$

where the distribution of the random variable $U_{1/2}$ is determined by (7.2) with $\gamma = 1/2$.

Case 3: $\alpha \in (1,2)$ **or** $\alpha = 1$**,** $\mu_1 < \infty$**.** Define $(a_t)_{t>0}$ by $\lim_{t\to\infty} t\, a_t^{-\alpha} \ell(a_t) = 1$. Since $t/a_t^2 \sim a_t^{\alpha-2}/\ell(a_t) \to 0$ and $\frac{N(t)}{t} \xrightarrow{\mathcal{D}} \Lambda$ as $t \to \infty$, it follows from Theorem 7.4(a), Slutsky's theorem and the continuous mapping theorem that:

$$\frac{\sqrt{N(t)}}{a_t} \widehat{\mathrm{CoVar}(X)} = \sqrt{\left(\frac{N(t)}{a_t}\right)^2 T_{N(t)} - \frac{t}{a_t^2}\frac{N(t)}{t}} \xrightarrow{\mathcal{D}} \frac{1}{\mu_1} \sqrt{U_{\alpha/2}}\, \Lambda^{1/\alpha} \quad \text{as} \quad t \to \infty$$

where the random variable $U_{\alpha/2}$ is independent of Λ and has a distribution determined by (7.2) with $\gamma = \alpha/2$.

Repeating the same arguments as above but using Theorem 7.4(b) instead of Theorem 7.4(a), we also get:

$$\frac{t}{a_t}\,\frac{\widehat{\mathrm{CoVar}(X)}}{\sqrt{N(t)}} = \sqrt{\left(\frac{t}{a_t}\right)^2 T_{N(t)} - \frac{t}{a_t^2}\,\frac{t}{N(t)}} \xrightarrow{\mathcal{D}} \frac{1}{\mu_1}\,\frac{\sqrt{U_{\alpha/2}}}{\Lambda^{1-1/\alpha}} \quad \text{as} \quad t \to \infty.$$

Case 4: $\alpha = 2$, $\mu_2 = \infty$. Define $(a'_t)_{t>0}$ by $\lim_{t\to\infty} t\, a_t'^{-2}\tilde{\ell}(a'_t) = 1$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,\mathrm{d}u \in \mathrm{RV}_0$. From $\mu_2 = \infty$, it follows that $\lim_{x\to\infty} \tilde{\ell}(x) = \infty$ so that $t/a_t'^2 \sim 1/\tilde{\ell}(a'_t) \to 0$ as $t \to \infty$. Moreover, $\frac{N(t)}{t} \xrightarrow{\mathcal{D}} \Lambda$ as $t \to \infty$. Theorem 7.5(a), Slutsky's theorem and the continuous mapping theorem then yield:

$$\frac{\sqrt{N(t)}}{a'_t}\,\widehat{\mathrm{CoVar}(X)} = \sqrt{\left(\frac{N(t)}{a'_t}\right)^2 T_{N(t)} - \frac{t}{a_t'^2}\,\frac{N(t)}{t}} \xrightarrow{\mathcal{D}} \frac{\sqrt{2}}{\mu_1}\sqrt{\Lambda} \quad \text{as} \quad t \to \infty.$$

By using Theorem 7.5(b) and the arguments above, we also get:

$$\frac{t}{a'_t}\,\frac{\widehat{\mathrm{CoVar}(X)}}{\sqrt{N(t)}} = \sqrt{\left(\frac{t}{a'_t}\right)^2 T_{N(t)} - \frac{t}{a_t'^2}\,\frac{t}{N(t)}} \xrightarrow{\mathcal{D}} \frac{\sqrt{2}}{\mu_1}\,\frac{1}{\sqrt{\Lambda}} \quad \text{as} \quad t \to \infty.$$

Case 5: $\alpha \in (2,4)$ or $\alpha = 2$, $\mu_2 < \infty$. Assume that $\{N(t);\, t \geqslant 0\}$ p-averages in time to the random variable Λ. Let $\ell_1^{\#} \in \mathrm{RV}_0$ be a de Bruijn conjugate of $\ell_1(x) := \ell^{-2/\alpha}(\sqrt{x}) \in \mathrm{RV}_0$. Note that $\ell_1^{\#}(x) = o(1)$ as $x \to \infty$ if $\alpha = 2$ and $\mu_2 < \infty$ (see Remark 7.6). Since $N(t)\,T_{N(t)} \xrightarrow{p} \mu_2/\mu_1^2$ as $t \to \infty$, the continuous mapping theorem gives:

$$\widehat{\mathrm{CoVar}(X)} \xrightarrow{p} \mathrm{CoVar}(X) \quad \text{as} \quad t \to \infty.$$

Now, define a sequence $(b_t)_{t>0}$ by $b_t := \frac{t^{1-2/\alpha}}{\ell_1^{\#}(t^{2/\alpha})}$. Denote $\sigma^2 := \mathbb{V}[X] < \infty$. For any $t > 0$, consider the decomposition:

$$b_t\Big(\widehat{\mathrm{CoVar}(X)} - \mathrm{CoVar}(X)\Big) = \underbrace{\frac{\mu_1}{2\sigma}\,b_t\left(N(t)\,T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right)}_{=:A_t} - \underbrace{\frac{\mu_1 b_t\left(N(t)\,T_{N(t)} - \frac{\mu_2}{\mu_1^2}\right)^2}{2\sigma\Big(\widehat{\mathrm{CoVar}(X)} + \mathrm{CoVar}(X)\Big)^2}}_{=:B_t}.$$

By using Theorem 7.6, Slutsky's theorem and the continuous mapping theorem, we easily deduce that $A_t \xrightarrow{\mathcal{D}} \frac{1}{2\mu_1\sigma}\,\frac{W_{\alpha/2}}{\Lambda^{1-2/\alpha}}$ and $B_t \xrightarrow{p} 0$ as $t \to \infty$, leading by another application of Slutsky's theorem to:

$$\frac{t^{1-2/\alpha}}{\ell_1^{\#}(t^{2/\alpha})}\Big(\widehat{\mathrm{CoVar}(X)} - \mathrm{CoVar}(X)\Big) \xrightarrow{\mathcal{D}} \frac{1}{2\mu_1\sigma}\,\frac{W_{\alpha/2}}{\Lambda^{1-2/\alpha}} \quad \text{as} \quad t \to \infty$$

where $W_{\alpha/2}$ is an $\alpha/2$-stable random variable independent of Λ.

Case 6: $\alpha = 4$**,** $\mu_4 = \infty$**.** Assume that $\{N(t);\, t \geqslant 0\}$ p-averages in time to the random variable Λ. Since $N(t)\, T_{N(t)} \xrightarrow{p} \mu_2/\mu_1^2$ as $t \to \infty$, we deduce by an application of the continuous mapping theorem that:

$$\widehat{\mathrm{CoVar}(X)} \xrightarrow{p} \mathrm{CoVar}(X) \quad \text{as} \quad t \to \infty.$$

Moreover, let $\ell_2(x) := \frac{1}{2\sqrt{\tilde{\ell}(\sqrt{x})}} \in \mathrm{RV}_0$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\, \mathrm{d}u \in \mathrm{RV}_0$. Define a sequence $(d_t)_{t>0}$ by $d_t := \frac{\sqrt{t}}{\ell_2^{\#}(\sqrt{t})}$ where $\ell_2^{\#} \in \mathrm{RV}_0$ is a de Bruijn conjugate of ℓ_2. For any $t > 0$, consider the splitting:

$$d_t\Big(\widehat{\mathrm{CoVar}(X)} - \mathrm{CoVar}(X)\Big) =: A_t - B_t$$

where the random variables A_t and B_t are defined as in Case 5 but with b_t replaced by d_t. By using Theorem 7.7, Slutsky's theorem and the continuous mapping theorem give $A_t \xrightarrow{\mathcal{D}} \frac{1}{2\mu_1\sigma} \frac{\mathrm{N}(0,1)}{\sqrt{\Lambda}}$ and $B_t \xrightarrow{p} 0$ as $t \to \infty$, leading by another application of Slutsky's theorem to:

$$\frac{\sqrt{t}}{\ell_2^{\#}\left(\sqrt{t}\right)} \Big(\widehat{\mathrm{CoVar}(X)} - \mathrm{CoVar}(X)\Big) \xrightarrow{\mathcal{D}} \frac{1}{2\mu_1\sigma} \frac{\mathrm{N}(0,1)}{\sqrt{\Lambda}} \quad \text{as} \quad t \to \infty$$

where $\mathrm{N}(0,1)$ is a standard normal random variable independent of Λ.

Case 7: $\mu_4 < \infty$**.** The proof of Theorem 7.8 can be repeated using the transformation $g(x,y) = \sqrt{\frac{y}{x^2} - 1}$ and this leads to:

$$\sqrt{t}\Big(\widehat{\mathrm{CoVar}(X)} - \mathrm{CoVar}(X)\Big) \xrightarrow{\mathcal{D}} \frac{\mathrm{N}\Big(0, \frac{\sigma_*^2 \mu_1^2}{4\sigma^2}\Big)}{\sqrt{\Lambda}} \quad \text{as} \quad t \to \infty \tag{7.18}$$

where $\mathrm{N}\Big(0, \frac{\sigma_*^2 \mu_1^2}{4\sigma^2}\Big)$ is a normal random variable independent of Λ with mean 0 and variance $\sigma_*^2 \mu_1^2/(4\sigma^2)$, with σ_*^2 defined in (7.16) and $\sigma^2 := \mathbb{V}[X] < \infty$.

Assume that $\mathbb{E}[\Lambda^{-1}] < \infty$. When $t\big(\widehat{\mathrm{CoVar}(X)} - \mathrm{CoVar}(X)\big)^2$ is uniformly integrable, the first two moments of the limiting distribution in (7.18) permit to determine the limiting behavior of $\mathrm{CoVar}\big(\widehat{\mathrm{CoVar}(X)}\big)$. Indeed, on the one hand, we have:

$$\lim_{t\to\infty} \sqrt{t}\Big(\mathbb{E}\Big[\widehat{\mathrm{CoVar}(X)}\Big] - \mathrm{CoVar}(X)\Big) = \lim_{t\to\infty} \mathbb{E}\Big[\sqrt{t}\Big(\widehat{\mathrm{CoVar}(X)} - \mathrm{CoVar}(X)\Big)\Big] = 0$$

which leads to:

$$\lim_{t\to\infty} \mathbb{E}\Big[\widehat{\mathrm{CoVar}(X)}\Big] = \mathrm{CoVar}(X). \tag{7.19}$$

On the other hand, we also get:

$$\lim_{t\to\infty} t\mathbb{V}\Big[\widehat{\mathrm{CoVar}(X)}\Big] = \lim_{t\to\infty} \mathbb{V}\Big[\sqrt{t}\Big(\widehat{\mathrm{CoVar}(X)} - \mathrm{CoVar}(X)\Big)\Big] = \frac{\sigma_*^2\, \mu_1^2\, \mathbb{E}[\Lambda^{-1}]}{4\sigma^2}$$

so that:

$$\mathbb{V}\Big[\widehat{\mathrm{CoVar}(X)}\Big] \sim \frac{\sigma_*^2\, \mu_1^2\, \mathbb{E}[\Lambda^{-1}]}{4\sigma^2} \frac{1}{t} \quad \text{as} \quad t \to \infty.$$

Consequently, under the above uniform integrability condition, the coefficient of variation of the sample coefficient of variation asymptotically behaves as:

$$\mathrm{CoVar}\big(\widehat{\mathrm{CoVar}(X)}\big) \sim \frac{\sigma_* \mu_1^2 \sqrt{\mathbb{E}[\Lambda^{-1}]}}{2\sigma^2} \frac{1}{\sqrt{t}} \quad \text{as} \quad t \to \infty.$$

In addition, it results from (7.18) and (7.19) that $\widehat{\mathrm{CoVar}(X)}$ is a consistent and asymptotically unbiased estimator for $\mathrm{CoVar}(X)$.

7.3.2 Sample Dispersion

Adapting the results of Section 7.2 to the random variable $C_{N(t)}$ defined in (7.6) permits us to derive limit distributions for the suitably normalized sample dispersion $\widehat{\mathrm{D}(X)}$ by means of (7.7). Different cases are considered as for the sample coefficient of variation in Subsection 7.3.1. In Cases 1-6, we assume that the distribution function F of X is of Pareto-type with index $\alpha > 0$, i.e. satisfies (7.1) for some $\ell \in \mathrm{RV}_0$. In Case 7, we assume that the distribution function F of X satisfies $\mu_4 < \infty$. Moreover, the counting process $\{N(t);\, t \geqslant 0\}$ is supposed to $\mathcal{D}$-average in time to the random variable Λ except in Cases 5-6 where it p-averages in time to Λ.

Case 1: $\alpha \in (0,1)$**.** Define $(a_t)_{t>0}$ by $\lim_{t\to\infty} t\, a_t^{-\alpha} \ell(a_t) = 1$. It follows from (7.9) and the continuous mapping theorem that:

$$\frac{1}{a_t} C_{N(t)} \xrightarrow{\mathcal{D}} \frac{U_{\alpha/2}}{U_\alpha} \Lambda^{1/\alpha} \quad \text{as} \quad t \to \infty$$

where the random vector $(U_{\alpha/2}, U_\alpha)'$ is independent of Λ and has a distribution determined by (7.8). We also have by (7.9) that $\frac{1}{a_t} \sum_{i=1}^{N(t)} X_i \xrightarrow{\mathcal{D}} U_\alpha \Lambda^{1/\alpha}$ as $t \to \infty$. Since $N(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$, Slutsky's theorem and the continuous mapping theorem then yield:

$$\frac{1}{a_t} \widehat{\mathrm{D}(X)} = \frac{1}{a_t} C_{N(t)} - \frac{1}{N(t)} \frac{1}{a_t} \sum_{i=1}^{N(t)} X_i \xrightarrow{\mathcal{D}} \frac{U_{\alpha/2}}{U_\alpha} \Lambda^{1/\alpha} \quad \text{as} \quad t \to \infty.$$

Case 2: $\alpha = 1$**,** $\mu_1 = \infty$**.** Define $(a_t)_{t>0}$ and $(a'_t)_{t>0}$ respectively by $\lim_{t\to\infty} t\, a_t^{-1} \ell(a_t) = 1$ and $\lim_{t\to\infty} t\, a_t'^{-1} \tilde{\ell}(a'_t) = 1$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\, \mathrm{d}u \in \mathrm{RV}_0$. It follows from (7.10) and the continuous mapping theorem that:

$$\frac{a'_t}{a_t^2} C_{N(t)} \xrightarrow{\mathcal{D}} U_{1/2}\, \Lambda \quad \text{as} \quad t \to \infty$$

where the random variable $U_{1/2}$ is independent of Λ and has a distribution determined by (7.2) with $\gamma = 1/2$.

Since $a'_t/a_t \sim \frac{(1/\tilde{\ell})^\#(t)}{(1/\ell)^\#(t)}$ as $t \to \infty$, where $(1/\tilde{\ell})^\# \in \mathrm{RV}_0$ and $(1/\ell)^\# \in \mathrm{RV}_0$ are de Bruijn conjugates of $1/\tilde{\ell} \in \mathrm{RV}_0$ and $1/\ell \in \mathrm{RV}_0$ respectively, it follows that $a'_t/a_t \in \mathrm{RV}_0$ and then $\lim_{t\to\infty} \frac{1}{t}\big(\frac{a'_t}{a_t}\big)^2 = 0$. Moreover, using the same independence and conditioning arguments as in the proof of Theorem 7.6, we obtain that at any continuity points x and y of the distribution function of Λ:

$$\lim_{t\to\infty} \mathbb{P}\left[\frac{N(t)}{t} \leqslant x, \frac{1}{a'_t} \sum_{i=1}^{N(t)} X_i \leqslant y\right] = \mathbb{P}[\Lambda \leqslant x]\ \mathbb{P}[\Lambda \leqslant y]$$

i.e., since $\frac{1}{a'_t}\sum_{i=1}^{N(t)} X_i \xrightarrow{\mathcal{D}} \Lambda$ as $t \to \infty$ by (7.10) and $\frac{N(t)}{t} \xrightarrow{\mathcal{D}} \Lambda$ as $t \to \infty$, that:

$$\left(\frac{N(t)}{t}, \frac{1}{a'_t}\sum_{i=1}^{N(t)} X_i\right)' \xrightarrow{\mathcal{D}} (\Lambda, \Lambda^\star)' \quad \text{as} \quad t \to \infty$$

where $\Lambda^\star$ is an independent copy of Λ. Using the continuous mapping theorem, we then deduce:

$$\frac{t}{N(t)} \frac{\sum_{i=1}^{N(t)} X_i}{a'_t} \xrightarrow{\mathcal{D}} \frac{\Lambda^\star}{\Lambda} \quad \text{as} \quad t \to \infty.$$

Hence, Slutsky's theorem gives:

$$\frac{a'_t}{a_t^2}\widehat{\mathrm{D}(X)} = \frac{a'_t}{a_t^2} C_{N(t)} - \frac{1}{t}\left(\frac{a'_t}{a_t}\right)^2 \frac{t}{N(t)} \frac{\sum_{i=1}^{N(t)} X_i}{a'_t} \xrightarrow{\mathcal{D}} U_{1/2}\,\Lambda \quad \text{as} \quad t \to \infty.$$

Case 3: $\alpha \in (1,2)$ **or** $\alpha = 1$**,** $\mu_1 < \infty$**.** Define $(a_t)_{t>0}$ by $\lim_{t\to\infty} t\, a_t^{-\alpha}\ell(a_t) = 1$. Since $\overline{X} \xrightarrow{p} \mu_1$ as $t \to \infty$, it follows from Theorem 7.4(a) and Slutsky's theorem that:

$$\frac{N(t)}{a_t^2} C_{N(t)} = \overline{X}\left(\frac{N(t)}{a_t}\right)^2 T_{N(t)} \xrightarrow{\mathcal{D}} \frac{1}{\mu_1} U_{\alpha/2}\,\Lambda^{2/\alpha} \quad \text{as} \quad t \to \infty$$

where the random variable $U_{\alpha/2}$ is independent of Λ and has a distribution determined by (7.2) with $\gamma = \alpha/2$. Since $t/a_t^2 \sim \frac{a_t^{\alpha-2}}{\ell(a_t)} \to 0$ and $\frac{N(t)}{t} \xrightarrow{\mathcal{D}} \Lambda$ as $t \to \infty$, Slutsky's theorem then leads to:

$$\frac{N(t)}{a_t^2}\widehat{\mathrm{D}(X)} = \frac{N(t)}{a_t^2} C_{N(t)} - \frac{t}{a_t^2}\frac{N(t)}{t}\overline{X} \xrightarrow{\mathcal{D}} \frac{1}{\mu_1} U_{\alpha/2}\,\Lambda^{2/\alpha} \quad \text{as} \quad t \to \infty.$$

By using (7.11) and the arguments above, we also get:

$$\frac{t}{a_t^2}\widehat{\mathrm{D}(X)} = \frac{t}{a_t^2} C_{N(t)} - \frac{t}{a_t^2}\overline{X} \xrightarrow{\mathcal{D}} \frac{1}{\mu_1}\frac{U_{\alpha/2}}{\Lambda^{1-\alpha/2}} \quad \text{as} \quad t \to \infty.$$

Case 4: $\alpha = 2$**,** $\mu_2 = \infty$**.** Define $(a'_t)_{t>0}$ by $\lim_{t\to\infty} t\, a_t'^{-2}\tilde{\ell}(a'_t) = 1$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,\mathrm{d}u \in \mathrm{RV}_0$. Since $\overline{X} \xrightarrow{p} \mu_1$ as $t \to \infty$, it follows from Theorem 7.5(a) and Slutsky's theorem that:

$$\frac{N(t)}{a_t'^2} C_{N(t)} = \overline{X}\left(\frac{N(t)}{a'_t}\right)^2 T_{N(t)} \xrightarrow{\mathcal{D}} \frac{2}{\mu_1}\,\Lambda \quad \text{as} \quad t \to \infty.$$

From $\mu_2 = \infty$, we get $\lim_{x\to\infty}\tilde{\ell}(x) = \infty$ so that $t/a_t'^2 \sim 1/\tilde{\ell}(a'_t) \to 0$ as $t \to \infty$. Since $\frac{N(t)}{t} \xrightarrow{\mathcal{D}} \Lambda$ as $t \to \infty$, Slutsky's theorem then yields:

$$\frac{N(t)}{a_t'^2}\widehat{\mathrm{D}(X)} = \frac{N(t)}{a_t'^2} C_{N(t)} - \frac{t}{a_t'^2}\frac{N(t)}{t}\overline{X} \xrightarrow{\mathcal{D}} \frac{2}{\mu_1}\,\Lambda \quad \text{as} \quad t \to \infty.$$

By using (7.12) and the arguments above, we also get:

$$\frac{t}{a_t'^2}\widehat{\mathrm{D}(X)} = \frac{t}{a_t'^2} C_{N(t)} - \frac{t}{a_t'^2}\overline{X} \xrightarrow{\mathcal{D}} \frac{2}{\mu_1} \quad \text{as} \quad t \to \infty.$$

Case 5: $\alpha \in (2,4)$ **or** $\alpha = 2$**,** $\mu_2 < \infty$**.** Assume that $\{N(t);\, t \geqslant 0\}$ p-averages in time to the random variable Λ. Define a sequence $(b_t)_{t>0}$ by $b_t := \frac{t^{1-2/\alpha}}{\ell_1^{\#}(t^{2/\alpha})}$ where $\ell_1^{\#} \in \mathrm{RV}_0$ is a de Bruijn conjugate of $\ell_1(x) := \ell^{-2/\alpha}(\sqrt{x}) \in \mathrm{RV}_0$. Note that if $\alpha = 2$ and $\mu_2 < \infty$, we have $\ell_1^{\#}(x) = o(1)$ as $x \to \infty$ (see Remark 7.6). For any $t > 0$, consider the decomposition:

$$\begin{aligned} b_t\Big(\widehat{\mathrm{D}(X)} - \mathrm{D}(X)\Big) &= b_t\left(C_{N(t)} - \frac{\mu_2}{\mu_1} - \overline{X} + \mu_1\right) \\ &= \underbrace{\frac{b_t}{\overline{X}}\left(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i^2 - \mu_2\right)}_{=:A_t} - \underbrace{b_t\big(\overline{X} - \mu_1\big)\left(1 + \frac{\mu_2}{\mu_1}\frac{1}{\overline{X}}\right)}_{=:B_t}. \end{aligned}$$

Since $\overline{X} \xrightarrow{p} \mu_1$, $\frac{N(t)}{t} \xrightarrow{p} \Lambda$ and $\frac{\ell_1^{\#}(N(t)^{2/\alpha})}{\ell_1^{\#}(t^{2/\alpha})} \xrightarrow{p} 1$ as $t \to \infty$, it follows from (7.13), Slutsky's theorem and the continuous mapping theorem that:

$$A_t = \frac{1}{\overline{X}}\left(\frac{t}{N(t)}\right)^{1-2/\alpha} \frac{\ell_1^{\#}(N(t)^{2/\alpha})}{\ell_1^{\#}(t^{2/\alpha})} \frac{N(t)^{1-2/\alpha}}{\ell_1^{\#}(N(t)^{2/\alpha})} \left(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i^2 - \mu_2\right) \xrightarrow{\mathcal{D}} \frac{1}{\mu_1}\frac{W_{\alpha/2}}{\Lambda^{1-2/\alpha}} \quad \text{as} \quad t \to \infty$$

where $W_{\alpha/2}$ is an $\alpha/2$-stable random variable independent of Λ.

It follows from (7.14) that:

$$\sqrt{t}\left(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i - \mu_1\right) = \sqrt{\frac{t}{N(t)}}\sqrt{N(t)}\left(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i - \mu_1\right) \xrightarrow{\mathcal{D}} \frac{\mathrm{N}(0,\sigma^2)}{\sqrt{\Lambda}} \quad \text{as} \quad t \to \infty \quad (7.20)$$

where $\mathrm{N}(0,\sigma^2)$ is a normal random variable independent of Λ with mean 0 and variance $\sigma^2 := \mathbb{V}[X] < \infty$. Since $\lim_{t\to\infty} b_t/\sqrt{t} = 0$, we then obtain:

$$B_t = \frac{b_t}{\sqrt{t}}\sqrt{t}\left(\frac{1}{N(t)}\sum_{i=1}^{N(t)} X_i - \mu_1\right)\left(1 + \frac{\mu_2}{\mu_1}\frac{1}{\overline{X}}\right) \xrightarrow{p} 0 \quad \text{as} \quad t \to \infty$$

by virtue of Slutsky's theorem and the continuous mapping theorem. A last application of Slutsky's theorem therefore gives:

$$\frac{t^{1-2/\alpha}}{\ell_1^{\#}(t^{2/\alpha})}\Big(\widehat{\mathrm{D}(X)} - \mathrm{D}(X)\Big) \xrightarrow{\mathcal{D}} \frac{1}{\mu_1}\frac{W_{\alpha/2}}{\Lambda^{1-2/\alpha}} \quad \text{as} \quad t \to \infty.$$

The latter relation shows in particular that:

$$\widehat{\mathrm{D}(X)} \xrightarrow{p} \mathrm{D}(X) \quad \text{as} \quad t \to \infty.$$

Case 6: $\alpha = 4$**,** $\mu_4 = \infty$**.** Assume that $\{N(t);\, t \geqslant 0\}$ p-averages in time to the random variable Λ. Define a sequence $(d_t)_{t>0}$ by $d_t := \frac{\sqrt{t}}{\ell_2^{\#}(\sqrt{t})}$ where $\ell_2^{\#} \in \mathrm{RV}_0$ is a de Bruijn conjugate of $\ell_2(x) := \frac{1}{2\sqrt{\tilde{\ell}(\sqrt{x})}} \in \mathrm{RV}_0$ with $\tilde{\ell}(x) := \int_0^x \frac{\ell(u)}{u}\,\mathrm{d}u \in \mathrm{RV}_0$. For any $t > 0$, consider the equality:

$$d_t\Big(\widehat{\mathrm{D}(X)} - \mathrm{D}(X)\Big) =: A_t - B_t$$

where the random variables A_t and B_t are defined as in Case 5 but with b_t replaced by d_t.

Since $\overline{X} \xrightarrow{p} \mu_1$, $\frac{N(t)}{t} \xrightarrow{p} \Lambda$ and $\frac{\ell_2^{\#}(\sqrt{N(t)})}{\ell_2^{\#}(\sqrt{t})} \xrightarrow{p} 1$ as $t \to \infty$, it follows from (7.15), Slutsky's theorem and the continuous mapping theorem that:

$$A_t = \frac{1}{\overline{X}} \sqrt{\frac{t}{N(t)}} \frac{\ell_2^{\#}\big(\sqrt{N(t)}\big)}{\ell_2^{\#}\big(\sqrt{t}\big)} \frac{\sqrt{N(t)}}{\ell_2^{\#}\big(\sqrt{N(t)}\big)} \left(\frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i^2 - \mu_2 \right) \xrightarrow{\mathcal{D}} \frac{1}{\mu_1} \frac{\mathrm{N}(0,1)}{\sqrt{\Lambda}} \quad \text{as} \quad t \to \infty$$

where $\mathrm{N}(0,1)$ is a standard normal random variable independent of Λ.

Since $\lim_{x\to\infty} \ell_2^{\#}(x) = \infty$ (see Remark 7.7), it follows from (7.20) that:

$$B_t = \frac{1}{\ell_2^{\#}\big(\sqrt{t}\big)} \sqrt{t} \left(\frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i - \mu_1 \right) \left(1 + \frac{\mu_2}{\mu_1} \frac{1}{\overline{X}} \right) \xrightarrow{p} 0 \quad \text{as} \quad t \to \infty$$

by virtue of Slutsky's theorem and the continuous mapping theorem. A last application of Slutsky's theorem therefore gives:

$$\frac{\sqrt{t}}{\ell_2^{\#}\big(\sqrt{t}\big)} \Big(\widehat{\mathrm{D}(X)} - \mathrm{D}(X) \Big) \xrightarrow{\mathcal{D}} \frac{1}{\mu_1} \frac{\mathrm{N}(0,1)}{\sqrt{\Lambda}} \quad \text{as} \quad t \to \infty.$$

The latter relation shows in particular that:

$$\widehat{\mathrm{D}(X)} \xrightarrow{p} \mathrm{D}(X) \quad \text{as} \quad t \to \infty.$$

Case 7: $\mu_4 < \infty$**.** Using $g(x,y) = \frac{y}{x} - x$ in the proof of Theorem 7.8 yields:

$$\sqrt{t}\Big(\widehat{\mathrm{D}(X)} - \mathrm{D}(X) \Big) \xrightarrow{\mathcal{D}} \frac{\mathrm{N}\big(0, \sigma_{**}^2\big)}{\sqrt{\Lambda}} \quad \text{as} \quad t \to \infty \tag{7.21}$$

where $\mathrm{N}\big(0, \sigma_{**}^2\big)$ is a normal random variable independent of Λ with mean 0 and variance σ_{**}^2 defined by:

$$\sigma_{**}^2 := \mu_2 - \mu_1^2 + \frac{\mu_2^3}{\mu_1^4} - 2\frac{\mu_3}{\mu_1} - 2\frac{\mu_2\mu_3}{\mu_1^3} + 2\left(\frac{\mu_2}{\mu_1}\right)^2 + \frac{\mu_4}{\mu_1^2}.$$

Assume that $\mathbb{E}[\Lambda^{-1}] < \infty$. When $t\big(\widehat{\mathrm{D}(X)} - \mathrm{D}(X)\big)^2$ is uniformly integrable, the first two moments of the limiting distribution in (7.21) permit to determine the limiting behavior of $\mathrm{D}\big(\widehat{\mathrm{D}(X)}\big)$. Indeed, on the one hand, we have:

$$\lim_{t\to\infty} \sqrt{t}\Big(\mathbb{E}\Big[\widehat{\mathrm{D}(X)}\Big] - \mathrm{D}(X) \Big) = \lim_{t\to\infty} \mathbb{E}\Big[\sqrt{t}\Big(\widehat{\mathrm{D}(X)} - \mathrm{D}(X) \Big) \Big] = 0$$

leading to:

$$\lim_{t\to\infty} \mathbb{E}\Big[\widehat{\mathrm{D}(X)}\Big] = \mathrm{D}(X). \tag{7.22}$$

Note that (7.21) together with (7.22) implies that $\widehat{\mathrm{D}(X)}$ is a consistent and asymptotically unbiased estimator for $\mathrm{D}(X)$.

On the other hand, we also get:

$$\lim_{t\to\infty} t\mathbb{V}\Big[\widehat{\mathrm{D}(X)}\Big] = \lim_{t\to\infty} \mathbb{V}\Big[\sqrt{t}\Big(\widehat{\mathrm{D}(X)} - \mathrm{D}(X)\Big)\Big] = \sigma_{**}^2\, \mathbb{E}[\Lambda^{-1}]$$

so that:

$$\mathbb{V}\Big[\widehat{\mathrm{D}(X)}\Big] \sim \sigma_{**}^2\, \mathbb{E}[\Lambda^{-1}]\, \frac{1}{t} \quad \text{as} \quad t \to \infty.$$

Consequently, under the above uniform integrability condition, the dispersion of the sample dispersion asymptotically behaves as:

$$\mathrm{D}\Big(\widehat{\mathrm{D}(X)}\Big) \sim \frac{\sigma_{**}^2\, \mu_1 \mathbb{E}[\Lambda^{-1}]}{\sigma^2}\, \frac{1}{t} \quad \text{as} \quad t \to \infty$$

where $\sigma^2 := \mathbb{V}[X] < \infty$.

7.4 Conclusion

In this chapter, we have derived limit distributions for the suitably normalized ratio $T_{N(t)}$ as $t \to \infty$ when the distribution function F of X_1 is of Pareto-type with index $\alpha > 0$ or is such that $\mu_4 < \infty$. In addition, the counting process $\{N(t);\, t \geqslant 0\}$ has been chosen to $\mathcal{D}$-average in time or, in two cases, to p-average in time. Different results have shown up according to the range of α and the (non)finiteness of the first moments of X_1. Then, we have exploited these results to derive limit distributions for two widely used risk measures, namely the sample coefficient of variation and the sample dispersion. The practical applicability of both risk measures depends on the existence of sufficiently many moments of the underlying distribution but our results help at illustrating their asymptotic behavior when such a moment condition is not satisfied.

Finally, we point out that by choosing the limiting random variable Λ to be degenerate at the point 1, we retrieve results of Albrecher et al. [1] where the counting process is non-random.

Conclusion

Part I of the thesis has been devoted to reinsurance which is considered as a form of risk sharing. Indeed, in an attempt to cope with excessively large claims and/or an unusually large number of claims, an insurer sells part of his portfolio and hence his risk to a reinsurer. Our results have been established in the context of the collective risk model where the insurance portfolio is regarded as a collective producing claims at random points in time. An emphasize has been put on the coverage against large claims. In this context, treaties of extreme value type have naturally been considered (ECOMOR, LC and QDXL reinsurance) together with the assumption of heavy-tailed claim size distributions. Most of our results have even been derived for claim size distributions that belong to a maximum domain of attraction. We hope that our results give an insight into the complex structure of these treaties based on the largest claim sizes in the portfolio in order to lay solid foundations for many practical applications. The concept of dependence has not been taken into account in this thesis although it may be relevant in practice. Hence, a line of future research would be to deal with asymptotic problems connected to the reinsurance of large claims while dropping the assumption of independence between the claim size process and the claim number process or allowing the claim sizes to be dependent. Another challenging research topic would be to focus on the ruin problem in the context of large claims reinsurance, i.e. to investigate the probability of ruin associated with the insurance risk reserve process after a reinsurance cover of extreme value type in the case of a heavy-tailed claim size distribution. The ruin probability with infinite horizon provides a measure of the global behavior of the insurance portfolio after reinsurance as time goes by. From a practical point of view, the ruin probability with finite horizon is a more attractive issue although it is seldom expressible in a closed form.

Part II of the thesis has been devoted to the asymptotic analysis of a random quantity defined as the ratio of the random sum of squares of a.s. positive i.i.d. random variables over the square of the random sum of these random variables whose distribution has a regularly varying right tail while the number of summands is governed by a counting process satisfying specific conditions. On the one hand, asymptotic expressions for the moments of the ratio have been obtained. On the other hand, limit distributions for the suitably normalized ratio have been derived. Some of our results have then been used to derive limit distributions for the sample coefficient of variation and the sample dispersion, both appropriately normalized, under the same assumptions as for the ratio. Our results reflects that the applicability of these risk measures depends on the existence of sufficiently many moments of the underlying distribution. Hence, the practitioner needs to verify statistically the range of the index of regular variation he might be working with. Fortunately, extreme value theory provides a wealth of statistical procedures to acquire sample information on this index. Finally, we emphasize that also the normalizing constants that ensure nonzero limit distributions are unknown in practice and therefore need to be estimated.

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