Metalogic, Metalanguage and Logical Geometry

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1 Introduction

Aristotelian diagrams visualize the logical relations among a finite set of elements from some logical, lexical or conceptual system. The oldest and most widely known example is undoubtedly the so-called 'square of opposition' (Parsons, 2012). These diagrams have a well-documented history in philosophical logic, and in recent years, they have come to serve "as a kind of *lingua franca*" (Jacquette, 2012, p. 81) that facilitates communication, research and teaching in a wide variety of disciplines that deal with logical reasoning in all its facets, including philosophy, cognitive science, law and computer science (Demey and Smessaert, 2018a).

Logical geometry studies Aristotelian diagrams as objects of independent interest. This research programme investigates the visual/diagrammatic properties of these diagrams (Demey and Smessaert, 2014a, 2016b, 2017b, 2018b), but also studies various aspects of their logical behavior (Demey, 2015, 2018; Pizzi, 2016; Demey and Smessaert, 2018a; Smessaert and Demey, 2014, 2017). From this latter perspective, it is clear that Aristotelian diagrams are directly related to a number of metalogical and metalinguistic issues.[1] Some of these issues have already been explored by authors such as Löbner (1987), Béziau (2012, 2013), Seuren (2014) and Diaconescu (2015). Furthermore, Demey (2017a) discusses their practical relevance in the context of teaching metalogic to certain groups of students. A systematic and comprehensive overview of work on metalogical Aristotelian diagrams is provided by Demey and Smessaert (2016a).

The present paper has two interrelated aims. On the one hand, I will provide a fully general and mathematically precise account of the Aristotelian relations, which takes into account their metalogical aspects, and explains how a single type of relations can hold between object-logical as well as between metalogical entities. On the other hand, I will argue for the theoretical fruitfulness of this general approach to the Aristotelian relations, by showing how it enables a unified pragmatic account of certain linguistic phenomena, regardless of whether they occur

[1] In this paper, I will draw a clear distinction between meta-*logic* and meta-*language*. A notion or question will be characterized as *metalogical* when we are primarily concerned with its abstract mathematical properties, regardless of how it is expressed in any specific language. By contrast, a metalogical notion will be specifically characterized as *metalinguistic* when we want to emphasize the specific details of how (if at all) this notion is expressed in natural language (e.g. by means of a single word). Roughly speaking, metalogical questions will mainly be addressed in Section 2 of this paper, while metalinguistic issues will mainly be addressed in Section 3.

at the object- or metalinguistic level. The paper thus fits within the larger project of constructing a bridge between logical geometry and metalogical and -linguistic considerations. As will become clear throughout the paper, this bridge accommodates two-way traffic between both domains: metalogical considerations are crucial for obtaining a complete account of the Aristotelian relations, and vice versa, Aristotelian diagrams can shed new light on certain metalinguistic expressions

The paper is organized as follows. Section 2 provides an in-depth analysis of the Aristotelian relations, focusing on their metalogical nature and showing how a single kind of relations can hold between object-logical as well as metalogical entities. Next, Section 3 describes some linguistic data regarding words such as *some* and *contrary*, and argues that highly similar Aristotelian diagrams and linguistic explanations are available for both the object-linguistic data on *some* and the metalinguistic data on *contrary*. Finally, Section 4 wraps things up, and offers some concluding thoughts on the heuristic importance of Aristotelian diagrams.

2 A Metalogical Perspective on the Aristotelian Relations

In this section I will argue that the Aristotelian relations are themselves fundamentally metalogical in nature, and explain how it is possible for a single type of relations to hold between metalogical as well as object-logical entities. The exact way in which the Aristotelian relations are defined turns out to be highly relevant in addressing these issues. In Subsections 2.1–2.3, I will therefore introduce a series of increasingly more abstract definitions, and discuss which metalogical considerations are taken into account in each of them. Finally, in Subsection 2.4, I will make some broader philosophical remarks regarding its unificatory power, and draw a connection with the cumulative hierarchy in set theory.

2.1 The Aristotelian Relations from an Informal Perspective

The oldest definition of the Aristotelian relations dates back to Aristotle himself, and has been used throughout the history of philophical logic (Ackrill, 1961; Parsons, 2012). In contemporary work on Aristotelian diagrams, too, it is still the most widely used definition (Béziau and Jacquette, 2012; Béziau and Basti, 2017). The formulation is entirely informal, and looks as follows:

Definition 1. Two statements φ and ψ are said to be

contradictory	iff	φ and ψ cannot be true together φ and ψ cannot be false together,	and
contrary	iff	φ and ψ cannot be true together φ and ψ can be false together,	and
subcontrary	iff	φ and ψ can be true together φ and ψ cannot be false together,	and
in subalternation	iff	φ entails ψ ψ does not entail φ.	and

This definition is clearly *modal* in nature. Consider, for example, the definition of the contradiction relation: for two statements to be contradictory to each other, it is not merely required that they are *actually* not true together, but rather that they *cannot* be true together. The modal verb *can(not)* is also explicitly present in the definition of (sub)contrariety, while the modal nature of subalternation is clear from the fact that entailment should itself be understood in modal terms: that φ entails ψ means that ψ cannot be false while φ is true (i.e. φ and the negation of ψ cannot be true together). Because of this modal aspect, the Aristotelian relation holding between two statements (if any at all) is not uniquely determined by those statements' actual truth values. For example, if the first statement is actually false and the second one actually true, they might turn out to be contradictory, contrary, subcontrary, in subalternation, or in no Aristotelian relation at all.[2]

Definition 1 involves the notions of truth and falsity — either explicitly, or implicitly via the notion of entailment (cf. supra). Since truth and falsity can exclusively be ascribed to statements,[3] it follows that the Aristotelian relations are restricted to *statements*. For example, it is meaningless to say that a set X is true, and a fortiori thus also to say that two sets X and Y can be true together, which would be required if we wished to say that X and Y are subcontrary to each other.

Most importantly, because of its informal nature, Definition 1 can apply both to *object*- and to *metalogical* statements. To illustrate this, consider once again the definition of contradiction, and note the ambiguity of the word *true* appearing in it: this word can stand for 'truth in a model' (in case two object-logical statements are said to be contradictory), or for 'absolute, informal truth' (in case two metalogical statements are said to be contradictory). Obviously, similar remarks apply to (sub)contrariety and subalternation (the latter being defined in terms of entailment, which itself also involves the notion of truth).

A problem with Definition 1 is that it makes the Aristotelian relations entirely *insensitive* to the 'background logic'. Which Aristotelian relation holds between

[2]The humanist Lorenzo Valla's criticisms of the scholastic definitions of the Aristotelian relations seem to be, at least partially, based on a misunderstanding of this modal aspect. For example, Valla argued that two propositions that are actually false cannot be said to be contrary to each other (Nauta, 2009; Copenhaver and Nauta, 2012).

[3]I do not mean to enter here into the philosophical debate whether truth and falsity should be ascribed to sentences, propositions, utterances, etc. (Glanzberg, 2013). I only wish to make the more fundamental point that, at the very least, truth and falsity should not be ascribed to predicates, sets, relations, etc., on pain of an avalanche of category mistakes.

two (object-logical) formulas partially depends on the logical system that is being assumed. The well-known issue of existential import can be seen as an illustration of this problem: in classical syllogistics, there is a subalternation from $\forall x(Sx \to Px)$ to $\exists x(Sx \wedge Px)$, but in contemporary predicate logic, these formulas stand in no Aristotelian relation at all (Demey and Smessaert, 2018a, Section 4).

2.2 The Aristotelian Relations from a Logical Perspective

In order to deal with the issue of logic-sensitivity, a new and more precise way of defining the Aristotelian relations has been proposed (Smessaert, 2012; Smessaert and Demey, 2014, 2015):

Definition 2. Let S be a logical system, which is assumed to have Boolean operators and a model-theoretic semantics $\models$. Two formulas $\varphi, \psi \in \mathcal{L}_\mathsf{S}$ are said to be

S-*contradictory*	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \varphi \vee \psi$,
S-*contrary*	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \not\models \varphi \vee \psi$,
S-*subcontrary*	iff	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \varphi \vee \psi$,
in S-*subalternation*	iff	$\mathsf{S} \models \varphi \to \psi$	and	$\mathsf{S} \not\models \psi \to \varphi$.

This definition stays very close to the original, more informal Definition 1 in several key aspects. For example, the condition that φ and ψ cannot be true together is formalized as $\mathsf{S} \models \neg(\varphi \wedge \psi)$, which means that for all S-models $\mathbb{M}$ it holds that $\mathbb{M} \models \neg(\varphi \wedge \psi)$, or equivalently: S has no model $\mathbb{M}$ in which φ and ψ are both true (i.e. $\mathbb{M} \models \varphi$ and $\mathbb{M} \models \psi$). Similarly, the condition that φ and ψ can be false together is formalized as $\mathsf{S} \not\models \varphi \vee \psi$, which means that there exists an S-model $\mathbb{M}$ such that $\mathbb{M} \not\models \varphi \vee \psi$, or equivalently: S has a model $\mathbb{M}$ in which φ and ψ are both false (i.e. $\mathbb{M} \not\models \varphi$ and $\mathbb{M} \not\models \psi$). The modal aspects of Definition 1 thus resurface here as the (non-)existence of models of the logical system S. This clearly shows how the Aristotelian relations are *metalogical* in nature: their definition involves quantifying over the entire class of models of S.[4,5]

Since it explicitly refers to the logical system S, Definition 2 is capable of dealing with the logic-sensitivity of the Aristotelian relations. For example, we can now say that two formulas are S_1-contrary, but S_2-contradictory, for distinct logical systems S_1 and S_2 that share the same object language. In terms of models, this means that (i) neither S_1 nor S_2 has any models in which both formulas are true, (ii) S_1 has at least one model in which both formulas are false, but (iii) all

[4]These modal/metalogical aspects are ignored by Price, who writes that "φ and ψ are contraries if they cannot be true together, and it follows from the truth tables that this is just to say that $\neg(\varphi \wedge \psi)$ is true" (1990, p. 226, notational conventions changed to those of the present paper). Again: the idea that φ and ψ cannot be true together does not correspond to $\neg(\varphi \wedge \psi)$ being *true*, but rather to $\neg(\varphi \wedge \psi)$ being a *tautology* (in the logical system under consideration). (Also recall Footnote 2.)

[5]Note that it follows directly from Definition 2 that the Aristotelian relations only hold *up to logical equivalence*, i.e. for all formulas $\varphi, \psi, \varphi', \psi' \in \mathcal{L}_\mathsf{S}$ and Aristotelian relations R_S, it holds that if $\varphi \equiv_\mathsf{S} \varphi'$ and $\psi \equiv_\mathsf{S} \psi'$, then $R_\mathsf{S}(\varphi, \psi)$ iff $R_\mathsf{S}(\varphi', \psi')$.

such models fail to qualify as models of S_2 (since the latter logical system has no models in which both formulas are false).

Just like the previous definition, Definition 2 is based on the notion of truth, and thus only applies to *statements* (it still does not make sense to speak of two sets being contrary to each other, for example). However, in contrast to Definition 1, the notion of truth is now explicitly understood as 'truth in a model (of S)', and hence, Definition 2 only applies to *object-logical* formulas. For example, conditions such as $\mathsf{S} \models \neg(\varphi \wedge \psi)$ and $\mathsf{S} \not\models \psi \rightarrow \varphi$ are only meaningful for formulas φ and ψ from the object language $\mathcal{L}_\mathsf{S}$ of the logical system S.

2.3 The Aristotelian Relations from a Boolean Perspective

A key insight of Definition 2 is that the Aristotelian relations are fully determined by the Boolean structure of the logical system S. This suggests a third and final way of defining these relations, which abstracts away from the concrete details of S, and only focuses on its Boolean structure:

Definition 3. Let $\mathbb{B} = \langle B, \wedge_\mathbb{B}, \vee_\mathbb{B}, \neg_\mathbb{B}, \top_\mathbb{B}, \bot_\mathbb{B}\rangle$ be a Boolean algebra. Two elements $x, y \in B$ are said to be

$\mathbb{B}$-*contradictory*	iff	$x \wedge_\mathbb{B} y = \bot_\mathbb{B}$	and	$x \vee_\mathbb{B} y = \top_\mathbb{B}$,
$\mathbb{B}$-*contrary*	iff	$x \wedge_\mathbb{B} y = \bot_\mathbb{B}$	and	$x \vee_\mathbb{B} y \neq \top_\mathbb{B}$,
$\mathbb{B}$-*subcontrary*	iff	$x \wedge_\mathbb{B} y \neq \bot_\mathbb{B}$	and	$x \vee_\mathbb{B} y = \top_\mathbb{B}$,
in $\mathbb{B}$-*subalternation*	iff	$x \wedge_\mathbb{B} y = x$	and	$x \wedge_\mathbb{B} y \neq y$.

Unlike the first two definitions, this third characterization is no longer explicitly modal or metalogical in nature. Rather, it should be seen as an abstract 'template': concrete definitions of the Aristotelian relations for specific contexts (which may or may not be metalogical in nature) can be obtained from it by plugging in concrete Boolean algebras for $\mathbb{B}$. I will now discuss some of the most important (families of) concrete instances of Definition 3.

The most prototypical cases arise when $\mathbb{B}$ is taken to be (a subalgebra of) the powerset $\wp(X)$ of some set X (Givant and Halmos, 2009). In such a Boolean algebra, two sets $A, B \subseteq X$ are said to be contrary iff $A \cap B = \emptyset$ and $A \cup B \neq X$ — in other words, iff A and B are disjoint but not exhaustive. For example, if we take X to be a set of possible worlds, then $\wp(X)$ consists of sets of possible worlds, i.e. *propositions*. In this case, the contrariety of two propositions A and B means that there is no possible world in which both propositions are true ($A \cap B = \emptyset$), while there is at least one possible world in which both propositions are false ($A \cup B \neq X$). By contrast, if we take X to be a set of individuals, then $\wp(X)$ consists of sets of individuals, i.e. *properties* or (interpretations of) *predicates*. In this case, the contrariety of two properties A and B means that there is no individual that has both properties ($A \cap B = \emptyset$), while there is at least one individual that lacks both properties ($A \cup B \neq X$).

This shows that unlike the first two definitions, Definition 3 is not restricted to *statements*, but also applies to properties, relations, arbitrary sets, etc.[6] Furthermore, the Aristotelian relations holding between statements, between properties, between sets, etc. are all analogous to each other, because all of them are special instances of one and the same template. This analogy was already noted by Keynes, who wrote: "These seven possible relations between *propositions* (taken in pairs) will be found to be precisely analogous to the seven possible relations between *classes* (taken in pairs)" (Keynes, 1906, p. 119, my emphases).[7]

Definition 3 also subsumes Definition 2 as a special case. After all, if S is a logical system as specified in Definition 2 (i.e. having Boolean connectives), then its Lindenbaum-Tarski algebra $\mathbb{B}(\mathsf{S}) := \mathcal{L}_\mathsf{S}/{\equiv_\mathsf{S}} = \{[\varphi]_\mathsf{S} \mid \varphi \in \mathcal{L}_\mathsf{S}\}$ (where $[\varphi]_\mathsf{S} := \{\psi \in \mathcal{L}_\mathsf{S} \mid \varphi \equiv_\mathsf{S} \psi\}$) constitutes a Boolean algebra. Since the Aristotelian relations hold up to logical equivalence (recall Footnote 5), one can easily show that the Aristotelian relations for the logical system S (as defined in Definition 2) correspond exactly to the Aristotelian relations for the Boolean algebra $\mathbb{B}(\mathsf{S})$ (as defined in Definition 3). For example, for formulas $\varphi, \psi \in \mathcal{L}_\mathsf{S}$ we have that

$$\begin{array}{lll} \varphi \text{ and } \psi \text{ are } \mathsf{S}\text{-contrary} & \text{iff} & \mathsf{S} \models \neg(\varphi \wedge \psi) \text{ and } \mathsf{S} \not\models \varphi \vee \psi \\ & \text{iff} & [\varphi \wedge \psi]_\mathsf{S} = \bot \text{ and } [\varphi \vee \psi]_\mathsf{S} \neq \top \\ & \text{iff} & [\varphi]_\mathsf{S} \wedge [\psi]_\mathsf{S} = \bot \text{ and } [\varphi]_\mathsf{S} \vee [\psi]_\mathsf{S} \neq \top \\ & \text{iff} & [\varphi]_\mathsf{S} \text{ and } [\psi]_\mathsf{S} \text{ are } \mathbb{B}(\mathsf{S})\text{-contrary.} \end{array}$$

In this way, Definition 3 is still able to deal with the logic-sensitivity of the Aristotelian relations. For example, recall that based on Definition 2, it is possible to have distinct logical systems S_1 and S_2 with the same object language $\mathcal{L}$, and formulas $\varphi, \psi \in \mathcal{L}$, such that φ and ψ are S_1-contrary, but S_2-contradictory. Since S_1 and S_2 are distinct logical systems, they yield distinct equivalence relations $\equiv_{\mathsf{S}_1}$ and $\equiv_{\mathsf{S}_2}$, and hence will have distinct Lindenbaum-Tarski algebras $\mathbb{B}(\mathsf{S}_1) = \mathcal{L}/{\equiv_{\mathsf{S}_1}}$ and $\mathbb{B}(\mathsf{S}_2) = \mathcal{L}/{\equiv_{\mathsf{S}_2}}$. Consequently, based on Definition 3 it is possible that $[\varphi]_{\mathsf{S}_1}$ and $[\psi]_{\mathsf{S}_1}$ are $\mathbb{B}(\mathsf{S}_1)$-contrary, whereas $[\varphi]_{\mathsf{S}_2}$ and $[\psi]_{\mathsf{S}_2}$ are $\mathbb{B}(\mathsf{S}_2)$-contradictory.

The first two (families of) instances of Definition 3 arise from taking $\mathbb{B}$ to be the powerset $\wp(X)$ of some set X, or the Lindenbaum-Tarski algebra $\mathbb{B}(\mathsf{S})$ of some logical system S. We can also combine both strategies, and take $\mathbb{B}$ to be $\wp(\mathbb{B}(\mathsf{S}))$, i.e. the powerset of the Lindenbaum-Tarski algebra of some logical system S. In this way, Definition 3 is able to accommodate Aristotelian relations between *metalogical* properties and statements. Consider, for example, the sets

[6] This definition also accommodates contemporary applications of the Aristotelian relations in artificial intelligence. For example, given a binary relation $R \subseteq X \times Y$ and a set $S \subseteq Y$, Ciucci et al. (2016, p. 355) define $R(S) := \{x \in X \mid \exists s \in S \colon xRs\}$, $R(\overline{S}) := \{x \in X \mid \exists s \in Y \backslash S \colon xRs\}$, and various other subsets of X. They then go on to analyze the Aristotelian relations between these sets; for example, under certain conditions, $R(S)$ and $R(\overline{S})$ are subcontrary to each other, because $R(S) \cap R(\overline{S}) \neq \emptyset$ and $R(S) \cup R(\overline{S}) = X$ (2016, p. 356). This can be seen as yet another instance of Definition 3, by taking $\mathbb{B}$ to be $\wp(X)$.

[7] Keynes talks about *seven* relations, because in addition to the four usual Aristotelian relations, he is considering three others. However, this difference is further irrelevant for our current purposes.

$A := \{[\varphi]_{\mathsf{S}} \mid \mathsf{S} \models \varphi\} = \{\top\}$ and $B := \{[\varphi]_{\mathsf{S}} \mid \mathsf{S} \models \neg\varphi\} = \{\bot\}$. If the logical system S is consistent, there exist no (equivalence classes of) formulas that are simultaneously S-tautologies and S-contradictions ($A \cap B = \emptyset$), while there is at least one (equivalence class of) formula(s) that is neither an S-tautology nor an S-contradiction ($A \cup B \neq \wp(\mathbb{B}(\mathsf{S}))$). This means exactly that the metalogical properties of being an S-tautology and being an S-contradiction are $\wp(\mathbb{B}(\mathsf{S}))$-contrary to each other.

Instances of Definition 3 at different 'levels' can also interact with each other. We have already shown that Definition 2 is a special case of Definition 3, by taking $\mathbb{B}$ to be $\mathbb{B}(\mathsf{S})$. However, it will also be interesting to take $\mathbb{B}$ to be $\wp(\mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S}))$, so that elements of $\mathbb{B}$ are subsets of $\mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S})$, i.e. binary relations over $\mathbb{B}(\mathsf{S})$. Consider, for example, the relations $A := \{([\varphi]_{\mathsf{S}}, [\psi]_{\mathsf{S}}) \mid [\varphi]_{\mathsf{S}} \text{ and } [\psi]_{\mathsf{S}} \text{ are } \mathbb{B}(\mathsf{S})\text{-contrary}\}$ and $B := \{([\varphi]_{\mathsf{S}}, [\psi]_{\mathsf{S}}) \mid [\varphi]_{\mathsf{S}} \text{ and } [\psi]_{\mathsf{S}} \text{ are } \mathbb{B}(\mathsf{S})\text{-subcontrary}\}$. There exist no pairs of equivalence classes of formulas that are simultaneously $\mathbb{B}(\mathsf{S})$-contrary and $\mathbb{B}(\mathsf{S})$-subcontrary to each other ($A \cap B = \emptyset$), while there is at least one pair of equivalence classes of formulas that are neither $\mathbb{B}(\mathsf{S})$-contrary nor $\mathbb{B}(\mathsf{S})$-subcontrary to each other ($A \cup B \neq \wp(\mathbb{B}(\mathsf{S}))$). This means exactly that the Aristotelian relations of $\mathbb{B}(\mathsf{S})$-contrariety and $\mathbb{B}(\mathsf{S})$-subcontrariety — which are themselves already metalogical in nature; cf. supra — are, 'at a higher level', $\wp(\mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S}))$-contrary to each other. Interestingly, a similar idea can already be found in the *Summulae Logicales* of the 13th-century philosopher Petrus Hispanus: after he has given the definitions (which he calls 'laws') of contrariety and subcontrariety, Hispanus writes that "the law of subcontraries is itself contrary to the law of contraries" (my translation; original Latin text: "lex subcontrariarum contrario modo se habet legi contrariarum"; Copenhaver et al. 2014, p. 112).[8]

2.4 Philosophical Discussion

In the previous subsections, I have discussed three, increasingly more abstract ways of defining the Aristotelian relations, and compared their various advantages and disadvantages. The results of this comparative analysis are summarized in the following table:

[8] However, it can be argued that Hispanus actually did not mean to suggest that contrariety and subcontrariety are *contrary* to each other, but rather that they are each other's *internal negation*. See Demey and Smessaert (2017a, 2018c) for the distinction between the two types of relations, and Copenhaver et al. (2014, p. 113, Footnote 16) and Demey and Smessaert (2016a, p. 275, Footnote 43) for further discussion about this subtle interpretation issue.

	Definition 1 (informal)	Definition 2 (relative to S)	Definition 3 (relative to $\mathbb{B}$)
modal nature	yes	yes	not explicit (but Def. 2 as special case)
logic-sensitivity	no	yes	yes
metalogical relata	yes	no	yes
scope	statements	statements	statements, predicates, sets, ⋮

With this overview in place, I will now finish this section by making some broader methodological and philosophical points.

First of all, based on the discussion and the table above, it should be clear that Definition 3 achieves the best balance between the *specificity* of the Aristotelian relations on the one hand, and the broad *diversity* of potential relata on the other. For example, it enables us to deal with Aristotelian relations holding between propositions (sets of possible worlds), properties (sets of individuals), sets induced by a binary relation (cf. Footnote 6), object-logical formulas, metalogical properties, etc., and it explains both the commonalities and the differences between these different types of relations (all of which arise by plugging in different concrete Boolean algebras for the abstract $\mathbb{B}$).

This unificatory power is not only important from a historical perspective (as is illustrated by the quotations by Hispanus and Keynes given above), but it also sheds new light on the widespread use of the Aristotelian relations (and the diagrams visualizing them) today. For example, Dubois et al. (2015, p. 2933) make use of a certain Aristotelian diagram to "exhibit fruitful parallelisms between different formalisms" in artificial intelligence, and Demey and Smessaert (2018a, p. 35) argue that Aristotelian diagrams constitute a language that enables us to "explore unexpected connections between *prima facie* unrelated areas of logic", comparing their role with that of category theory in the field of mathematics (Landry, 1999). In order to fulfill this heuristic role, it is absolutely crucial that Aristotelian diagrams be very broadly applicable, while maintaining the specific characteristics of the relations that they visualize. Definition 3 shows exactly how the Aristotelian relations achieve this balance between specificity and broad applicability.

By focusing on the abstract notion of a Boolean algebra, Definition 3 also provides the mathematical background for the technique of *bitstring semantics*, which plays a central role in logical geometry, and is based on representations of finite Boolean algebras (Demey and Smessaert, 2018a; Smessaert and Demey, 2017). Furthermore, Definition 3 naturally opens up the way for alternative, more general versions of the Aristotelian relations. Several authors have recently proposed to study Aristotelian relations in the context of non-classical logics, which yield Lindenbaum-Tarski algebras that are *not* Boolean algebras; for example, Mélès

(2012) considers the case of intuitionistic logic, whereas Ciucci et al. (2016) consider various many-valued logics. Based on Definition 3, it is to be expected that the natural mathematical settings to study such generalizations will be those of *Heyting algebras* and *MV-algebras*, respectively.

The final observation concerns the importance of the *powerset* operation with respect to object- and metalogical applications of the Aristotelian relations.[9] We have seen that Aristotelian relations between object-logical formulas (of some logical system S) can be obtained from Definition 3 by plugging in the Lindenbaum-Tarski algebra $\mathbb{B}(\mathsf{S})$. Furthermore, we have also seen that Aristotelian relations between metalogical properties (with respect to the same system S) are obtained by plugging in $\wp(\mathbb{B}(\mathsf{S}))$. By applying the powerset operation (to the Lindenbaum-Tarski algebra $\mathbb{B}(\mathsf{S})$),[10] we have thus jumped from the object- to the metalogical level. Of course, we can keep on repeating this process, thereby climbing higher and higher in the hierarchy of metalanguages (for S).

The powerset operation plays an analogous role in axiomatic set theory, where it is used (in the successor ordinal case of a transfinite induction) to define the *cumulative hierarchy of sets* (Devlin 1993, p. 38; Jech 2003, p. 64):[11]

$$\begin{array}{lcll} V_0 & = & \emptyset & \\ V_{\alpha+1} & = & \wp(V_\alpha) & \\ V_\alpha & = & \bigcup_{\beta<\alpha} V_\beta & \quad \text{(if } \alpha \text{ is a limit ordinal)} \end{array}$$

We thus have a hierarchy of logical languages on the one hand, and a hierarchy of sets on the other, with the powerset operation playing a crucial role in moving from one level to the next in both of these hierarchies. This fundamental analogy between the semantics of metalanguages and set theory is also noted by Priest (2006). After discussing "the metalinguistic ascent [:] the constructions inherent in our semantic concepts force us, given any semantically open theory, to ascend to a stronger metalanguage to express certain facts about it" (2006, p. 38), and noting that "given any well founded totality, constructions inherent in our set theoretic concepts, and in particular the powerset operation, force us into a similar ascent, this time, in effect, up the cumulative hierarchy" (2006, p. 38), Priest draws the following conclusion:

> these two ascents, despite different appearances, are closely related. For, since Tarski, we know what set theoretic machinery we need to

[9] I would like to thank the audience of the workshop in Louvain-la-Neuve for some very useful discussion about this point.

[10] Note that I am here *not* concerned with applying the powerset operation to some set X of possible worlds or of individuals. The main difference is that, for the Lindenbaum-Tarski algebra $\mathbb{B}(\mathsf{S})$, both $\mathbb{B}(\mathsf{S})$ itself and $\wp(\mathbb{B}(\mathsf{S}))$ are Boolean algebras (and can thus both be plugged into Definition 3), whereas for a set X of possible worlds/individuals, $\wp(X)$ is a Boolean algebra, but X itself is not.

[11] It is standardly assumed that the cumulative hierarchy follows from a certain philosophical perspective on what sets are, viz. the *iterative conception of set* (Boolos, 1971; Incurvati, 2012). However, Forster (2008) argues that the iterative conception of set is actually broader than (i.e. compatible with other sorts of sets than just those in) the cumulative hierarchy. Since this discussion is irrelevant for our current purposes, I will not go into it any further.

define appropriate semantic notions for a theory T: second order T, that is, the theory whose intended interpretation is the powerset of T. Hence, in a sense, there is only one construction [viz. the powerset operation, LD] which pushes us ever on to bigger and better things (if we wish to remain consistent), which may manifest either a set theoretic or a semantic aspect (2006, p. 38)

In light of this discussion, then, it should be clear that the distinction between plugging in $\mathbb{B}(\mathsf{S})$ versus $\wp(\mathbb{B}(\mathsf{S}))$ into Definition 3 is "just our old friend, the language/metalanguage distinction set theoretically writ" (2006, p. 38).

3 The Pragmatics of Metalanguage

I have just discussed a theoretical perspective on the metalogical aspects of Aristotelian diagrams and relations. In this section, I will show how this theoretical perspective sheds new light on certain linguistic issues (e.g. implicatures, homophony, lexicalization, etc.) that crop up in the metalinguistic terminology we use to describe the Aristotelian relations. For example, Horn (2004, p. 11) characterizes the contrariety relation as follows: "Contraries [are formulas that] cannot be simultaneously true (though they may be simultaeously false)". The linguistic phenomena to be addressed in this section are related to the ambiguity that Horn creates by putting the second condition between brackets: is this condition an essential part of the definition of contrariety, or can it also be left out?[12]

In Subsection 3.1 I will present some basic natural language phenomena, describe the (neo-)Gricean pragmatic theory concerning these phenomena, and emphasize the important role of Aristotelian diagrams in this theory. Next, in Subsection 3.2 I will show that these linguistic phenomena also occur in the technical jargon of a logical metalanguage, and argue that the same pragmatic theory and the same type of Aristotelian diagrams also apply to such a metalanguage.

3.1 Aristotelian Diagrams in Pragmatics

The natural language quantifier *some* is famously ambiguous between a *unilateral* and a *bilateral* reading. On the unilateral interpretation, *some* means *at least one*, which is formalized in standard first-order logic as the existential quantifier ($\exists$). On this reading, *some* is compatible with *all*, i.e. the truth of *some As are B* does not entail that *all As are B* is false. However, in many everyday contexts, people often seem to prefer the bilateral interpretation of *some*, taking this word to mean *at least one but not all*. Obviously, on its bilateral reading, *some* is effectively incompatible with *all*. Nevertheless, it is unclear how strong this incompatibility

[12] Horn's definition is informal, and thus most in line with our Definition 1. However, the same ambiguity (and thus the same types of linguistic issues) also arise for the more formal Definitions 2 and 3. In particular, in Subsection 3.2 I will mainly work with the Aristotelian relations as characterized in Definition 2.

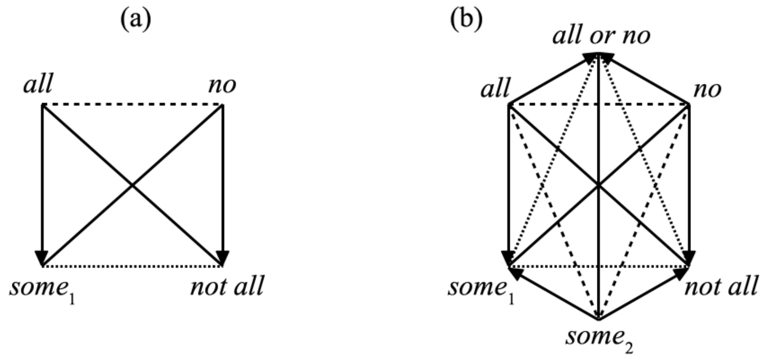

Figure 1: (a) Square of opposition and (b) JSB hexagon for the natural language quantifiers.

actually is. For example, the *some* in sentence (1) below is strongly incompatible with *all*: this sentence clearly seems to imply that not all students passed the test. By contrast, the *some* in (2) does seem to be compatible with *all*: if, as a matter of fact, all students get infected, then (2) still applies, and the school should still be closed.

Some students passed the test. (1)

If some of our students get infected, we'll close the school. (2)

Finally, it is well-known among linguists that, whereas *not some* is lexicalized as *no* (i.e. *it is not the case that some As are B* $\equiv$ *no As are B*), the expression *not all* is not 'primitively' (i.e. by means of a single word) lexicalized at all in English. Furthermore, this observation seems to generalize to all natural languages spoken across the world throughout history. For example, it already applied to Latin (which has *nullus* for *non aliquis*, but no single word for *non omnis*), as was first observed by Thomas Aquinas in his commentary on Aristotle's *De Interpretatione* (Oesterle, 1962; Horn, 1989).

In order to explain this cluster of observations, linguists and philosophers often make use of Aristotelian diagrams. The four quantifiers discussed above give rise to a square of opposition, as shown in Figure 1(a). Note that the word *some* appears in this square with its unilateral reading, as is indicated by the 1-subscript. For ease of notation, the square only displays the quantifier expressions themselves, but these can easily be expanded into full sentences (e.g. *all As are B*, *some As are B*, etc.). Since the elements of this square are entirely informal, natural language sentences, the Aristotelian relations between these sentences are defined according to the informal Definition 1 from Section 2.

The subalternation on the left side of this square of opposition says that *all* implies *some*, but not vice versa.[13] In terms of information contents, this means that

[13]It is well-known that the implication from *all As are B* to *some As are B* rests on the assumption

all is strictly more informative than *some* (informally: *all As are B* is true in strictly fewer possible worlds than *some As are B*).[14] Based on Grice's (1967 / 1989) principle of cooperation, and in particular the maxim of quantity that he derives from it, it follows that if someone says *some*, this generates an implicature of the form *not all*. After all, if the speaker meant *all*, then she should have explicitly said so, in order to make her utterance as informative as possible (following the maxim of quantity). Given that the speaker chose to use the less informative *some*, we are pragmatically entitled to conclude *not all*. This is a pragmatic implicature, rather than a deductive inference, and can thus be cancelled without yielding a logical contradiction. All of this can be summarized by saying that $\langle$*all*, *some*$_1\rangle$ constitutes a *Horn scale* (Horn, 1989, 2004). More generally, if we have a Horn scale $\langle A_1, A_2, \ldots, A_n\rangle$, with each element A_i strictly more informative than A_{i+1} (for $1 \leq i < n$), and someone utters A_i, then this generates the *scalar implicature* that $\neg A_j$ (for each $1 \leq j < i$). From a pragmatic perspective, the left side of the square of opposition in Figure 1(a) thus constitutes a Horn scale (van der Auwera, 1996). Similar remarks were already made by Doyle (1951, p. 382), in his comparison of the square of opposition with other Aristotelian diagrams for the natural language quantifiers.

If we consider the conjunction of the unilateral *some*$_1$ with its scalar implicature *not all*, we find exactly *some*$_1$ *but not all*, i.e. the bilateral interpretation *some*$_2$. In other words, based on the theory of scalar implicatures, we find that the bilateral interpretation of *some* incorporates the implicature of its unilateral interpretation, i.e. the former is the *pragmatic strengthening* of the latter (Traugott, 1988). This explains the homophony and co-lexicalization of both meanings (Cruse 1986, p. 256; Seuren and Jaspers 2014, p. 608). The unilateral and bilateral interpretations of *some* agree that *at least one* is (part of) the semantic content of *some*, but they disagree on the linguistic status of the *not all* meaning aspect: the bilateral reading (*some*$_2$) also includes it in the semantic content of *some*, whereas the unilateral reading (*some*$_1$) derives it as a pragmatic implicature:

	at least one	*not all*
unilateral interpretation	semantic content	pragmatic implicature
bilateral interpretation	semantic content	semantic content

The bilateral *some*$_2$ is often added to the square of opposition, together with its negation, *all or no* (in order to maintain closure under negation). The Aristotelian diagram that is obtained in this way is a so-called Jacoby-Sesmat-Blanché (JSB) hexagon, as shown in Figure 1(b).[15] From a logical perspective, this hexagon is the

of *existential import*, i.e. the assumption that there actually exists at least one A. However, in most everyday contexts, this assumption is fairly unproblematic, and it is further irrelevant for our current purposes in this paper.

[14]This illustrates the *inverse correlation* between degree of information and range of application, as studied in the philosophy of information (Smessaert and Demey, 2014).

[15]The JSB hexagon is named after Jacoby (1950), Sesmat (1951) and Blanché (1966). See Jaspers and Seuren (2016) for more historical background.

Boolean closure of the square of opposition in Figure 1(a), i.e. it contains precisely the contingent Boolean combinations of elements from the square.

The pragmatic theory outlined above can also be used to explain the non-lexicalization of *not all*, i.e. the so-called O-corner of the square of opposition (Horn, 1989, 2012). However, in light of the extension of this square into a JSB hexagon (in order to incorporate $some_2$), it is theoretically desirable to be able to explain the non-lexicalization of *all or no* (i.e. the so-called U-corner of the hexagon) as well (Jaspers, 2012). Seuren and Jaspers (2014) propose a theory that simultaneously explains the non-lexicalization of *not all* as well as *all or no*. Their theory is based on a *recursive partitioning process* of logical space. Based on various types of linguistic evidence, Seuren and Jaspers argue that the most primitive distinction in the realm of quantification is the binary distinction between the negative quantifier *no* and the positive quantifier $some_1$. Next, within the positive 'subuniverse' corresponding to $some_1$, there is a further binary distinction between *all* and $some_2$.[16] This recursive partitioning process ultimately yields a tripartition of logical space, as shown in Figure 2(a).[17] The key prediction is now that a meaning is primitively lexicalized iff it occurs at any stage of this partitioning process. This accounts for the lexicalization of *no*, $some_1$, *all* and $some_2$. These four expressions jointly constitute the lexicalized 'kite' diagram shown in Figure 2(b), which can be seen as a subdiagram of the JSB hexagon shown in Figure 1(b) (Seuren and Jaspers, 2014, p. 621ff.). By contrast, meanings that can only be obtained by combining an element from the positive subuniverse with the negative element *no* are not lexicalized. In particular, the O-corner (*not all* $\equiv$ $some_2 \vee no$) and the U-corner ($all \vee no$) are not lexicalized.

In this subsection I have focused exclusively on the lexical field of quantifier expressions. However, the theoretical framework outlined above also applies to various other lexical fields in natural language, such as the connectives (*and*, *or*, *neither ... nor*), the alethic modalities (*necessary*, *possible*, *impossible*), the deontic modalities (*obligatory*, *permitted*, *forbidden*), and even to less 'logic-oriented' domains, such as those describing living things (*human*, *animal*, *plant*) and sexual orientations (*lesbian*, *gay*, *straight*). In the latter domains, the account can be used to explain the ambiguity of words such as *animal* (as either including or excluding humans) and *gay* (as either comprising all homosexuality, or only male

[16] In Seuren and Jaspers's informal approach, logical space can be seen as a class of possible worlds, each of which makes every given sentence either true or false. The first binary distinction of the partitioning process then corresponds to the distinction between the possible worlds that make *no As are B* true and those that make $some_1$ *As are B* true. The second binary distinction takes place within the subclass of possible worlds that make $some_1$ *As are B* true, and distinguishes between the possible worlds that make *all As are B* true and those that make $some_2$ *As are B* true.

[17] The end result being a tripartition is entirely natural, since it is well-known in logical geometry that the Boolean closure of a square of opposition corresponds precisely to a tripartition of some underlying class (Demey and Smessaert, 2018a). Although all examples given in Seuren and Jaspers (2014) are based on tripartitions, this is not a hard limitation of their theory. For example, Roelandt (2016) has extended the theory to partitioning processes that yield more fine-grained partitions of logical space, and used them to analyze lexical fields such as the measure adjectives.

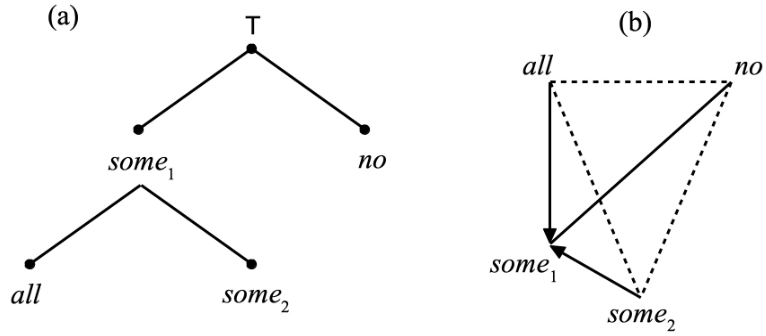

Figure 2: (a) Recursive partitioning process and (b) kite diagram for the lexicalized natural language quantifiers.

homosexuality; i.e. as either including or excluding lesbianism), and to explain the absence of (primitive) lexicalizations of meanings such as *human or plant* and *lesbian or straight*. Further examples and more detailed discussion (including the corresponding Aristotelian diagrams) can be found in Seuren and Jaspers (2014, p. 626ff.).

3.2 Pragmatics in Aristotelian Diagrams

I now turn to the metalinguistic terminology that is used for the Aristotelian relations.[18] In particular, I will focus on (the term for) the relation of *contrariety*, but similar remarks apply to *subcontrariety* and *subalternation*. Furthermore, I will only discuss the formal characterization of the Aristotelian relations as given in Definition 2, but again, similar remarks could be made about the characterizations provided by Definitions 1 and 3 (also recall Footnote 12).

Throughout the history of philosophical logic, the Aristotelian relation of contrariety has been defined in two clearly different, yet interrelated ways, yielding a *strong* and a *weak* notion of contrariety. In more linguistic terms: the word *contrary* is ambiguous between a strong and a weak interpretation. The strong interpretation was traditionally the most popular one (especially in medieval logic), and has recently been used by authors such as Smessaert (2009) and Parsons (2012). The weak interpretation has mainly been used from the 20th century onwards, by authors such as Bochenski (1959), McCall (1967) and Seuren (2010). The strong definition of contrariety, which was already introduced in Section 2, consists of a $\models$- and a $\not\models$-condition; the weak definition keeps the former condition, but discards the latter.

[18]We have seen in Section 2 that the Aristotelian relations are *metalogical* in nature. Consequently, the specific terminology used to describe these relations is itself *metalinguistic* in nature. (Also recall Footnote 1.)

Definition 4. Let S be a logical system as in Definition 2. Two formulas $\varphi, \psi \in \mathcal{L}_\mathsf{S}$ are said to be

$$\begin{array}{lll} \textit{strongly } \mathsf{S}\textit{-contrary} & \text{iff} & \mathsf{S} \models \neg(\varphi \wedge \psi) \quad \text{and} \quad \mathsf{S} \not\models \varphi \vee \psi, \\ \textit{weakly } \mathsf{S}\textit{-contrary} & \text{iff} & \mathsf{S} \models \neg(\varphi \wedge \psi). \end{array}$$

From Definitions 2 and 4, it follows immediately that the weak notion of contrariety is compatible with contradiction, i.e. two formulas being weakly S-contrary to each other does not entail that these formulas cannot be S-contradictory to each other. By contrast, the strong notion of contrariety is incompatible with contradiction: if two formulas are strongly S-contrary to each other, then they cannot be S-contradictory.

Furthermore, it is well-known that the negation of weak contrariety can alternatively be expressed in terms of *compatibility*: saying that φ and ψ are not weakly contrary to each other is equivalent to saying that φ and ψ are compatible with each other (formally: $\mathsf{S} \not\models \neg(\varphi \wedge \psi)$; informally: φ and ψ can be true together).[19] By contrast, the negation of contradiction is not lexicalized at all in our metalinguistic jargon: there does not seem to be a single term which expresses that φ and ψ are not contradictory to each other, nor a single term which expresses that φ and ψ are not strongly contrary to each other.[20]

In recent years, the distinction between strong and weak contrariety has itself become the topic of several logical investigations. For example, Humberstone (2011) links the differences between these two notions to the differences between 'traditionalist' and 'modernist' approaches to logic. Furthermore, Béziau (2012) and Demey and Smessaert (2014b, 2016a) show that the strong and weak notions of (sub)contrariety give rise to new, metalogical decorations for various Aristotelian diagrams. Two typical examples include the metalogical square of opposition and JSB hexagon that are shown in Figure 3. Note that the term *contrary* appears in the square with its weak interpretation, as is indicated by the *w*-subscript; in the JSB hexagon it also appears with its strong interpretation, as is indicated by the *s*-subscript.

Before we continue, it is important to emphasize that the Aristotelian diagrams shown in Figure 3 are fully in line with the theoretical perspective on the Aristotelian relations that was described in Section 2. Consider, for example, the relations of S-contradiction and strong S-contrariety, which appear in resp. the upper left vertex and the lower vertex of the JSB hexagon in Figure 3(b). For ease of notation, we will abbreviate these relations as *CD* and C_s, respectively. In line with Definition 2, these are binary relations over $\mathcal{L}_\mathsf{S}$, but since they are defined up to

[19] Weak compatibility is thus seen as the negation of contrariety. Vice versa, one can also view weak contrariety as the negation of compatibility, i.e. two formulas are weakly contrary to each other iff they are not compatible with each other. From this perspective, *contrariety* (on its weak reading) is thus synonymous to *incompatibility*.

[20] Smessaert and Demey (2014) define the relation of *non-contradiction*, but that notion is strictly stronger than the negation of contradiction. In particular, φ and ψ not being contradictory means that not[$\mathsf{S} \models \neg(\varphi \wedge \psi)$ and $\mathsf{S} \models \varphi \vee \psi$], i.e. $\mathsf{S} \not\models \neg(\varphi \wedge \psi)$ *or* $\mathsf{S} \not\models \varphi \vee \psi$, whereas φ and ψ being non-contradictory is defined as $\mathsf{S} \not\models \neg(\varphi \wedge \psi)$ *and* $\mathsf{S} \not\models \varphi \vee \psi$.

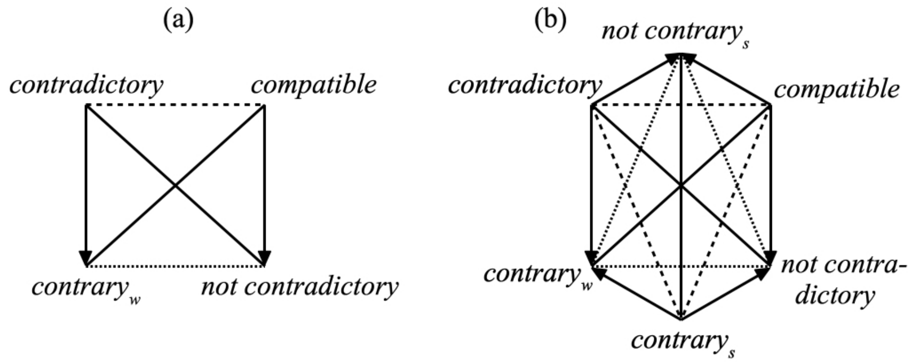

Figure 3: (a) Square of opposition and (b) JSB hexagon for strong/weak contrariety and related notions.

logical equivalence, we can also view them as binary relations over $\mathbb{B}(\mathsf{S})$ — recall from Section 2 that $\mathbb{B}(\mathsf{S}) = \mathcal{L}_\mathsf{S}/\equiv_\mathsf{S}$ is the Lindenbaum-Tarski algebra of S — i.e. we have $CD, C_s \subseteq \mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S})$, and thus also $CD, C_s \in \wp(\mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S}))$. It is easy to show that (i) $CD \cap C_s = \emptyset$ and (ii) $CD \cup C_s \neq \mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S})$,[21] which means exactly (by Definition 3) that *CD* and C_s are contrary to each other in the Boolean algebra $\wp(\mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S}))$. It is precisely this $\wp(\mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S}))$-contrariety that is represented by the edge that connects the vertices for *contradictory* and *contrary*$_s$ in the JSB hexagon.

The diagrams in Figure 3 thus represent Aristotelian relations from two distinct logical levels. On the one hand, there are the Aristotelian relations (and their complements) for the logical system S — or equivalently, for the Boolean algebra $\mathbb{B}(\mathsf{S})$ —, which appear on the *vertices* of the diagrams. On the other hand, there are the Aristotelian relations for the Boolean algebra $\wp(\mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S}))$, which appear on the *edges* of the diagrams. At the level of $\mathbb{B}(\mathsf{S})$, we are interested in both the strong and the weak notion of contrariety, and in their logico-linguistic interplay; cf. the vertices for *contrary*$_s$ and *contrary*$_w$ in the diagrams. However, at the level of $\wp(\mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S}))$, we exclusively use the *strong* notion of contrariety, as defined in Definition 3.[22]

Now that the metalogical status of the Aristotelian diagrams in Figure 3 has been clarified, I turn to their logico-linguistic significance. After all, upon visual inspection of Figures 1 and 3, it should immediately be obvious that there are clear similarities between the Aristotelian diagrams for the unilateral/bilateral interpretations of *some* and those for the weak/strong interpretations of *contrary*. This

[21]For (ii), take any formulas $\varphi, \psi \in \mathcal{L}_\mathsf{S}$ that are S-compatible with each other; it then follows easily that $([\varphi]_\mathsf{S}, [\psi]_\mathsf{S}) \notin CD$ and $([\varphi]_\mathsf{S}, [\psi]_\mathsf{S}) \notin C_s$.

[22]Of course, in choosing for the strong interpretations of the Aristotelian relations in arbitrary Boolean algebras (i.e. in Definition 3), I am aligning myself with what Humberstone (2011) would call the 'traditionalist' approach to logic. However, one could also opt for the weak interpretations of the Aristotelian relations in Definition 3. Nothing of importance in the remainder of my argument hinges on this choice.

strongly suggests that similar linguistic principles are at work in both lexical fields, and in particular, that the pragmatic account described in Subsection 3.1 is also applicable here.

The subalternation on the left side of the square of opposition in Figure 3(a) can once again be cast in terms of information levels — with contradiction being strictly more informative than weak contrariety (Smessaert and Demey, 2014) —, and thus once again constitutes a Horn scale: $\langle$*contradictory*, *weakly contrary*$\rangle$. If someone says that the formulas φ and ψ are weakly contrary to each other, this generates the scalar implicature that φ and ψ are not contradictory to each other. After all, if the speaker meant that φ and ψ are effectively contradictory, then she should have explicitly said so, in order to make her utterance as informative as possible (following the maxim of quantity). Given that the speaker chose to use the less informative notion of *weak contrariety*, we are pragmatically entitled to conclude *not contradictory*.

If we take the conjunction of weak $contrary_w$ with its scalar implicature *not contradictory*, we obtain exactly the strong interpretation $contrary_s$. After all, by means of straightforward Boolean reasoning, we find that

$$
\begin{array}{ll}
 & \varphi \text{ and } \psi \text{ are weakly S-contrary and not S-contradictory} \\
\text{iff} & \mathsf{S} \models \neg(\varphi \wedge \psi) \text{ and not}[\mathsf{S} \models \neg(\varphi \wedge \psi) \text{ and } \mathsf{S} \models \varphi \vee \psi] \\
\text{iff} & \mathsf{S} \models \neg(\varphi \wedge \psi) \text{ and } [\mathsf{S} \not\models \neg(\varphi \wedge \psi) \text{ or } \mathsf{S} \not\models \varphi \vee \psi] \\
\text{iff} & \mathsf{S} \models \neg(\varphi \wedge \psi) \text{ and } \mathsf{S} \not\models \varphi \vee \psi \\
\text{iff} & \varphi \text{ and } \psi \text{ are strongly S-contrary.}
\end{array}
$$

In other words, based on the theory of scalar implicatures, we find that the strong interpretation of *contrary* is exactly the pragmatic strengthening of its weak interpretation (Traugott, 1988). Just like in the case of the bi- and unilateral interpretations of *some*, this explains the homophony and co-lexicalization of both meanings (Cruse 1986, p. 256; Seuren and Jaspers 2014, p. 608). The strong and weak interpretations agree that the $\models$-condition is (part of) the semantic content of *contrary*, but they disagree on the linguistic status of the $\not\models$-condition: the strong reading ($contrary_s$) also includes it in the semantic content, whereas the weak reading ($contrary_w$) derives it as a pragmatic implicature:

	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	$\mathsf{S} \not\models \varphi \vee \psi$
weak interpretation	semantic content	pragmatic implicature
strong interpretation	semantic content	semantic content

Adding the strong $contrary_s$ (and its negation) to the square of opposition in Figure 3(a) leads to the JSB hexagon in Figure 3(b), which is the Boolean closure of the square. The O-corner (*not contradictory*) and the U-corner (*not* $contrary_s$) of this hexagon are not primitively lexicalized in our metalogical jargon, which can once again be explained by the recursive partitioning theory of Seuren and Jaspers (2014).

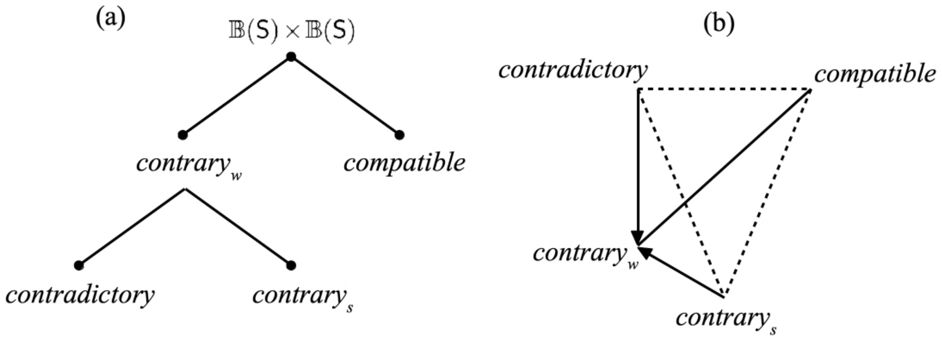

Figure 4: (a) Recursive partitioning process and (b) kite diagram for the lexicalized relations of contradiction, strong/weak contrariety and compatibility.

We begin by positing that the most primitive distinction in the metalogical realm of binary relations over $\mathbb{B}(\mathsf{S})$ is the binary distinction between the relations of weak contrariety (*contrary*$_w$) and compatibility. Next, within the 'subuniverse' corresponding to *contrary*$_w$, there is a further binary distinction between *contradictory* and *contrary*$_s$. This recursive partitioning process ultimately yields a tripartition of metalogical space, as shown in Figure 4(a). The key prediction is now, once again, that a meaning is primitively lexicalized iff its occurs at any stage of this partitioning process. This accounts for the lexicalization of *compatible*, *contrary*$_w$, *contradictory* and *contrary*$_s$. These four expressions jointly constitute the lexicalized kite diagram shown in Figure 4(b), which can be seen as a subdiagram of the JSB hexagon shown in Figure 3(b) (Seuren and Jaspers, 2014, p. 621ff.). By contrast, meanings that can only be obtained by combining elements from inside and outside of the subuniverse are not lexicalized; in particular, the O-corner (*not contradictory* $\equiv$ *contrary*$_s$ $\vee$ *compatible*) and the U-corner (*not contrary*$_s$ $\equiv$ *contradictory* $\vee$ *compatible*) are not lexicalized.

To conclude, I will address a potential worry regarding the analogy between Seuren and Jaspers's (2014) account of *some* (as described in Subsection 3.1) and the account of *contrary* presented here. In Seuren and Jaspers's recursive partitioning process for *some*, the two outcomes of the initial distinction are clearly marked as positive (*some*$_1$) and negative (*no*). Furthermore, Seuren and Jaspers explicitly state that it is (the subuniverse corresponding to) the *positive* element that needs to be further partitioned. By contrast, in the recursive partitioning process for *contrary*, it does not seem to make much sense to label *contrary*$_w$ as positive and *compatible* as negative (recall Footnote 19). However, this discrepancy is not a major issue, since the positive/negative polarity is not as central within Seuren and Jaspers's account as one might be tempted to think. For example, they themselves also offer several examples where the initial binary distinction does not yield a clearly positive and a clearly negative element (2014, p. 626), and argue that in such cases, (the subuniverses corresponding to) both elements can be further partitioned, leading to “two distinct but complementary kites” (2014, p. 627).

4 Conclusion

In this paper I have discussed metalogic and metalanguage from the perspective of logical geometry, i.e. insofar as they pertain to Aristotelian relations and diagrams. In Section 2, I have argued that the Aristotelian relations are themselves metalogical in nature, by providing a sequence of increasingly more abstract definitions. The final of these definitions, which is formulated in terms of Boolean algebras, strikes the right balance between the specificity of the Aristotelian relations and the diversity of their relata; in particular, it clearly shows how a single type of relations can hold between object-logical as well as between metalogical entities. Furthermore, this definition also suggests a clear analogy between the Tarski hierarchy of metalanguages and the cumulative hierarchy of sets, and the role of the powerset operation in both hierarchies. Next, in Section 3, I have shown that, since the Aristotelian relations and diagrams apply to meta- as well as to object-logical entities, these relations and diagrams can also be used to analyze the logico-linguistic behavior of both types of entities. In particular, I have described some issues related to the natural language quantifier *some* (e.g. homophony, lexicalization, implicatures), and emphasized the important role of Aristotelian diagrams in Horn’s and Seuren and Jaspers’s linguistic theorizing about these issues. I then noted that the same issues occur with metalinguistic terms such as *contrary*, and argued that they can be explained using similar Aristotelian diagrams and linguistic theories.

In recent years, several authors have highlighted the important heuristic role that Aristotelian diagrams can play, by enabling us to draw parallels between *prima facie* unrelated logical systems and knowledge representation formalisms (Yao, 2013; Dubois et al., 2015; Demey and Smessaert, 2018a; Demey, 2017b). So far, this claim has only been illustrated by means of examples that are entirely at the *object-logical* level, such as (the connection between) Russell’s theory of definite descriptions and public announcement logic (Demey, 2017b). However, in this paper I have argued that Aristotelian diagrams can also fruitfully be used to study parallels that cut *across* the object-/meta-level divide, such as the striking connection between the object-linguistic ambiguity of *some* and the metalinguistic ambiguity of *contrary*. Examples such as these show that the applicability of Aristotelian diagrams is significantly broader than might initially be thought, and thus provide further support for their heuristic importance in contemporary research.

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