The computation of rational Szegő quadrature formulas

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Summary

- Gauss & Szegő quadrature
  Orthogonal polynomials
  Eigenvalue problem

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Gauss QF $\longrightarrow$ Szegő QF $\longrightarrow$ Rational Szegő QF

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Gauss-Christoffel Quadrature

- n-point formula:

$$\int f(x)d\mu(x) = \sum_{k=1}^{n} \lambda_k f(z_k)$$

$\lambda_k > 0$ and z_k in support of μ.

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$$\lambda_k^{-1} = \sum_{j=0}^{n-1} |\varphi_j(z_k)|^2 = k_{n-1}(z_k, z_k)$$

  then
  polynomial domain of validity is maximal ($N = 2n - 1$).

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Gauss



Stieltjes



Christoffel

- basis $B = [b_0, b_1, \ldots]$, $\quad \mathcal{L}_n = \mathrm{span}\{B_n(x)\}$

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- reproducing kernel: $k_n(x,y) = \sum_{j=0}^{n} \varphi_j(x)\varphi_j(y) =$

$$\det \begin{bmatrix} M_{n,n} & B_n(x) \\ B_n(y)^* & 0 \end{bmatrix} / \det M_{n,n}$$

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recurrence relation

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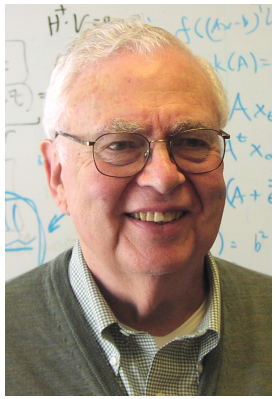
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- $\lambda_k^{-1} = \sum_{j=0}^{n-1} |\varphi_j(z_k)|^2$, hence (assuming $\varphi_0 = 1$)
  the weights are the squares of the first components of the normalized (left) eigenvectors

Gauss quadrature



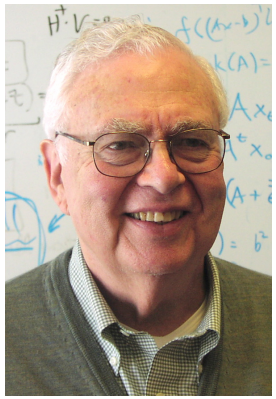
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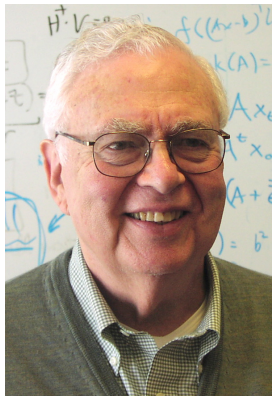
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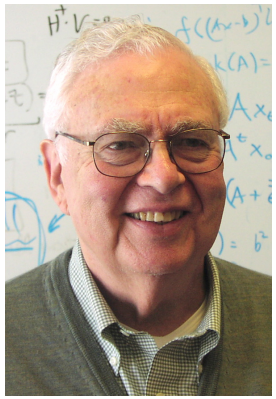
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- max domain of validity = polynomials up to degree $2n - 1$

Gauss vs. Szegő quadrature

- inner product $\langle f, g \rangle_\mu$:
  $\int_{[a,b]} f(x)g(x)d\mu(x) \leftrightarrow \int_{\mathbb{T}} \overline{f(z)}g(z)d\mu(z)$



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- zeros φ_n on $[a, b] \leftrightarrow$ zeros φ_n in $\mathbb{D}$
  Szegő QF needs para-orthogonal polynomials
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  requires a unitary modification of the Hessenberg
- exact in $\mathcal{P}_{2n-1} \leftrightarrow$ exact in $\Lambda_{n-1,n-1}$ (same dimension)
  QF keeps inner product in $\mathcal{P}_{n-1}$



Gauss



Szegő

Szegő quadrature (Hessenberg matrix)

Szegő



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Szegő quadrature (CMV matrix)

(Cantero-Moral-Velazquez 2003 = Ammar-Gragg-Reichel 1988)

Rearrangement of Givens rotations (even/odd splitting):

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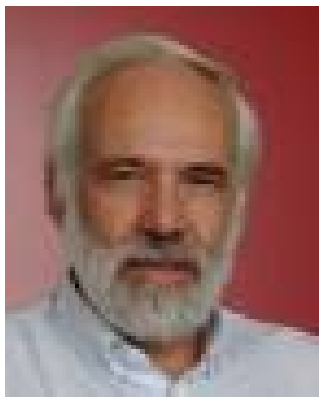
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eigenvalueproblem gives again knots and weights of Szegő QF.

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- max domain of validity: QF retains inner product in $\mathcal{L}_{n-1}$: in $\mathcal{M}_{n-1} = \operatorname{span}\{B_{n-1}^{-1}, \ldots, B_1^{-1}, B_0, B_1, \ldots, B_{n-1}\}$

Rational Szegő quadrature (Hessenberg matrix)

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$$c_n \begin{bmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{bmatrix} = \begin{bmatrix} 1 & s_n \\ \overline{s}_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z)\varphi_{n-1}(z) \\ \varphi_{n-1}^*(z) \end{bmatrix}, \quad s_n = \frac{\varphi_n(\alpha_{n-1})}{\varphi_n^*(\alpha_{n-1})}$$

Givens:



$$\Theta_k = \begin{bmatrix} -s_k & c_k \\ c_k & \overline{s}_k \end{bmatrix}, \quad \hat{\Theta}_k = \begin{bmatrix} I_{k-1} & & \\ & \Theta_k & \\ & & I \end{bmatrix}$$

$$c_k = \sqrt{1 - |s_k|^2}$$

Hessenberg:



$$H = \hat{\Theta}_1 \hat{\Theta}_2 \cdots \hat{\Theta}_n \cdots \qquad \varphi_n^*(z) = B_n(z)\overline{\varphi_n(1/\overline{z})}$$

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BUT!: $z\Phi(z) \neq \Phi(z)H$ multiply with different ζ_k for each φ_k

$\Phi(z)Z(z) = \Phi(z)H, \quad Z(z) = \mathrm{diag}(\zeta_0, \zeta_1(z), \ldots)$

Summary

- Gauss & Szegő quadrature
  Orthogonal polynomials
  Eigenvalue problem
- Rational Szegő quadrature
  Orthogonal rational functions
- (Generalized) eigenvalue problems

Eigenvalue problem

L. Velazquez (2008) rewrites

$\Phi(z)Z = \Phi(z)H, \quad Z = \mathrm{diag}(\zeta_0, \zeta_1, \ldots), \;\; \zeta_k = \frac{z-\alpha_k}{1-\overline{\alpha}_k z}$

using $A = \mathrm{diag}(\alpha_0, \alpha_1, \ldots)$ and $\eta_A = \sqrt{I - AA^*}$

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Back in business: eigenvalue problem



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Similarly for CMV.

(Generalized) Eigenvalue problems

Solve eigenvalue problem for

- $(A_n + H_n^u)(I_n + A_n^* H_n^u)^{-1}$

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- $(C_{en}^u + C_{on}^u A_n^*, C_{on}^u + C_{en}^u A_n^*)$

Examples

$d\mu(\theta) = d\theta/2\pi$ $\quad s_k = 0$ for all k

$\varphi_k(z) = \frac{\sqrt{1-|\alpha_k|^2}}{1-\overline{\alpha}_k z} B_{n-1}(z)$

only 2 points, then exact for $\int_{-\pi}^{\pi} \frac{az+b+c/z}{|z-\alpha_1|^2} \frac{d\theta}{2\pi}$

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$d\mu(\theta) = \frac{1-|r|^2}{|e^{i\theta}-r|^2} \frac{d\theta}{2\pi}$, $r \in \mathbb{D}$ $\quad s_1 = -r$, $s_k = 0$ for all $k > 1$

$\int_{-\pi}^{\pi} f(e^{i\theta}) d\mu(\theta)$,

$f(z) = \frac{1+z}{1-z/2} + \frac{z}{2-z} + \frac{z^2+3z-z^3}{(z-5)(1-z/6)(1-2z)}$

$\alpha_k = 1/(k+1)$, $k \geq 1$

Result exact for all $n \geq 5$, for all $r \in \mathbb{D}$ and for all $u_n \in \mathbb{T}$.

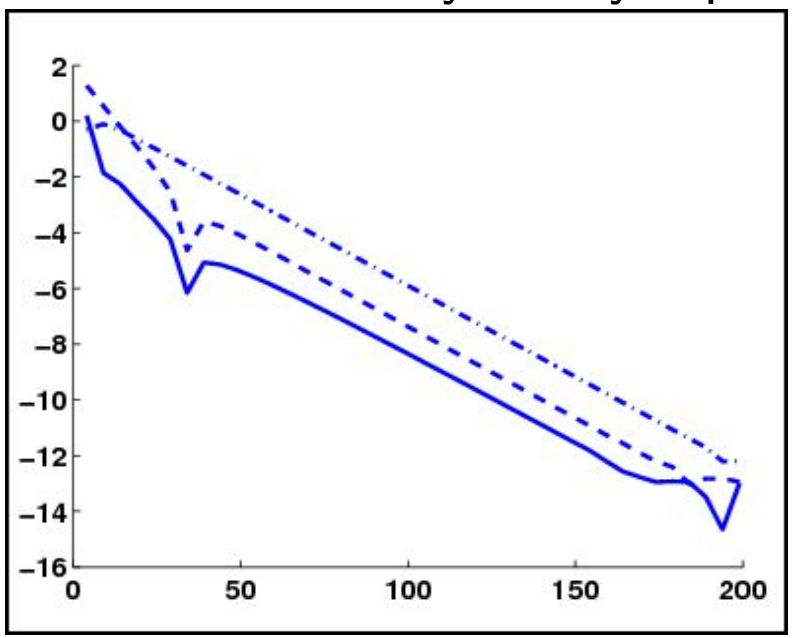
Examples

$d\mu(\theta) = (1 - \nu \cos\theta)\frac{d\theta}{2\pi}$, $\nu \in \{1, -1\}$ (Chebyshev)

s_n are known.

$\int_{-\pi}^{\pi} f(e^{i\theta}) d\mu(\theta)$, $\quad f(z) = \frac{z^p}{\sin(z^5)+0.2}$.

Approximate poles: $z^5 + 0.2 = 0$, cyclically repeated.



$\nu = 1$, $p = -1, 0, 1$, $\log_{10}$ error as function of $n = 5(5)200$

Examples

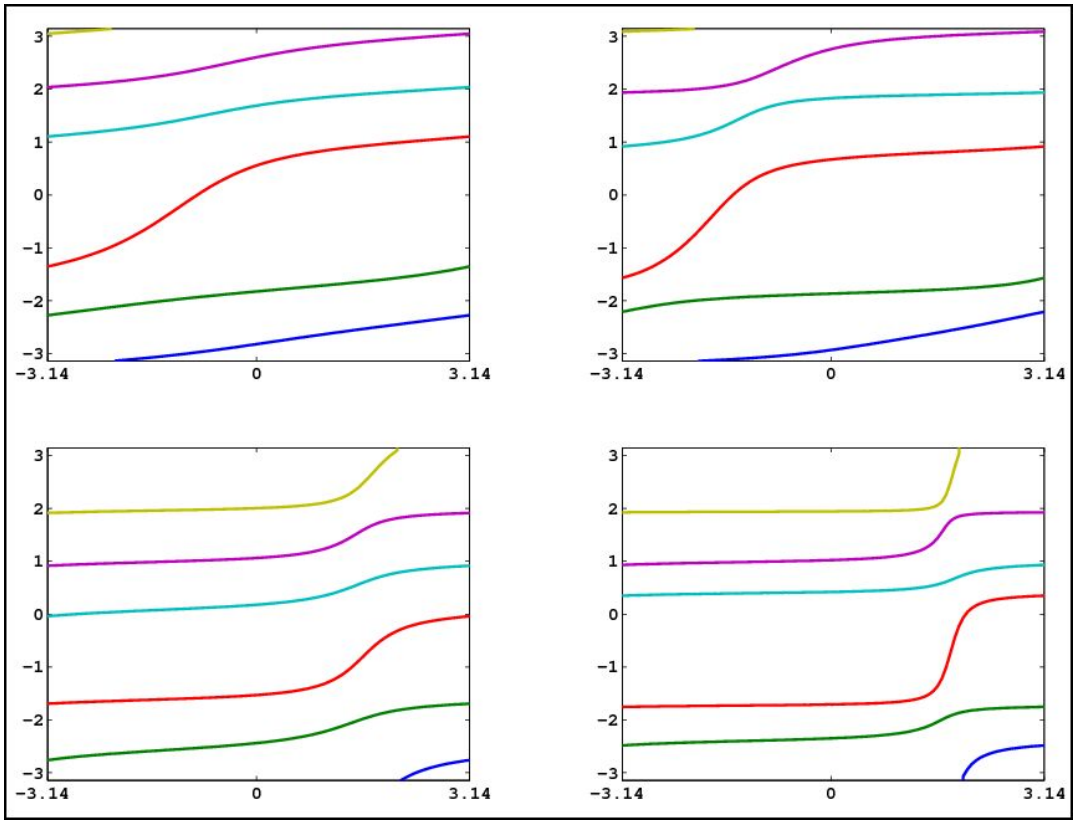


Figure: The nodes of the of the quadrature formula with $n = 5$ points as a function of $\theta \in [-\pi, \pi]$ where $v = e^{i\theta}$. The measure is $d\mu(\theta) = (1 - \nu\cos(\theta))d\theta/(2\pi)$. The poles are given by $r(-1)^{1/5}$. On the left for $r = 0.72$ and on the right for $r = 0.9$. On top for $\nu = 1$, at the bottom for $\nu = -1$.

Examples

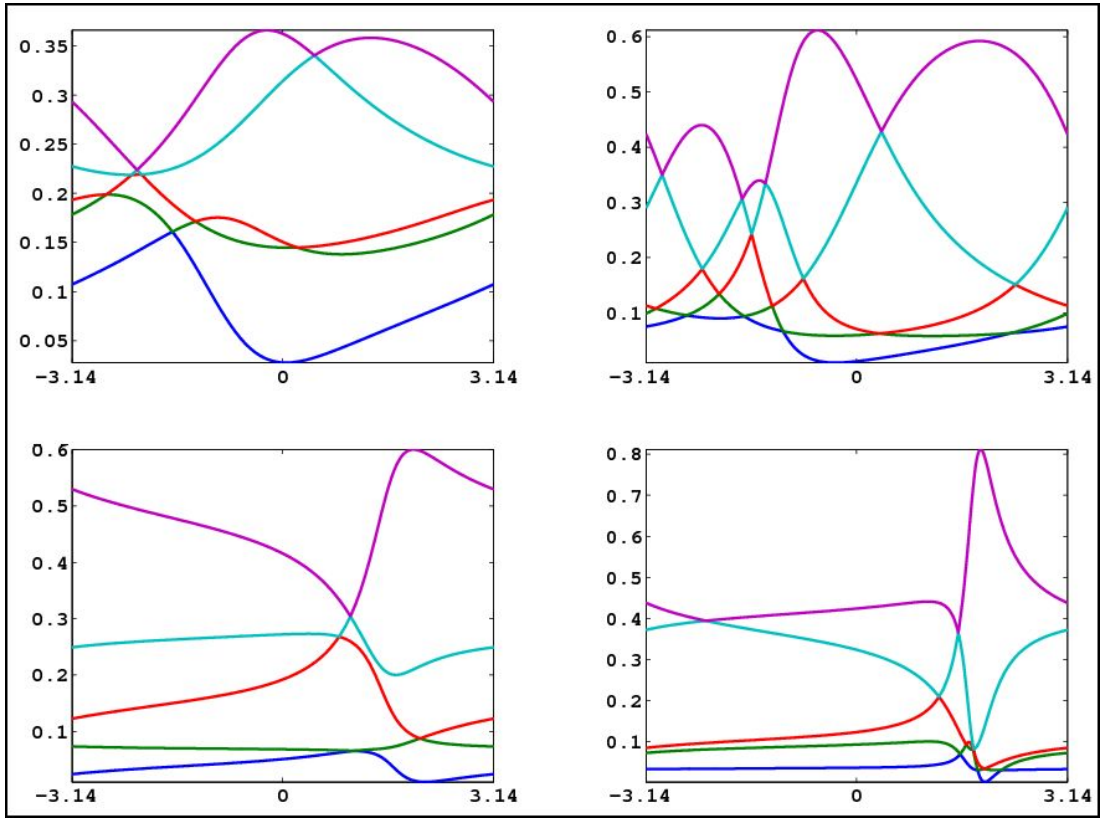


Figure: The weights of the of the quadrature formula with $n = 5$ points as a function of $\theta \in [-\pi, \pi]$ where $v = e^{i\theta}$. The measure is $d\mu(\theta) = (1 - \nu\cos(\theta))d\theta/(2\pi)$. The poles are given by $r(-1)^{1/5}$. On the left for $r = 0.72$ and on the right for $r = 0.9$. On top for $\nu = 1$, at the bottom for $\nu = -1$.