Orthogonal rational functions and quadrature on the real half line

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Abstract

In this paper we generalize the notion of orthogonal Laurent polynomials to orthogonal rational functions. Orthogonality is considered with respect to a measure on the positive real line. From this, Gauss-type quadrature formulas are derived and multipoint Padé approximants for the Stieltjes transform of the measure. Convergence of both the quadrature formula and the multipoint Padé approximants is discussed.

1 Introduction

Laurent polynomials can be considered as rational functions with poles at zero and infinity. In this paper we generalize this notion to rational functions that may have several negative real poles, which can be conveniently thought of as poles that are grouped around zero and infinity.

When for the integral

$$I_\mu(f) = \int_0^\infty f(x)\mathrm{d}\mu(x),$$

we consider quadrature formulas of the form

$$I_n(f) = \sum_{j=1}^{n} A_j f(x_j),$$

we may look for nodes x_j and weights A_j such that the quadrature formula is exact for a space of functions that is as large as possible. In the polynomial case, one obtains exactness in the space of polynomials of degree up to $2n-1$ by choosing the nodes as the zeros of the nth orthogonal polynomial with respect to the measure μ and taking the corresponding

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Gauss-Christoffel numbers as weights. These are the well known Gauss formulas. See [13, 16]. In [3] (see also [10]) the polynomials were replaced by Laurent polynomials having a finite number of positive and negative powers of z. In that situation, exactness can be obtained for certain spaces of Laurent polynomials of dimension $2n-1$, if we take as nodes the zeros of orthogonal Laurent polynomials. As mentioned above, we now will generalize this idea to rational functions whose poles need not all be located at zero and infinity. Quadrature formulas for the integral over the positive half line exact in certain spaces of rational functions have been considered before. See [7] and [18]. For related results, see also [15, 9, 11]

It is also well known that these so called Gauss-type quadrature formulas are closely related to rational approximants for the Stieltjes transform $F_\mu(z) = I_\mu(1/(z-x))$. More precisely, in the case of Laurent polynomials this quadrature formula $F_n(z) = I_n(1/(z-x))$ yields a two-point Padé approximant for F_μ as was extensively studied in [10]. In the present situation of more general rational functions we obtain multipoint Padé approximants.

The outline of the paper is as follows. In Section 2 we introduce our spaces of rational functions. The quadrature formulas are given in Section 3 where several expressions for the weights are proved. Error formulas and convergence for the Gauss-type formulas are discussed in Section 4 where we make use of results by Lopez et al [14]. Section 5 introduces the multipoint Padé approximants and elaborates further on the connection between the quadrature formulas and the multipoint Padé approximants, which will give rise to new error estimates.

2 The spaces of rational functions

Let p and q be two given integers such that $p \leq q$. As usual, by a Laurent polynomial (L-polynomial in short) we mean a function of the form,

$$L(z) = \sum_{j=p}^{q} \gamma_j z^j; \quad \gamma_j \in \mathbb{C}. \tag{2.1}$$

We denote by $\Lambda_{p,q}$ the space of Laurent polynomials as given by (2.1) and by Λ the space of all Laurent polynomials. Also, Π_k (k being a nonnegative integer) will denote the space of polynomials of degree k at most, (observe that $\Pi_k = \Lambda_{0,k}$) and Π the space of all the polynomials. Laurent polynomials can be viewed as rational functions with prescribed poles at the origin and/or at infinity. Here, we will deal with more general rational functions with prescribed poles at points $a_1, a_2, \ldots$. As will be explained later on, we will assume that the sequence of the poles $a = \{a_k\}_1^\infty$ is in $\hat{\mathbb{C}} \setminus (0, \infty)$. (Here and in the sequel $\hat{\mathbb{C}}$ denotes the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.) Furthermore, in order to reproduce the behavior of the Laurent polynomials (two-poles) we will assume that the sequence a is conveniently distributed around the origin and infinity. More precisely, assume that $a = \alpha \cup \beta$ with $\alpha = \{\alpha_j\}_{j=1}^\infty$ and $\beta = \{\beta_j\}_{j=1}^\infty$ so that both of them are contained in $\hat{\mathbb{C}} \setminus (0, \infty)$ with $\alpha_j \neq \infty$ and $\beta_k \neq 0$ for any $j, k \geq 1$. Although some of the results derived below can be obtained in this more general setting, we shall restrict ourselves in this paper to the situation where all the α's are in $(-1, 0]$ (these are the points around the origin) and all the β's are in $[-\infty, -1)$ (giving the points near infinity). In fact, the introduction of two separate notations α'a and β's is not strictly necessary for our results. Any non-specified basis that "respects the nesting of the $\mathcal{L}_n$ spaces" would do, but since some formulations become simpler with the

introduction of this notation, we shall use it in the rest of this text.
Now, for any nonnegative integer n, we define

$$U_0 = 1 \quad \text{and} \quad U_n(x) = \frac{x - \alpha_n}{1 - x/\beta_n}; \quad n \geq 1. \tag{2.2}$$

In the rest of this paper, we set $B_0(x) = 1$ while for $n \geq 1$

$$B_n(x) = U_n(x)B_{n-1}(x) \quad \text{and} \quad B_{-n}(x) = \frac{1}{B_n(x)}. \tag{2.3}$$

By introducing the polynomials

$$\begin{array}{lll} \omega_0(x) = 1, & \omega_n(x) = (x - \alpha_1)\cdots(x - \alpha_n), & n \geq 1 \\ \pi_0(x) = 1, & \pi_n(x) = (1 - x/\beta_1)\cdots(1 - x/\beta_n), & n \geq 1 \end{array}$$

we can write $B_n(x) = \omega_n(x)/\pi_n(x)$ for $n = 0, 1, 2, \ldots$

On the other hand, for p and q nonnegative integers we define

$$\mathcal{L}_{-p,q} = \text{span}\{B_k : -p \leq k \leq q\}. \tag{2.4}$$

Thus,

$$\mathcal{L}_{-p,q} = \left\{ \frac{P(x)}{\omega_p(x)\pi_q(x)} : \quad P \in \Pi_{p+q} \right\}$$

so that its dimension is $p + q + 1$.
Clearly, when all the α_n coincide at the origin and all the β_n at infinity, then $B_k(x) = x^k$; $k = 0, \pm 1, \pm 2, \ldots$ and $\mathcal{L}_{-p,q}$ coincides with $\Lambda_{-p,q}$.
In order to construct a sequence of nested spaces as defined by (2.4) we will start from two nondecreasing sequences $\{p(n)\}$ and $\{q(n)\}$ of nonnegative integers such that $p(n)+q(n) = n$ (obviously $p(0) = q(0) = 0$). Thus, we define

$$\mathcal{L}_n = \mathcal{L}_{-p(n),q(n)} = \left\{ \frac{P(x)}{\omega_{p(n)}(x)\pi_{q(n)}(x)} : P \in \Pi_n \right\}.$$

So, we see that $\mathcal{L}_n \subset \mathcal{L}_{n+1}$ and $\dim(\mathcal{L}_n) = n + 1$. We denote $\mathcal{L} = \bigcup_{n=0}^{\infty} \mathcal{L}_n$.
Setting $D_n(x) = \omega_{p(n)}(x)\pi_{q(n)}(x)$, then for any $f \in \mathcal{L}_n$, one can write $f(x) = P(x)/D_n(x)$, $P \in \Pi_n$. Now, in order to simplify our forthcoming statements it will be convenient to introduce the factor

$$\lambda_n(x) = \begin{cases} x - \alpha_{p(n)} & \text{if} \quad p(n) > p(n-1) \\ 1 - x/\beta_{q(n)} & \text{if} \quad q(n) > q(n-1). \end{cases} \tag{2.5}$$

Thus, $D_n(x) = \lambda_n(x)D_{n-1}(x)$ for $n \geq 1$ with $D_0(x) = 1$.

3 Quadrature and Orthogonality

Let μ be a finite positive measure with an infinite support contained in $[0, \infty)$. Moreover assume that the integrals $I_\mu(f) = \int_0^\infty f(x)\mathrm{d}\mu(x)$ exist for all $f \in \mathcal{L}\cdot\mathcal{L}$. Then we can consider

an inner product in the spaces $\mathcal{L}_n$, defined by $\langle f, g\rangle = \int_0^\infty f(x)g(x)\mathrm{d}\mu(x)$, and construct an orthogonal basis. Our aim is to approximate the integral

$$I_\mu(f) = \int_0^\infty f(x)\mathrm{d}\mu(x) \tag{3.1}$$

by a quadrature formula like

$$I_n(f) = \sum_{j=1}^{n} A_j f(x_j) \tag{3.2}$$

with distinct nodes $\{x_j\}_{j=1}^n \subset (0,\infty)$ and coefficients or weights $\{A_j\}_{j=1}^n$ to be determined so that $I_n(f)$ exactly matches $I_\mu(f)$ for f in $\mathcal{L}_N$ with $N = N(n)$ as large as possible. We see we have at our disposal $2n$ unknown parameters so that it seems natural to ask for quadrature formulas exact in subspaces of dimension $2n$. Recall that the poles $\alpha = \{\alpha_n\} \subset (-1, 0]$ while $\beta = \{\beta_n\} \subset [-\infty, -1)$. Taking into account that $\dim(\mathcal{L}_{p+q}) = p+q+1$ we have the following general result (which actually holds under the weaker assumption that all the poles are in $[-\infty, 0]$).

Theorem 3.1 *[7] Let p and q be nonnegative integers such that $p + q = 2n - 1$. Then, $I_n(f) = \sum_{j=1}^n A_j f(x_j) = I_\mu(f)$ for any $f \in \mathcal{L}_{-p,q}$, if and only if,*

i) $\{x_j\}$ are the zeros of the n-th orthogonal polynomial with respect to the positive measure

$$\mathrm{d}\mu_{p,q}(x) = \frac{\mathrm{d}\mu(x)}{\omega_p(x)\pi_q(x)}. \tag{3.3}$$

ii) $A_j = C_j\omega_p(x_j)\pi_q(x_j) > 0$, $\{C_j\}_{j=1}^n$ being the n-th Christoffel numbers for the measure $\mathrm{d}\mu_{p,q}(x)$.

Remark 3.1 The quadrature formula obtained in Theorem 3.1 is "optimal" in the sense that there does not exist a quadrature formula with n nodes in $(0,\infty)$ that is exact in $\mathcal{L}_{-p,q}$ with $p + q = 2n$.
Indeed, assume $I_n(f) = \sum_{j=1}^n A_j f(x_j) = I_n(f)$ for any $f \in \mathcal{L}_{-p,q}$ with $p + q = 2n$. Let $Q_n(x) = (x-x_1)\cdots(x-x_n)$ and define $R(x) = Q_n^2(x)/[\omega_p(x)\pi_q(x)]$. Then, clearly $R \in \mathcal{L}_{-p,q}$ and satisfies $I_\mu(R) > 0$ and $I_n(R) = 0$.

From Theorem 3.1, we see that when quadrature formulas of the form $I_n(f) = \sum_{j=1}^n A_j f(x_j)$, exact in $\mathcal{L}_{2n-1}$ are required, then the nodes are the zeros of the nth orthogonal polynomial with respect to $\mathrm{d}\mu(x)/[\omega_{p(2n-1)}(x)\pi_{q(2n-1)}(x)]$. So, zeros of orthogonal polynomials with respect to "varying" measures immediately arise. This is what some authors (see e.g. [18]) call the "Gaussian approach".
On the other hand, one could think of using "orthogonal rational functions". In this respect, as usual, the measure μ induces an inner product on $\mathcal{L}$, defined by

$$\langle f, g\rangle = \int_0^\infty f(x)g(x)d\mu(x), \quad f, g \in \mathcal{L}.$$

Let $\{\varphi_k\}_0^n$ be the orthonormal basis for $\mathcal{L}_n$ obtained by applying the Gram-Schmidt orthogonalization process to the basis $\{B_k : -p(n) \le k \le q(n)\}$. We suppose thereby that the ordering of the $\mathcal{L}_n$ is respected, i.e., $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$. When the process is repeated for

$n = 1, 2, 3, \ldots$ an essentially unique orthonormal sequence $\{\varphi_n\}_0^\infty$ is achieved. Each φ_n can be expressed as

$$\varphi_n(x) = \frac{P_n(x)}{\omega_{p(n)}(x)\pi_{q(n)}(x)} = \frac{P_n(x)}{D_n(x)}, \quad P_n \in \Pi_n. \tag{3.4}$$

Since $\varphi_n \perp \mathcal{L}_{n-1}$ it follows that P_n represents (up to a multiplicative factor) the nth orthonormal polynomial with respect to the positive measure

$$\frac{\mathrm{d}\mu(x)}{D_n(x)D_{n-1}(x)}. \tag{3.5}$$

Thus, the polynomial $P_n(x)$ and hence $\varphi_n(x)$ has n distinct zeros in $(0, \infty)$ so that the sequence $\{\varphi_n\}_0^\infty$ is regular (see [4]).
Let $x_1, \ldots, x_n$ be the zeros of $\varphi_n(x)$. Since $\mathcal{L}_{n-1}$ is a Chebyshev space (i.e., spanned by a Chebyshev system) of dimension n in $(0, \infty)$, we can determine a unique set of weights $A_1, \ldots, A_n$ so that

$$I_n(f) = \sum_{j=1}^{n} A_j f(x_j) = I_\mu(f), \quad \forall f \in \mathcal{L}_{n-1}. \tag{3.6}$$

Furthermore, it also holds that

$$I_n(f) = I_\mu(R_{n-1}(f, \cdot)), \tag{3.7}$$

$R_{n-1}(f, x)$ being the unique element in $\mathcal{L}_{n-1}$ interpolating f at nodes $x_1, \ldots, x_n$.
Formula (3.6) corresponds to the so-called "orthogonal approach" (see [18]) and it is known [2], [12] that in general both approaches, "the Gaussian" and "the orthogonal" do not coincide, i.e. that $I_n(f)$ given by (3.6) is not, in general exact in $\mathcal{L}_{2n-1}$ (observe that this fact does hold in the polynomial situation i.e. $p(n) = 0$ for any n ($q(n) = n$) and $\beta_k = \infty$ so that now $\mathcal{L}_n = \Pi_n$ and the well known Gauss-Christoffel formulas, exact in $\mathcal{L}_{2n-1} = \Pi_{2n-1}$, arise).
However, taking into account that Π_{2n-1} can be written as $\Pi_n \cdot \Pi_{n-1}$, paralleling the polynomial situation (see [13]) we can prove the following

Theorem 3.2 *An n-point quadrature formula of the form $I_n(f) = \sum_{j=1}^n A_j f(x_j)$ with distinct nodes in $(0, \infty)$ is exact in $\mathcal{L}_n \cdot \mathcal{L}_{n-1}$, if and only if*

i) $I_n(f)$ is exact in $\mathcal{L}_{n-1}$

ii) $\{x_j\}_{j=1}^n$ are the zeros of the nth orthogonal function $\varphi_n(x)$ with respect to $\mathrm{d}\mu$.

Proof. "$\Rightarrow$"

i) Since $\mathcal{L}_{n-1} \subset \mathcal{L}_n \cdot \mathcal{L}_{n-1}$ this part trivially follows.

ii) Set $Q_n(x) = (x - x_1) \cdots (x - x_n)$ and define

$$R_n(x) = \frac{Q_n(x)}{\omega_{p(n)}(x)\pi_{q(n)}(x)} = \frac{Q_n(x)}{D_n(x)} \in \mathcal{L}_n.$$

Since for any $r \in \mathcal{L}_{n-1}$, $r \cdot R_n \in \mathcal{L}_n \cdot \mathcal{L}_{n-1}$, it follows that

$$\langle r, R_n \rangle = I_\mu(r \cdot R_n) = \sum_{j=1}^{n} A_j r(x_j) R_n(x_j) = 0.$$

Thus, $R_n(x)$ coincides with $\varphi_n(x)$ up to a multiplicative factor.
"$\Leftarrow$".
Let $\{x_j\}_{j=1}^n$ be the zeros of $\varphi_n(x)$. As already said, weights or coefficients $A_1, \ldots, A_n$ can be uniquely determined so that

$$I_n(f) = \sum_{j=1}^{n} A_j f(x_j) = I_\mu(f), \quad \forall f \in \mathcal{L}_{n-1}. \tag{3.8}$$

Take now $R \in \mathcal{L}_n \cdot \mathcal{L}_{n-1}$, then one can write

$$R(x) = \frac{P(x)}{D_n(x) D_{n-1}(x)}, \quad P \in \Pi_{2n-1}$$

and

$$I_\mu(R) = \int_0^\infty R(x) \mathrm{d}\mu(x) = \int_0^\infty P(x) \frac{\mathrm{d}\mu(x)}{D_n(x) D_{n-1}(x)} = \sum_{j=1}^{n} \tilde{B}_j P(x_j)$$

$\tilde{B}_j$ being the nth Christoffel numbers for the measure $\mathrm{d}\mu(x)/[D_n(x)D_{n-1}(x)]$.
Hence, $\tilde{I}_n(R) = I_\mu(R)$ if we define

$$\tilde{I}_n(R) = \sum_{j=1}^{n} \tilde{A}_j R(x_j), \quad \tilde{A}_j = \tilde{B}_j D_n(x_j) D_{n-1}(x_j)$$

since

$$I_\mu(R) = \sum_{j=1}^{n} \frac{\tilde{B}_j D_n(x_j) D_{n-1}(x_j)}{D_n(x_j) D_{n-1}(x_j)} P(x_j) = \sum_{j=1}^{n} \tilde{A}_j R(x_j) = \tilde{I}_n(R).$$

Thus, a quadrature formula with nodes the zeros $\{x_j\}_{j=1}^n$ of $\varphi_n(x)$ being exact in $\mathcal{L}_n \cdot \mathcal{L}_{n-1}$ has been constructed. Thus, by (3.8) it follows that

$$\tilde{I}_n(f) = I_\mu(f) = I_n(f), \quad \forall f \in \mathcal{L}_{n-1}.$$

Since both quadrature formulas $I_n(f)$ and $\tilde{I}_n(f)$ have the same nodes it holds that $A_j = \tilde{A}_j$; $j = 1, \ldots, n$. □

Remark 3.2 The quadrature formula deduced from Theorem 3.2 is also "optimal" in the sense that there can not exist quadrature formulas $I_n(f) = \sum_{j=1}^n A_j f(x_j)$ with nodes in $(0, \infty)$ being exact in $\mathcal{L}_n \cdot \mathcal{L}_n$.
In the sequel, we are going to restrict ourselves to these quadratures and they will be called of "Gauss-type" for $\mathcal{L}_n \cdot \mathcal{L}_{n-1}$ or more simply, when no confusion is possible of "Gauss-type".

Next, from Theorem 3.2, we can deduce the following proposition where positivity of the weights is clearly displayed.

Proposition 3.3 *Let $I_n(f) = \sum_{j=1}^n A_j f(x_j)$ be the n-point Gauss-type formula for $I_\mu(f)$ and let $\varphi_n(x)$ be the nth orthonormal rational functions as defined above, then*

$$A_j = \int_0^\infty \left[\frac{\lambda_n(x)\varphi_n(x)}{\lambda_n(x_j)(x-x_j)\varphi_n'(x_j)}\right]^2 \mathrm{d}\mu(x), \quad j = 1, \ldots, n \tag{3.9}$$

with $\lambda_n(x)$ as given by (2.5).

Proof. For $1 \le j \le n$, let $L_j \in \mathcal{L}_{n-1}$ be such that

$$L_j(x_k) = \delta_{j,k} = \begin{cases} 1 & \text{if} \quad j = k \\ 0 & \text{if} \quad j \neq k \end{cases} \tag{3.10}$$

where $\{x_k\}_{k=1}^n$ are the zeros of $\varphi_n(x)$. Clearly, $L_j^2 \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}$. Therefore, by Theorem 3.2

$$I_\mu(L_j) = I_n(L_j) \quad \text{and} \quad I_\mu(L_j^2) = I_n(L_j^2), \quad j = 1, 2, \ldots, n$$

and from this, we obtain

$$A_j = \int_0^\infty L_j(x)\mathrm{d}\mu(x) = \int_0^\infty L_j^2(x)\mathrm{d}\mu(x); \quad j = 1, 2, \ldots, n. \tag{3.11}$$

On the other hand, $L_j \in \mathcal{L}_{n-1}$ is uniquely determined by condition (3.10) and it can be checked easily that

$$L_j(x) = \frac{\lambda_n(x)}{\lambda_n(x_j)} \frac{\varphi_n(x)}{(x-x_j)\varphi_n'(x_j)}; \quad j = 1, 2, \ldots, n. \tag{3.12}$$

Thus, from (3.11)-(3.12) the proof follows. □

To end this section, let us introduce the associated rational functions,

$$\sigma_n(z) = \int_0^\infty \frac{\varphi_n(z) - \varphi_n(x)}{z - x} \mathrm{d}\mu(x), \quad n = 1, 2, \ldots \tag{3.13}$$

Then, we have

Proposition 3.4 *The weights of the n-point Gauss-type quadrature formula are given by*

$$A_j = \frac{\sigma_n(x_j)}{\varphi_n'(x_j)}; \quad j = 1, \ldots, n, \tag{3.14}$$

where φ_n is the nth orthonormal rational function and σ_n is the associated rational function.

Proof. Let $L_j \in \mathcal{L}_{n-1}$ be as in the proof of Proposition 3.3, i.e. $L_j(x_k) = \delta_{j,k}$. Hence

$$A_j = \int_0^\infty L_j(x)\mathrm{d}\mu(x) = \int_0^\infty \frac{\lambda_n(x)}{\lambda_n(x_j)} \frac{\varphi_n(x)}{(x-x_j)} \frac{\mathrm{d}\mu(x)}{\varphi_n'(x_j)} \mathrm{d}\mu(x), \quad j = 1, \ldots, n$$

with $\lambda_n(x)$ given by (2.5). Thus

$$A_j = \frac{1}{\lambda_n(x_j)\varphi_n'(x_j)} \int_0^\infty \frac{1}{(x_j - x)} [\varphi_n(x_j) - \varphi_n(x)\lambda_n(x)] \mathrm{d}\mu(x). \tag{3.15}$$

Assume first that $\lambda_n(x) = x - \alpha_{p(n)}$, then

$$\begin{aligned}\int_0^\infty \frac{\varphi_n(x)}{x_j - x}\lambda_n(x)\mathrm{d}\mu(x) &= \int_0^\infty \frac{\varphi_n(x)}{x_j - x}(x - \alpha_{p(n)} - x_j + x_j)\mathrm{d}\mu(x) \\ &= \lambda_n(x_j)\int_0^\infty \frac{\varphi_n(x)}{x_j - x}\mathrm{d}\mu(x). \end{aligned} \tag{3.16}$$

So, (3.15)-(3.16) implies (3.14).
Suppose next that $\lambda_n(x) = 1 - x/\beta_{q(n)}$. If $\beta_{q(n)} = \infty$, then $\lambda_n(x) = 1$ and (3.13) trivially follows. If $\beta_{q(n)}$ is a finite number, then $\lambda_n(x)/\lambda_n(x_j) = (x - \beta_{q(n)})/(x_j - \beta_{q(n)})$. Hence

$$\begin{aligned}\frac{1}{\lambda_n(x_j)}\int_0^\infty \frac{\varphi_n(x)\lambda_n(x)}{x_j - x}\mathrm{d}\mu(x) &= \frac{1}{x_j - \beta_{q(n)}}\int_0^\infty \frac{\varphi_n(x)}{x_j - x}(x - \beta_{q(n)})\mathrm{d}\mu(x) \\ &= \frac{1}{x_j - \beta_{q(n)}}\int_0^\infty \frac{\varphi_n(x)}{x_j - x}(x - x_j + x_j - \beta_{q(n)})\mathrm{d}\mu(x) \\ &= \int_0^\infty \frac{\varphi_n(x)}{x_j - x}\mathrm{d}\mu(x).\end{aligned}$$

Again, from (3.15) we obtain

$$A_j = \frac{1}{\varphi_n'(x_j)}\int_0^\infty \frac{\varphi_n(x_j) - \varphi_n(x)}{x_j - x}\mathrm{d}\mu(x) = \frac{\sigma_n(x_j)}{\varphi_n'(x_j)}.$$

□

4 Errors and convergence for Gauss-type formulas

In this section we will first deal with the error expression for an n-point Gauss-type formula, that is, we shall be concerned with

$$E_n(f) = \int_0^\infty f(x)\mathrm{d}\mu(x) - \sum_{j=1}^n A_j f(x_j) = I_\mu(f) - I_n(f). \tag{4.1}$$

Following the polynomial case (see e.g. [13]) we will start with a certain rational-Hermite interpolant in $\mathcal{L}_n \cdot \mathcal{L}_{n-1}$. Indeed, let $t_1, \ldots, t_n$ be n distinct nodes in $(0, \infty)$ and f a function admitting a derivative on $(0, \infty)$. Since $\mathcal{L}_n \cdot \mathcal{L}_{n-1}$ represents a Chebyshev space of dimension $2n$, there exists a unique $R_{2n-1}(f, x) \in \mathcal{L}_n \cdot \mathcal{L}_{n-1}$ satisfying the following interpolation conditions

$$\left.\begin{array}{l} R_{2n-1}(f, t_j) = f(t_j) \\ R'_{2n-1}(f, t_j) = f'(t_j) \end{array}\right\}, \quad j = 1, 2, \ldots, n. \tag{4.2}$$

Recall now that for any $R \in \mathcal{L}_n \cdot \mathcal{L}_{n-1}$,

$$R(x) = \frac{P(x)}{D_n(x)D_{n-1}(x)} = \frac{P(x)}{C_{2n-1}(x)}$$

with $C_{2n-1}(x) = D_n(x)D_{n-1}(x) \in \Pi_{2n-1}$ and $P \in \Pi_{2n-1}$.
Let $L_{j,n}(x)$ and $\tilde{L}_{j,n}(x)$ be polynomials of degree $2n-1$ at most, satisfying

$$\begin{array}{llll} L_{j,n}(t_k) = \delta_{j,k}, & L'_{j,n}(t_k) = 0, & j, k = 1, \ldots, n \\ \tilde{L}_{j,n}(t_k) = 0, & \tilde{L}'_{j,n}(t_k) = \delta_{j,k}, & j, k = 1, \ldots, n. \end{array}$$

Then, by virtue of uniqueness for the interpolant $R_{2n-1}(f,x)$ we have

$$R_{2n-1}(f,x) = \sum_{j=1}^{n} H_{j,n}(x) f(t_j) + \sum_{j=1}^{n} \tilde{H}_{j,n}(x) f'(t_j) \tag{4.3}$$

where

$$H_{j,n}(x) = \frac{L_{j,n}(x) C_{2n-1}(t_j) + \tilde{L}_{j,n}(x) C'_{2n-1}(t_j)}{C_{2n-1}(x)} \in \mathcal{L}_n \cdot \mathcal{L}_{n-1} \tag{4.4}$$

and

$$\tilde{H}_{j,n}(x) = \frac{\tilde{L}_{j,n}(x) C_{2n-1}(t_j)}{C_{2n-1}(x)} \in \mathcal{L}_n \cdot \mathcal{L}_{n-1}. \tag{4.5}$$

Therefore, by integrating with respect to $\mathrm{d}\mu$ one obtains,

$$\begin{aligned} I_\mu(R_{2n-1}(f,\cdot)) &= \sum_{j=1}^{n} I_\mu(H_{j,n}) f(t_j) + \sum_{j=1}^{n} I_\mu(\tilde{H}_{j,n}) f'(t_j) \\ &= \sum_{j=1}^{n} \tilde{A}_j f(t_j) + \sum_{j=1}^{n} \tilde{B}_j f'(t_j) \end{aligned} \tag{4.6}$$

with $\tilde{A}_j = I_\mu(H_{j,n})$ and $\tilde{B}_j = I_\mu(\tilde{H}_{j,n})$, $j = 1, \ldots, n$.
In short, (4.6) represents a quadrature formula for $I_\mu(f)$ involving values of $f(x)$ and its derivative and exactly integrating any functions in $\mathcal{L}_n \cdot \mathcal{L}_{n-1}$.
Now, it should be taken into account that we have started from n distinct points $t_1, \ldots, t_n$ on $(0, \infty)$. Let us see what happens when the zeros $\{x_1, \ldots, x_n\}$ of $\varphi_n(x)$ are taken as nodes in the above approach.
Setting as usual $\varphi_n(x) = P_n(x)/D_n(x)$ and using the fact that $\tilde{L}_{j,n}(x_k) = 0$ for $k = 1, 2, \ldots, n$, it follows that

$$\tilde{L}_{j,n}(x) = P_n(x) q(x), \quad q \in \Pi_{n-1}.$$

Thus,

$$\tilde{B}_j = C_{2n-1}(x_j) \int_0^\infty \frac{P_n(x)}{D_n(x)} \mathrm{d}\mu(x) = C_{2n-1}(x_j) \int_0^\infty \varphi_n(x) r(x) \mathrm{d}\mu = 0, \quad j = 1, \ldots, n,$$

since $r(x) = q(x)/D_{n-1}(x) \in \mathcal{L}_{n-1}$. Therefore, formula (4.6) becomes,

$$I_\mu(R_{2n-1}(f,\cdot)) = \sum_{j=1}^{n} \tilde{A}_j f(x_j). \tag{4.7}$$

Thus, $\tilde{I}_n(f) = \sum_{j=1}^{n} \tilde{A}_j f(x_j)$ represents an n-point quadrature formula exact in $\mathcal{L}_n \cdot \mathcal{L}_{n-1}$. By virtue of Theorem 3.2, $\tilde{I}_n(f)$ must coincide with the n-point Gauss-type formula. Therefore

$$E_n(f) = I_\mu(f) - I_n(f) = I_\mu(f - R_{2n-1}(f,\cdot)). \tag{4.8}$$

Now, from (4.8) and using the error term in the Hermite interpolation (see e.g. [5]), the following can be easily proved.

Theorem 4.1 *Let $f(x)$ be a real function for which $f^{(2n)}(x)$ exists on $[0,\infty)$. Then*

$$E_n(f) = \frac{(f(x)C_{2n-1}(x))^{(2n)}(\theta_n)}{(2n)!}\int_0^\infty \tilde{\varphi}_n^2(x)\lambda_n(x)\mathrm{d}\mu(x), \quad 0 \le \theta_n \le \infty, \tag{4.9}$$

where $C_{2n-1}(x) = D_n(x)D_{n-1}(x)$, $\lambda_n(x)$ is given by (2.5) and $\tilde{\varphi}_n(x) = \tilde{P}_n(x)/D_n(x)$ represents the nth orthogonal function in $\mathcal{L}_n$, normalized so that its numerator $\tilde{P}_n(x)$ is a monic polynomial of degree n.

Remark 4.1 When $\alpha_k = 0$ and $\beta_j = \infty$ for all $k, j = 1, 2, \ldots$, then $C_{2n-1}(x) = x^{p(n)+p(n-1)}$, $\lambda_n(x) = x^{p(n)-p(n-1)}$ and $\tilde{\varphi}_n(x) = x^{-p(n)}\tilde{P}_n(x) \in \Lambda_{-p(n),q(n)}$. Thus, from expression (4.9) we can recover the error formula in the n-point Gauss-type quadrature formula for Laurent polynomials as given in [3] (see e.g. also [10]). On the other hand, when taking $p(n) = 0$ for each n ($q(n) = n$) and $\beta_k = \infty$ for any $k \ge 1$, then $\mathcal{L}_n = \Lambda_{0,n} = \Pi_n, \tilde{\varphi}_n(x) = \tilde{P}_n(x)$ (monic) and (4.9) now yields (recall that $\lambda_n(x) = 1$) the well-known formula

$$E_n(f) = \frac{f^{(2n)}(\theta_n)}{(2n)!}\int_0^\infty \tilde{P}_n^2(x)\mathrm{d}\mu(x)$$

which gives the error in the n-point Gauss-Christoffel formula.

Concerning convergence of the sequence $\{I_n(f)\}$ of Gauss-type formulas we first need the following lemma (which is proved in [14] as a direct consequence of [1]).

Lemma 4.2 *Let $\{p(n)\}$ and $\{q(n)\}$ be two nondecreasing sequences of nonnegative integers such that $p(n) + q(n) = n$, $n = 0, 1, \ldots$. Let $\alpha = \{\alpha_n\} \subset (-1, 0]$ and $\beta = \{\beta_n\} \subset [-\infty, -1)$ be two sequences. Assume that at least one of the following two conditions hold*

$$\lim_{n\to\infty} p(n) = \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{\sqrt{|\alpha_n|}}{1+|\alpha_n|} = +\infty \tag{4.10}$$

or

$$\lim_{n\to\infty} q(n) = \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{\sqrt{|\beta_n|}}{1+|\beta_n|} = +\infty. \tag{4.11}$$

Then, $\mathcal{L} = \bigcup_0^\infty \mathcal{L}_n$ is dense (with respect to the uniform norm) in the class $C_{+\infty} = \{f : [0,\infty) \to \mathbb{R} : f$ a continuous function and $\lim_{x\to+\infty} f(x)$ exists and is finite $\}$.

Note that these conditions mean that the sequences α_n and β_n may tend to 0 or infinity but not too fast. If they tend to zero faster than $1/n^2$ or to infinity faster than n^2, then the infinite sums above would converge and the conditions would not be satisfied.

Then, from the density of $\mathcal{L}$ in $C_{+\infty}$ and the positivity of the weights for the Gauss-type formulas, one can prove the following.

Theorem 4.3 *Under the same conditions as in Lemma 4.2, it holds that*

$$\lim_{n\to\infty} I_n(f) = I_\mu(f) = \int_0^\infty f(x)\mathrm{d}\mu(x)$$

for any function f in $C_{+\infty}$ where $\{I_n(f)\}$ is the sequence of Gauss-type quadrature formulas as defined in Section 3.

To conclude this section, we prove the following theorem.

Theorem 4.4 *Let g be an increasing function on $[0,\infty)$ such that $\mathrm{d}\mu = \mathrm{d}g$ and suppose that f is a bounded function on $[0,\infty)$ for which $\lim_{t\to\infty} f(t)$ exists and such that its improper Riemann-Stieltjes integral $\int_0^\infty f(x)\mathrm{d}g(x)$ converges. Then, under the same hypothesis as in Lemma 4.2, it holds that*

$$\lim_{n\to\infty} I_n(f) = \int_0^\infty f(x)\mathrm{d}g(x)$$

where $\{I_n(f)\}$ is the sequence of Gauss-type formulas as defined in Section 3.

Proof. The proof follows closely the proof of Theorem 2 in [14]. The main ingredients are Theorem 1.5.4 in [17], the density of the the space $\mathcal{L}$ of rational functions in the space of continuous functions on $[0,\infty)$, and the positivity of the weights of our quadrature formulas. Because [14] is not easily available, we include the proof for the sake of completeness.

Take $a > 0$ and consider the change of variable $t = \varphi(x) = \frac{1}{x+a}$. Now define,

$$h(t) = \begin{cases} (f\circ\varphi^{-1})(t), & 0 < t \le 1/a \\ \lim_{t\to 0}(f\circ\varphi^{-1})(t), & t = 0. \end{cases}$$

Thus, $h(t)$ represents a bounded Riemann-Stieltjes integrable function on $[0,1/a)$ with respect to $(g\circ\varphi^{-1})$ satisfying

$$\int_0^{1/a} h(t)\mathrm{d}(g\circ\varphi^{-1})(t) = -\int_0^\infty f(x)\mathrm{d}g(x) = -\int_0^\infty f(x)\mathrm{d}\mu(x).$$

Now, making use of [17, Th. 1.5.4], for a given $\epsilon > 0$, there exist polynomials P and p such that

$$p(t) \le h(t) \le P(t) \quad \text{in} \quad [0,1/a]$$

and

$$\left(\int_0^{1/a} (P(t)-p(t))\mathrm{d}(g\circ\varphi^{-1})(t)\right) < \epsilon.$$

Hence, one can write

$$p\left(\frac{1}{x+a}\right) \le f(x) \le P\left(\frac{1}{x+a}\right) \quad \text{for} \quad x \ge 0 \tag{4.12}$$

and

$$\left|\int_0^\infty (P-p)\left(\frac{1}{x+a}\right)\mathrm{d}g(x)\right| < \epsilon. \tag{4.13}$$

Now, because of the positivity of the weights in $I_n(f)$ from (4.12) it follows

$$I_n\left(p\left(\frac{1}{x+a}\right)\right) \le I_n(f) \le I_n\left(P\left(\frac{1}{x+a}\right)\right). \tag{4.14}$$

On the other hand, since $(P\circ\varphi)(x)$ and $(p\circ\varphi)(x)$ belong to $C_{+\infty}$ from Theorem 4.3 and (4.14) one obtains

$$\int_0^\infty p\left(\frac{1}{x+a}\right)\mathrm{d}g(x) \le \liminf_{n\to\infty} I_n(f) \le \limsup_{n\to\infty} I_n(f) \le \int_0^\infty P\left(\frac{1}{x+a}\right)\mathrm{d}g(x). \tag{4.15}$$

Thus,

$$0 \leq \limsup_{n\to\infty} I_n(f) - \liminf_{n\to n} I_n(f) \leq \int_0^\infty \left(P\left(\frac{1}{x+a}\right) - p\left(\frac{1}{x+a}\right)\right) \mathrm{d}g(x) < \epsilon$$

by (4.13). Since ϵ is arbitrary, we conclude that $\lim_{n\to\infty} I_n(f)$ exists. Finally by (4.12) this limit must coincide with $I_\mu(f)$. □

5 Multipoint Padé Approximants

Throughout this section we shall be concerned with the function in the variable z

$$F_\mu(z) = I_\mu\left(\frac{1}{z-x}\right) = \int_0^\infty \frac{\mathrm{d}\mu(x)}{z-x}, \quad z \notin [0,\infty). \tag{5.1}$$

Here, $F_\mu(z)$ represents the well known Stieltjes transform of the measure $\mathrm{d}\mu$. For this purpose, suppose that in order to approximate $F_\mu(z)$ we make use of the n-point Gauss-type formula defined in Section 3. Thus, we have (z is a parameter)

$$I_n\left(\frac{1}{z-x}\right) = \sum_{j=1}^n \frac{A_j}{z-x_j} = F_n(z). \tag{5.2}$$

Clearly, $F_n(z)$ represents a rational function of type $(n-1,n)$ i.e. the numerator is a polynomial of degree $n-1$ at most and the denominator a polynomial of degree n at most. On the other hand, let $R_{n-1}(x,z)$ be the interpolant to $1/(z-x)$ in $\mathcal{L}_{n-1}$, satisfying

$$R_{n-1}(x_j,z) = \frac{1}{z-x_j}, \quad j = 1,\ldots,n. \tag{5.3}$$

Now, by recalling that the nodes $x_1,\ldots,x_n$ are the zeros of $\varphi_n(x)$, it can be easily checked that

$$R_{n-1}(x,z) = \frac{1}{z-x}\left(1 - \frac{\varphi_n(x)}{\varphi_n(z)}\frac{\lambda_n(x)}{\lambda_n(z)}\right). \tag{5.4}$$

It follows that

$$I_\mu(R_{n-1}(x,z)) = I_n\left(\frac{1}{z-x}\right) = F_n(z). \tag{5.5}$$

Therefore we may write

$$\begin{aligned} F_n(z) = I_\mu(R_{n-1}(x,z)) &= I_\mu\left(\frac{1}{z-x}\right) - I_\mu\left(\frac{\varphi_n(x)\lambda_n(x)}{(z-x)\varphi_n(z)\lambda_n(z)}\right) \\ &= F_\mu(z) - I_\mu\left(\frac{\varphi_n(x)\lambda_n(x)}{(z-x)\varphi_n(z)\lambda_n(z)}\right). \end{aligned} \tag{5.6}$$

We also get

$$\begin{aligned} F_n(z) &= I_\mu\left(\frac{\varphi_n(z)\lambda_n(z) - \varphi_n(x)\lambda_n(x)}{(z-x)\varphi_n(z)\lambda_n(z)}\right) \\ &= \frac{1}{\varphi_n(z)\lambda_n(z)} I_\mu\left(\frac{[\varphi_n(z)-\varphi_n(x)]\lambda_n(z) - \varphi_n(x)[\lambda_n(x)-\lambda_n(z)]}{z-x}\right) \\ &= \frac{1}{\varphi_n(z)} I_\mu\left(\frac{\varphi_n(z)-\varphi_n(x)}{z-x}\right) + \frac{1}{\varphi_n(z)\lambda_n(z)} I_\mu\left(\frac{\lambda_n(z)-\lambda_n(x)}{z-x}\varphi_n(x)\right). \end{aligned}$$

Notice that $(\lambda_n(z) - \lambda_n(x))/(z-x)$ is constant. So by the orthogonality of the φ_n we have

$$I_\mu\left(\frac{\lambda_n(z)-\lambda_n(x)}{z-x}\varphi_n(x)\right) = 0$$

for $n = 1, 2, 3, \ldots$. Therefore

$$F_n(z) = \frac{1}{\varphi_n(z)} I_\mu\left(\frac{\varphi_n(z)-\varphi_n(x)}{z-x}\right) = \frac{\sigma_n(z)}{\varphi_n(z)}, \quad n = 1, 2, 3, \ldots \tag{5.7}$$

$\sigma_n(z)$ being the "associated" function as defined by (3.13).

Remark 5.1 (5.7) can also be obtained from (3.14) and the partial fraction decomposition of $F_n(z)$.
Conversely, (3.14) follows directly from (5.2) and (5.7).

Considering the error $F_\mu(z) - F_n(z)$, for $z \notin [0, \infty)$, it follows from (5.6) that

$$F_\mu(z) - F_n(z) = \int_0^\infty \frac{\varphi_n(x)}{\varphi_n(z)} \frac{\lambda_n(x)}{\lambda_n(z)} \frac{\mathrm{d}\mu(x)}{(z-x)}. \tag{5.8}$$

On the other hand, the function (in the variable x)

$$\frac{(\varphi_n(z)-\varphi_n(x))}{z-x}\frac{\lambda_n(x)}{\lambda_n(z)}$$

is in $\mathcal{L}_{n-1}$. Therefore, because of orthogonality,

$$\int_0^\infty \left(\frac{\varphi_n(z)-\varphi_n(x)}{z-x}\right)\frac{\lambda_n(x)}{\lambda_n(z)}\varphi_n(x)\mathrm{d}\mu(x) = 0$$

From here and (5.8) one arrives at the following expression for $z \notin [0, \infty)$

$$F_\mu(z) - F_n(z) = \frac{1}{\lambda_n(z)\varphi_n^2(z)} \int_0^\infty \frac{\varphi_n^2(x)}{(z-x)}\lambda_n(x)\mathrm{d}\mu(x). \tag{5.9}$$

On the other hand, since $1 \in \mathcal{L}_{n-1}$, it holds that $\sum_{j=1}^n A_j = \int_0^\infty \mathrm{d}\mu(x) = c_0$. Hence,

$$F_n(z) = \frac{A_1}{z-x_1} + \frac{A_2}{z-x_2} + \ldots + \frac{A_n}{z-x_n} = \frac{A_1+\ldots+A_n}{z} + O\left(\frac{1}{z^2}\right) = \frac{c_0}{z} + O\left(\frac{1}{z^2}\right).$$

The $O(z^{-2})$ terms hold of course for $z \to \infty$. Thus, since $F_\mu(z) = c_0 z^{-1} + O(z^{-2})$, which is an asymptotic expansion for $z \to \infty$ in the cut plane $\mathbb{S} = \mathbb{C}\setminus[0,\infty)$, we see that $F_\mu(z) - F_n(z) = O(z^{-2})$ asymptotically for $z \to \infty$ in $\mathbb{S}$. In this sense $F_n(z)$ interpolates $F_\mu(z)$ twice at infinity. Interpolating the value zero at infinity is implied by the fact that the rational interpolant F_n is of type $(n-1, n)$. At the same time, from (5.9), we see that $F_n(z)$ also interpolates $F_\mu(z)$ at the points $\{\alpha_1, \alpha_1, \ldots, \alpha_{p(n-1)}, \alpha_{p(n-1)}, \beta_1, \beta_1, \ldots, \beta_{q(n-1)}, \beta_{q(n-1)}, \gamma_n\}$ where

$$\gamma_n = \begin{cases} \alpha_{p(n)} & \text{if} \quad p(n) > p(n-1) \\ \beta_{q(n)} & \text{if} \quad q(n) > q(n-1). \end{cases}$$

This gives an additional $2(q(n-1)+p(n-1))+1 = 2n-1$ interpolation conditions. Since F_n is of type $(n-1, n)$, we have these $2n-1$ conditions plus 1 additional interpolation condition

at infinity, giving exactly $2n$ interpolation conditions for the $2n$ parameters. In other words, $F_n(z)$ represents a multipoint Padé approximant (MPA) to $F_\mu(z)$ at points of the tables $\alpha = \{\alpha_n\}_1^\infty$ and $\tilde{\beta} = \beta \cup \{\infty\} = \{\beta_k\}_1^\infty \cup \{\infty\}$. We have $p(n) + p(n-1)$ interpolation conditions for α and $2n - (p(n) + p(n-1)) = q(n) + q(n-1) + 1$ for $\tilde{\beta}$. For this reason and according to the notation used in the two-point situation (see e.g. [6]) i.e. $\alpha = \{0\}$ and $\tilde{\beta} = \{\infty\}$, we will sometimes write,

$$F_n(z) = \frac{\sigma_n(z)}{\varphi_n(z)} = [p(n) + p(n-1)/n]_{F_\mu}(z). \tag{5.10}$$

Remark 5.2 When taking $\beta_n = \infty$ and $p(n) = 0$ for each $n \geq 0$, then we have seen that $\mathcal{L}_n = \Pi_n$ so that $\varphi_n(z) = P_n(z)$ coincides with the nth orthonormal polynomial with respect to the measure $\mathrm{d}\mu$. Then, (5.9) becomes ($\lambda_n(z) = 1$)

$$F_\mu(z) - F_n(z) = \frac{1}{P_n^2(z)} \int_0^\infty \frac{P_n^2(x)}{z - x} \mathrm{d}\mu(x) \tag{5.11}$$

and the well-known formula for the errors in the one-point Padé approximant (at ∞) to $F_\mu(z)$ is recovered.
On the other hand, if we take $\beta_k = \infty$ and $\alpha_k = 0$ for any $k \geq 1$, then

$$\varphi_n(z) = \frac{P_n(z)}{z^{p(n)}} \in \Lambda_{-p(n),q(n)}$$

and (5.9) reduces to

$$F_\mu(z) - F_n(z) = \frac{z^{p(n)+p(n-1)}}{P_n^2(z)} \int_0^\infty \frac{P_n^2(x)}{x^{p(n)+p(n-1)}} \frac{\mathrm{d}\mu(x)}{(z - x)}. \tag{5.12}$$

Thus, the error formula (see e.g. [3]) for the $[p(n) + p(n-1)/n]$ two point Padé approximant to F_μ is also recovered.

Connections between multipoint Padé approximants and Gauss-type quadrature formulas will be given in the following results. We first have,

Theorem 5.1 *Under the same assumptions as in Lemma 4.2, the following two statements are equivalent:*

(a) $\lim_{n\to\infty} I_n(f) = I_\mu(f)$, $\forall f \in C_{+\infty}$, *where* $\{I_n(f)\}$ *is the sequence of Gauss-type quadrature formulas for* $\mathrm{d}\mu$.

(b) $\lim_{n\to\infty} [p(n) + p(n-1)/n]_{F_\mu}(z) = F_\mu(z) = \int_0^\infty \frac{\mathrm{d}\mu(x)}{z-x}$ *uniformly on any compact set* $K \subset \mathbb{C} \setminus [0, \infty)$.

Proof. (a) $\Rightarrow$ (b). We know that $[p(n) + p(n-1)/n]_{F_\mu}(z) = I_n\left(1/(z-x)\right)$. Since for $z \notin [0, \infty)$, $1/(z-x) \in C_{+\infty}$, the pointwise convergence of the MPA is assured for any $z \in \mathbb{C} \setminus [0, \infty)$.
Let K be compact in $\mathbb{C} \setminus [0, \infty)$ and take $z \in K$. Then,

$$\begin{aligned} \left|[p(n) + p(n-1)/n]_{F_\mu}(z)\right| &= \left|I_n\left(\frac{1}{z-x}\right)\right| = \left|\sum_{j=1}^n \frac{A_j}{z - x_j}\right| \\ &\leq \sum_{j=1}^n \frac{A_j}{|z - x_j|} \leq \frac{c_0}{\mathrm{dist}(K, [0, \infty))}. \end{aligned}$$

Recall that c_0 is the zeroth moment $c_0 = \int_0^\infty \mathrm{d}\mu(x)$. Thus, we see that the sequence of MPA is normal and the conclusion follows by applying the Stieltjes-Vitali Theorem [8, Th. 1.5.3.1].

(b) $\Rightarrow$ (a). Let $\{z_k\}$ be a sequence of complex numbers in $\mathbb{C} \setminus [0,\infty)$ satisfying $z_j \neq z_k$ if $j \neq k$ and

$$\sum_{\mathrm{Im}\sqrt{z_n}>0} \frac{\mathrm{Im}\sqrt{z_n}}{1+|z_n|} = +\infty.$$

Then, it is known (see [1]) that span $\left\{1, \frac{1}{z_1-x}, \frac{1}{z_2-x}, \ldots\right\}$ is dense in $C_{+\infty}$ with respect to the uniform norm. Thus, for $f \in C_{+\infty}$ and for given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$|f(x) - S_N(x)| < \epsilon, \quad \forall x \in [0,\infty)$$

where for certain values of $a_0, \ldots, a_N$

$$S_N(x) = a_0 + \frac{a_1}{z_1 - x} + \frac{a_2}{z_2 - x} + \ldots + \frac{a_N}{z_N - x}.$$

Clearly, $\lim_{n\to\infty} I_n(S_N) = I_\mu(S_N)$. Therefore, it holds that

$$|I_\mu(S_N) - I_n(S_N)| < \epsilon, \quad \forall n > n_0.$$

Thus, by taking $n > n_0$, one can write

$$\begin{aligned} |I_\mu(f) - I_n(f)| &= |I_\mu(f) - I_\mu(S_N) + I_\mu(S_n) - I_n(f)| \\ &\leq |I_\mu(f - S_N)| + |I_\mu(S_N) - I_n(f)| \\ &\leq c_0\epsilon + |I_\mu(S_N) - I_n(S_N) + I_n(S_N) - I_n(f)| \\ &\leq c_0\epsilon + |I_\mu(S_N) - I_n(S_N)| + |I_n(S_N - f)| \\ &< (2c_0 + 1)\epsilon. \end{aligned}$$

□

Theorem 5.2 *Let f be an analytic function in a simply connected domain G which contains the half line $[0,\infty]$ and whose boundary Γ is a rectifiable Jordan curve. Let $I_n(f)$ be the n-point Gauss-type formula for $\mathcal{L}_n \cdot \mathcal{L}_{n-1}$. Then,*

$$I_\mu(f) - I_n(f) = \frac{1}{2\pi i} \int_\Gamma \left[F_\mu(z) - [p(n) + p(n-1)/n]_{F_\mu}(z)\right] f(z)\mathrm{d}z. \tag{5.13}$$

Proof. By Cauchy's formula for the exterior of Γ in $\hat{\mathbb{C}}$, we have that for $x \in [0,\infty]$

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - x}\mathrm{d}z = f(x) - f(\infty).$$

Now we can prove our theorem following standard arguments. Let μ_n be the measure on $[0,\infty) = \mathbb{R}^+$ corresponding to the quadrature formula I_n. By the Fubini theorem we have

$$\begin{aligned} \frac{1}{2\pi i} \int_\Gamma f(z)F_\mu(z)\mathrm{d}z &= \frac{1}{2\pi i} \int_\Gamma \left(\int_{\mathbb{R}^+} f(z) \frac{1}{z-x} \mathrm{d}\mu(x) \right) \mathrm{d}z \\ &= \int_{\mathbb{R}^+} \left(\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-x}\mathrm{d}z \right) \mathrm{d}\mu(x) \\ &= \int_{\mathbb{R}^+} \Big(f(x) - f(\infty) \Big) \mathrm{d}\mu(x) = I_\mu(f) - c_0 f(\infty). \end{aligned}$$

Similarly

$$\frac{1}{2\pi i}\int_\Gamma f(z)F_n(z)\mathrm{d}z = \int_{\mathbb{R}^+}\Big(f(x)-f(\infty)\Big)\mathrm{d}\mu_n(x) = I_n(f) - c_0 f(\infty).$$

Hence

$$E_n(f) = I_\mu(f) - I_n(f) = \frac{1}{2\pi i}\int_\Gamma f(z)[F_\mu(z) - F_n(z)]\mathrm{d}z. \tag{5.14}$$

Finally, observe that $F_n = [p(n) + p(n-1)/n]_{F_\mu}$, and that this result also holds when Γ is replaced by any rectifiable curve which is homotopic to Γ in $G \setminus [0,\infty]$. □

Thus when considering analytic integrands, the errors for the Gauss-type formulas are "controlled" by the errors of the MPA. Indeed, by (5.9) and (5.14) it readily follows that the following is true.

Corollary 5.3 *Under the same assumptions as in Theorem 5.1, it holds that*

$$|I_\mu(f) - I_n(f)| \le \frac{C(f,\Gamma)}{\inf_{z\in\Gamma}|\lambda_n(z)\varphi_n^2(z)|}\int_0^\infty \varphi_n^2(x)\lambda_n(x)\mathrm{d}\mu(x) \tag{5.15}$$

$C(f,\Gamma)$ being a positive constant depending on f, Γ, and $\lambda_n(x)$ as given by (2.5).

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