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FIRST-ORDER LINEAR RECURRENCE SYSTEMS AND GENERAL N-FRACTIONS

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Abstract. In this paper we define a natural generalization of ordinary continued fractions and n-fractions (de Bruin [2]). This so-called general n-fraction is associated with a first-order recurrence system, and we look at some of its applications: computation of eigenvectors, stable computation of non-dominant solutions of recurrence systems, and calculation of vector rational interpolants.

Key words: continued fraction, n-fraction, recurrence system

1. Introduction

It is well known ([5], [8]) that there is a close relationship between ordinary continued fractions and second-order linear recurrence relations: the sequence of numerators (denominators) of a continued fraction is a solution of such a recurrence relation, and if the continued fraction converges, its value is related to the minimal solution of this recurrence relation. A similar relationship exists between some generalizations of continued fractions and higher-order linear recurrence relations, for instance n-fractions, introduced by de Bruin [2], are natural generalizations of ordinary continued fractions, and they are related to $(n+1)$-th-order linear recurrence relations. Both ordinary continued fractions and n-fractions have many interesting applications in the field of applied mathematics: calculation of special functions [15], (vector) Padé approximation [2], (vector) rational interpolation [7],... .
In this paper we look at a continued fraction-like construction which has continued fractions and n-fractions as special cases. The idea is based on the following observation: an ordinary continued fraction $K(a_k/b_k)$ is of the form

$$K(a_k/b_k) = \frac{a_0}{\lfloor b_0} + \frac{a_1}{\lfloor b_1} + \ldots + \frac{a_k}{\lfloor b_k} + \ldots .$$

Its k-th approximant is given by A_k/B_k where both numerator and denominator are solutions of the following second-order recurrence relation:

$$y_k = b_k y_{k-1} + a_k y_{k-2}, \quad k = 0, 1, 2, \ldots .$$

Another way to calculate A_k and B_k is by using a product of 2×2-matrices:

$$\begin{pmatrix} A_{k-1} & B_{k-1} \\ A_k & B_k \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a_k & b_k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ a_{k-1} & b_{k-1} \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} 0 & 1 \\ a_0 & b_0 \end{pmatrix}$$

(This can already be found in Milne-Thomson [11]). This matrix-product is closely related to the first-order recurrence system:

$$\begin{pmatrix} y_k^{(1)} \\ y_k^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a_k & b_k \end{pmatrix} \begin{pmatrix} y_{k-1}^{(1)} \\ y_{k-1}^{(2)} \end{pmatrix}, \quad k = 0, 1, \ldots .$$

Note that the matrix of this recurrence system has a special form. The generalization of a continued fraction we will discuss in this paper is in the same way associated with a first-order recurrence system, but the matrix of this system will be a full $p \times p$-matrix.

In the second section we define this *general n-fraction* by its sequence of approximants and we show that its approximants may be calculated using the composition of linear fractional transformations with some of the coefficients being matrices. Next we define convergence and we prove a Pincherle [4]-Van der Cruyssen [14]-like theorem that relates the convergence of the general n-fractions to the existence of (in some way) non-dominant solutions of the associated recurrence system.

In the third section we look at some of the applications. These include: the calculation of eigenvectors of a matrix (a scaled version of the power method), the (stable) calculation of certain non-dominant solutions of first-order recurrence systems, and the solution of a simultaneous rational interpolation problem.

2. General n-Fractions: Definitions and Notations

The general n-fraction associated with the first-order recurrence system

$$Y_k = \mathcal{A}_k Y_{k-1} \quad , \quad k = 0, 1, \ldots,$$

or, written componentwise,

$$\begin{pmatrix} y_k^{(1)} \\ y_k^{(2)} \\ \vdots \\ y_k^{(n+1)} \end{pmatrix} = \begin{pmatrix} c_k^{(1,1)} & c_k^{(1,2)} & \cdots & c_k^{(1,n)} & d_k^{(1)} \\ \vdots & \vdots & & \vdots & \vdots \\ c_k^{(n,1)} & c_k^{(n,2)} & \cdots & c_k^{(n,n)} & d_k^{(n)} \\ a_k^{(1)} & a_k^{(2)} & \cdots & a_k^{(n)} & b_k \end{pmatrix} \begin{pmatrix} y_{k-1}^{(1)} \\ y_{k-1}^{(2)} \\ \vdots \\ y_{k-1}^{(n+1)} \end{pmatrix} \tag{1}$$

is defined by its sequence of n-tuples of approximants

$$\{ \frac{\mathcal{G}_k^{(n+1,j)}}{\mathcal{G}_k^{(n+1,n+1)}}, \ j = 1, \ldots, n \}$$

where $(\mathcal{G}_k^{(1,j)} \ \ldots \ \mathcal{G}_k^{(n+1,j)})^T$, $(j = 1, \ldots, n+1)$ is the solution of (1) with initial value

$$(\mathcal{G}_{-1}^{(1,j)} \ \ldots \ \mathcal{G}_{-1}^{(n+1,j)})^T = (0 \ \ldots \ 0 \ 1 \ 0 \ \ldots \ 0)$$

with the 1 in the j-th position.

Example. Let

$$\begin{pmatrix} c_k^{(1,1)} & \cdots & c_k^{(1,n)} \\ \vdots & & \vdots \\ c_k^{(n,1)} & \cdots & c_k^{(n,n)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} d_k^{(1)} \\ \vdots \\ d_k^{(n-1)} \\ d_k^{(n)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{2}$$

If we put $z_{k+j} = y_k^{(j)}$ in (1), then the first-order system (1) is equivalent with the $(n+1)$-th-order linear recurrence relation

$$z_{k+n+1} = b_k z_{k+n} + a_k^{(n)} z_{k+n-1} + \cdots + a_k^{(1)} z_k.$$

In this case the general n-fraction reduces to the n-fraction (see de Bruin [2]) associated with this recurrence relation.

Using the notation

$$\begin{pmatrix} C_k & D_k \\ A_k & B_k \end{pmatrix}$$

for the matrix of the system (1), we shall from now on assume that it is nonsingular for all k.
The approximants of the general n-fraction associated with the system (1) may be calculated from the composition of linear fractional transformations :

$$\begin{aligned} &s_k(W) = \frac{A_k + WC_k}{B_k + WD_k} && (k = 0, 1, ...) \\ &S_0(W) = s_0(W) \qquad \text{and} \quad S_k(W) = S_{k-1}(s_k(W)) && (k = 1, 2, ...) \end{aligned} \tag{3}$$

where $W = (W^{(1)} \ ... \ W^{(n)})$ is a $1 \times n-$matrix (note: the denominator in the definition of s_k is a scalar).
We have the following theorem :

Theorem 1

$$S_k(W) = \frac{\tilde{A}_k + W\tilde{C}_k}{\tilde{B}_k + W\tilde{D}_k}$$

with

$$\begin{pmatrix} \tilde{C}_k & \tilde{D}_k \\ \tilde{A}_k & \tilde{B}_k \end{pmatrix} = \begin{pmatrix} C_k & D_k \\ A_k & B_k \end{pmatrix} \begin{pmatrix} C_{k-1} & D_{k-1} \\ A_{k-1} & B_{k-1} \end{pmatrix} \cdot \ ... \ \cdot \begin{pmatrix} C_0 & D_0 \\ A_0 & B_0 \end{pmatrix}$$

Proof : By induction on k, using simple algebra.

If

$$\begin{pmatrix} \tilde{C}_k & \tilde{D}_k \\ \tilde{A}_k & \tilde{B}_k \end{pmatrix} = \begin{pmatrix} \tilde{c}_k^{(1,1)} & \tilde{c}_k^{(1,2)} & \cdots & \tilde{c}_k^{(1,n)} & \tilde{d}_k^{(1)} \\ \vdots & \vdots & & \vdots & \vdots \\ \tilde{c}_k^{(n,1)} & \tilde{c}_k^{(n,2)} & \cdots & \tilde{c}_k^{(n,n)} & \tilde{d}_k^{(n)} \\ \tilde{a}_k^{(1)} & \tilde{a}_k^{(2)} & \cdots & \tilde{a}_k^{(n)} & \tilde{b}_k \end{pmatrix}$$

it is easy to see from (1) and (3) that $\mathcal{G}_k^{(n+1,j)} = \tilde{a}_k^{(j)}$ and $\mathcal{G}_k^{(n+1,n+1)} = \tilde{b}_k$. Hence

$$S_k(0) = \frac{\tilde{A}_k}{\tilde{B}_k} = \left(\frac{\tilde{a}_k^{(1)}}{\tilde{b}_k} \; \frac{\tilde{a}_k^{(2)}}{\tilde{b}_k} \; \ldots \; \frac{\tilde{a}_k^{(n)}}{\tilde{b}_k} \right)$$

is an n-tuple of approximants of the general n-fraction associated with (1).
The general n-fraction associated with (1) is said to converge if

$$\lim_{k\to\infty} \frac{\tilde{A}_k}{\tilde{B}_k} \in \mathbb{C}^n.$$

We have the following generalization of a result by Van der Cruyssen [14]:

Theorem 2 *The recurrence system (1) has a fundamental system of solutions*

$$\{(X_k^{(1,j)} \; \ldots \; X_k^{(n+1,j)})^T, \; j = 1, \ldots, n+1\}$$

satisfying

$$(a) \quad \det \begin{pmatrix} X_{-1}^{(1,1)} & \cdots & X_{-1}^{(1,n)} \\ \vdots & & \vdots \\ X_{-1}^{(n,1)} & \cdots & X_{-1}^{(n,n)} \end{pmatrix} \neq 0$$

$$(b) \quad \lim_{k\to\infty} \frac{X_k^{(n+1,j)}}{X_k^{(n+1,n+1)}} = 0 \quad \textit{for} \quad j = 1, \ldots, n$$

iff the general n-fraction associated with (1) converges, i.e.

$$\lim_{k\to\infty} \frac{\tilde{A}_k}{\tilde{B}_k} \in \mathbb{C}^n.$$

Proof : Let us first assume that (a) and (b) are satisfied. Since

$$\{(\mathcal{G}_k^{(1,j)} \; \ldots \; \mathcal{G}_k^{(n+1,j)})^T, \; j = 1, \ldots, n+1\} \quad \text{and} \quad \{(X_k^{(1,j)} \; \ldots \; X_k^{(n+1,j)})^T, \; j = 1, \ldots, n+1\}$$

are two fundamental systems of solutions of (1) there exists a constant non-singular $(n+1) \times (n+1)$−matrix F for which

$$\begin{pmatrix} \mathcal{G}_k^{(1,1)} & \cdots & \mathcal{G}_k^{(1,n)} & \mathcal{G}_k^{(1,n+1)} \\ \vdots & & \vdots & \vdots \\ \mathcal{G}_k^{(n,1)} & \cdots & \mathcal{G}_k^{(n,n)} & \mathcal{G}_k^{(n,n+1)} \\ \mathcal{G}_k^{(n+1,1)} & \cdots & \mathcal{G}_k^{(n+1,n)} & \mathcal{G}_k^{(n+1,n+1)} \end{pmatrix} = \begin{pmatrix} X_k^{(1,1)} & \cdots & X_k^{(1,n)} & X_k^{(1,n+1)} \\ \vdots & & \vdots & \vdots \\ X_k^{(n,1)} & \cdots & X_k^{(n,n)} & X_k^{(n,n+1)} \\ X_k^{(n+1,1)} & \cdots & X_k^{(n+1,n)} & X_k^{(n+1,n+1)} \end{pmatrix} \cdot F \; .$$

If we write F in block form :

$$F = \begin{pmatrix} F^{(1,1)} & F^{(1,2)} \\ F^{(2,1)} & F^{(2,2)} \end{pmatrix}$$

with $F^{(1,1)}$ an $n \times n-$matrix, we have

$$(\mathcal{G}_k^{(n+1,1)} \ \ldots \ \mathcal{G}_k^{(n+1,n)}) = (X_k^{(n+1,1)} \ \ldots \ X_k^{(n+1,n)}) \cdot F^{(1,1)} + X_k^{(n+1,n+1)} \cdot F^{(2,1)}$$

and

$$\mathcal{G}_k^{(n+1,n+1)} = (X_k^{(n+1,1)} \ \ldots \ X_k^{(n+1,n)}) \cdot F^{(1,2)} + X_k^{(n+1,n+1)} \cdot F^{(2,2)} \ .$$

Hence

$$\begin{pmatrix} \frac{\mathcal{G}_k^{(n+1,1)}}{\mathcal{G}_k^{(n+1,n+1)}} & \cdots & \frac{\mathcal{G}_k^{(n+1,n)}}{\mathcal{G}_k^{(n+1,n+1)}} \end{pmatrix} = \frac{\begin{pmatrix} \frac{X_k^{(n+1,1)}}{X_k^{(n+1,n+1)}} & \cdots & \frac{X_k^{(n+1,n)}}{X_k^{(n+1,n+1)}} \end{pmatrix} \cdot F^{(1,1)} + F^{(2,1)}}{\begin{pmatrix} \frac{X_k^{(n+1,1)}}{X_k^{(n+1,n+1)}} & \cdots & \frac{X_k^{(n+1,n)}}{X_k^{(n+1,n+1)}} \end{pmatrix} \cdot F^{(1,2)} + F^{(2,2)}}$$

and we have immediately from (b) that

$$\lim_{k\to\infty} \begin{pmatrix} \frac{\mathcal{G}_k^{(n+1,1)}}{\mathcal{G}_k^{(n+1,n+1)}} & \cdots & \frac{\mathcal{G}_k^{(n+1,n)}}{\mathcal{G}_k^{(n+1,n+1)}} \end{pmatrix} = \frac{F^{(2,1)}}{F^{(2,2)}}$$

if $F^{(2,2)} \neq 0$. To prove this observe that

$$\begin{pmatrix} \mathcal{G}_k^{(1,n+1)} \\ \vdots \\ \mathcal{G}_k^{(n,n+1)} \end{pmatrix} = \begin{pmatrix} X_k^{(1,1)} & \cdots & X_k^{(1,n)} \\ \vdots & & \vdots \\ X_k^{(n,1)} & \cdots & X_k^{(n,n)} \end{pmatrix} \cdot F^{(1,2)} + \begin{pmatrix} X_k^{(1,n+1)} \\ \vdots \\ X_k^{(n,n+1)} \end{pmatrix} \cdot F^{(2,2)}$$

and for $k = -1$ this reduces to

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} X_{-1}^{(1,1)} & \cdots & X_{-1}^{(1,n)} \\ \vdots & & \vdots \\ X_{-1}^{(n,1)} & \cdots & X_{-1}^{(n,n)} \end{pmatrix} \cdot F^{(1,2)} + \begin{pmatrix} X_{-1}^{(1,n+1)} \\ \vdots \\ X_{-1}^{(n,n+1)} \end{pmatrix} \cdot F^{(2,2)} \ .$$

$F^{(2,2)} = 0$ together with (a) would imply that $F^{(1,2)} = (0 \ \ldots \ 0)^T$ which is impossible since F is non-singular.

Let us now assume that the general n-fraction associated with (1) converges and that

$$\lim_{k\to\infty} \begin{pmatrix} \frac{\mathcal{G}_k^{(n+1,1)}}{\mathcal{G}_k^{(n+1,n+1)}} & \cdots & \frac{\mathcal{G}_k^{(n+1,n)}}{\mathcal{G}_k^{(n+1,n+1)}} \end{pmatrix} = (t^{(1)} \ \ldots \ t^{(n)}) = T \ .$$

The column-vectors of the matrix

$$\begin{pmatrix} \tilde{C}_k - \tilde{D}_k \cdot T & \tilde{D}_k \\ \tilde{A}_k - \tilde{B}_k \cdot T & \tilde{B}_k \end{pmatrix} = \begin{pmatrix} \tilde{C}_k & \tilde{D}_k \\ \tilde{A}_k & \tilde{B}_k \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -T & 1 \end{pmatrix}$$

(I_n is the identity matrix) form a fundamental system of solutions of (1) satisfying (b) since

$$\lim_{k\to\infty} \frac{\tilde{A}_k - \tilde{B}_k \cdot T}{\tilde{B}_k} = \lim_{k\to\infty} \frac{\tilde{A}_k}{\tilde{B}_k} - T = 0$$

and

$$\det \begin{pmatrix} \tilde{C}_{-1} - \tilde{D}_{-1} \cdot T & \tilde{D}_{-1} \\ \tilde{A}_{-1} - \tilde{B}_{-1} \cdot T & \tilde{B}_{-1} \end{pmatrix} = 1 \neq 0 \ .$$

□

Note that as a consequence a solution of (1) from the subspace spanned by the columns of the matrix

$$\begin{pmatrix} \tilde{C}_k - \tilde{D}_k \cdot T \\ \tilde{A}_k - \tilde{B}_k \cdot T \end{pmatrix}$$

has initial values $(Z_{-1}^{(1)} \ \ldots \ Z_{-1}^{(n)} \ Z_{-1}^{(n+1)})^T$ which satisfy

$$Z_{-1}^{(n+1)} = -t^{(n)} Z_{-1}^{(n)} - \ldots - t^{(1)} Z_{-1}^{(1)} \ . \tag{4}$$

In 1917 O. Perron [12] proved the following generalization to first-order recurrence systems of the well-known Poincaré-Perron theorem for linear recurrence relations of Poincaré-type. We shall use this result later on.

Theorem 3 *If the recurrence system (1) has the property that :* $\lim_{k\to\infty} \mathcal{A}_k = \mathcal{A}$, *with for all* k $\det \mathcal{A}_k \neq 0$, *and if the eigenvalues of* $\mathcal{A}$ *are all different in modulus :*

$$|\alpha_1| < |\alpha_2| < \cdots < |\alpha_{n+1}|, \tag{5}$$

then for each $j \in \{1, 2, \ldots, n+1\}$ *the recurrence system (1) has a solution*

$$(X_k^{(1,j)} \ \ldots \ X_k^{(n+1,j)})^T$$

for which

$$\lim_{k\to\infty} \frac{X_{k+1}^{(i,j)}}{X_k^{(i,j)}} = \alpha_j$$

for all $i \in \{1, 2, \ldots, n+1\}$ *for which the eigenvector* $(P^{(1,j)}, \ldots, P^{(n+1,j)})^T$ *corresponding to the eigenvalue* α_j *has* i*-th component different from zero, i.e.* $P^{(i,j)} \neq 0$.

3. Applications

3.1. Numerical Calculation of Non-Dominant Solutions of a Recurrence System

Theorem 2 provides us with a method to calculate solutions of the recurrence system (1) which in a certain sense are non-dominant, and cannot be calculated numerically from (1) using forward recurrence:
if the condition (b) of Theorem 2 holds, then a solution of (1) from the subspace

spanned by $\{(X_k^{(1,j)} \ \ldots \ X_k^{(n+1,j)})^T, \ j = 1, \ldots, n\}$ has $(n+1)$−st component dominated by the $n+1$−st component of $(X_k^{(1,n+1)} \ \ldots \ X_k^{(n+1,n+1)})^T$. To see this we look at the following example. If we take $n = 1$ and

$$C_k = (2 \cdot 8^{k+1} - 1)/(8^{k+1} - 1) \ , \quad D_k = -2^{k+1}/(8^{k+1} - 1)$$
$$A_k = -3 \cdot 4^{k+1}/(8^{k+1} - 1) \ , \quad B_k = (4 \cdot 8^{k+1} - 1)/(8^{k+1} - 1)$$

in (1), then this recurrence system has the solutions $(2^{k+1} \ 1)^T$, and $(1 \ 4^{k+1})^T$. Hence every solution of the system can be written as :

$$r \begin{pmatrix} 2^{k+1} \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 4^{k+1} \end{pmatrix} .$$

If we want to calculate the solution $(2^{k+1} \ 1)^T$ using forward recurrence, a rounding error in the initial value will result in an $s \neq 0$, and so the second component of $(1 \ 4^{k+1})^T$ will eventually become dominant in the calculated solution.
Let us now assume that the conditions (a) and (b) of Theorem 2 hold, but with (a) replaced by

$$\det \begin{pmatrix} X_k^{(1,1)} & \cdots & X_k^{(1,n)} \\ \vdots & & \vdots \\ X_k^{(n,1)} & \cdots & X_k^{(n,n)} \end{pmatrix} \neq 0$$

for all k. Then the general n-fraction associated with the recurrence system

$$Y_k = \mathcal{A}_{k+p} Y_{k-1}$$

converges for each $p = 0, 1, \ldots$. Let its value be $T_p = (t_p^{(1)} \ \ldots \ t_p^{(n)})$.
A solution of the subspace spanned by $\{(X_k^{(1,j)} \ \ldots \ X_k^{(n+1,j)})^T, \ j = 1, \ldots, n\}$ satisfies

$$Z_{k-1}^{(n+1)} = -T_k \cdot \begin{pmatrix} Z_{k-1}^{(1)} \\ \vdots \\ Z_{k-1}^{(n)} \end{pmatrix}$$
$$\begin{pmatrix} Z_k^{(1)} \\ \vdots \\ Z_k^{(n)} \end{pmatrix} = (C_k - D_k \cdot T_k) \begin{pmatrix} Z_{k-1}^{(1)} \\ \vdots \\ Z_{k-1}^{(n)} \end{pmatrix} \tag{6}$$

for $k = 0, 1, \ldots$. (This follows immediately from (4)). We can use this to calculate solutions of this subspace which are dominant in this subspace : for some index N we calculate $S_N(0)$ using (3). This gives us an approximation $T_0(N) = S_N(0)$ for the value T_0 of the general n-fraction associated with (1), and also approximations $T_k(N)$ for T_k $(k = 1, 2, \ldots, N)$. If the solution we want to calculate is $(Z_k^{(1)} \ \ldots \ Z_k^{(n)} \ Z_k^{(n+1)})^T$ where $Z_{-1}^{(i)}$, $i = 1, \ldots, n$, are given initial values, we get an approximation for this

solution from :

$$Z_{k-1}^{(n+1)} = -T_k(N) \cdot \begin{pmatrix} Z_{k-1}^{(1)} \\ \vdots \\ Z_{k-1}^{(n)} \end{pmatrix}$$
$$\begin{pmatrix} Z_k^{(1)} \\ \vdots \\ Z_k^{(n)} \end{pmatrix} = (C_k - D_k \cdot T_k(N)) \begin{pmatrix} Z_{k-1}^{(1)} \\ \vdots \\ Z_{k-1}^{(n)} \end{pmatrix} \tag{7}$$

for $k = 0, 1, ..., N$. For the example above we get with $N = 80$ the results given in the following table (for the solution $(2^{k+1} \quad 1)^T$). In the table the results using forward recursion and using (7) are compared. We note that other authors have

TABLE I

k	$Z_k^{(1)}$ using (7)	$Z_k^{(1)}$ forward	$Z_k^{(2)}$ using(7)	$Z_k^{(2)}$ forward
-1	0.200000E+01	0.200000E+01	0.100000E+01	0.100000E+01
10	0.409600E+04	0.409600E+04	0.100000E+01	0.100000E+01
20	0.419430E+07	0.419430E+07	0.100000E+01	0.100000E+01
30	0.429497E+10	0.429497E+10	0.100000E+01	0.100000E+01
40	0.439805E+13	0.439805E+13	0.100000E+01	0.112500E+01
50	0.450360E+16	0.450360E+16	0.100000E+01	0.131073E+06
60	0.461169E+19	0.461169E+19	0.100000E+01	0.137439E+12
70	0.472237E+22	0.472237E+22	0.999999E+00	0.144115E+18
80	0.483570E+25	0.483570E+25	0.000000E+00	0.151116E+24

already given algorithms to compute non-dominant solutions of (1): Mattheij in the papers [9], [10], and Zahar [16]. Our method is equivalent with Zahar's method, the differences being that Zahar's algorithm is more general and it is a forward-backward algorithm, while ours is a backward-forward algorithm. However, it is possible to adapt our algorithm to the more general setting of Zahar's work. We also note that the theoretical method behind these two algorithms is known in the literature as *method of embedding* (see [1]).

3.2. The 1-Periodic Case

Let us assume that for all k $\mathcal{A}_k = \mathcal{A}$ with $\det\mathcal{A} \neq 0$, that the eigenvalues α_j of $\mathcal{A}$ satisfy (5) and that the corresponding eigenvectors $(P^{(1,j)}, ..., P^{(n+1,j)})^T$ satisfy

$$\det \begin{pmatrix} P^{(1,1)} & \dots & P^{(1,n)} \\ \vdots & & \vdots \\ P^{(n,1)} & \dots & P^{(n,n)} \end{pmatrix} \neq 0 \ ,$$

and that $P^{(n+1,n+1)} \neq 0$. The conditions of Theorem 3 are satisfied with

$$X_k^{(i,j)} = \alpha_j^{k+1} P^{(i,j)}$$

for $i,j = 1,...,n+1$ and the general n-fraction associated with (1) converges to $T = (t^{(1)} \ \ldots \ t^{(n)})$ where $(t^{(1)} \ \ldots \ t^{(n)} \ 1)^T$ is an eigenvector of $\mathcal{A}^T$ corresponding to the eigenvalue α_{n+1} : if the general n-fraction converges to T, then it is easy to see from (3) that

$$T = \frac{A+TC}{B+TD} = \frac{(T \ \ 1)\begin{pmatrix} C \\ A \end{pmatrix}}{(T \ \ 1)\begin{pmatrix} D \\ B \end{pmatrix}}$$

and hence with $\lambda = (T \ \ 1)\begin{pmatrix} D \\ B \end{pmatrix}$

$$(\lambda T \ \ \lambda) = (T \ \ 1)\begin{pmatrix} C & D \\ A & B \end{pmatrix}$$

or

$$\begin{pmatrix} C & D \\ A & B \end{pmatrix}^T \begin{pmatrix} T^T \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} T^T \\ 1 \end{pmatrix} .$$

Furthermore $(T \ \ 1)$ satisfies (4) :

$$(T \ \ 1)\begin{pmatrix} P^{(1,j)} \\ \vdots \\ P^{(n+1,j)} \end{pmatrix} = 0 \tag{8}$$

for $j = 1,...,n$. Now the only eigenvector of $\mathcal{A}^T$ satisfying (8) is the one corresponding to the eigenvalue α_{n+1}.
It is easy to see that for the periodic case the algorithm presented in the previous section is just a scaled version of the power method for the computation of the dominant eigenvalue (and the corresponding eigenvector) of a matrix.

3.3. The Limit-Periodic Case

Let us assume that for all k $\lim_{k\to\infty} \mathcal{A}_k = \mathcal{A}$ with $\det\mathcal{A}_k \neq 0$ for all k, that the eigenvalues α_j of $\mathcal{A}$ satisfy (5) and that the corresponding eigenvectors $(P^{(1,j)},...,P^{(n+1,j)})^T$ satisfy

$$\det \begin{pmatrix} P^{(1,1)} & \ldots & P^{(1,n)} \\ \vdots & & \vdots \\ P^{(n,1)} & \ldots & P^{(n,n)} \end{pmatrix} \neq 0 , \tag{9}$$

and that $P^{(n+1,j)} \neq 0$ for $j = 1,...,n+1$. Then the conditions of Theorem 3 are satisfied and the recurrence system (1) has a fundamental system of solutions $\{(X_k^{(1,j)} \ \ldots \ X_k^{(n+1,j)})^T$ for which

$$\lim_{k\to\infty} \frac{X_{k+1}^{(n+1,j)}}{X_k^{(n+1,j)}} = \alpha_j$$

and hence

$$\lim_{k\to\infty} \frac{X_k^{(n+1,j)}}{X_k^{(n+1,n+1)}} = 0$$

as a consequence of (5). Furthermore from the proof of Theorem 3 (see [12]) it follows that

$$\lim_{k\to\infty} \frac{X_k^{(i,j)}}{X_k^{(n+1,j)}} = \frac{P^{(i,j)}}{P^{(n+1,j)}} \,. \tag{10}$$

From the determinantal condition (9) together with (10) we get that

$$\det \begin{pmatrix} X_k^{(1,1)} & \cdots & X_k^{(1,n)} \\ \vdots & & \vdots \\ X_k^{(n,1)} & \cdots & X_k^{(n,n)} \end{pmatrix} \neq 0$$

from some $k = k_0$ on, hence the general n-fraction associated with $Y_k = \mathcal{A}_{k+p} Y_{k-1}$ will converge for $k = k_0, k_0 + 1, \ldots$ to the value T_p. Since

$$T_p = \frac{A_p + T_{p+1} C_p}{B_p + T_{p+1} D_p}$$

(as is easy to prove using Theorem 1), and (from (4))

$$t_p^{(j)} = -\frac{\det \begin{pmatrix} X_{p-1}^{(1,1)} & \cdots & X_{p-1}^{(j-1,1)} & X_{p-1}^{(n+1,1)} & X_{p-1}^{(j+1,1)} & \cdots & X_{p-1}^{(n,1)} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ X_{p-1}^{(1,n)} & \cdots & X_{p-1}^{(j-1,n)} & X_{p-1}^{(n+1,n)} & X_{p-1}^{(j+1,n)} & \cdots & X_{p-1}^{(n,n)} \end{pmatrix}}{\det \begin{pmatrix} X_{p-1}^{(1,1)} & \cdots & X_{p-1}^{(n,1)} \\ \vdots & & \vdots \\ X_{p-1}^{(1,n)} & \cdots & X_{p-1}^{(n,n)} \end{pmatrix}} \,,$$

we get using (10) that $\lim_{p\to\infty} t_p^{(j)} = \tilde{t}^{(j)}$ with $\tilde{T} = (\tilde{t}^{(1)} \quad \ldots \quad \tilde{t}^{(n)})$ satisfying

$$\tilde{T} = \frac{A + \tilde{T}C}{B + \tilde{T}D} \quad \text{and} \quad P^{(n+1,j)} = -\tilde{t}^{(n)} P^{(n,j)} - \ldots - \tilde{t}^{(1)} P^{(1,j)} \,.$$

Now it is easy to see that in this case too $\lim_{p\to\infty} T_p$ is the eigenvector of $\mathcal{A}^T$ corresponding to the eigenvalue α_{n+1}.

We note that this result can be used to accelerate the convergence of the algorithm discussed in section 3.1 in the limit-periodic case. For similar results on ordinary continued fractions and n-fractions we refer to [8], [3], [6].

3.4. A New Approach to the Simultaneous Rational Interpolation Problem

In their paper [13] Van Barel and Bultheel describe an algorithm to solve the vector case of the rational interpolation problem. We shall reformulate this algorithm and

show that the solution of the problem may be given in the form of a general n-fraction (for more details we refer to [13]).
Let us first state the problem:

Definition 1 (proper simultaneous rational interpolation problem)
Find all polynomial $(n+1)$*-tuples* $(p^{(1)}, p^{(2)}, \ldots, p^{(n+1)})$ *such that*
– *the following rational interpolation conditions are satisfied:*

$$(p^{(1)}(z) \quad \cdots \quad p^{(n)}(z))/p^{(n+1)}(z) = F(z) + (O(\pi_\sigma(z)) \quad \cdots \quad O(\pi_\sigma(z)))$$

with $F(z)$ *a vector Newton series using the sequence of points* $z_1, z_2, \ldots$, *and* $\pi_\sigma(z) = \prod_{j=1}^{\sigma}(z - z_j)$,
– *the following degree conditions are satisfied:*

$$\deg(p^{(1)} \quad \cdots \quad p^{(n+1-r)}\ p^{(n+2-r)} \quad \cdots \quad p^{(n+1)}) \leq (\mu \quad \cdots \quad \mu \overbrace{\mu+1 \quad \cdots \quad \mu+1}^{r}) \tag{11}$$

with $n\sigma + 1 = (\mu+1)(n+1) + r$ *and* $0 \leq r < n+1$.

Note that the number of parameters, i.e. the total number of coefficients $(\mu+1)(n+1)+r$, is one more than the number of conditions $n\sigma$. Increasing the order σ by one, i.e. $\sigma' = \sigma + 1$, transforms μ and r as follows: if $r = 0$ then $\mu' = \mu$ and $r' = n$; if $0 < r < n+1$ then $\mu' = \mu + 1$ and $r' = (r+n)\mathbf{mod}(n+1)$.

Definition 2 (linearized simultaneous rational interpolation problem)
Find all polynomial $(n+1)$*-tuples* $(p^{(1)}, p^{(2)}, \ldots, p^{(n+1)})$ *such that:*
– *the following linearized rational interpolation conditions are satisfied:*

$$(p^{(1)}(z) \quad \cdots \quad p^{(n)}(z)\ p^{(n+1)}(z)) \begin{pmatrix} -I_n \\ F(z) \end{pmatrix} = (O(\pi_\sigma(z)) \quad \cdots \quad O(\pi_\sigma(z))) \ , \tag{12}$$

with the same notation as before,
– *the same degree conditions as for the proper case are satisfied.*

Note that this linearized problem has at least one solution with one degree of freedom (which disappears when the polynomial tuple is considered as a vector rational function). When $p^{(n+1)}(z_j) \neq 0$, for $j = 1, 2, \ldots, \sigma$, the solution of the linearized problem is also a solution of the proper problem. It is easy to see that all solutions of the proper problem form a subset of all solutions of the linearized problem. Hence, we first compute all solutions of the linearized problem.

Theorem 4 *All polynomial* $(n+1)$*-tuples satisfying the linearized rational interpolation conditions (12) form a submodule* S_σ *of the* $\mathbf{F}[z]$*-module* $\mathbf{F}[z]^{n+1}$ *with a set of* $(n+1)$ *basis vectors.*

Proof : see [13].
Let us give an efficient algorithm to compute these basis vectors in a recursive way, i.e. for increasing σ.
– **Initialization:** The rows of the $(n+1) \times (n+1)$ identity matrix form a set of basis vectors for S_0.

– **Recursive step:** Suppose we have a basis for S_σ. We can compute a basis for $S_{\sigma+1}$ as follows:

= Using the basis vectors $g_\sigma^{(1)}, \ldots, g_\sigma^{(n+1)}$ for S_σ, we can determine the corresponding residual vectors $r_\sigma^{(1)}, \ldots, r_\sigma^{(n+1)}$:

$$\begin{pmatrix} g_\sigma^{(1)}(z) \\ \vdots \\ g_\sigma^{(n+1)}(z) \end{pmatrix} \begin{pmatrix} -I_n \\ F(z) \end{pmatrix} = \begin{pmatrix} r_\sigma^{(1)} \\ \vdots \\ r_\sigma^{(n+1)} \end{pmatrix} \pi_\sigma(z) + \begin{pmatrix} O(\pi_{\sigma+1}(z)) \cdots O(\pi_{\sigma+1}(z)) \\ \vdots \qquad\qquad \vdots \\ O(\pi_{\sigma+1}(z)) \cdots O(\pi_{\sigma+1}(z)) \end{pmatrix}$$

where we have written the polynomial $(n+1)$-tuples rowwise.

= The set of homogeneous linear equations

$$\left(v_{\sigma+1}^{(1)} \; v_{\sigma+1}^{(2)} \cdots v_{\sigma+1}^{(n+1)}\right) \begin{pmatrix} r_\sigma^{(1)} \\ r_\sigma^{(2)} \\ \vdots \\ r_\sigma^{(n+1)} \end{pmatrix} = 0 \tag{13}$$

always has at least one non-zero solution (which can always be multiplied by a scalar). Let us assume that there always exists a solution $v_{\sigma+1}$ of (12) with $v_{\sigma+1}^{(1)} \neq 0$, $\sigma = 0, 1, \ldots$.

= Once we have determined the non-zero solution $v_{\sigma+1}$ of (12) it is easy to show that the following set of polynomial $(n+1)$-tuples forms a basis for $S_{\sigma+1}$:

$$\begin{pmatrix} g_{\sigma+1}^{(1)}(z) \\ \vdots \\ g_{\sigma+1}^{(n+1)}(z) \end{pmatrix} = \mathcal{A}_\sigma \begin{pmatrix} g_\sigma^{(1)}(z) \\ \vdots \\ g_\sigma^{(n+1)}(z) \end{pmatrix}$$

or

$$G_{\sigma+1} = \mathcal{A}_\sigma G_\sigma \tag{14}$$

with

$$\mathcal{A}_\sigma = \begin{pmatrix} 0 & z - z_{\sigma+1} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & z - z_{\sigma+1} \\ v_{\sigma+1}^{(1)} & v_{\sigma+1}^{(2)} & \cdots & v_{\sigma+1}^{(n)} & v_{\sigma+1}^{(n+1)} \end{pmatrix}.$$

From the fact that $(r_0^{(1)} \; r_0^{(2)} \cdots r_0^{(n)})^T = -I_n$, it can easily be shown that $r_\sigma^{(n+1)} \neq 0$, $v_{\sigma+1}^{(n+1)} \neq 0$ and $r_\sigma^{(1)}, \ldots, r_\sigma^{(n)}$ are linearly independent, i.e. the solution of (13) is unique except for a scalar multiple.

The set S_σ of all solutions of (12) is generated by the $n+1$ polynomial $(n+1)$-tuples $g_\sigma^{(1)}, g_\sigma^{(2)}, \ldots, g_\sigma^{(n+1)}$. We want to extract only those solutions satisfying the additional degree conditions (11). It is easy to prove that the last basis vector $g_\sigma^{(n+1)}$

satisfies the degree conditions with equality for at least one component. On the other hand, the other basis vectors do not satisfy the degree conditions. It is also true that the last component of $g_\sigma^{(n+1)}(z_j) \neq 0$, $j = 1, 2, \ldots, \sigma$. Hence, the linearized as well as the proper simultaneous rational interpolation problem has one and just one solution, the last basis vector $g_\sigma^{(n+1)}$.
Now it is easy to see from (14) that, in the notation of the second paragraph,

$$\tilde{A}_k = (g_{k+1}^{(1)}, g_{k+1}^{(2)}, \ldots, g_{k+1}^{(n)}) \ , \quad \tilde{B}_k = g_{k+1}^{(n+1)}$$

and hence that the solution of our first problem is given by the approximants of the general n-fraction associated with this recurrence system. Because the matrix of this system has a special form, it is easy to transform it in the form of an n-fraction (see (2)). This is done by transforming the general n-fraction represented by $\cdots \mathcal{A}_\sigma \mathcal{A}_{\sigma-1} \cdots \mathcal{A}_1 \mathcal{A}_0$ into the following form $\cdots W_\sigma W_{\sigma-1} \cdots W_1 W_0$ where all W_σ, $\sigma > 0$ have the form

$$\begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ w_{\sigma+1}^{(1)} & w_{\sigma+1}^{(2)} & \cdots & w_{\sigma+1}^{(n)} & w_{\sigma+1}^{(n+1)} \end{pmatrix}.$$

Suppose we have already transformed the first σ blocks, i.e.

$$\mathcal{A}_{\sigma-1} \cdots \mathcal{A}_0 = E_{\sigma-1} W_{\sigma-2} \cdots W_0 \tag{15}$$

with $E_{\sigma-1}$ having the following form

$$E_{\sigma-1} = \begin{pmatrix} 0 & u_\sigma^{(1)} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & u_\sigma^{(n)} \\ s_\sigma^{(1)} & s_\sigma^{(2)} & \cdots & s_\sigma^{(n)} & s_\sigma^{(n+1)} \end{pmatrix}$$

with $s_\sigma^{(j)}$ and $u_\sigma^{(j)}$ polynomials. These polynomials together with the polynomials $w_\sigma^{(j)}$ can be determined by using the equality $\mathcal{A}_\sigma E_{\sigma-1} = E_\sigma W_{\sigma-1}$ giving the following equations

$$\begin{aligned} w_\sigma^{(i)} &= s_\sigma^{(i)}, \quad i = 1, 2, \ldots n+1 \\ s_{\sigma+1}^{(i)} &= u_\sigma^{(i)} v_{\sigma+1}^{(i)}, \quad i = 1, 2, \ldots, n+1 \\ u_{\sigma+1}^{(i)} &= u_\sigma^{(i+1)} \cdot (z - z_{\sigma+1}), \quad i = 1, 2, \ldots, n \end{aligned}$$

with $u_\sigma^{(n+1)} = 1$ and initial values $u_1^{(i)} = z - z_1$, $i = 1, 2, \ldots, n$ and $s_1^{(i)} = v_1^{(i)}$, $i = 1, 2, \ldots, n+1$. Using these recurrence relations, it is easy to prove that

$$\begin{aligned} u_\sigma^{(i)}(z) &= \textstyle\prod_{j=0}^{n-i} (z - z_{\sigma-j}), \quad i = 1, 2, \ldots, n \\ w_\sigma^{(i)}(z) &= u_{\sigma-1}^{(i)}(z) v_\sigma^{(i)}, \quad i = 1, 2, \ldots, n+1 \\ s_\sigma^{(i)}(z) &= w_\sigma^{(i)}(z), \quad i = 1, 2, \ldots, n+1 \end{aligned}$$

where the factors $z - z_k = 1$ for $k \leq 0$. Because we can always scale the initial solution v_1 such that $v_1^{(n+1)} = 1$, the solutions of the proper (and linearized) simultaneous rational interpolation problem can also be written as the convergents of the n-fraction associated with (1).
A similar n-fraction was found in [7] using another method.

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