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Cartan subalgebras in II_1 factors

Anna Sofie Krogager

Supervisor:
Prof. dr. Stefaan Vaes

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Anna Sofie KROGAGER

Examination committee:
Prof. dr. Nero Budur, chair
Prof. dr. Stefaan Vaes, supervisor
Prof. dr. Johan Quaegebeur
Prof. dr. Johannes Nicaise
Dr. Jorge Castillejos
Prof. dr. Magdalena Musat
(University of Copenhagen)
Prof. dr. Martijn Caspers
(Delft University of Technology)

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Uitgegeven in eigen beheer, Anna Sofie Krogager, Celestijnenlaan 200A box 2402, B-3001 Leuven (Belgium)

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Abstract

The group measure space construction of Murray and von Neumann associates to every free ergodic probability measure preserving group action $\Gamma \curvearrowright (X, \mu)$ a crossed product II_1 factor $L^\infty(X) \rtimes \Gamma$. It is a fundamental problem in the theory of II_1 factors to classify these crossed products in terms of the underlying group action.

The subalgebra $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ plays an important role in such classification results; it is a so-called *Cartan subalgebra.* When a crossed product II_1 factor $L^\infty(X) \rtimes \Gamma$ has a *unique* Cartan subalgebra, we can partially recover information about the group action $\Gamma \curvearrowright X$ from the associated II_1 factor. In the most extreme case, we can completely recover the underlying action $\Gamma \curvearrowright X$, in the following sense: whenever $L^\infty(X) \rtimes \Gamma$ is isomorphic with another crossed product $L^\infty(Y) \rtimes \Lambda$, then the action $\Lambda \curvearrowright Y$ must be conjugate with $\Gamma \curvearrowright X$. A group action $\Gamma \curvearrowright X$ satisfying this property is called *W*-superrigid.* The first W*-superrigid actions were discovered in 2009 in [Pe09, PV09].

In this thesis, we construct the first II_1 factors having exactly two group measure space decompositions up to unitary conjugacy. Also, for every positive integer n, we construct a II_1 factor M that has exactly n group measure space decompositions up to conjugacy by an automorphism of M.

Our second main result is concerned with the existence of Cartan subalgebras inside a given II_1 factor. It is a wide open problem to give an intrinsic criterion for a II_1 factor M to admit a Cartan subalgebra A. When $A \subset M$ is a Cartan subalgebra, the A-bimodule $L^2(M)$ is "simple" in the sense that the left and right action of A generate a maximal abelian subalgebra of $B(L^2(M))$. A II_1 factor M that admits such a subalgebra A is said to be s-thin. Recently, Popa discovered an intrinsic local criterion for a II_1 factor M to be s-thin and left open the question whether all s-thin II_1 factors admit a Cartan subalgebra. We answer this question negatively by constructing s-thin II_1 factors without Cartan subalgebras.

Contents

Chapter 1

Introduction

This thesis is based on my two publications [KV15] and [KV16], which are joint work with Stefaan Vaes. In particular, parts of this introduction have already appeared in these articles.

A *von Neumann algebra* is an algebra of bounded linear operators on a Hilbert space that is stable under taking the adjoint of an operator and that is closed in the strong operator topology. The commutative von Neumann algebras are of the form $L^\infty(X, \mu)$ for some measure space (X, μ). Therefore, von Neumann algebras can be considered as non-commutative measure spaces.

The most important examples of von Neumann algebras arise from discrete groups and their actions on probability spaces. Given a countable discrete group Γ, the *group von Neumann algebra* $L(\Gamma)$ is the von Neumann algebra generated by the left regular representation of Γ on the Hilbert space $\ell^2(\Gamma)$. If Γ acts on a probability space (X, μ), then we can associate a *group measure space von Neumann algebra* $L^\infty(X) \rtimes \Gamma$, generated by a copy of $L^\infty(X)$ and unitary elements $(u_g)_{g\in\Gamma}$ satisfying $u_g u_h = u_{gh}$ and $u_g^* f u_g = f(g\,\cdot)$ for $g, h \in \Gamma$ and $f \in L^\infty(X)$.

The simple objects among von Neumann algebras are called *factors*. More precisely, a factor is a von Neumann algebra with trivial center. Factors are exactly the von Neumann algebras that cannot be written as a direct sum of two. Moreover, Murray and von Neumann [vN49] showed that any von Neumann algebra can be written as a "generalized direct sum" of factors, thereby theoretically reducing the study of von Neumann algebras to the study of factors.

Murray and von Neumann furthermore divided factors into 3 different types. The type that we will focus on in this thesis is type II_1. A factor M is said to be of *type II_1* if M is infinite-dimensional and admits a finite *trace* τ, i.e., a positive linear functional $\tau\colon M \to \mathbb{C}$ satisfying $\tau(xy) = \tau(yx)$ for all $x, y \in M$. Factors of type II_1 should be thought of as "continuous" analogues of the matrix algebras $M_n(\mathbb{C})$, equipped with their usual trace tr_n. Indeed, if M is a II_1 factor with its normalized trace τ, then $\tau(p)$ can take any value in the interval $[0, 1]$, when p ranges through the set of projections in M. On the other hand, tr_n can only take a discrete set of values on the projections in $M_n(\mathbb{C})$.

The main examples of II_1 factors are given by group von Neumann algebras $L(\Gamma)$ and the group measure space construction $L^\infty(X) \rtimes \Gamma$ introduced above. Indeed, $L(\Gamma)$ is a II_1 factor whenever Γ is an icc group, meaning that all nontrivial conjugacy classes of Γ are infinite. The group measure space von Neumann algebra $L^\infty(X) \rtimes \Gamma$ is a II_1 factor whenever the action $\Gamma \curvearrowright (X, \mu)$ is free, ergodic and probability measure preserving (pmp).

One of the core problems in operator algebras is to classify these II_1 factors associated with groups and group actions. A natural question to ask is to what extent the II_1 factor $L(\Gamma)$ or $L^\infty(X) \rtimes \Gamma$ "remembers" the group Γ or the group action $\Gamma \curvearrowright X$, respectively. For amenable icc groups Γ, Connes [Co75] showed that all group von Neumann algebras and all group measure space II_1 factors are isomorphic. In fact, he proved that there is a unique amenable II_1 factor R. This means that, within the class of amenable groups, all information about the group gets lost when passing to the von Neumann algebra level.

For non-amenable groups Γ, the situation is much more complicated. Rigidity phenomena appear and we are sometimes able to recover structural properties of the group Γ or group action $\Gamma \curvearrowright X$ only by looking at the associated von Neumann algebra. Proving this kind of classification results for non-amenable II_1 factors is, however, an extremely difficult problem, and not much progress was made until the early 2000's, when Popa developed his *deformation/rigidity theory*, [Po01, Po03, Po04]. This was a major breakthrough and has led to far reaching classification theorems. In particular, a surprisingly strong rigidity property for II_1 factors was discovered, called *W*-superrigidity*: in certain cases, the group measure space II_1 factor $L^\infty(X) \rtimes \Gamma$ entirely remembers Γ and its action on X. More precisely, a group action $\Gamma \curvearrowright X$ is called W*-superrigid if whenever $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$ for any other action $\Lambda \curvearrowright Y$, we must have that the two actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are *conjugate*: there exists a group isomorphism $\varphi\colon \Gamma \to \Lambda$ and a measure space isomorphism $\Phi\colon X \to Y$ such that $\Phi(g \cdot x) = \varphi(g) \cdot \Phi(x)$ for almost every $x \in X$ and $g \in \Gamma$. This extreme form of rigidity was first discovered by Peterson in [Pe09] and since then, concrete examples of W*-superrigid actions were found in [PV09, Io10, IPV10, GIT16], to name a few.

In these results, the subalgebra $L^\infty(X)$ of $L^\infty(X) \rtimes \Gamma$ plays a special role. Indeed, by [Si55], if there is an isomorphism $\pi\colon L^\infty(X) \rtimes \Gamma \to L^\infty(Y) \rtimes \Lambda$ of group measure space II_1 factors satisfying $\pi(L^\infty(X)) = L^\infty(Y)$, then the two group actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ must have the same orbit structure. More precisely, the actions must be *orbit equivalent*, meaning that there is a measure space isomorphism $\Phi\colon X \to Y$ such that $\Phi(\Gamma \cdot x) = \Lambda \cdot \Phi(x)$ almost everywhere. So, in order to prove that a given action $\Gamma \curvearrowright X$ is W*-superrigid, one must solve two different problems. First, one shows that the subalgebra $L^\infty(X)$ of $L^\infty(X) \rtimes \Gamma$ is unique, in some sense. If that is the case, then any isomorphism $\pi\colon L^\infty(X) \rtimes \Gamma \to L^\infty(Y) \rtimes \Lambda$ must automatically satisfy $\pi(L^\infty(X)) = L^\infty(Y)$, so that the actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are automatically orbit equivalent. Secondly, one can apply methods from measured group theory to deduce that the two actions are actually conjugate.

The subalgebra $A = L^\infty(X)$ of $M = L^\infty(X) \rtimes \Gamma$ is *Cartan:* it is maximal abelian and the normalizer $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ generates M. Note that a general Cartan subalgebra $A \subset M$ need not be of group measure space type, i.e., there need not exist a group Γ complementing A in such a way that $M = A \rtimes \Gamma$. This is closely related to the phenomenon that a countable pmp equivalence relation need not be the orbit equivalence relation of a group action that is free. For the purpose of proving W*-superrigidity, it is enough to show that a given II_1 factor M has a unique group measure space Cartan subalgebra *up to conjugacy by an automorphism*, in the sense that for any two group measure space Cartan subalgebras $A, B \subset M$, there exists an automorphism $\varphi \in \mathrm{Aut}(M)$ such that $\varphi(A) = B$. However, the methods used to prove such uniqueness results will usually yield a much stronger result, namely that the given II_1 factor M has a unique *general* Cartan subalgebra *up to unitary conjugacy*. Here, we say that two Cartan subalgebras $A, B \subset M$ are unitarily conjugate if there exists a unitary $u \in M$ such that $uAu^* = B$. The first actual uniqueness theorems for Cartan subalgebras up to unitary conjugacy were only obtained in [OP07], where it was proved in particular that A is the unique Cartan subalgebra of $A \rtimes \Gamma$ whenever $\Gamma = \mathbb{F}_n$ is a free group and $\mathbb{F}_n \curvearrowright A$ is a free ergodic pmp action that is *profinite*. More recently, in [PV11], it was shown that A is the unique Cartan subalgebra of $A \rtimes \Gamma$ for *arbitrary* free ergodic pmp actions of the free groups $\Gamma = \mathbb{F}_n$. A group Γ satisfying this property is called *Cartan-rigid* or *$\mathcal{C}$-rigid*.

Since the work in [PV11], more and more groups have been shown to be $\mathcal{C}$-rigid. However, it is at the moment highly unclear how widespread the phenomenon of $\mathcal{C}$-rigidity is and there are no conjectures on a possible characterization of $\mathcal{C}$-rigid groups. The main reason for this is a lack of a wide variety of counterexamples, i.e., groups Γ that admit a crossed product $L^\infty(X) \rtimes \Gamma$ with at least two non unitarily conjugate Cartan subalgebras. All amenable groups serve as such counterexamples. Indeed, any crossed product by an amenable group Γ is isomorphic to the unique hyperfinite II_1 factor R, and it is known since [FM75] that R has at least two Cartan subalgebras that are not unitarily conjugate. By [Pa85], there are in fact uncountably many Cartan subalgebras up to unitary conjugacy. On the other hand, all Cartan subalgebras of R are conjugate by an automorphism, [CFW81]. The first example of a II_1 factor with at least two Cartan subalgebras that are not conjugate by an automorphism was obtained in [CJ82]. Later, several explicit examples of this phenomenon were given in [OP08, PV09, SV11]. In all of these examples, the "second" Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$ comes from an abelian, normal subgroup of Γ. In Chapter 3, we show that in fact any group Γ that contains an infinite, abelian, almost normal subgroup is not $\mathcal{C}$-rigid, up to taking the quotient by a finite normal subgroup of Γ. This is done by generalizing the constructions in [PV09] and [SV11] to construct a concrete action of Γ for which the associated crossed product has at least two non-conjugate Cartan subalgebras.

Despite all the progress on uniqueness of Cartan subalgebras, there are so far no results describing *all* Cartan subalgebras of a II_1 factor M once uniqueness fails. However, in [KV15] we proved such a result for the special class of group measure space Cartan subalgebras. The following is our main theorem.

Theorem A. (1) *For every integer $n \geq 0$, there exist II_1 factors M that have exactly 2^n group measure space Cartan subalgebras up to unitary conjugacy.*

(2) *For every integer $n \geq 1$, there exist II_1 factors M that have exactly n group measure space Cartan subalgebras up to conjugacy by an automorphism of M.*

Two free ergodic pmp actions are called *W*-equivalent* if they have isomorphic crossed product von Neumann algebras. Thus, a free ergodic pmp action $G \curvearrowright (X, \mu)$ is W*-superrigid if every action that is W*-equivalent to $G \curvearrowright (X, \mu)$ must be conjugate to $G \curvearrowright (X, \mu)$. Theorem A(2) can then be rephrased in the following way: we construct free ergodic pmp actions $G \curvearrowright (X, \mu)$ with the property that $G \curvearrowright (X, \mu)$ is W*-equivalent to exactly n group actions, up to orbit equivalence of the actions (and actually also up to conjugacy of the actions, see Theorem 4.24).

The proof of Theorem A will be presented in Chapter 4. We will provide a concrete construction of crossed product II_1 factors M for which all possible group measure space decompositions $M = L^\infty(X) \rtimes \Gamma$ with $\Gamma \curvearrowright X$ can be characterized. We also give concrete examples and computations, thus proving Theorem A. A crucial ingredient in the proof is a version of *Popa's spectral gap rigidity* [Po06b], which is a powerful method for proving W*-rigidity results for crossed products arising from Bernoulli actions of product groups. In Appendix A, we present a generalization of these methods and results for co-induced actions. The proofs are easy adaptations of the proofs in [BV14, Theorems 3.1 and 3.3], which in turn were very close to the original proofs in [Po06b].

Note that in Theorem A, we can only describe the group measure space Cartan subalgebras of M. The reason for this is that our method entirely relies on a technique of [PV09], using the so-called *dual coaction* that is associated to a group measure space decomposition $M = B \rtimes \Lambda$, i.e., the normal $*$-homomorphism $\Delta\colon M \to M \mathbin{\overline{\otimes}} M$ given by $\Delta(bv_s) = bv_s \otimes v_s$ for all $b \in B$, $s \in \Lambda$. When $B \subset M$ is an arbitrary Cartan subalgebra, we do not have such a structural $*$-homomorphism.

Our second main result is concerned with the existence of Cartan subalgebras inside a given II_1 factor M. Not all II_1 factors have a Cartan subalgebra, as was first discovered by Voiculescu in [Vo95]. He showed that the group von Neumann algebras $L(\Gamma)$ of free groups $\Gamma = \mathbb{F}_n$ with $n \geq 2$ have no Cartan subalgebras. Having a Cartan subalgebra can be seen as a decomposability property. Indeed by [FM75], when M admits a Cartan subalgebra, then M can be realized as the von Neumann algebra $L_\Omega(\mathcal{R})$ associated with a countable equivalence relation $\mathcal{R}$, possibly twisted by a scalar 2-cocycle Ω (see Definition 2.2). If moreover this Cartan subalgebra is unique in the appropriate sense, this decomposition $M = L_\Omega(\mathcal{R})$ is canonical.

Although a lot of progress on the existence and uniqueness of Cartan subalgebras has been made, there is so far no intrinsic local criterion to check whether a given II_1 factor admits a Cartan subalgebra. However, Popa recently found such a criterion for the existence of a different kind of maximal abelian subalgebra (MASA), called an *s-MASA*. We say that $A \subset M$ is an s-MASA if A is a MASA and if the A-bimodule ${}_A L^2(M)_A$ is *cyclic*, i.e., there exists a vector $\xi \in L^2(M)$ such that $A\xi A$ spans a dense subset of $L^2(M)$. In [Po16], Popa proved that a II_1 factor M admits an s-MASA if and only if M satisfies the s-thin approximation property: for every finite partition of the identity $p_1, \ldots, p_n$ in M, every finite subset $\mathcal{F} \subset M$ and every $\varepsilon > 0$, there exists a finer partition of the identity $q_1, \ldots, q_m$ and a single vector $\xi \in L^2(M)$ such that every element in $\mathcal{F}$ can be approximated up to ε in $\|\cdot\|_2$ by linear combinations of the $q_i \xi q_j$. So, the existence of an s-MASA in a II_1 factor M is an intrinsic local property.

Any Cartan subalgebra is also an s-MASA, but the converse is far from being true. Indeed, s-MASAs are quite often singular, meaning that $\mathcal{N}_M(A) = \mathcal{U}(A)$, and in [Po16] it is even proved that every s-thin II_1 factor admits uncountably many non-conjugate singular s-MASAs. However, all examples of s-MASAs so far were inside II_1 factors that also admit a Cartan subalgebra. Therefore, Popa poses as [Po16, Problem 5.1.2] to give examples of s-thin factors without Cartan subalgebras. We solved this problem in [KV16] by constructing s-thin II_1 factors M that are even *strongly solid:* whenever $B \subset M$ is a diffuse amenable von

Neumann subalgebra, the von Neumann algebra generated by the normalizer $\mathcal{N}_M(B)$ stays amenable. Clearly, non-amenable strongly solid II_1 factors have no Cartan subalgebras.

We obtain this new class of strongly solid II_1 factors by applying Popa's deformation/rigidity theory to Shlyakhtenko's A-valued semicircular systems (see [Sh97] and Section 5.1 below). When A is abelian, this provides a rich source of examples of MASAs with special properties, like MASAs satisfying the s-thin approximation property of [Po16]. This work will be presented in Chapter 5.

Very interesting examples arise by taking $A = L^\infty(K, \mu)$ where K is a second countable compact group with Haar probability measure μ. In this case, any symmetric probability measure ν on K whose support topologically generates K can be used to construct a II_1 factor M_ν via Shlyakhtenko's A-valued semicircular systems (see Section 5.5). We show that M_ν is strongly solid whenever ν is a c_0 probability measure, meaning that the convolution operator $\lambda(\nu)$ on $L^2(K)$ is compact (see Definition 5.46 and Proposition 5.47). On the other hand, when the measure ν is concentrated on a subset of the form $F \cup F^{-1}$, where $F \subset K$ is *free* in the sense that every reduced word with letters from $F \cup F^{-1}$ defines a nontrivial element of K, then $A \subset M_\nu$ is an s-MASA.

In Theorem 5.49, we construct a compact group K, a free subset $F \subset K$ generating K and a symmetric c_0 probability measure ν with support $F \cup F^{-1}$. For this, we use results of [AR92, GHSSV07] on the spectral gap and girth of a random Cayley graph of the finite groups $\text{PGL}(2, \mathbb{Z}/p\mathbb{Z})$. As a consequence, we obtain the first examples of s-thin II_1 factors that have no Cartan subalgebras, solving [Po16, Problem 5.1.2], which was the motivation for our work.

Theorem B. *Taking a compact group K and a symmetric probability measure ν on K as above, the associated II_1 factor M_ν is non-amenable, strongly solid and the canonical subalgebra $A \subset M_\nu$ is an s-MASA.*

We finally make some concluding remarks on the existence of c_0 probability measures supported on free subsets of a compact group. On an *abelian* compact group K, a probability measure ν is c_0 if and only if its Fourier transform $\widehat{\nu}$ tends to zero at infinity as a function from $\widehat{K}$ to $\mathbb{C}$. Of course, no two elements of an abelian group are free, but the abelian variant of being free is the so-called independence property: a subset F of an abelian compact group K is called independent if any linear combination of distinct elements in F with coefficients in $\mathbb{Z} \setminus \{0\}$ defines a nonzero element in K. It was proved in [Ru60] that there exist closed independent subsets of the circle group $\mathbb{T}$ that carry a c_0 probability measure. It would be very interesting to get a better understanding of which, necessarily non-abelian, compact groups admit c_0 probability measures supported on a free subset and we conjecture that these exist on the groups $\mathrm{SO}(n)$, $n \geq 3$.

Chapter 2

Preliminaries

2.1 Von Neumann algebras

Given a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$, we denote by $B(H)$ the algebra of bounded linear operators on H. The *strong operator topology* is the weakest topology on $B(H)$ for which the seminorms $T \mapsto \|T\xi\|$ are continuous for all $\xi \in H$.

A *von Neumann algebra* is a $*$-subalgebra of $B(H)$ that contains the identity operator 1_H and that is closed in the strong operator topology. Here, a $*$-subalgebra of $B(H)$ means a subalgebra that is closed under taking the adjoint of an operator, $T \mapsto T^*$. It is a basic result in functional analysis that we may replace the strong operator topology with other operator topologies, such as the *weak operator topology* (generated by the seminorms $T \mapsto |\langle T\xi, \eta \rangle|$, $\xi, \eta \in H$), the *ultraweak operator topology* or the *ultrastrong operator topology.*

The notion of a von Neumann algebra was first introduced by John von Neumann in [vN29]. Motivated by his work on quantum mechanics and operator theory, he introduced von Neumann algebras as a mathematical abstraction of quantum mechanics. Together with Francis Murray, they developed the basic theory on von Neumann algebras in a series of papers called "Rings of Operators".

One of the most celebrated results from the early days of von Neumann algebra theory is the double commutant theorem of von Neumann. It states that a von Neumann algebra can be characterized purely algebraically as a "double commutant" in the following sense. Given a subset $S \subset B(H)$, we denote by

$S' \subset B(H)$ the set of operators that commute with S, i.e.,

$$S' = \{x \in B(H) \mid xy = yx \text{ for all } y \in S\}.$$

It is not hard to see that if S is a self-adjoint set, meaning that $S = S^*$, then S' is always a von Neumann algebra. The converse also holds by the double commutant theorem, i.e., von Neumann algebras can be characterized as commutants.

Theorem 2.1 (The double commutant theorem, [vN29])**.** *Let $M \subset B(H)$ be a $*$-subalgebra with $1_H \in M$. Then M is a von Neumann algebra if and only if $M'' = M$.*

We also note that von Neumann algebras can be characterized abstractly without referring to the underlying Hilbert space, namely as C^*-algebras that have a predual[1] [Sa56]. For this reason, we will often talk about von Neumann algebras without referring to a specific Hilbert space.

Two von Neumann algebras M and N are said to be *isomorphic* if there exists a $*$-isomorphism $M \to N$. Such a $*$-isomorphism is automatically *normal*, meaning that its restriction to the unit ball of M is continuous with respect to the weak (or strong) operator topologies on M and N (see [AP16, Proposition 2.5.8 and Corollary 2.5.9]). In particular, the weak/strong operator topologies on the unit ball of M do not depend on the concrete representation of M on a Hilbert space.

The most obvious example of a von Neumann algebra is $B(H)$ itself, in particular the matrix algebras $M_n(\mathbb{C})$, $n \geq 1$. Another basic example is $L^\infty(X, \mu)$ for any measure space (X, μ) with X being locally compact, where $L^\infty(X, \mu)$ is viewed as multiplication operators on the Hilbert space $L^2(X, \mu)$. In fact, any abelian von Neumann algebra is isomorphic with $L^\infty(X, \mu)$ for some locally compact space X equipped with a measure μ (see for example [Di81, Theorem 1 and Theorem 2, p. 132]). For this reason, von Neumann algebras are sometimes referred to as "non-commutative measure spaces". More interesting examples of von Neumann algebras can be constructed from groups and group actions, as we shall see later on.

A von Neumann algebra M is called a *factor* if it has trivial center, i.e., if

$$\mathcal{Z}(M) := M \cap M' = \mathbb{C}1.$$

Note that M is a factor if and only if M does not decompose as a direct sum of two von Neumann algebras $M_1 \oplus M_2$. In [vN49], it was showed that any von

[1] A *predual* of a C*-algebra A is a Banach space B such that $A = B^*$, where B^* denotes the dual Banach space of B.

Neumann algebra acting on a separable Hilbert space can be written as a direct integral of factors. In this way, factors can be seen as the basic building blocks in the theory of von Neumann algebras.

In this thesis, we will usually consider von Neumann algebras M acting on a separable Hilbert space. In this case, M is separable in the strong operator topology and for this reason, we say that M is a *separable* von Neumann algebra.

2.1.1 Type classification of factors

One important feature of a von Neumann algebra M is that it always contains many projections, i.e., elements $p \in M$ such that $p = p^* = p^2$. For example, if $x \in M$ is a positive element, meaning that $x = y^*y$ for some $y \in M$, then any spectral projection of x belongs to M. In fact, M is equal to the norm-closed linear span of its projections, by the Borel functional calculus. By studying the projections of a von Neumann algebra, Murray and von Neumann classified factors into three different types [MvN36]. We will here present a perhaps more intuitive version of this type classification, where the types are defined in terms of traces.

We denote by $M_+ = \{x^*x \mid x \in M\}$ the set of positive elements of a von Neumann algebra M. A *state* on M is a linear functional $\varphi\colon M \to \mathbb{C}$ that is positive, in the sense that $\varphi(x) \geq 0$ for all $x \in M_+$, and such that $\varphi(1_M) = 1$. If φ moreover satisfies $\varphi(xy) = \varphi(yx)$ for all $x, y \in M$, then φ is called a *trace*. Note that this definition generalizes the definition of the usual (normalized) trace on the matrix algebras $M_n(\mathbb{C})$. It turns out that von Neumann algebras that have a trace are particularly well-behaved, as we will come back to soon. We will usually require our traces to have two additional properties. A state φ on M is called *faithful* if $\varphi(x^*x) = 0$ implies that $x = 0$. Moreover, φ is called *normal* if φ is weak operator continuous on the unit ball of M. Now, a *tracial von Neumann algebra* (M, τ) is a von Neumann algebra M equipped with a faithful normal trace τ.

Given a tracial von Neumann algebra (M, τ), we can define a pre-inner product on M given by $\langle x, y\rangle_2 := \tau(xy^*)$ for $x, y \in M$. The associated norm $\|x\|_2 := \tau(x^*x)^{1/2}$ is called the 2-*norm* on M. The completion of M with respect to the 2-norm gives us a Hilbert space that we denote by $L^2(M)$. Since the 2-norm satisfies the inequality $\|xy\|_2 \leq \|x\|\|y\|_2$ for $x, y \in M$, we get a well-defined normal $*$-representation $\pi\colon M \to B(L^2(M))$ induced by left multiplication of M on itself: $\pi(x)(y) = xy$ for $x \in M$, $y \in M \subset L^2(M)$. This representation is called the *GNS representation* of M with respect to τ, or also the *standard representation*. We usually do not write π explicitly but simply view $M \subset B(L^2(M))$ as left multiplication operators.

A more general setting is the case where M has no trace but it does have a so-called tracial weight. This is a generalization of the infinite trace Tr on $B(H)$ when H is infinite-dimensional. A *weight* on M is a linear functional $\omega\colon M_+ \to [0,\infty]$ that may take the value ∞. We say that ω is *semifinite* if $\operatorname{span}\{x \in M_+ \mid \omega(x) < \infty\}$ is dense in M in the ultraweak operator topology. The functional ω is called *tracial* if $\omega(x^*x) = \omega(xx^*)$ for all $x \in M$, *faithful* if $\omega(x) = 0$ implies $x = 0$, and *normal* if $\omega(\sup_{i\in I} x_i) = \sup_{i\in I} \omega(x_i)$ for any bounded increasing net $(x_i)_{i\in I} \subset M_+$. A von Neumann algebra is called *semifinite* if it admits a normal faithful semifinite tracial weight.

We are now ready to give the type classification of Murray and von Neumann. Let M be a factor. Then M is said to be of

- type I if $M \cong B(H)$ for some Hilbert space H;
- type II_1 if M is infinite-dimensional and M has a normal faithful trace;
- type II_∞ if M has a normal faithful semifinite tracial weight Tr with $\mathrm{Tr}(1_M) = \infty$ and $M \ncong B(H)$ for any Hilbert space H;
- type III if M has no nontrivial tracial weight.

When M is a II_1 factor, the normal faithful trace is even unique by [MvN37]. Moreover, we have that two projections $p, q \in M$ are *Murray-von Neumann equivalent* (written $p \sim q$), meaning that $p = vv^*$ and $q = v^*v$ for some partial isometry $v \in M$, if and only if $\tau(p) = \tau(q)$ (see [AP16, Corollary 2.4.11]).

One of the reasons why having a trace or a tracial weight is of great interest, is that it allows for a certain dimension theory for the projections of the von Neumann algebra. Recall that in the case of a matrix algebra $M_n(\mathbb{C})$, the trace of a projection gives you its rank. In particular, $\mathrm{tr}_n(p) \in \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$ for any projection $p \in M_n(\mathbb{C})$, where tr_n denotes the normalized trace on $M_n(\mathbb{C})$. In the case of a II_1 factor M with its trace τ, we have that $\{\tau(p) \mid p \in \mathrm{Proj}(M)\} = [0,1]$ (see [AP16, Proposition 4.1.6]). So, II_1 factors are continuous dimensional analogues of the matrix algebras. Also, II_∞ factors are the continuous dimensional analogues of $B(H)$ when $\dim H = \infty$. On the other hand, type III factors have no good “dimension function” at all.

The fact that the trace τ on a II_1 factor M takes values in a continuous interval means that M has no minimal projections. We say that M is *diffuse*. For tracial von Neumann algebras M, being diffuse is equivalent with the existence of a net of unitaries in M that converges to 0 weakly (this follows from Theorem 2.12 below). When $M = L^\infty(X,\mu)$ is an abelian von Neumann algebra, we have that M is diffuse if and only if μ is atomless.

On the other hand, a von Neumann algebra that has no diffuse direct summand is called *atomic*. The atomic factors are exactly the factors of type I. When $M = L^\infty(X,\mu)$ is an abelian von Neumann algebra, then M is atomic if and only if μ is a purely atomic measure.

The main focus of this thesis is II_1 factors. By the theory of Connes [Co72], any von Neumann algebra can be build out of type II_1 factors via constructions such as tensor products and crossed products. So, II_1 factors can be seen as the building blocks of all von Neumann algebras.

The main source of examples of II_1 factors comes from groups and group actions on measure spaces. To any countable discrete group Γ, we can associate a group von Neumann algebra $L(\Gamma)$ that encodes the group structure. More generally, the group measure space construction of Murray and von Neumann associates a von Neumann algebra to any group action $\Gamma \curvearrowright (X,\mu)$ on a measure space (X,μ). We will present these constructions in the following two subsections.

2.1.2 Group von Neumann algebras

Let Γ be a countable discrete group and consider the left regular representation $\lambda\colon \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ given by

$$\lambda_g(\delta_h) = \delta_{gh} \quad \text{for} \;\; g,h \in \Gamma,$$

where $(\delta_h)_{h\in\Gamma}$ denotes the canonical basis of $\ell^2(\Gamma)$.

The *group von Neumann algebra* $L(\Gamma)$ associated with Γ is defined by

$$L(\Gamma) := \{\lambda_g \mid g \in \Gamma\}'' \subset B(\ell^2(\Gamma)).$$

We will usually denote the unitaries λ_g by u_g. It is easy to check that $L(\Gamma)$ is a factor if and only if Γ is *icc*, i.e., the conjugacy class of every nontrivial element is infinite.

The group von Neumann algebra is equipped with a canonical faithful normal trace defined by $\tau(x) = \langle x\delta_e, \delta_e\rangle$ for $x \in L(\Gamma)$. Therefore, $L(\Gamma)$ is a II_1 factor whenever Γ is an icc group. Note that the GNS Hilbert space $L^2(L(\Gamma))$ is isomorphic with $\ell^2(\Gamma)$ via the isomorphism $x \mapsto x\delta_e$, $x \in L(\Gamma)$.

Any element $x \in L(\Gamma)$ can be written uniquely as a 2-norm converging sum $x = \sum_{g\in\Gamma} x_g u_g$ with $x_g \in \mathbb{C}$. Indeed, we can write $x\delta_e = \sum_{g\in\Gamma} x_g \delta_g$ with $x_g = \langle x\delta_e, \delta_g\rangle = \tau(x u_g^*)$ for all $g \in \Gamma$ and the family $(x_g)_{g\in\Gamma}$ uniquely determines x. The decomposition $x = \sum_{g\in\Gamma} x_g u_g$ is called the *Fourier decomposition* of x.

If Γ is an abelian group, then its dual $\widehat{\Gamma}$ is a compact second countable abelian group and we have a canonical isomorphism $L(\Gamma) \cong L^\infty(\widehat{\Gamma})$ implemented by the Fourier transform $\mathcal{F}\colon \ell^2(\Gamma) \to L^2(\widehat{\Gamma}) : \ \mathcal{F}(\delta_g)(\chi) = \chi(g)$.

2.1.3 The group measure space construction

A Borel space $(X, \mathcal{B})$ is called *standard* if its σ-algebra $\mathcal{B}$ can be generated by a Polish topology on X (i.e., a separable and completely metrizable topology) as its Borel σ-algebra. Now, a standard probability space is simply a standard Borel space $(X, \mathcal{B})$ equipped with a probability measure μ. Standard probability spaces have a simple structure: they are always isomorphic to the interval $[0, 1]$ with a combination of the Lebesgue measure and a countable number of atoms, [Ke95, Theorem 17.41]. It is natural to only consider standard probability spaces in the theory of von Neumann algebras since there is a one-to-one correspondence between standard probability spaces (X, μ) and separable abelian von Neumann algebras $L^\infty(X, \mu)$ (see [Di81, Theorem 1 and Theorem 2, p. 132]).

Let (X, μ) be a standard probability space and let $\sigma\colon \Gamma \curvearrowright (X, \mu)$ be a probability measure preserving (pmp) action of a countable discrete group Γ. We will often use the notation $g \cdot x$ to denote the action $\sigma_g(x)$ for $g \in \Gamma$, $x \in X$. Also, we will sometimes view σ as an action on $L^2(X)$ (or $L^\infty(X)$) via the formula $\sigma_g(\xi)(x) = \xi(g^{-1} \cdot x)$ for $\xi \in L^2(X)$, $g \in \Gamma$, $x \in X$.

To such an action $\sigma\colon \Gamma \curvearrowright (X, \mu)$, we can associate a von Neumann algebra denoted by $L^\infty(X) \rtimes \Gamma$ as follows. Consider the Hilbert space $H = L^2(X) \otimes \ell^2(\Gamma)$. We can represent $L^\infty(X)$ as operators on H given by $f(\xi \otimes \delta_s) = f\xi \otimes \delta_s$ for $\xi \in L^2(X)$, $s \in \Gamma$. We can also represent Γ on H via the unitary representation $u\colon \Gamma \to \mathcal{U}(H) : \ u_g(\xi \otimes \delta_s) = \sigma_g(\xi) \otimes \delta_{gs}$. The *group measure space construction* $L^\infty(X) \rtimes \Gamma$ is then defined as

$$L^\infty(X) \rtimes \Gamma := \{f u_g \mid f \in L^\infty(X), \ g \in \Gamma\}'' \subset B(H).$$

Note that f and u_g satisfy the relation $u_g f u_g^* = \sigma_g(f)$. So, we can simply view $L^\infty(X) \rtimes \Gamma$ as being generated by a copy of $L^\infty(X)$ and a copy of Γ, in terms of unitaries $(u_g)_{g \in \Gamma}$ with $u_g u_h = u_{gh}$, that encode the action σ in the sense that $u_g f u_g^* = \sigma_g(f)$ for $f \in L^\infty(X)$.

An action $\Gamma \curvearrowright (X, \mu)$ is called *ergodic* if any Γ-invariant subset of X is either null or co-null. Moreover, the action is called *(essentially) free* if the set of fixed points $\{x \in X \mid g \cdot x = x\}$ has measure zero for all $g \in \Gamma$. Whenever the action $\Gamma \curvearrowright (X, \mu)$ is free, we have that $L^\infty(X)$ is a maximal abelian subalgebra of $L^\infty(X) \rtimes \Gamma$, in the sense that $L^\infty(X)' \cap (L^\infty(X) \rtimes \Gamma) = L^\infty(X)$. Consequently, for a free action $\Gamma \curvearrowright (X, \mu)$ of an infinite group Γ, we have that $L^\infty(X) \rtimes \Gamma$ is a factor if and only if $\Gamma \curvearrowright (X, \mu)$ is ergodic (see [AP16, Proposition 1.4.5]).

The von Neumann algebra $L^\infty(X) \rtimes \Gamma$ has a canonical faithful normal trace defined by $\tau(x) = \langle x(1_X \otimes \delta_e), 1_X \otimes \delta_e\rangle$. This means that $\tau(fu_g) = 0$ when $g \neq e$ and $\tau(fu_e) = \int_X f \, \mathrm{d}\mu$. So, $L^\infty(X) \rtimes \Gamma$ is a II_1 factor whenever $\Gamma \curvearrowright X$ is a free ergodic pmp action and Γ is infinite.

As for group von Neumann algebras, we also have a Fourier decomposition for the group measure space construction. Indeed, any element $x \in L^\infty(X) \rtimes \Gamma$ can be written uniquely as a 2-norm convergent sum $x = \sum_{g\in\Gamma} x_g u_g$ with $x_g \in L^\infty(X)$ for all g (see [AP16, Section 1.4]).

More generally, we can associate a *crossed product von Neumann algebra* $P \rtimes \Gamma$ to any trace-preserving action $\Gamma \curvearrowright P$ on a tracial von Neumann algebra (P, τ). An action of Γ on P is a homomorphism $\sigma\colon \Gamma \to \mathrm{Aut}(P)$, where $\mathrm{Aut}(P)$ denotes the group of $*$-automorphisms of P. The action σ is called *trace-preserving* if $\tau \circ \sigma_g = \tau$ for all $g \in \Gamma$. Given a trace-preserving action $\sigma\colon \Gamma \curvearrowright P$, we can represent both Γ and P on the Hilbert space $L^2(P) \otimes \ell^2(\Gamma)$ analogous to the group measure space case, in such a way that $u_g a u_g^* = \sigma_g(a)$ for $g \in \Gamma$ and $a \in P$. The crossed product $P \rtimes \Gamma$ is then defined to be the von Neumann algebra generated by P and $(u_g)_{g\in\Gamma}$ inside $B(L^2(P) \otimes \ell^2(\Gamma))$. We again have a canonical trace τ' on $P \rtimes \Gamma$ defined uniquely by the formula $\tau'(au_g) = \tau(a)\delta_{g,e}$ for $a \in P$, $g \in \Gamma$. Moreover, $P \rtimes \Gamma$ is a factor whenever $\Gamma \curvearrowright P$ is ergodic and properly outer in the following sense. The action $\Gamma \curvearrowright P$ is called *ergodic* if any Γ-invariant element of P is contained in $\mathbb{C}1$, and the action is called *properly outer* if α_g is an outer automorphism of P for all $g \in \Gamma \setminus \{e\}$, i.e., α_g is not of the form $\alpha_g = \mathrm{Ad}\, u$ for some unitary $u \in \mathcal{U}(P)$. A proof of all of these facts can be found in [AP16, Section 5.2].

2.1.4 II_1 factors arising from equivalence relations

Any group action $\Gamma \curvearrowright (X, \mu)$ gives rise to an equivalence relation on X, where the equivalence classes are given by the orbits of the action. In [FM75], Feldman and Moore introduced a generalization of the group measure space construction, where they constructed II_1 factors out of certain equivalence relations.

Let $\mathcal{R} \subset X \times X$ be an equivalence relation on a standard probability space (X, μ). The *full pseudogroup* of $\mathcal{R}$, denoted by $[[\mathcal{R}]]$, is defined to be the set of partial automorphisms of X whose graph is contained in $\mathcal{R}$. A *partial automorphism* of X is a Borel isomorphism $\varphi\colon A \to B$ where $A, B \subset X$ are non-negligible Borel subsets.

The equivalence relation $\mathcal{R}$ is said to be

- *Borel* if $\mathcal{R}$ is a Borel subset of $X \times X$;

- *countable* if the equivalence classes of $\mathcal{R}$ are countable;
- *pmp* if any partial automorphism $\varphi \in [[\mathcal{R}]]$ preserves the measure μ;
- *ergodic* if any $\mathcal{R}$-invariant subset of X is either null or co-null. A subset $A \subset X$ is called $\mathcal{R}$-invariant if A is equal (up to null sets) to its $\mathcal{R}$-saturation $[A]_{\mathcal{R}} := \{x \in X \mid (x, a) \in \mathcal{R} \text{ for some } a \in A\}$.

An equivalence relation which is Borel, countable and pmp is said to be of *type II_1*. By [FM75, Theorem 1], any equivalence relation of type II_1 is given by the orbits of some pmp action $\Gamma \curvearrowright X$ of a countable discrete group, but the action need not be essentially free. The first example of an equivalence relation that is not generated by a free group action was found by Adams in [Ad88].

The construction of Feldman and Moore associates a II_1 factor $L(\mathcal{R})$ to any ergodic equivalence relation $\mathcal{R}$ of type II_1. More generally, the construction can be “twisted” by a scalar 2-cocycle.

Let $\mathcal{R} \subset X \times X$ be a II_1 equivalence relation. We define a σ-finite measure $\mu^{(1)}$ on $\mathcal{R}$ by

$$\mu^{(1)}(A) := \int_X \#\{y \in X \mid (x, y) \in A\} \, \mathrm{d}\mu(x),$$

for $A \subset \mathcal{R}$ Borel. By [AP16, Lemma 1.5.2], using that $\mathcal{R}$ is pmp, we can define this measure equivalently by

$$\mu^{(1)}(A) = \int_X \#\{x \in X \mid (x, y) \in A\} \, \mathrm{d}\mu(y).$$

We denote by $\mathcal{R}^{(2)} \subset X \times X \times X$ the set of all 3-tuples (x, y, z) with $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$. Again since $\mathcal{R}$ is pmp, we can define a measure $\mu^{(2)}$ on $\mathcal{R}^{(2)}$ by

$$\begin{aligned} \mu^{(2)}(A) &:= \int_X \#\{(y, z) \in X \times X \mid (x, y, z) \in A\} \, \mathrm{d}\mu(x) \\ &= \int_X \#\{(x, z) \in X \times X \mid (x, y, z) \in A\} \, \mathrm{d}\mu(y) \\ &= \int_X \#\{(x, y) \in X \times X \mid (x, y, z) \in A\} \, \mathrm{d}\mu(z), \end{aligned}$$

for $A \subset \mathcal{R}^{(2)}$ Borel.

A *scalar 2-cocycle* for $\mathcal{R}$ is a Borel map $\Omega \colon \mathcal{R}^{(2)} \to \mathbb{T}$ satisfying

$$\Omega(y, z, t)\Omega(x, z, t)^{-1}\Omega(x, y, t)\Omega(x, y, z)^{-1} = 1 \qquad \mu^{(2)}\text{-a.e.}$$

Definition 2.2 ([FM75])**.** Let $\mathcal{R} \subset X \times X$ be a II_1 equivalence relation on a standard probability space (X, μ), and let $\Omega \colon \mathcal{R}^{(2)} \to \mathbb{T}$ be a scalar 2-cocycle for $\mathcal{R}$. For every partial automorphism $\varphi \in [[\mathcal{R}]]$, we define an operator L_φ^Ω on $L^2(\mathcal{R}, \mu^{(1)})$ by

$$(L_\varphi^\Omega \xi)(x,y) = \mathbb{1}_{\mathrm{Ran}\,\varphi}(x)\Omega(x, \varphi^{-1}(x), y)\xi(\varphi^{-1}(x), y) \quad \text{for } \xi \in L^2(\mathcal{R}, \mu^{(1)}).$$

The von Neumann algebra $L_\Omega(\mathcal{R})$ associated with $(\mathcal{R}, \Omega)$ is defined by

$$L_\Omega(\mathcal{R}) := \{L_\varphi^\Omega \mid \varphi \in [[\mathcal{R}]]\}''.$$

If $\mathcal{R}$ is ergodic, then $L_\Omega(\mathcal{R})$ is a II_1 factor (see [AP16, Proposition 1.5.5]). Moreover, we have that $L_\Omega(\mathcal{R})$ contains $L^\infty(X)$ as a maximal abelian von Neumann subalgebra, by viewing $L^\infty(X)$ as operators on $L^2(\mathcal{R}, \mu^{(1)})$ via the representation

$$(f\xi)(x,y) = f(x)\xi(x,y) \quad \text{for } f \in L^\infty(X),\ \xi \in L^2(\mathcal{R}, \mu^{(1)}).$$

When Ω is the trivial 2-cocycle, we simply write $L(\mathcal{R})$ instead of $L_\Omega(\mathcal{R})$.

As a final remark, note that whenever $\mathcal{R}$ is the orbit equivalence relation given by an ergodic pmp action $\Gamma \curvearrowright X$ that is *free*, then the II_1 factors $L(\mathcal{R})$ and $L^\infty(X) \rtimes \Gamma$ coincide. Indeed, the map $\phi \colon X \times \Gamma \to \mathcal{R}$ given by $\phi(x,g) = (x, g^{-1} \cdot x)$ is a measure space isomorphism and thus induces a unitary operator $V \colon \xi \mapsto \xi \circ \phi$ from $L^2(\mathcal{R}, \mu^{(1)})$ to $L^2(X \times \Gamma) \cong L^2(X) \otimes \ell^2(\Gamma)$. It is now easy to check that $\mathrm{Ad}\, V \colon L^\infty(X) \rtimes \Gamma \to L(\mathcal{R})$ is an isomorphism satisfying $V^* u_g V = L_g$ for all $g \in \Gamma$ and $V^* f V = f$ for all $f \in L^\infty(X)$.

2.1.5 Cartan subalgebras

In the introduction, we mentioned that the subalgebra $L^\infty(X)$ of a crossed product II_1 factor $L^\infty(X) \rtimes \Gamma$ plays a special role. This is because of Singer's theorem [Si55], which states that any isomorphism of crossed product II_1 factors $\pi \colon L^\infty(X) \rtimes \Gamma \to L^\infty(Y) \rtimes \Lambda$ satisfying $\pi(L^\infty(X)) = L^\infty(Y)$ must come from an orbit equivalence of the actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$.

A Cartan subalgebra of a II_1 factor is an abstraction of subalgebras of the form $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$. The definition is as follows.

Definition 2.3. Let M be a II_1 factor and $A \subset M$ a von Neumann subalgebra. Then A is called a *Cartan subalgebra* of M if

(1) A is *maximal abelian* (A is a *MASA*), i.e., $A' \cap M = A$;

(2) A is *regular*, i.e., its normalizer $\mathcal{N}_M(A) := \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ generates M as a von Neumann algebra.

Whenever $\Gamma \curvearrowright (X,\mu)$ is a free ergodic pmp action, then $L^\infty(X)$ is a maximal abelian subalgebra of the II_1 factor $L^\infty(X) \rtimes \Gamma$. Since $L^\infty(X)$ by definition is normalized by the canonical unitaries $(u_g)_{g\in\Gamma}$, we also have that $L^\infty(X)$ is regular. So, $L^\infty(X)$ is indeed a Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$. Any Cartan subalgebra $A \subset M$ that arises from a group measure space decomposition in this way is called a *group measure space Cartan subalgebra*, or a gms Cartan subalgebra for short.

Not all Cartan subalgebras are gms Cartan subalgebras. More generally, it is not hard to see that $L^\infty(X) \subset L_\Omega(\mathcal{R})$ is a Cartan subalgebra for any ergodic II_1 equivalence relation $\mathcal{R} \subset X \times X$ and any scalar 2-cocycle Ω for $\mathcal{R}$. This is in fact the most general construction of Cartan subalgebras inside separable II_1 factors.

Theorem 2.4 ([FM75]). *Let M be a separable II_1 factor. Then $A \subset M$ is a Cartan subalgebra of M if and only if there exists an ergodic II_1 equivalence relation $\mathcal{R}$ on a standard probability space (X,μ) and a scalar 2-cocycle Ω for $\mathcal{R}$ such that the inclusion $A \subset M$ is isomorphic with $L^\infty(X) \subset L_\Omega(\mathcal{R})$.*

Let us briefly explain how one associates an equivalence relation $\mathcal{R}$ to a Cartan inclusion $A \subset M$, as in Theorem 2.4. Write $A = L^\infty(X)$ for some standard probability space (X,μ). Any normalizing unitary $u \in \mathcal{N}_M(A)$ gives rise to an automorphism $\theta_u \in \mathrm{Aut}(X,\mu)$ given by $a \circ \theta_u = u^*au$ for $a \in A = L^\infty(X)$. Since M is separable, we can take a countable $\|\cdot\|_2$-dense subgroup $\Gamma < \mathcal{N}_M(A)$. The equivalence relation $\mathcal{R}$ associated with the Cartan inclusion $A \subset M$ is given by $x \sim y$ if and only if $x = \theta_u(y)$ for some $u \in \Gamma$. This equivalence relation is of type II_1 and does not depend on the choice of Γ.

Finally, we show the following well-known lemma, which states that $A \subset M$ being a Cartan subalgebra is preserved under taking corners. A *corner* of a von Neumann algebra M is a von Neumann algebra of the form pMp for some projection $p \in M$. When M is a II_1 factor, then any corner of M is also a II_1 factor.

Lemma 2.5. *Let M be a II_1 factor and let $A \subset M$ be a Cartan subalgebra. Then $Ap \subset pMp$ is also a Cartan subalgebra, for any projection $p \in A$.*

Proof. By [AP16, Proposition 4.2.2], we have that $(Ap)' \cap pMp = p(A' \cap M)p = Ap$ so $Ap \subset pMp$ is maximal abelian. By [Po03, Lemma 3.5], $Ap \subset pMp$ is also regular. □

2.1.6 Tensor products, amalgamated free products and ultraproducts

We will here introduce some more constructions of von Neumann algebras, namely the tensor product, the free product with amalgamation and the ultraproduct.

Tensor products. The tensor product is a very fundamental construction and corresponds to taking the direct product of groups. Let $M \subset B(H)$ and $N \subset B(K)$ be von Neumann algebras. Given $x \in M$ and $y \in N$, there is a unique operator $x \otimes y$ on the Hilbert space $H \otimes K$ given by $(x \otimes y)(\xi \otimes \eta) = x\xi \otimes y\eta$ for $\xi \in H$, $\eta \in K$. The *von Neumann algebra tensor product* of M and N is defined to be the von Neumann algebra generated by these operators, i.e.,

$$M \mathbin{\overline{\otimes}} N = \{x \otimes y \mid x \in M,\ y \in N\}'' \subset B(H \otimes K).$$

One can show that the tensor product is independent of the chosen representations of M and N, up to isomorphism. In the case where M or N is finite-dimensional, the von Neumann algebra tensor product coincides with the algebraic tensor product and is simply denoted by $M \otimes N$. When Γ and Λ are countable groups, we have that $L(\Gamma) \mathbin{\overline{\otimes}} L(\Lambda) \cong L(\Gamma \times \Lambda)$.

Let us mention a few basic properties of the tensor product (see [AP16, Proposition 5.1.3] for a proof). We have that $M \mathbin{\overline{\otimes}} N$ is a factor if and only if M and N are both factors. Moreover, when (M, τ_M) and (N, τ_N) are tracial, we can define a faithful trace on $M \mathbin{\overline{\otimes}} N$ by the formula $\tau(x \otimes y) = \tau_M(x)\tau_N(y)$ for $x \in M$, $y \in N$. In particular, $M \mathbin{\overline{\otimes}} N$ is a II_1 factor whenever M and N are II_1 factors.

We can also define the tensor product of infinitely many tracial von Neumann algebras $(M_i, \tau_i)_{i\in\mathbb{N}}$. Let H denote the closed linear span of all elements of the form $\otimes_{i\in\mathbb{N}}\xi_i$ with $\xi_i \in L^2(M_i, \tau_i)$ for all $i \in \mathbb{N}$ and $\xi_i = 1_{M_i}$ for all but finitely many $i \in \mathbb{N}$, where the closure is taken with respect to the inner product given by $\langle \otimes_{i\in\mathbb{N}}\xi_i, \otimes_{i\in\mathbb{N}}\eta_i\rangle := \prod_{i\in\mathbb{N}}\langle \xi_i, \eta_i\rangle$. Then $\overline{\bigotimes}_{\mathbb{N}}(M_i, \tau_i)$ is defined to be the von Neumann algebra on H generated by the operators $\bigotimes_{i\in\mathbb{N}} x_i$ with $x_i \in M_i$ for all $i \in \mathbb{N}$ and $x_i = 1$ for all but finitely many $i \in \mathbb{N}$. Note that $\overline{\bigotimes}_{\mathbb{N}}(M_i, \tau_i)$ is a tracial von Neumann algebra with trace τ given by $\tau(\otimes_{i\in\mathbb{N}} x_i) = \prod_{i\in\mathbb{N}} \tau_i(x_i)$. As in the finite case, $\overline{\bigotimes}_{\mathbb{N}}(M_i, \tau_i)$ is a factor whenever each component M_i is a factor.

Amalgamated free products. The amalgamated free product of von Neumann algebras is a construction that has its roots in group theory: given two groups Γ, Λ that both contain a copy of the same subgroup Σ, the amalgamated free product group $\Gamma *_\Sigma \Lambda$ is defined as a certain quotient of the free product group $\Gamma * \Lambda$, where elements of Σ inside Γ get identified with the corresponding elements of Σ inside Λ. We can mimic this construction in the setting of tracial von Neumann algebras. Given two tracial von Neumann algebras (M, τ_M) and (N, τ_N) that both contain a common von Neumann subalgebra A in such a way that $\tau_M = \tau_N$ on A, we can define a tracial von Neumann algebra $M *_A N$ generated by all "reduced words" in M and N, where elements of A get identified. In the following, we will make this construction precise.

First, we need to define the notion of a conditional expectation. Given a von Neumann algebra M and a von Neumann subalgebra $A \subset M$, a *conditional expectation from M onto A* is a linear map $E\colon M \to A$ satisfying the following properties.

- E is positive: $E(M_+) \subset A_+$.
- E is a projection onto A: $E(a) = a$ for all $a \in A$.
- E is A-bimodular: $E(axb) = aE(x)b$ for all $a, b \in A$ and $x \in M$.

Such a conditional expectation may not always exist. However, when (M, τ) is a tracial von Neumann algebra and $A \subset M$ is a von Neumann subalgebra, then we can always find a unique trace-preserving conditional expectation of M onto A (see for instance [BO08, Lemma 1.5.11]). This conditional expectation is usually denoted by E_A (even though it depends on the choice of trace τ). We use the notation $M \ominus A$ to denote the set of elements $x \in M$ with $E_A(x) = 0$.

Let (M_1, τ_1) and (M_2, τ_2) be tracial von Neumann algebras and assume that $A \subset M_i$ for both $i = 1, 2$ and that $\tau_1|_A = \tau_2|_A$. The *amalgamated free product* $M = M_1 *_A M_2$ is the unique von Neumann algebra equipped with a faithful normal conditional expectation $E\colon M \to A$ such that M is generated by M_1 and M_2, and M_1 and M_2 satisfy the following freeness condition with respect to E:

$$E(x_1 \cdots x_n) = 0 \quad \text{whenever } \ x_k \in M_{i_k} \ominus A \ \text{ and } \ i_k \neq i_{k+1}.$$

The details of the construction can be found in [Po93, VDN92]. Using the conditional expectation E, we can define a trace τ on $M = M_1 *_A M_2$, extending τ_1 and τ_2, by $\tau = \tau_1 \circ E = \tau_2 \circ E$.

Ultraproducts. The last construction that we will introduce is the ultraproduct of tracial von Neumann algebras, which is a very useful tool for studying the asymptotic behavior of sequences in a tracial von Neumann algebra.

Let I be any set. An *ultrafilter* on I is a collection $\mathcal{U}$ of subsets of I satisfying the following properties:

- $\emptyset \notin \mathcal{U}$;
- if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$;
- if $A \in \mathcal{U}$ and $A \subset B \subset I$, then also $B \in \mathcal{U}$;
- for any $A \subset I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

An ultrafilter $\mathcal{U}$ is called *free* if it is non-principal, i.e., if it is not of the form $\mathcal{U} = \{B \subset I \mid i_0 \in B\}$ for some fixed $i_0 \in I$.

Intuitively, subsets of I belonging to an ultrafilter $\mathcal{U}$ should be thought of as being "large". Given $(x_i)_{i\in I} \in \ell^\infty(I)$, the limit of $(x_i)_{i\in I}$ along an ultrafilter $\mathcal{U}$ is the unique $x \in X$ such that $\{i \in I \mid |x_i - x| < \varepsilon\} \in \mathcal{U}$ for every $\varepsilon > 0$. This limit is denoted by $\lim_{i\to\mathcal{U}} x_i$.

Fix a free ultrafilter $\mathcal{U}$ on I and a tracial von Neumann algebra (M, τ). Let $\ell^\infty(I, M)$ denote the algebra consisting of all bounded sequences $(x_i)_{i\in I}$ with $x_i \in M$, equipped with the norm $\|(x_i)_{i\in I}\| := \sup_{i\in I} \|x_i\|$. The subspace $\mathcal{J} := \{(x_i)_{i\in I} \mid \lim_{i\to\mathcal{U}} \|x_i\|_2 = 0\}$ is a closed two-sided ideal of $\ell^\infty(I, M)$, and the quotient $\ell^\infty(I, M)/\mathcal{J}$ is a (non-separable) von Neumann algebra (see for instance [BO08, Appendix A]). We denote this von Neumann algebra by $M^{\mathcal{U}}$ and call it the *ultraproduct* of M. Note that $M^{\mathcal{U}}$ has a normal faithful trace $\tau_{\mathcal{U}}$ given by $\tau_{\mathcal{U}}((x_i)_{i\in I}) := \lim_{i\to\mathcal{U}} \tau(x_i)$. We view M as a von Neumann subalgebra of $M^{\mathcal{U}}$ by identifying M with the constant sequences in $M^{\mathcal{U}}$.

Let $M = B \rtimes \Lambda$ be a crossed product von Neumann algebra and denote by $\Lambda^{\mathcal{U}}$ the ultraproduct group given by the quotient $\Lambda^{\mathcal{U}} = \Lambda^I/K$, where $K = \{(g_i)_{i\in I} \mid \lim_{i\to\mathcal{U}} g_i = e\}$. Note that $\Lambda^{\mathcal{U}} < \mathcal{U}(M^{\mathcal{U}})$ by identifying $s = (s_i)_{i\in I} \in \Lambda^{\mathcal{U}}$ with the unitary element $v_s := (v_{s_i})_{i\in I} \in M^{\mathcal{U}}$. Also note that $\Lambda^{\mathcal{U}}$ and $B^{\mathcal{U}}$ are in crossed product position inside $M^{\mathcal{U}}$. Indeed, $\Lambda^{\mathcal{U}}$ normalizes $B^{\mathcal{U}}$ and

$$E_{B^{\mathcal{U}}}(v_s) = (E_B(v_{s_i}))_{i\in I} = (\tau(v_{s_i}))_{i\in I} = \tau_{\mathcal{U}}(v_s)$$

for $s = (s_i)_i \in \Lambda^{\mathcal{U}}$.

2.2 Bimodules

The notion of a bimodule is due to Connes and serves as the appropriate notion of a representation in the theory of von Neumann algebras. The subject was further developed by Popa and we refer to [AP16] for a thorough treatment of the subject.

Let (M, τ) be a tracial von Neumann algebra. A *left M-module* ${}_M H$ is a Hilbert space H equipped with a normal $*$-representation $\pi_\ell \colon M \to B(H)$. This representation is referred to as the left M-action on H and we use the notation $x\xi := \pi_\ell(x)\xi$ for $x \in M$ and $\xi \in H$. Similarly, a *right M-module* H_M is a Hilbert space H equipped with a normal $*$-representation of the opposite algebra $\pi_r \colon M^{\mathrm{op}} \to B(H)$, referred to as the right M-action. Again, we will use the notation $\xi x := \pi_r(x^{\mathrm{op}})\xi$ for $x \in M$ and $\xi \in H$.

Given two tracial von Neumann algebras (M, τ) and (N, τ), an *M-N-bimodule* ${}_M H_N$ is a Hilbert space H equipped with two commuting normal $*$-representations $\pi_\ell \colon M \to B(H)$ and $\pi_r \colon N^{\mathrm{op}} \to B(H)$. We say that two bimodules ${}_M H_N$ and ${}_M K_N$ are *isomorphic* if there exists a unitary operator $U \colon H \to K$ that intertwines the actions in the sense that $U(x\xi y) = xU(\xi)y$ for $x \in M$, $y \in N$.

Given an M-N-bimodule ${}_M H_N$, the *contragredient bimodule* ${}_N \overline{H}_M$ is defined by “flipping” the left and right actions. More precisely, $\overline{H}$ equals the conjugate Hilbert space of H with left and right actions given by

$$y \cdot \overline{\xi} \cdot x = \overline{x^* \xi y^*}, \qquad x \in M,\ y \in N,\ \xi \in H.$$

The simplest example of an M-M-bimodule is given by the GNS representation. Recall that the GNS Hilbert space $L^2(M)$ was defined to be the completion of M with respect to the inner product $\langle x, y\rangle_2 = \tau(xy^*)$. Now, $L^2(M)$ is an M-M-bimodule when equipped with the commuting left and right M-actions induced by left and right multiplication of M on itself. This bimodule is called the *trivial M-bimodule*. Note that we have a canonical anti-linear involution operator J on $L^2(M)$ defined by $J(x) = x^*$ for $x \in M$. With this notation, we have that $JxJ \in B(L^2(M))$ exactly equals the right action of x^* on $L^2(M)$.

More generally, when $\alpha \colon M \to N$ is a normal $*$-homomorphism, we get an M-N-bimodule $H(\alpha) := \alpha(1)L^2(N)$ with left and right actions given by $x \cdot \xi \cdot y := \alpha(x)\xi y$ for $x \in M$, $y \in N$ and $\xi \in \alpha(1)L^2(N)$. This construction can be generalized further by considering normal $*$-homomorphisms from M into an *amplification* of N. An amplification of N is a von Neumann algebra of the form $p(B(\ell^2(I)) \mathbin{\overline{\otimes}} N)p$ for some (possibly uncountable) index set I and a nonzero projection $p \in B(\ell^2(I)) \mathbin{\overline{\otimes}} N$. Note that we can view $B(\ell^2(I)) \mathbin{\overline{\otimes}} N$ as the von Neumann algebra of infinite-dimensional matrices (indexed over I) with entries in N. Given a normal $*$-homomorphism $\alpha \colon M \to p(B(\ell^2(I)) \mathbin{\overline{\otimes}} N)p$, we define $H(\alpha) := p(\ell^2(I) \otimes L^2(N))$ with M-N-bimodular actions given by $x \cdot \xi \cdot y := \alpha(x)\xi(1 \otimes y)$ for $x \in M$, $y \in N$. This is in fact the most general example of an M-N-bimodule. Therefore, M-N-bimodules can be seen as generalized $*$-homomorphisms between M and N.

Proposition 2.6 (See e.g. [JS97, Theorem 2.2.2]). *Let* (M,τ) *and* (N,τ) *be tracial von Neumann algebras and let* ${}_M H_N$ *be an* M*-*N*-bimodule. Then there exists a set* I*, a projection* $p \in B(\ell^2(I)) \mathbin{\overline{\otimes}} N$ *and a normal* $*$*-isomorphism* $\alpha\colon M \to p(B(\ell^2(I)) \mathbin{\overline{\otimes}} N)p$ *such that* ${}_M H_N$ *is isomorphic with* ${}_M H(\alpha)_N$*.*[2]

The proof of this proposition follows from the fact that any right N-module H_N can be written as $p(\ell^2(I) \otimes L^2(N))$ for some projection $p \in B(\ell^2(I)) \mathbin{\overline{\otimes}} N$, where N acts diagonally from the right. If H is also a left M-module, then such an isomorphism $H_M \cong p(\ell^2(I) \otimes L^2(N))$ transforms the left M-action into a normal $*$-homomorphism $\alpha\colon M \to p(B(\ell^2(I)) \mathbin{\overline{\otimes}} N)p$.

Whenever we write a right M-module H_M as $p(\ell^2(I) \otimes L^2(M))$, the number $(\mathrm{Tr} \otimes \tau)(p) \in [0,\infty]$ does not depend on the choice of projection p. We call this number the *right* M*-dimension* of H and use the notation $\dim_{-M}(H) := (\mathrm{Tr} \otimes \tau)(p)$. A similar construction for left M-modules allows us to define the *left* M*-dimension* $\dim_{M-}(H)$ of a left M-module ${}_M H$. Note however that these notions of left and right dimensions depend on the trace τ on M. When M is a II_1 factor, the trace τ is unique and therefore the notions of left and right M-dimensions are intrinsic. Moreover, the left/right M-dimension is a complete invariant for left/right M-modules in this case.

When N is a subfactor of a II_1 factor M, we define the *Jones index* $[M:N]$ to be the right N-dimension of the module $L^2(M)_N$.

The finitely generated right M-modules are of the form $p(\mathbb{C}^n \otimes L^2(M))$ for some $n \in \mathbb{N}$ and some projection $p \in M_n(\mathbb{C}) \otimes M$ (similarly for left modules). When M is a II_1 factor, we have that H_M is finitely generated if and only if H_M has finite right dimension. For general tracial von Neumann algebras (M,τ) this is not too far from being true: H_M has finite right dimension if and only if there exists a central projection $z \in M$ arbitrarily close to 1 such that $(Hz)_M$ is finitely generated.

One very important example of an M-M-bimodule is the *coarse* M*-bimodule* $L^2(M) \otimes L^2(M)$ with M-bimodular actions given by

$$x \cdot (\xi \otimes \eta) \cdot y := x\xi \otimes \eta y \quad \text{for } x,y \in M, \;\; \xi,\eta \in L^2(M).$$

It plays the same role as the regular representation does in the context of unitary group representations. In fact, given any unitary representation $\pi\colon \Gamma \to \mathcal{U}(K)$ of a countable discrete group, one can construct an $L(\Gamma)$-bimodule as follows. Let $H(\pi) = K \otimes \ell^2(\Gamma)$ and consider the following $L(\Gamma)$-bimodular actions on $H(\pi)$:

$$u_g \cdot (\xi \otimes \delta_s) \cdot u_h := \pi_g(\xi) \otimes \delta_{gsh}.$$

[2] In the case where M and N are separable, we may take $I = \mathbb{N}$.

Here, $(u_g)_{g\in\Gamma}$ denotes the canonical unitaries of $L(\Gamma)$ as usual. Using Fell's absorption principle (see e.g. [Pe11, Lemma 1.1.6]), it follows that both the left and the right actions extend to all of $L(\Gamma)$, so that $H(\pi)$ is a well-defined $L(\Gamma)$-bimodule. Via this construction, the trivial representation $\iota\colon \Gamma \to \mathbb{C}1$ gives rise to the trivial $L(\Gamma)$-bimodule and the left regular representation $\lambda\colon \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ gives rise to the coarse $L(\Gamma)$-bimodule.

Just as for group representations, there is a notion of weak containment for bimodules.

Definition 2.7. Let M and N be tracial von Neumann algebras and let ${}_M H_N$ and ${}_M K_N$ be two M-N-bimodules. Then ${}_M H_N$ is said to be *weakly contained* in ${}_M K_N$ if every coefficient of ${}_M H_N$ can be approximated by a finite sum of coefficients of ${}_M K_N$, i.e., for every $\xi \in H$, every finite subsets $F \subset M$ and $E \subset N$ and every $\varepsilon > 0$, there exist $\eta_1, \dots, \eta_n \in K$ such that

$$|\langle x\xi y, \xi\rangle - \sum_{i=1}^{n} \langle x\eta_i y, \eta_i\rangle| \leq \varepsilon \quad \text{for all } \; x \in F \; \text{ and } \; y \in E.$$

When an M-bimodule ${}_M H_M$ contains the trivial M-bimodule via an M-bimodular embedding $\iota\colon L^2(M) \hookrightarrow H$, we obviously have an M-central and tracial vector inside H, i.e., a vector $\xi \in H$ such that $x\xi = \xi x$ and $\langle x\xi, \xi\rangle = \tau(x)$ for all $x \in M$. Indeed, we simply take $\xi = \iota(1_M)$. Having an M-central and tracial vector is in fact equivalent to containing the trivial M-bimodule. Similarly, we can characterize weak containment of the trivial M-bimodule by the existence of *almost* M-central and *almost* tracial vectors, in the following sense.

Lemma 2.8 ([AP16, Proposition 12.3.11])**.** *Let (M, τ) be a tracial von Neumann algebra and let $P \subset M$ be a von Neumann subalgebra. An M-P-bimodule ${}_M H_P$ weakly contains the trivial M-P-bimodule ${}_M L^2(M)_P$ if and only if there exists a net of vectors $(\xi_i)_{i\in I}$ in H such that*

(1) $\lim_i \langle x\xi_i, \xi_i\rangle = \tau(x)$ *for all* $x \in M$*;*

(2) $\lim_i \|y\xi_i - \xi_i y\| = 0$ *for all* $y \in P$*.*

The following lemma contains two easy observations that we will need later on.

Lemma 2.9. *Let (M, τ) be a tracial von Neumann algebra and let $Q \subset M$ be a von Neumann subalgebra.*

(1) *The coarse M-bimodule $L^2(M) \otimes L^2(M)$ restricted to Q is contained in a multiple of the coarse Q-bimodule $L^2(Q) \otimes L^2(Q)$.*

(2) *Let $q \in Q' \cap M$ be a nonzero projection and let $z \in \mathcal{Z}(Q)$ be the support projection of $E_Q(q)$. Then $L^2(Q)z$ is isomorphic with $L^2(Q)q$, as Q-bimodules.*

Proof. (1) We have that $L^2(M)$ is contained in a multiple of $L^2(Q)$, as either a left Q-module or a right Q-module. So, $L^2(M) \otimes L^2(M)$ is contained in $L^2(Q) \otimes \ell^2(I) \otimes \ell^2(I) \otimes L^2(Q)$ as Q-bimodules for some index set I, and this is simply a multiple of the coarse Q-bimodule.

(2) Define a bounded linear map $T\colon L^2(Q)z \to L^2(Q)q$ by $T(\xi) = \xi q$. Note that T is Q-bimodular and that the image of T is dense in $L^2(Q)q$. Moreover, T is injective since $\xi q = 0$ implies that $\xi E_Q(q) = 0$ and hence $\xi z = 0$, for $\xi \in L^2(Q)$. Letting $U\colon L^2(Q)z \to L^2(Q)q$ be the partial isometry coming from the polar decomposition of T, we have that U is a Q-bimodular unitary as wanted. $\square$

We end this section by introducing a way of composing bimodules, namely Connes's tensor product of bimodules. For this, we first need to discuss the notion of left and right bounded vectors.

Let (M, τ) be a tracial von Neumann algebra. A vector ξ in a right (resp. left) M-module H is said to be right (resp. left) bounded if there exists a $\kappa > 0$ such that $\|\xi a\| \leq \kappa \|a\|_2$ (resp. $\|a\xi\| \leq \kappa\|a\|_2$) for all $a \in M$. We will use the notation H_0 (resp. ${}_0H$) to denote the set of all right (resp. left) bounded vectors of H. For the trivial M-bimodule $H = L^2(M)$, we have that $H_0 = {}_0H = M$. In general, when H is a right M-module, the subspace $H_0 \subset H$ is always dense and similarly for left M-modules.

Whenever ξ is right bounded, we denote by $\ell(\xi)$ the map $L^2(M) \to H\colon a \mapsto \xi a$. Similarly, when ξ is left bounded, we denote by $r(\xi)$ the map $L^2(M) \to H\colon a \mapsto a\xi$. Given right bounded vectors ξ, η, the operator $\ell(\xi)^*\ell(\eta)$ belongs to M and is denoted $\langle \xi, \eta\rangle_M$. This defines an M-valued scalar product associated with the right M-module H. Similarly, if $\xi, \eta \in H$ are left bounded vectors, we define an M-valued scalar product associated with the left M-module H by ${}_M\langle \xi, \eta\rangle = Jr(\xi)^*r(\eta)J \in M$. Here, J denotes the canonical involution on $L^2(M)$ given by $J(x) = x^*$ for $x \in M$.

When H_M is a right M-module and ${}_MK$ is a left M-module, we can define a positive sesquilinear form on the algebraic tensor product $H_0 \odot K$ by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle = \langle \eta_1, \langle \xi_1, \xi_2\rangle_M \, \eta_2\rangle_K .$$

The Hilbert space obtained by separation and completion of $H_0 \odot K$ with respect to this sesquilinear form is denoted by $H \otimes_M K$, and the image of $\xi \otimes \eta \in H_0 \odot K$ inside $H \otimes_M K$ is denoted by $\xi \otimes_M \eta$. By construction, we have the following important property:

$$\xi a \otimes_M \eta = \xi \otimes_M a\eta \quad \text{for } \; a \in M.$$

If H is an N-M-bimodule and K is an M-P-bimodule for some tracial von Neumann algebras N and P, then $H \otimes_M K$ becomes a N-P-bimodule when equipped with the bimodular actions

$$x \cdot (\xi \otimes_M \eta) \cdot y := x\xi \otimes_M \eta y, \quad x \in N, \; y \in P.$$

The N-P-bimodule $H \otimes_M K$ is called the *Connes tensor product* of ${}_N H_M$ and ${}_M K_P$.

Note that this construction of the Connes tensor product could also be carried out using left bounded vectors of K instead of right bounded vectors of H. However, these two constructions lead to isomorphic bimodules.

2.3 Amenability and relative amenability

Recall that a countable discrete group Γ is called *amenable* if it admits a left-invariant mean, i.e., a finitely additive probability measure on the power set $\mathcal{P}(\Gamma)$ that is invariant under the left translation action of Γ on itself. A similar notion of amenability exists for II_1 factors and is defined in such a way that $L(\Gamma)$ is amenable if and only if Γ is amenable.

Following [Co75], a von Neumann algebra $M \subset B(H)$ is said to be *amenable* (or *injective*) if there exists a conditional expectation of $B(H)$ onto M. This definition is independent of the chosen representation of M as operators on a Hilbert space (see e.g. [AP16, Proposition 10.2.2]). When (M, τ) is a tracial von Neumann algebra, we have that M is amenable if and only if there exists a state φ on $B(L^2(M))$ that is M-central, in the sense that $\varphi(xT) = \varphi(Tx)$ for all $x \in M$ and $T \in B(L^2(M))$, and such that $\varphi|_M = \tau$ (see e.g. [AP16, Proposition 10.2.5]). Such a state is called a *hypertrace* for M and is the analogue of a left-invariant mean on a group. One can easily show that a countable discrete group Γ is amenable if and only if $L(\Gamma)$ is amenable (see e.g. [AP16, Chapter 10]). Even more, when Γ is a non-amenable group, the group von Neumann algebra $L(\Gamma)$ has no amenable direct summand.

In [Co75], Connes showed that a separable II_1 factor is amenable if and only if it is hyperfinite, meaning that there is an increasing sequence of finite dimensional von Neumann subalgebras $M_n \subset M$ such that $M = (\bigcup_{n\in\mathbb{N}} M_n)''$. It has been known since the work of Murray and von Neumann [MvN43] that there is a *unique hyperfinite II_1 factor*, which is usually denoted by R. So by the work of Connes, there is a unique separable amenable II_1 factor. In particular, any amenable icc groups give rise to the same group von Neumann algebra and any free ergodic pmp actions of amenable groups give rise to the same crossed product von Neumann algebra.

As for countable groups, there are many equivalent definitions of amenability for von Neumann algebras. We will mention one very useful characterization, which involves bimodules. Recall that a group Γ is amenable if and only if the trivial representation of Γ is weakly contained in the left regular representation of Γ (see e.g. [BO08, Theorem 2.6.8]). The following is the analogue in the setting of tracial von Neumann algebras.

Proposition 2.10 ([Po86, Theorem 3.1.2])**.** *A tracial von Neumann algebra (M,τ) is amenable if and only if the trivial M-bimodule $L^2(M)$ is weakly contained in the coarse M-bimodule $L^2(M) \otimes L^2(M)$.*

In [OP07, Section 2.2], the concept of relative amenability was introduced. The definition makes use of Jones' basic construction: Given a tracial von Neumann algebra (M,τ) and a von Neumann subalgebra $N \subset M$, we can consider the orthogonal projection $e_N \colon L^2(M) \to L^2(N)$. The *Jones basic construction* $\langle M, e_N\rangle$ is defined as the von Neumann algebra inside $B(L^2(M))$ generated by M and e_N. Equivalently, $\langle M, e_N\rangle$ equals the commutant of the right N-action on $L^2(M)$, i.e., $\langle M, e_N\rangle = B(L^2(M)) \cap (N^{\mathrm{op}})'$.

Let $Q \subset (M,\tau)$ be a von Neumann subalgebra and $p \in M$ a projection. Following [OP07, Definition 2.2], we say that a von Neumann subalgebra $P \subset pMp$ is *amenable relative to Q inside M* if there exists a positive functional φ on $p\langle M, e_Q\rangle p$ that is P-central and satisfies $\varphi|_{pMp} = \tau$. By definition, we have that P is amenable if and only if P is amenable relative to $\mathbb{C}1$ inside M. The following proposition states a few equivalent characterizations of relative amenability in terms of bimodules.

Proposition 2.11 ([OP07, Theorem 2.1], [PV11, Proposition 2.4])**.** *Let (M,τ) be a tracial von Neumann algebra, let $p \in M$ be a projection and let $P \subset pMp$, $Q \subset M$ be von Neumann subalgebras. The following are equivalent.*

- *P is amenable relative to Q inside M.*
- *${}_{pMp}L^2(pMp)_P$ is weakly contained in ${}_{pMp}(pL^2(M) \otimes_Q L^2(M)p)_P$.*
- *${}_{pMp}L^2(pMp)_P$ is weakly contained in ${}_{pMp}(pL^2(M) \otimes_Q K)_P$ for some Q-P-bimodule K.*
- *There exists a net $(\xi_i)_{i\in I}$ in $pL^2(M)\otimes_Q L^2(M)p$ such that $\langle x\xi_i, \xi_i\rangle \to \tau(x)$ for all $x \in pMp$ and $\|a\xi_i - \xi_i a\|_2 \to 0$ for all $a \in P$.*

By combining Propositions 2.10 and 2.11, we see that when Q is amenable, then P is amenable relative to Q if and only if P is amenable. Indeed, this follows from the fact that the M-bimodule $L^2(M) \otimes_Q L^2(M)$ is weakly contained in the coarse M-bimodule, when Q is amenable.

We say that P is *strongly non-amenable relative to* Q if Pq is non-amenable relative to Q for every nonzero projection $q \in P' \cap pMp$. Note that in that case, also p_0Pp_0 is strongly non-amenable relative to Q for all nonzero projections $p_0 \in P$ (see [DHI16, Lemma 2.6.2]).

2.4 Popa's intertwining-by-bimodules

A general technique in the theory of II_1 factors is to try to "locate" certain subalgebras of a given II_1 factor M. The strongest way to obtain this is to show that one subalgebra $A \subset M$ can be unitarily conjugated into another subalgebra $B \subset M$, in the sense that $uAu^* \subset B$ for some $u \in \mathcal{U}(M)$. Note that in this case, $H = u^*L^2(B)$ is an A-B-subbimodule of $L^2(M)$ with $\dim_{-B}(H) < \infty$ (in fact $\dim_{-B}(H) = 1$). Such a bimodule is called an *intertwining bimodule* of A into B. Having an intertwining bimodule is in general much weaker than unitary conjugacy. However, Popa showed in [Po03] that the existence of an intertwining bimodule of A into B exactly means that a corner of A can be conjugated into a corner of B via a partial isometry. This is made precise in the following theorem.

Theorem 2.12 ([Po03, Theorem 2.1 and Corollary 2.3])**.** *Let (M,τ) be a tracial von Neumann algebra. Let $p,q \in M$ be projections and let $P \subset pMp$ and $Q \subset qMq$ be von Neumann subalgebras. The following are equivalent.*

(1) *There exists a nonzero P-Q-subbimodule $H \subset pL^2(M)q$ with* $\dim_{-Q}(H) < \infty$.

(2) *There exist nonzero projections $p_0 \in P$ and $q_0 \in Q$, a normal unital $*$-homomorphism $\theta\colon p_0Pp_0 \to q_0Qq_0$ and a nonzero partial isometry $v \in p_0Mq_0$ such that $xv = v\theta(x)$ for all $x \in p_0Pp_0$.*

(3) *There exists a nonzero projection $r \in M_n(\mathbb{C}) \otimes Q$, a normal unital $*$-homomorphism $\theta\colon P \to r(M_n(\mathbb{C}) \otimes Q)r$ and a nonzero partial isometry $v \in M_{1,n}(\mathbb{C}) \otimes pMq$ such that $xv = v\theta(x)$ for all $x \in P$.*

(4) *There is no net of unitaries $w_i \in \mathcal{U}(P)$ satisfying $\|E_Q(x^*w_iy)\|_2 \to 0$ for all $x,y \in pMq$. Here, E_Q denotes the unique trace-preserving conditional expectation of M onto Q.*

If one of the four equivalent conditions from Theorem 2.12 holds, then we say that P *intertwines into* Q *inside* M and we denote this by $P \prec_M Q$, or simply $P \prec Q$ if there is no doubt about the ambient von Neumann algebra M.

When P and Q are Cartan subalgebras of a II_1 factor M, then the intertwining criterion $P \prec Q$ is equivalent with a true unitary conjugacy of P and Q. Therefore, Theorem 2.12 is extremely useful for proving unitary conjugacy of Cartan subalgebras.

Theorem 2.13 ([Po01, Theorem A.1]). *Let M be a II_1 factor and let $P, Q \subset M$ be Cartan subalgebras. Then $P \prec_M Q$ if and only if P and Q are unitarily conjugate.*

The intertwining relation $\prec$ is not transitive. For example, if M is diffuse then $M \prec_{M\oplus\mathbb{C}} M \oplus \mathbb{C}$ and $M \oplus \mathbb{C} \prec_{M\oplus\mathbb{C}} \mathbb{C}$ but $M \not\prec_{M\oplus\mathbb{C}} \mathbb{C}$. However, there is a stronger notion, called full intertwining, for which we do have transitivity. Given von Neumann subalgebras $P \subset pMp$ and $Q \subset qMq$, we say that P *fully intertwines into* Q and write $P \prec_M^f Q$ if $Pp_0 \prec_M Q$ for all nonzero projections $p_0 \in P' \cap pMp$. It is not hard to show that if $A, B, C \subset M$ are (possibly non-unital) subalgebras, then $A \prec_M^f B$ and $B \prec_M^f C$ implies $A \prec_M^f C$. Also, $A \prec_M B$ and $B \prec_M^f C$ implies $A \prec_M C$ (see [Va07, Lemma 3.7] for a proof of these facts).

Note that a tracial von Neumann algebra M is atomic if and only if $M \prec^f \mathbb{C}1$, and M is diffuse if and only if $M \not\prec \mathbb{C}1$.

We are particularly interested in the case where M is a crossed product $M = A \rtimes \Gamma$ by a trace-preserving action $\Gamma \curvearrowright (A, \tau)$. Given a subset $F \subset \Gamma$, we denote by P_F the orthogonal projection of $L^2(M)$ onto the closed linear span of $\{au_g \mid a \in A,\, g \in F\}$, where $\{u_g\}_{g\in\Gamma}$ denote the canonical unitaries in $L(\Gamma)$. By [Va10, Lemma 2.5], a von Neumann subalgebra $P \subset pMp$ satisfies $P \prec_M^f A$ if and only if for every $\varepsilon > 0$, there exists a finite subset $F \subset \Gamma$ such that

$$\|x - P_F(x)\|_2 \leq \|x\|\varepsilon \quad \text{for all } \ x \in P \ .$$

We also need the following elementary lemma.

Lemma 2.14. *Let $\Gamma \curvearrowright (A, \tau)$ be a trace-preserving action and put $M = A \rtimes \Gamma$. If $P \subset M$ is a diffuse von Neumann subalgebra such that $P \prec^f A$, then $P \not\prec L(\Gamma)$.*

Proof. Let $\varepsilon > 0$ be given and assume $P \prec^f A$. As explained above, we can take a finite set $F \subset \Gamma$ such that $\|u - P_F(u)\|_2 \leq \frac{\varepsilon}{2}$ for all $u \in \mathcal{U}(P)$. Moreover, since P is diffuse, we can choose a net of unitaries $(w_i) \subset \mathcal{U}(P)$ tending to 0 weakly. We will prove that $\|E_{L(\Gamma)}(xw_iy)\|_2 \to 0$ for all $x, y \in M$, meaning that $P \not\prec L(\Gamma)$. Note that it suffices to consider $x, y \in (A)_1$.

Take $x, y \in (A)_1$. Write $w_i = \sum_{g\in\Gamma}(w_i)_g u_g$ with $(w_i)_g \in A$. Then,

$$\|E_{L(\Gamma)}(P_F(xw_iy))\|_2^2 = \sum_{g\in F} |\tau(x(w_i)_g\sigma_g(y))|^2 \to 0$$

since $w_i \to 0$ weakly. Take i_0 large enough such that

$$\|E_{L(\Gamma)}(P_F(xw_iy))\|_2 \leq \frac{\varepsilon}{2} \quad \text{for all} \;\; i \geq i_0 \; .$$

Since P_F is A-bimodular, we have that

$$\begin{aligned}\|E_{L(\Gamma)}(P_F(xw_iy)) - E_{L(\Gamma)}(xw_iy)\|_2 &\leq \|P_F(xw_iy) - xw_iy\|_2 \\ &= \|x(P_F(w_i) - w_i)y\|_2 \leq \frac{\varepsilon}{2}\end{aligned}$$

for all i. We conclude that $\|E_{L(\Gamma)}(xw_iy)\|_2 \leq \varepsilon$ for all $i \geq i_0$. □

On the level of group von Neumann algebras, the notion of intertwining-by-bimodules simply translates into the following finite index criterion.

Lemma 2.15. *Let Γ be a countable discrete group and let $\Lambda_1, \Lambda_2 < \Gamma$ be subgroups. Then, $L(\Lambda_1) \prec_{L(\Gamma)} L(\Lambda_2)$ if and only if there exists $g \in \Gamma$ such that $[\Lambda_1 : \Lambda_1 \cap g\Lambda_2 g^{-1}] < \infty$.*

Proof. Assume that $[\Lambda_1 : \Lambda_1 \cap g\Lambda_2 g^{-1}] < \infty$ for some $g \in \Gamma$. Then $\{g^{-1}sg\Lambda_2 \mid s \in \Lambda_1\}$ is a finite subset of Γ/Λ_2 and hence

$$H := \overline{\text{span}}\,\{\ell^2(s\Lambda_2) \mid s \in g^{-1}\Lambda_1 g\}$$

is a $L(g^{-1}\Lambda_1 g)$-$L(\Lambda_2)$-subbimodule of $\ell^2(\Gamma)$ with finite right $L(\Lambda_2)$-dimension. This means that $L(g^{-1}\Lambda_1 g) \prec_{L(\Gamma)} L(\Lambda_2)$. Since $L(g^{-1}\Lambda_1 g) = u_g^* L(\Lambda_1) u_g$, we then also have that $L(\Lambda_1) \prec_{L(\Gamma)} L(\Lambda_2)$.

Conversely, assume that $[\Lambda_1 : \Lambda_1 \cap g\Lambda_2 g^{-1}] = \infty$ for all $g \in \Gamma$. Given any finite subset $F \subset \Gamma$, we claim that there exists an $s \in \Lambda_1$ such that $s \notin g\Lambda_2 h$ for all $g, h \in F$. Indeed, assume that this is not the case. Then

$$\Lambda_1 \subset \bigcup_{g,h\in F} g\Lambda_2 h = \bigcup_{g,h\in F} (g\Lambda_2 g^{-1})gh.$$

By [Ne54, Lemma 4.1], this contradicts the fact that $g\Lambda_2 g^{-1} \cap \Lambda_1$ has infinite index in Λ_1 for all $g \in \Gamma$.

Let I be the directed set consisting of all finite subsets of Γ, ordered by inclusion. Given $F \in I$, we define $u_F := u_s \in L(\Lambda_1)$ where $s \in \Lambda_1$ is an element such that $s \notin \bigcup_{g,h\in F} g\Lambda_2 h$. This gives us a net $(u_i)_{i\in I}$ of unitaries in $L(\Lambda_1)$ such that

$$\|E_{L(\Lambda_2)}(xu_iy)\|_2 \to 0 \quad \text{for all } \ x, y \in L(\Gamma).$$

By Theorem 2.12, this means that $L(\Lambda_1) \not\prec_{L(\Gamma)} L(\Lambda_2)$. □

We end this section with the following elementary result, which shows the relationship between relative amenability and intertwining-by-bimodules.

Proposition 2.16. *Let (M, τ) be a tracial von Neumann algebra and $Q, P_1, P_2 \subset M$ be von Neumann subalgebras with $P_1 \subset P_2$. Assume that Q is strongly non-amenable relative to P_1. Then the following holds.*

(1) *If $Q \prec_M P_2$, there exist projections $q \in Q$, $p \in P_2$, a nonzero partial isometry $v \in qMp$ and a normal unital $*$-homomorphism $\theta : qQq \to pP_2p$ such that $xv = v\theta(x)$ for all $x \in qQq$ and such that, inside P_2, we have that $\theta(qQq)$ is non-amenable relative to P_1.*

(2) *We have $Q \not\prec_M P_1$.*

Proof. (1) Assume that $Q \prec P_2$. By Theorem 2.12, we can take projections $q \in Q$, $p \in P_2$, a nonzero partial isometry $v \in qMp$ and a normal unital $*$-homomorphism $\theta\colon qQq \to pP_2p$ such that $xv = v\theta(x)$ for all $x \in qQq$. Assume that $\theta(qQq)$ is amenable relative to P_1 inside P_2. We can then take a positive functional φ on $p\langle P_2, e_{P_1}\rangle p$ that is $\theta(qQq)$-central and satisfies $\varphi|_{pP_2p} = \tau$. Denote by e_{P_2} the orthogonal projection of $L^2(M)$ onto $L^2(P_2)$. Observe that $e_{P_2}\langle M, e_{P_1}\rangle e_{P_2} = \langle P_2, e_{P_1}\rangle$. We can then define the positive functional ω on $q\langle M, e_{P_1}\rangle q$ given by

$$\omega(T) = \varphi(e_{P_2}v^*Tve_{P_2}) \quad \text{for all } \ T \in q\langle M, e_{P_1}\rangle q \ .$$

By construction, ω is qQq-central and $\omega(x) = \tau(v^*xv)$ for all $x \in qMq$. Writing $q_0 = vv^*$, we have $q_0 \in (Q' \cap M)q$ and it follows that $qQqq_0$ is amenable relative to P_1. This contradicts the strong non-amenability of Q relative to P_1.

Finally, note that (2) follows from (1) by taking $P_1 = P_2$. □

2.5 Mixing properties

Recall that a unitary representation $\pi\colon \Gamma \to \mathcal{U}(H)$ of a countable discrete group Γ is said to be *weakly mixing* if $\{0\}$ is the only finite-dimensional subrepresentation.

This is equivalent with the existence of a sequence $(g_n)_{n\in\mathbb{N}} \subset \Gamma$ such that $\langle \pi(g_n)\xi, \eta\rangle \to 0$ for all $\xi, \eta \in H$. The representation π is called *mixing* if this limit holds for any sequence $(g_n)_{n\in\mathbb{N}} \subset \Gamma$ with $g_n \to \infty$ (in the sense that g_n escapes all finite subsets of Γ). The following proposition is classical, see for example [Pe11, Proposition 1.5.6].

Proposition 2.17. *Let $\pi\colon \Gamma \to \mathcal{U}(H)$ be a unitary representation. The following are equivalent.*

- *π is weakly mixing.*
- *The representation $\pi \otimes \overline{\pi}$ contains no nonzero invariant vectors.*
- *The representation $\pi \otimes \rho$ contains no nonzero invariant vectors for any (finite-dimensional) unitary representation $\rho\colon \Gamma \to \mathcal{U}(K)$.*

Similar notions of mixing and weak mixing exist for group actions $\Gamma \curvearrowright (X, \mu)$ and for bimodules ${}_N H_M$ of tracial von Neumann algebras N and M.

Definition 2.18. A pmp action $\Gamma \curvearrowright (X, \mu)$ is called

- *weakly mixing* if for any $\varepsilon > 0$ and any finite collection of measurable subsets $A_1, \dots, A_n \subset X$, there exists a $g \in \Gamma$ such that
$$|\mu(A_i \cap gA_j) - \mu(A_i)\mu(A_j)| < \varepsilon \quad \text{for all } \; i, j = 1, \dots, n;$$
- *mixing* if for any measurable subsets $A, B \subset X$ and any sequence $g_n \in \Gamma$ with $g_n \to \infty$, we have that
$$\mu(A \cap g_n B) \to \mu(A)\mu(B) \quad \text{as } \; n \to \infty.$$

It is clear from the definition that a weakly mixing action is in particular ergodic. More generally, we can define (weak) mixing for trace-preserving actions $\Gamma \curvearrowright (B, \tau)$ in an analogous way.

Definition 2.19. Let $\Gamma \curvearrowright (B, \tau)$ be a trace-preserving action on a tracial von Neumann algebra (B, τ). The action is called

- *weakly mixing* if for every finite set $a_1, \dots, a_n \in B$ and every $\varepsilon > 0$, there exists $g \in \Gamma$ such that
$$|\tau(a_i \sigma_g(a_j)) - \tau(a_i)\tau(a_j)| < \varepsilon \quad \text{for all } \; i, j = 1, \dots, n;$$

- *mixing* if for every $a, b \in B$ and every sequence $g_n \in \Gamma$ with $g_n \to \infty$, we have that
$$\tau(a\sigma_{g_n}(b)) \to \tau(a)\tau(b) \quad \text{as } n \to \infty.$$

When $\Gamma \curvearrowright (X, \mu)$ is a pmp action, we have that $\Gamma \curvearrowright X$ is (weakly) mixing if and only if $\Gamma \curvearrowright L^\infty(X)$ is (weakly) mixing. The following proposition states some equivalent formulations of being weakly mixing.

Proposition 2.20 ([Va06, Proposition D.2]). *Let $\sigma\colon \Gamma \curvearrowright (B, \tau)$ be a trace-preserving action. The following are equivalent.*

(i) The action σ is weakly mixing.

(ii) $\mathbb{C}1$ is the only finite-dimensional invariant subspace of B.

(iii) The associated unitary representation of Γ on $L^2(B) \ominus \mathbb{C}1$ is weakly mixing.

(iv) The diagonal action $\Gamma \curvearrowright B \mathbin{\overline{\otimes}} B\colon g \cdot (a \otimes b) = \sigma_g(a) \otimes \sigma_g(b)$ is ergodic.

Example 2.21. Given a countable group Γ and a standard probability space (X, μ), we can form the *Bernoulli action* $\Gamma \curvearrowright (X, \mu)^\Gamma$ given by $g \cdot (x_s)_{s \in \Gamma} = (x_{gs})_{s \in \Gamma}$. More generally, given any action $\Gamma \curvearrowright I$ on a countable set I, the *generalized Bernoulli action* $\Gamma \curvearrowright (X, \mu)^I$ is given by $g \cdot (x_i)_{i \in I} = (x_{g \cdot i})_{i \in I}$. It is a classical result that the generalized Bernoulli action $\Gamma \curvearrowright (X, \mu)^I$ is weakly mixing if and only if the action $\Gamma \curvearrowright I$ has infinite orbits (see [PV06, Proposition 2.3 and Lemma 2.4]).

When $\sigma\colon \Gamma \curvearrowright (B, \tau)$ is a trace-preserving action that globally preserves a von Neumann subalgebra $B_0 \subset B$, we have that the subspace $B \ominus B_0$ is also globally preserved. Even though $B \ominus B_0$ is not a subalgebra, we still say that σ restricts to an action of Γ on $B \ominus B_0$. We say that this action is weakly mixing if the associated unitary representation of Γ on $L^2(B) \ominus L^2(B_0)$ is weakly mixing. Exactly as in the proof of Proposition 2.20, it can be shown that $\Gamma \curvearrowright B \ominus B_0$ is weakly mixing if and only if $B \ominus B_0$ contains no nontrivial finite-dimensional Γ-invariant subspace. For completeness, we provide a proof of this fact.

Proposition 2.22. *Let $\sigma\colon \Gamma \curvearrowright (B, \tau)$ be a trace-preserving action that globally preserves a von Neumann subalgebra $B_0 \subset B$. The following are equivalent.*

(i) $\sigma\colon \Gamma \curvearrowright B \ominus B_0$ is weakly mixing.

(ii) $B \ominus B_0$ contains no nontrivial finite-dimensional Γ-invariant subspaces.

(iii) The diagonal action $\Gamma \curvearrowright (B \mathbin{\overline{\otimes}} B) \ominus (B_0 \mathbin{\overline{\otimes}} B_0)$ is ergodic.

Proof. (i) ⇒ (ii) is clear from the definition.

To show (ii) ⇒ (iii), suppose that $x \in B \mathbin{\overline{\otimes}} B$ is a Γ-invariant element. Denote by ι the canonical embedding $B \to L^2(B)$. Given $\xi, \eta \in L^2(B)$, we denote by $\omega_{\xi,\eta}$ the vector functional on B given by $\omega_{\xi,\eta}(a) = \langle a\xi, \eta\rangle$. Define a Hilbert-Schmidt operator $T\colon L^2(B) \to \overline{L^2(B)}$ by $T\xi = \iota((\mathrm{id} \otimes \omega_{\iota(1),\xi})(x))$. Then TT^* is a trace-class operator that commutes with the unitary representation of Γ on $L^2(B)$ associated with σ. So, any spectral projection of TT^* has finite rank and still commutes with σ. Since the image of T is contained in $\iota(B)$, we obtain using (ii) that $x \in B_0 \mathbin{\overline{\otimes}} B$. By symmetry, we also obtain $x \in B \mathbin{\overline{\otimes}} B_0$ and hence $x \in B_0 \mathbin{\overline{\otimes}} B_0$. So, $(B \mathbin{\overline{\otimes}} B) \ominus (B_0 \mathbin{\overline{\otimes}} B_0)$ has no Γ-invariant vectors.

To show (iii) ⇒ (i), assume that $\sigma\colon \Gamma \curvearrowright B \ominus B_0$ is not weakly mixing. Then there exists $\varepsilon > 0$ and $b_1, \dots, b_n \in B \ominus B_0$ such that

$$\sum_{i,j=1}^{n} |\tau(\sigma_g(b_i)b_j^*)|^2 \geq \varepsilon \qquad \text{for all } \ g \in \Gamma.$$

Define $x = \sum_{i=1}^{n} b_i \otimes b_i^* \in (B \mathbin{\overline{\otimes}} B) \ominus (B_0 \mathbin{\overline{\otimes}} B_0)$. Let $x_1 \in B \mathbin{\overline{\otimes}} B$ be the unique element of minimal 2-norm in the weakly closed convex hull of $\{(\sigma_g \otimes \sigma_g)(x) \mid g \in \Gamma\}$. Note that $x_1 \in (B \mathbin{\overline{\otimes}} B) \ominus (B_0 \mathbin{\overline{\otimes}} B_0)$. For any $g \in \Gamma$, we have that

$$\tau((\sigma_g \otimes \sigma_g)(x)x^*) = \sum_{i,j=1}^{n} |\tau(\sigma_g(b_i)b_j^*)|^2 \geq \varepsilon.$$

It follows that $\tau(x_1 x^*) \geq \varepsilon$ and hence $x_1 \neq 0$. By uniqueness of x_1, we also have that x_1 is $(\sigma_g \otimes \sigma_g)_{g\in\Gamma}$-invariant. This contradicts (iii). □

Given a tracial crossed product $M = A \rtimes \Gamma$ and given a subgroup $\Lambda < \Gamma$, we will often consider the conjugation action $\{\mathrm{Ad}\, u_g\}_{g\in\Lambda}$ on M. This action has an obvious non weakly mixing part, which we can describe as follows. We denote by $vC_\Gamma(\Lambda)$ the *virtual centralizer* of Λ inside Γ, i.e., $vC_\Gamma(\Lambda)$ consists of all elements $s \in \Gamma$ that commute with a finite index subgroup of Λ. Let $A_0 \subset A$ be the von Neumann subalgebra generated by the set of all $a \in A$ such that $\{\sigma_s(a) \mid s \in \Lambda\}$ spans a finite-dimensional subspace of A. Note that $vC_\Gamma(\Lambda)$ is a subgroup that normalizes A_0. Now, it is clear that the action $\{\mathrm{Ad}\, u_s\}_{s\in\Lambda}$ is not weakly mixing on neither A_0 nor $vC_\Gamma(\Lambda)$. The following lemma shows that the crossed product $A_0 \rtimes vC_\Gamma(\Lambda) \subset M$ is the only non weakly mixing part of M.

Lemma 2.23. *Let $\sigma\colon \Gamma \curvearrowright (A,\tau)$ be a trace-preserving action and put $M = A \rtimes \Gamma$. Fix a subgroup $\Lambda < \Gamma$ and let $K := vC_\Gamma(\Lambda)$ be its virtual centralizer. Let $A_0 \subset A$ be as above, i.e., A_0 is generated by the set of all $a \in A$ such that* $\mathrm{span}\{\sigma_s(a) \mid s \in \Lambda\}$ *is finite-dimensional. Then the unitary representation* $\{\mathrm{Ad}\, u_s\}_{s\in\Lambda}$ *is weakly mixing on $L^2(M) \ominus L^2(A_0 \rtimes K)$.*

Proof. Let $H = L^2(M) \ominus L^2(A_0 \rtimes K)$ and recall that $L^2(M) = L^2(A) \otimes \ell^2(\Gamma)$. Put

$$H_1 = L^2(A) \otimes (\ell^2(\Gamma) \ominus \ell^2(K)), \quad H_2 = L^2(A \ominus A_0) \otimes \ell^2(\Gamma).$$

Then H_1 and H_2 are closed subspaces of H that are invariant under the representation $\{\mathrm{Ad}\, u_s\}_{s\in\Lambda}$, and such that $H = \overline{\mathrm{span}}\, H_1 \cup H_2$. It is enough to show that $\{\mathrm{Ad}\, u_s\}_{s\in\Lambda}$ is weakly mixing on H_1 and H_2 since then, any finite-dimensional invariant subspace $F \subset H$ satisfies $F \subset H_1^\perp \cap H_2^\perp = \{0\}$.

To prove that $\{\mathrm{Ad}\, u_s\}_{s\in\Lambda}$ is weakly mixing on H_2, it suffices to show that the representation $\{\sigma_s\}_{s\in\Lambda}$ is weakly mixing on $L^2(A \ominus A_0)$. By construction, $A \ominus A_0$ contains no nonzero finite-dimensional $\{\sigma_s\}_{s\in\Lambda}$-invariant subspace. By Proposition 2.22, this exactly means that the representation $\{\sigma_s\}_{s\in\Lambda}$ is weakly mixing on $L^2(A \ominus A_0)$.

To prove that $\{\mathrm{Ad}\, u_s\}_{s\in\Lambda}$ is weakly mixing on H_2, it suffices to show that $\{\mathrm{Ad}\, u_s\}_{s\in\Lambda}$ is weakly mixing on $\ell^2(\Gamma - K)$. But this follows from [PV06, Lemma 2.4] since $\{sgs^{-1} \mid s \in \Lambda\}$ is infinite whenever $g \in \Gamma - K$. □

Note that it in particular follows from Lemma 2.23 that the conjugation action $\Gamma \curvearrowright L(\Gamma)$ is weakly mixing if and only if Γ is icc.

Finally, we introduce the notion of a (weakly) mixing bimodule. Let (M, τ) be a tracial von Neumann algebra and let $A, B \subset M$ be von Neumann subalgebras. Popa's non-intertwinability condition (see Theorem 2.12) saying that $B \not\prec_M A$ is equivalent with the existence of a net of unitaries $b_i \in \mathcal{U}(B)$ such that $\lim_i \|E_A(xb_iy)\|_2 = 0$ for all $x, y \in M$, can be viewed as a weak mixing condition for the B-A-bimodule ${}_BL^2(M)_A$ (cf. the notions of relative (weak) mixing in [Po05, Definition 2.9]). This then naturally lead to the notions of mixing and weakly mixing bimodules in [PS12].

Recall from Section 2.2 the notion of left and right bounded vectors in a B-A-bimodule H, as well as the A-valued inner product $\langle \xi, \eta \rangle_A$ for right bounded vectors $\xi, \eta \in H$.

Definition 2.24 ([PS12]). Let (A, τ) and (B, τ) be tracial von Neumann algebras and ${}_BH_A$ a B-A-bimodule.

(1) ${}_BH_A$ is called *left weakly mixing* if there exists a net of unitaries $b_i \in \mathcal{U}(B)$ such that for all right bounded vectors $\xi, \eta \in H$, we have

$$\lim_i \|\langle b_i\xi, \eta\rangle_A\|_2 = 0 .$$

(2) ${}_B H_A$ is called *left mixing* if every net $b_i \in \mathcal{U}(B)$ tending to 0 weakly satisfies

$$\lim_n \|\langle b_i \xi, \eta \rangle_A\|_2 = 0$$

for all right bounded vectors $\xi, \eta \in H$.

We similarly define the notions of *right (weak) mixing.* When ${}_A H_A$ is a *symmetric* A-bimodule in the sense of Definition 5.4, left (weak) mixing is equivalent with right (weak) mixing and we simply refer to these properties as (weak) mixing.

Example 2.25. If $\pi \colon \Gamma \to \mathcal{U}(K)$ is a weakly mixing group representation, then the associated $L(\Gamma)$-bimodule $H = \ell^2(\Gamma) \otimes K$ is left weakly mixing. Indeed, if $(g_n)_{n \in \mathbb{N}} \subset \Gamma$ is a sequence such that $\langle \pi(g_n)\xi, \eta \rangle \to 0$ for all $\xi, \eta \in K$, then $u_{g_n} \in L(\Gamma)$ is a sequence of unitaries witnessing the left weak mixing of H.

In [Po03, Section 2], Popa proved that the intertwining relation $B \prec_M A$ is equivalent with the existence of a nonzero B-A-subbimodule of $L^2(M)$ having finite right A-dimension (see Theorem 2.12). In the same way, one gets the following characterization of weakly mixing bimodules. For details, see [PS12] and [Bo14, Theorem A.2.2].

Proposition 2.26 ([Po03, PS12, Bo14])**.** *Let (A, τ) and (B, τ) be tracial von Neumann algebras and ${}_B H_A$ a B-A-bimodule. The following are equivalent.*

(1) *${}_B H_A$ is left weakly mixing.*

(2) *$\{0\}$ is the only B-A-subbimodule of ${}_B H_A$ of finite A-dimension.*

(3) *${}_B(H \otimes_A \overline{H})_B$ has no nonzero B-central vectors.*

(4) *${}_B(H \otimes_A K)_B$ has no nonzero B-central vectors for any A-B-bimodule K.*

2.6 (Relative) strong solidity and class $\mathcal{C}_{\text{rss}}$

In the study of non-amenable II_1 factors, it is natural to consider different kinds of indecomposability properties. One such property is primeness, which means that a given II_1 factor cannot be written as a nontrivial tensor product. More precisely, a II_1 factor M is called *prime* if any tensor product decomposition $M = M_1 \mathbin{\overline{\otimes}} M_2$ forces either M_1 or M_2 to be finite-dimensional. A similar property is the impossibility of writing M as a crossed product or more generally as $L(\mathcal{R})$ for some equivalence relation $\mathcal{R}$. Recall from Theorem 2.4 that this exactly amounts to M not having a Cartan subalgebra. Therefore, absence of Cartan subalgebras can also be seen as an indecomposability property.

In [Oz03], Ozawa discovered a new property that generalizes primeness. This property was called solidity and is defined as follows. A von Neumann algebra M is called *solid* if the relative commutant $A' \cap M$ is amenable for any diffuse von Neumann subalgebra $A \subset M$. It is clear from the definition that any solid von Neumann algebra is also prime. Ozawa proved that the group von Neumann algebra $L(\Gamma)$ is solid for any nonelementary hyperbolic group Γ, in particular for any free group $\mathbb{F}_n$ with $n \geq 2$.

The strongest indecomposability property for von Neumann algebras is strong solidity, which was introduced by Ozawa and Popa in [OP07].

Definition 2.27 ([OP07])**.** Let (M, τ) be a diffuse tracial von Neumann algebra. We say that M is *strongly solid* if for any diffuse amenable von Neumann subalgebra $A \subset M$, the normalizer $\mathcal{N}_M(A)'' = \{u \in \mathcal{U}(M) \mid uAu^* = A\}''$ stays amenable.

Note that, as the name suggests, strong solidity implies solidity. Indeed, if $A \subset M$ is a diffuse von Neumann subalgebra, then A contains a diffuse amenable von Neumann subalgebra $B \subset A$. Since $A' \cap M \subset \mathcal{N}_M(B)''$, it follows that strong solidity of M implies solidity of M. It is immediate that strong solidity also implies absence of Cartan subalgebras for non-amenable von Neumann algebras M. Therefore, strong solidity is a very interesting property and has become a key concept in the context of Popa's deformation/rigidity theory.

Ozawa and Popa showed that the free group factors $L(\mathbb{F}_n)$ with $n \geq 2$ are strongly solid, thereby generalizing previous results by Voiculescu [Vo95] (absence of Cartan), Ge [Ge98] (primeness) and Ozawa [Oz03] (solidity). In fact, they showed an even stronger property: for every free ergodic profinite[3] action $\mathbb{F}_n \curvearrowright X$, the crossed product $M = L^\infty(X) \rtimes \mathbb{F}_n$ is strongly solid relative to $L^\infty(X)$ in the following sense. For any diffuse amenable subalgebra $A \subset M$, either $\mathcal{N}_M(A)''$ stays amenable or $A \prec L^\infty(X)$. The results of Ozawa and Popa were generalized by Popa and Vaes in [PV11, PV12], and they obtained the following remarkably strong result. For *any* trace-preserving action of a hyperbolic group Γ on a tracial von Neumann algebra B, the associated crossed product $M = B \rtimes \Gamma$ is strongly solid relative to B, i.e., for any diffuse subalgebra $A \subset M$ that is amenable relative to B, either $\mathcal{N}_M(A)''$ stays amenable relative to B or $A \prec_M B$. This led to the notion of relative strong solidity, which is defined as follows.

Definition 2.28. A group Γ is called *relatively strongly solid* if for any tracial crossed product $M = P \rtimes \Gamma$ and any von Neumann subalgebra $Q \subset pMp$ that is

[3] An action $\Gamma \curvearrowright X$ is called *profinite* if $L^\infty(X)$ is a limit of an increasing sequence of finite-dimensional Γ-invariant subalgebras of $L^\infty(X)$.

amenable relative to P, we have that either $Q \prec P$ or the normalizer $\mathcal{N}_{pMp}(Q)''$ stays amenable relative to P.

As in [CIK13, Definition 2.7], we denote by $\mathcal{C}_{\text{rss}}$ the class of all non-amenable relatively strongly solid groups. The class $\mathcal{C}_{\text{rss}}$ is quite large. Indeed, by [PV11, Theorem 1.6], all weakly amenable groups that admit a proper 1-cocycle into an orthogonal representation weakly contained in the regular representation belong to $\mathcal{C}_{\text{rss}}$. In particular, the free groups $\mathbb{F}_n$ with $2 \leq n \leq \infty$ belong to $\mathcal{C}_{\text{rss}}$ and more generally, all free products $\Lambda_1 * \Lambda_2$ of amenable groups Λ_1, Λ_2 with $|\Lambda_1| \geq 2$ and $|\Lambda_2| \geq 3$ belong to $\mathcal{C}_{\text{rss}}$. By [PV12, Theorem 1.4], all weakly amenable, non-amenable, bi-exact groups belong to $\mathcal{C}_{\text{rss}}$ and thus $\mathcal{C}_{\text{rss}}$ contains all nonelementary hyperbolic groups.

Lemma 2.29. *Let Γ be a group in $\mathcal{C}_{\text{rss}}$ and $M = P \rtimes \Gamma$ any tracial crossed product. If $Q_1, Q_2 \subset pMp$ are commuting von Neumann subalgebras, then either $Q_1 \prec_M P$ or Q_2 is amenable relative to P.*

Proof. Assume that $Q_1 \not\prec_M P$. By [BO08, Corollary F.14], there exists a diffuse abelian von Neumann subalgebra $A \subset Q_1$ such that $A \not\prec_M P$. Because $\Gamma \in \mathcal{C}_{\text{rss}}$, we get that $\mathcal{N}_{pMp}(A)''$ is amenable relative to P. Since $Q_2 \subset \mathcal{N}_{pMp}(A)''$, also Q_2 is amenable relative to P. □

From Lemma 2.29, it follows that for groups Γ in $\mathcal{C}_{\text{rss}}$, the centralizer $C_\Gamma(L)$ of an infinite subgroup $L < \Gamma$ is amenable. So, torsion-free groups Γ in $\mathcal{C}_{\text{rss}}$ have the property that $C_\Gamma(g)$ is amenable for every $g \neq e$. As a consequence, torsion-free groups Γ in $\mathcal{C}_{\text{rss}}$ are icc and even have the property that every non-amenable subgroup $L < \Gamma$ is *relatively icc* in the sense that $\{hgh^{-1} \mid h \in L\}$ is an infinite set for every $g \in \Gamma$, $g \neq e$. Finally note that torsion-free groups Γ in $\mathcal{C}_{\text{rss}}$ have no nontrivial amenable normal subgroups. In particular, every nontrivial normal subgroup of Γ is relatively icc.

Chapter 3

Counterexamples to $\mathcal{C}$-rigidity

Recall from the introduction the notion of $\mathcal{C}$-rigidity.

Definition 3.1 ([PV11, Definition 1.4])**.** A countable group Γ is called *$\mathcal{C}$-rigid* if for any free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$, the associated crossed product $L^\infty(X) \rtimes \Gamma$ has a unique Cartan subalgebra up to unitary conjugacy.

In [PV11], Popa and Vaes gave the first examples of $\mathcal{C}$-rigid groups. In fact, they provided a rather large class of $\mathcal{C}$-rigid groups, including free groups $\mathbb{F}_n$ with $n \geq 2$. Since then, more and more groups have been shown to be $\mathcal{C}$-rigid. On the other hand, there are not a lot of counterexamples to $\mathcal{C}$-rigidity. So far, all known counterexamples have an infinite amenable almost normal subgroup.

Definition 3.2. Let Γ be a countable discrete group. A subgroup $\Lambda < \Gamma$ is called *almost normal* if $g\Lambda g^{-1} \cap \Lambda$ has finite index in Λ for all $g \in \Gamma$.

It is immediate from the definition that a subgroup $\Lambda < \Gamma$ is almost normal if and only if the left translation action $\Lambda \curvearrowright \Gamma/\Lambda$ has finite orbits. A typical example of a group with an almost normal subgroup is the Baumslag-Solitar group $\mathrm{BS}(n, m) = \langle a, b \mid ba^n b^{-1} = a^m \rangle$ with $n, m \in \mathbb{Z} \setminus \{0\}$, where the subgroup generated by a is almost normal.

In this chapter, we will prove that any group with an infinite abelian almost normal subgroup is non-$\mathcal{C}$-rigid, up to taking the quotient by a finite normal subgroup (this work is unpublished). The proof is inspired by [PV09, Example 5.8], where it is shown that any semidirect product group $H \rtimes G$ with H infinite abelian is non-$\mathcal{C}$-rigid.

Theorem 3.3. *Let Γ be a countable discrete group with an infinite abelian almost normal subgroup Λ, and let $F = \bigcap_{g\in\Gamma} g\Lambda g^{-1}$.*

(1) *If F is infinite, then Γ is non-$\mathcal{C}$-rigid.*

(2) *If F is finite, then Γ/F is non-$\mathcal{C}$-rigid.*

It is a natural question to ask whether Γ/F being $\mathcal{C}$-rigid implies that Γ itself is $\mathcal{C}$-rigid, when $F < \Gamma$ is a finite normal subgroup. This is related to the question whether $\mathcal{C}$-rigidity passes to finite index restrictions/extensions, in the following sense. Given a finite index inclusion of II_1 factors $N \subset M$, is it true that N has a unique Cartan subalgebra if and only if M has a unique Cartan subalgebra? A priori, there is no reason why such stability properties should hold true. For instance, if $N \subset M$ is a finite index inclusion of II_1 factors and if $A \subset N$ is a Cartan subalgebra, there is no reason why A should also be a Cartan subalgebra of M. In order to get better stability properties, the notion of $\mathcal{C}_s$-rigidity was introduced in [PV11] (based on work of Ozawa and Popa in [OP07]). Recall that a subfactor $N \subset M$ is said to be of finite index if the Jones index $[M : N] = \dim_{-N} L^2(M)$ is finite.

Definition 3.4 ([OP07, Proposition 4.12], [PV11, Definition 1.4])**.** A group Γ is called $\mathcal{C}_s$-rigid if for any free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$, the II_1 factor $M = L^\infty(X) \rtimes \Gamma$ has the following property:

(a) Every MASA $A \subset M$ whose normalizer $\mathcal{N}_M(A)''$ is a finite index subfactor of M, is unitarily conjugate to $L^\infty(X)$.

Clearly, any $\mathcal{C}_s$-rigid group is also $\mathcal{C}$-rigid. In [OP07, Proposition 4.12], it was proved that the property (a) of Definition 3.4 above is stable under amplifications and under finite index restrictions/extensions of II_1 factors.

In [VV14], another strengthening of $\mathcal{C}$-rigidity was introduced, called class $\mathcal{C}$, this time being stable under group extensions and commensurability[1]. In order to define class $\mathcal{C}$, we first need the notion of a virtual core subalgebra. This plays the role of a generalized Cartan subalgebra in the definition of class $\mathcal{C}$.

Definition 3.5 ([VV14])**.** A von Neumann subalgebra $A \subset M$ is called a *virtual core subalgebra* if $A' \cap M = \mathcal{Z}(A)$ and if the inclusion $\mathcal{N}_M(A)'' \subset M$ has essentially finite index, in the sense of Definition 5.26.

Definition 3.6 ([VV14, Definition 4.2])**.** An infinite group Γ is said to belong to class $\mathcal{C}$ if for every trace-preserving cocycle action $\Gamma \curvearrowright (B, \tau)$ and every amenable virtual core subalgebra $A \subset p(B \rtimes \Gamma)p$, we have that $A \prec B$.

[1] Two groups are called commensurable if they contain isomorphic finite index subgroups.

Here, a *cocycle action* is a map $\alpha\colon \Gamma \to \mathrm{Aut}(B)$ together with a 2-cocycle $v\colon \Gamma \times \Gamma \to \mathcal{U}(B)$ such that

$$\alpha_e = \mathrm{id}, \qquad \alpha_g \circ \alpha_h = \mathrm{Ad}\, v_{g,h} \circ \alpha_{gh},$$

$$v_{g,e} = v_{e,g} = 1, \qquad v_{g,h}v_{gh,k} = \alpha_g(v_{h,k})v_{g,hk}, \qquad g,h,k \in \Gamma.$$

So far, all known examples of $\mathcal{C}$-rigid groups are also in class $\mathcal{C}$ and we have no tools to distinguish between the notions of $\mathcal{C}$-rigidity, $\mathcal{C}_s$-rigidity and class $\mathcal{C}$. Indeed, all tools currently available are insensitive to finite index issues.

Using the notion of class $\mathcal{C}$, we get a nicer formulation of Theorem 3.3. We leave it as an open question whether the same conclusion holds for $\mathcal{C}_s$-rigidity or even $\mathcal{C}$-rigidity.

Corollary 3.7. *If* Γ *is a countable group containing an infinite, abelian, almost normal subgroup, then* Γ *is not in class* $\mathcal{C}$*.*

Proof. Using Theorem 3.3, it is enough to show that if Γ is in class $\mathcal{C}$, then Γ/F is also in class $\mathcal{C}$ for any finite normal subgroup $F < \Gamma$.

Assume that Γ is in class $\mathcal{C}$ and let $F < \Gamma$ be a finite normal subgroup. Let $\Gamma/F \curvearrowright (B,\tau)$ be an arbitrary cocycle action and put $M = B \rtimes (\Gamma/F)$. By composing with the quotient map, we can lift this to a cocycle action of Γ on (B,τ). Put $N = B \rtimes \Gamma$. Now, $z := \frac{1}{|F|}\sum_{h\in F} u_h \in B \rtimes \Gamma$ is a central projection in N, and we have that $z(B \rtimes \Gamma) \cong B \rtimes (\Gamma/F)$ via the isomorphism $z(bu_g) \mapsto bu_{gF}$.

Let $A \subset pMp$ be a virtual core subalgebra. Since $M \cong zN$, A can be seen as a virtual core subalgebra of $zpNp$. Since Γ is in class $\mathcal{C}$, it follows that $A \prec_N B$ and equivalently $A \prec_M B$. □

In the rest of this chapter, we will prove Theorem 3.3. This is done by constructing an explicit free ergodic pmp action $\Gamma/F \curvearrowright X$ such that the crossed product $L^\infty(X) \rtimes (\Gamma/F)$ has two Cartan subalgebras that are not unitarily conjugate. The action $\Gamma/F \curvearrowright X$ will be constructed explicitly as a certain *co-induced action.*

3.1 Co-induced actions

Let G be a locally compact second countable group and let H be an open subgroup of G such that H acts on a probability space (X,μ). We can "co-induce" this H-action to an action of G as follows.

Note that the quotient $H\backslash G$ is countable since H is an open subgroup of G. Choose a section $\theta\colon H\backslash G \to G$ such that $\theta(H) = e$ and let $r\colon G \to H$ be the unique map satisfying $g = r(g)\theta(Hg)$. Note that $r(e) = e$ and $r(hg) = hr(g)$ for $h \in H$, $g \in G$. We define $\Omega\colon H\backslash G \times G \to H$ by $\Omega(Ht, g) = r(t)^{-1}r(tg)$. Then Ω is a 1-cocycle for the action $G \curvearrowright H\backslash G$, in the sense that

$$\Omega(Ht, gh) = \Omega(Ht, g)\Omega(Htg, h), \quad g, h, t \in G.$$

A different choice of section θ gives rise to a cocycle cohomologous with Ω.

The formula

$$(g \cdot x)_{Ht} = \Omega(Ht, g) \cdot x_{Htg}, \qquad x = (x_{Ht})_{t\in\theta(H\backslash G)} \in X^{H\backslash G}$$

then gives a well-defined action of G on $X^{H\backslash G}$ called the *co-induced action* of $H \curvearrowright X$ to G. It is independent of the choice of section, in the sense that a different choice of section leads to a conjugate action.

It is clear that the co-induced action $G \curvearrowright X^{H\backslash G}$ is pmp whenever $H \curvearrowright X$ is pmp. In the case where G is a countable discrete group, the following well-known lemma gives sufficient conditions for when $G \curvearrowright X^{H\backslash G}$ is free and ergodic (even weakly mixing).

Lemma 3.8. *Let Γ be a countable discrete group with $\Lambda < \Gamma$ a subgroup of infinite index, and let $\Lambda \curvearrowright (X, \mu)$ be a pmp action on a standard probability space. Then the co-induced action $\Gamma \curvearrowright X^{\Lambda\backslash\Gamma}$ is weakly mixing. If the action $\Lambda \curvearrowright X$ is free, then $\Gamma \curvearrowright X^{\Lambda\backslash\Gamma}$ is also free.*

Proof. Let $A_1, \dots, A_n \subset X^{\Lambda\backslash\Gamma}$ be Borel sets and let $\varepsilon > 0$. We have to find a $g \in \Gamma$ such that

$$|\nu(A_i \cap gA_j) - \nu(A_i)\nu(A_j)| < \varepsilon, \quad i, j = 1, \dots, n, \tag{3.1}$$

where ν denotes the product measure on $X^{\Lambda\backslash\Gamma}$. We will first assume that the A_i's are product sets, i.e., $A_i = \prod_{t\in\Lambda\backslash\Gamma} A_i^t$ where $A_i^t = X$ for cofinitely many t. So, we have a finite subset $\mathcal{F} \subset \Lambda\backslash\Gamma$ such that $A_i^t = X$ for all $t \notin \mathcal{F}$ and all $i = 1, \dots, n$.

Note that $gA_j = \prod_{t\in\Lambda\backslash\Gamma} \Omega(t, g) \cdot A_j^{tg}$. Since $\Lambda\backslash\Gamma$ is infinite, there exists $g \in \Gamma$ such that $tg \notin \mathcal{F}$ for all $t \in \mathcal{F}$. Then $\nu(A_i \cap gA_j) = \nu(A_i)\nu(A_j)$ for all $i, j = 1, \dots, n$, proving (3.1) in the case where the A_i's are product sets. In the general case, we simply approximate the A_i's with product sets, so we conclude that the co-induced action $\Gamma \curvearrowright X^{\Lambda\backslash\Gamma}$ is weakly mixing.

Assume next that $\Lambda \curvearrowright X$ is free. Denote by $\mathrm{Fix}(g)$ the set of fixed points for $g \in \Gamma$, i.e., the set of $x \in X^{\Lambda\backslash\Gamma}$ such that $g \cdot x = x$. For $g \in \Gamma \setminus \{e\}$, we have

$$\mathrm{Fix}(g) = \{(x_t)_{t\in\Lambda\backslash\Gamma} \mid \Omega(t,g)\cdot x_{tg} = x_t \text{ for all } t\}$$

$$\subset \{(x_t)_{t\in\Lambda\backslash\Gamma} \mid x_{\Lambda g} = \Omega(\Lambda,g)^{-1}\cdot x_\Lambda\}.$$

If $g \in \Lambda$, then this is a null-set by freeness of the action $\Lambda \curvearrowright X$. If $g \notin \Lambda$, then this is a null-set by Fubini's Theorem. We conclude that the action $\Gamma \curvearrowright X^{\Lambda\backslash\Gamma}$ is free. □

We can also define co-induced actions in the more general setting of tracial von Neumann algebras. Assume that $\Lambda < \Gamma$ are countable groups and that $\Lambda \curvearrowright (A,\tau)$ is a trace-preserving action. Let $A^{\Lambda\backslash\Gamma} = \overline{\bigotimes}_{t\in\Lambda\backslash\Gamma}(A,\tau)$ denote the infinite tensor product and let $\pi_t \colon A \to A^{\Lambda\backslash\Gamma}$, $t \in \Lambda\backslash\Gamma$, denote the embedding as the t'th tensor factor. Then, the co-induced action $\Gamma \curvearrowright A^{\Lambda\backslash\Gamma}$ is defined by

$$g\cdot\pi_{\Lambda t}(a) = \pi_{\Lambda t g^{-1}}(\Omega(\Lambda t, g^{-1})^{-1}\cdot a), \quad a \in A,\ g \in \Gamma,$$

where $\Omega\colon \Lambda\backslash\Gamma \times \Gamma \to \Lambda$ denotes the cocycle defined above. As in the proof of Lemma 3.8, one sees that the co-induced action $\Gamma \curvearrowright A^{\Lambda\backslash\Gamma}$ is weakly mixing whenever Λ has infinite index in Γ.

3.2 Proof of Theorem 3.3

Let $\Lambda < \Gamma$ be an infinite abelian almost normal subgroup. Given a free ergodic pmp action $\Gamma \curvearrowright X$, we denote by $L^\infty(X)^\Lambda$ the subalgebra of Λ-invariant functions inside $L^\infty(X)$. Note that $A = L^\infty(X)^\Lambda \vee L(\Lambda)$ is an abelian subalgebra of $M = L^\infty(X) \rtimes \Gamma$. The following proposition gives a criterion for when $A \subset M$ is a MASA.

Proposition 3.9. *Let $\Lambda < \Gamma$ be an abelian almost normal subgroup and let $\Gamma \curvearrowright X$ be a free ergodic pmp action. Take a standard probability space (Y,ν) and a surjective Borel map $\pi\colon X \to Y$ such that $L^\infty(Y) \cong L^\infty(X)^\Lambda$ and such that the inclusion $\iota\colon L^\infty(Y) \hookrightarrow L^\infty(X)$ is given by $\iota(f)(x) = f(\pi(x))$.*

Assume that $\pi(g\cdot x) \neq \pi(x)$ for all $g \in \Gamma - \Lambda$ and for almost every $x \in X$. Then, $A := L^\infty(X)^\Lambda \vee L(\Lambda)$ is a MASA of $M := L^\infty(X) \rtimes \Gamma$.

Proof. Let $b \in A' \cap M$ and let $b = \sum_{g\in\Gamma} b_g u_g$ with $b_g \in L^\infty(X)$ be its Fourier decomposition. Then $f b_g = (g\cdot f) b_g$ for all $f \in L^\infty(X)^\Lambda$ and all $g \in \Gamma$. Hence $\mathbb{1}_{U_g} f = \mathbb{1}_{U_g}(g\cdot f)$ where U_g denotes the set $U_g = \{x \in X \mid b_g(x) \neq 0\}$. We need to show that $\mu(U_g) = 0$ for $g \notin \Lambda$. By our assumption that $\pi(g \cdot x) \neq \pi(x)$ almost everywhere, we may replace U_g with the set

$$\{x \in X \mid b_g(x) \neq 0,\ \pi(x) \neq \pi(g^{-1} \cdot x)\}.$$

Take a separating sequence of non-negligible Borel sets $(V_n)_{n\in\mathbb{N}} \subset Y$, i.e., for any $x \neq y$ in Y, there exists $n \in \mathbb{N}$ such that $x \in V_n$ and $y \notin V_n$. Put $\widetilde{V}_n = \pi^{-1}(V_n) \subset X$. Then $\widetilde{V}_n$ is a non-negligible and Λ-invariant subset of X. Moreover, for any $x \in U_g$, we can find $n \in \mathbb{N}$ such that $\pi(x) \in V_n$ and $\pi(g^{-1} \cdot x) \notin V_n$, i.e., $x \in \widetilde{V}_n \cap (g \cdot \widetilde{V}_n)^c$. Hence

$$U_g = \bigcup_{n\in\mathbb{N}} U_g \cap \widetilde{V}_n \cap (g \cdot \widetilde{V}_n)^c.$$

In order to show that $\mu(U_g) = 0$, it therefore suffices to show that $\mu(U_g \cap \widetilde{V}_n \cap (g \cdot \widetilde{V}_n)^c) = 0$ for all n. Above, we saw that $\mathbb{1}_{U_g} f = \mathbb{1}_{U_g}(g \cdot f)$ for all Λ-invariant functions $f \in L^\infty(X)^\Lambda$. Since $\mathbb{1}_{\widetilde{V}_n}$ is Λ-invariant for all n, it follows that $\mathbb{1}_{U_g\cap\widetilde{V}_n} = \mathbb{1}_{U_g\cap g\cdot\widetilde{V}_n}$. Hence, $\mu(U_g \cap \widetilde{V}_n \cap (g \cdot \widetilde{V}_n)^c) = 0$ for all n so that $\mu(U_g) = 0$.

We conclude that $b_g = 0$ for all $g \in \Gamma - \Lambda$. Moreover, b_g is Λ-invariant for all $g \in \Lambda$ since

$$\sum_{g\in\Lambda} b_g u_g = b = u_h b u_h^* = \sum_{g\in\Lambda} (h \cdot b_g) u_g, \quad h \in \Lambda.$$

Hence, $b \in A$ and thus $A' \cap M \subset A$. This means that A is maximal abelian. $\square$

We will prove the two statements (1) and (2) of Theorem 3.3 separately, using Proposition 3.9.

To prove (1), assume that $F := \bigcap_{g\in\Gamma} g\Lambda g^{-1}$ is infinite. By replacing Λ with F, we may assume that Λ is an infinite abelian *normal* subgroup of Γ. We will also assume that Λ has infinite index in Γ since otherwise, Γ would be an amenable group, which is already known to be non-$\mathcal{C}$-rigid.

Let K be a compact second countable abelian group into which Λ embeds densely and assume that the conjugation action $\alpha\colon \Gamma \curvearrowright \Lambda$, $\alpha_g(s) = gsg^{-1}$, extends to an action on K. For example, this can be obtained by considering the embedding $\iota\colon \Lambda \hookrightarrow \mathbb{T}^{\widehat{\Lambda}}$ given by $\iota(s) = (\omega(s))_{\omega\in\widehat{\Lambda}}$ and letting $K = \overline{\iota(\Lambda)}$. We will consider the action $\Gamma \curvearrowright K^{\Gamma/\Lambda}$ co-induced from the left translation action $\Lambda \curvearrowright K$. This action is free, ergodic and pmp by Lemma 3.8.

Put $X = K^{\Gamma/\Lambda}$. Since Λ is a normal subgroup of Γ, we can consider Λ as a subgroup of X via the embedding $\Lambda \hookrightarrow K^{\Gamma/\Lambda}$ given by $g \mapsto (tgt^{-1})_{t\in\Gamma/\Lambda}$.

Let $\overline{\Lambda}$ denote the closure of Λ inside X under this embedding. Note that $\overline{\Lambda} = \{x \in X \mid x_{\Lambda t} = \alpha_t(x_\Lambda) \text{ for all } t \in \Gamma\}$, where α denotes the extension of the conjugation action of Γ on Λ. Now, the restricted action $\Lambda \curvearrowright X$ is simply given by left multiplication of Λ, and the set of Λ-invariant functions in $L^\infty(X)$ is equal to $L^\infty(X/\overline{\Lambda})$.

In order to apply Proposition 3.9, we need to show that the quotient map $\pi\colon X \to X/\overline{\Lambda}$ satisfies $\pi(g \cdot x) \neq \pi(x)$ almost everywhere for all $g \in \Gamma - \Lambda$. So, for fixed $g \in \Gamma - \Lambda$, we need to show that

$$\mu(\{x \in X \mid (g \cdot x)x^{-1} \in \overline{\Lambda}\}) = 0.$$

Note that $(g \cdot x)x^{-1} \in \overline{\Lambda}$ if and only if

$$\Omega(\Lambda t, g)x_{\Lambda tg}(x_{\Lambda t})^{-1} = \alpha_t(\Omega(\Lambda, g)x_{\Lambda g}(x_\Lambda)^{-1}) \quad \text{for all } \ t \in \Gamma.$$

By Fubini's theorem, it follows that $\{x \in X \mid (g \cdot x)x^{-1} \in \overline{\Lambda}\}$ is indeed a null-set.

Using Proposition 3.9, we conclude that $A := L^\infty(X)^\Lambda \vee L(\Lambda)$ is a MASA in $M := L^\infty(X) \rtimes \Gamma$.

Next, we will show that $A \subset M$ is regular. Since Λ is a normal subgroup of Γ, we clearly have that u_g normalizes A for all $g \in \Gamma$. Moreover, $L^\infty(X)$ is generated by the characters on the group $X = K^{\Gamma/\Lambda}$ so it is enough to check that these normalize A. When ω is a character on $K^{\Gamma/\Lambda}$, we have that $h \cdot \omega = \omega(h^{-1})\omega$ for all $h \in \Lambda$, since Λ acts by left multiplication on $K^{\Gamma/\Lambda}$. Hence,

$$\omega u_h \omega^* = \omega(h \cdot \omega^*)u_h = \omega(h)u_h \in A \quad \text{for all } \ h \in \Lambda.$$

We conclude that A is a Cartan subalgebra.

That A is not unitarily conjugate to the canonical Cartan subalgebra $L^\infty(X)$ follows from the fact that Λ is infinite. Indeed, let $h_n \in \Lambda$ be a sequence tending to infinity. For fixed $g \in \Gamma$, we then have $E_{L^\infty(X)}(u_{h_n}u_g) = 0$ for n so large that h_n escapes $\{g^{-1}\}$. Thus it follows from Popa's intertwining-by-bimodules theorem, Theorem 2.12, that $A \not\prec L^\infty(X)$. Since A and $L^\infty(X)$ are Cartan subalgebras, this means that they are not unitarily conjugate, by Theorem 2.13.

This finishes the proof of Theorem 3.3 (1).

To prove (2), assume instead that $F = \bigcap_{g\in\Gamma} g\Lambda g^{-1}$ is finite. Again, we will assume that Λ has infinite index in Γ. In order to construct an action $\Gamma/F \curvearrowright X$ satisfying the assumption of Proposition 3.9, we will work with the relative profinite completion of Γ with respect to Λ. This is a construction introduced in [Sc80] that generalizes the notion of a quotient group by a normal subgroup to the case of an almost normal subgroup. We start by recalling this construction.

Denote by $\mathrm{Sym}(\Lambda\backslash\Gamma)$ the group of all permutations of the countable set $\Lambda\backslash\Gamma$ endowed with the topology of pointwise convergence. We have a homomorphism $\pi\colon \Gamma \to \mathrm{Sym}(\Lambda\backslash\Gamma)$ induced by right multiplication on $\Lambda\backslash\Gamma$. Note that $\ker(\pi) = \bigcap_{g\in\Gamma} g\Lambda g^{-1} = F$. Let G denote the closure of $\pi(\Gamma)$ inside $\mathrm{Sym}(\Lambda\backslash\Gamma)$ and let K denote the closure of $\pi(\Lambda)$. Then G is a locally compact group called *the relative profinite completion* of Γ with respect to Λ. Note that if Λ is a normal subgroup of Γ, then $\ker\pi = \Lambda$ so that $K = \{e\}$ and $G \cong \Gamma/\Lambda$. However, since we assumed that F is finite, we have that $\ker\pi \neq \Lambda$ so that K is nontrivial.

Lemma 3.10 ([Sc80])**.** *Let $K < G$ be as above. Then K is a compact, open, almost normal subgroup of G. Moreover, the map $\Lambda\backslash\Gamma \to K\backslash G$ given by $\Lambda g \mapsto K\pi(g)$ is a bijection.*

Proof. Let $\lambda_n \in \Lambda$ be any sequence and let $\{x_1, \dots, x_k\} \subset \Lambda\backslash\Gamma$ be a finite set. Since the right action $\Lambda \curvearrowright \Lambda\backslash\Gamma$ has finite orbits, the set $\{\pi(\lambda_n)x_1\}_{n\in\mathbb{N}}$ is finite. So, we can choose $y_1 \in \Lambda\backslash\Gamma$ such that $N_1 := \{n \in \mathbb{N} \mid \pi(\lambda_n)x_1 = y_1\}$ is an infinite set. Inductively, we get $y_1, \dots, y_k \in \Lambda\backslash\Gamma$ and an infinite set $N_k \subset \mathbb{N}$ such that $\pi(\lambda_n)x_i = y_i$ for all $i = 1, \dots, k$ and all $n \in N_k$. This means that the sequence $(\pi(\lambda_n))_{n\in\mathbb{N}}$ has an accumulation point. We conclude that $\pi(\Lambda)$ is precompact and hence K is compact.

Next, note that for $\varphi \in G$, we have that $\varphi \in K$ if and only if $\varphi(\Lambda) = \Lambda$. Hence K is open inside G. It now follows that $K \cap gKg^{-1}$ is an open subgroup of the compact group K for all $g \in G$, hence of finite index. This means that K is almost normal in G.

Finally, we show that the map $\Lambda\backslash\Gamma \to K\backslash G\colon \Lambda g \mapsto K\pi(g)$ is bijective. Assume that $K\pi(g) = K\pi(h)$ for $g, h \in \Gamma$. Then $\pi(gh^{-1})(\Lambda) = \Lambda$ so that $\Lambda g = \Lambda h$. Thus, the map $\Lambda g \mapsto K\pi(g)$ is injective. To show surjectivity, let $\varphi \in G$ be arbitrary and take a sequence $(g_n)_{n\in\mathbb{N}} \subset \Gamma$ such that $\pi(g_n) \to \varphi$. Since K is open, we have that $\pi(g_n)\varphi^{-1} \in K$ for n large enough. But this means that $K\varphi = K\pi(g_n)$. □

We will consider the action of Γ on $X := K^{\Lambda\backslash\Gamma}$ co-induced from the translation action $\Lambda \curvearrowright K$ given by $\lambda \cdot k = \pi(\lambda)k$, $\lambda \in \Lambda$, $k \in K$. This is an ergodic pmp action by Lemma 3.8, but not necessarily free since π need not be injective. However, as in the proof of Lemma 3.8, we see that if $\mathrm{Fix}(g)$ has positive measure, then g must belong to $\ker\pi = F$. So, by taking the quotient with the finite normal subgroup F, we get a free ergodic pmp action $\Gamma/F \curvearrowright X$. Note that Γ/F still has an infinite abelian almost normal subgroup, namely Λ/F. So, we may replace Γ with Γ/F and assume that F is trivial so that $\Gamma \curvearrowright X$ is a free ergodic pmp action.

Put $M = L^\infty(X) \rtimes \Gamma$ and let $A \subset M$ be the subalgebra generated by $L(\Lambda)$ and the Λ-invariant functions $L^\infty(X)^\Lambda$, as in Proposition 3.9. We will show that A is a Cartan subalgebra of M that is not unitarily conjugate with $L^\infty(X)$.

In order to show that A is a Cartan subalgebra, we will also consider the action $G \curvearrowright K^{K\backslash G}$ co-induced from the left translation action $K \curvearrowright K$. Since $\Lambda\backslash\Gamma = K\backslash G$ via the bijection $\Lambda g \mapsto K\pi(g)$, we get that the action of Γ on $K^{\Lambda\backslash\Gamma}$ is just the composition of $\pi\colon \Gamma \to G$ and the co-induced action $G \curvearrowright K^{K\backslash G}$, i.e.,

$$g \cdot x = \pi(g) \cdot x \quad \text{for } \ g \in \Gamma,\ x \in K^{\Lambda\backslash\Gamma} = K^{K\backslash G}.$$

We first show that the co-induced action $G \curvearrowright K^{K\backslash G}$ is *free*, in the sense that $\{x \in K^{K\backslash G} \mid \mathrm{Stab}_G(x) = \{e\}\}$ has measure 1. Here, $\mathrm{Stab}_G(x)$ denotes the stabilizer subgroup of G, i.e.,

$$\mathrm{Stab}_G(x) := \{g \in G \mid g \cdot x = x\}.$$

Lemma 3.11. *The action $G \curvearrowright K^{K\backslash G}$, co-induced from the translation action $K \curvearrowright K$, is free.*

Proof. Let $\theta\colon K\backslash G \to G$ be a section with $\theta(K) = e$ and let $r\colon G \to K$ be the unique map satisfying $g = r(g)\theta(Kg)$. Let $\Omega\colon K\backslash G \times G \to K$ be the associated 1-cocycle. Recall that the co-induced action is defined by the formula

$$(g \cdot x)_{Kt} = \Omega(Kt, g) \cdot x_{Ktg}, \qquad g \in G,\ x = (x_{Kt})_{t \in K\backslash G} \in K^{K\backslash G}.$$

Clearly, $K \curvearrowright K^{K\backslash G}$ is free since $(k \cdot x)_K = kx_K$ for $k \in K$, $x \in K^{K\backslash G}$. We proceed by showing that the action is free on every right coset, i.e., that the set $U_g := \{x \in K^{K\backslash G} \mid g \cdot x \in K \cdot x\}$ has measure zero for all $g \in G - K$.

Fix $g \in G - K$. First, note that if $g \cdot x = k \cdot x$ for some $k \in K$, then k is uniquely determined from g and x by the formula $k = (g \cdot x)_K x_K^{-1} = r(g) x_{Kg} x_K^{-1}$. Hence

$$U_g = \{x \in K^{K\backslash G} \mid x_K x_{Kg}^{-1} r(g)^{-1} g \cdot x = x\}.$$

To show that this set has measure zero, we will consider the sets

$$U_{k,\ell} = \{x \in K^{K\backslash G} \mid x_K = k,\ x_{Kg} = \ell \ \text{ and } \ k\ell^{-1} r(g)^{-1} g \cdot x = x\}$$

for fixed $k, \ell \in K$. To show that U_g has measure zero, it is by Fubini's Theorem enough to show that $U_{k,\ell}$ has measure zero for all $k, \ell \in K$, when considered as a subset of $K^{K\backslash G - \{K, Kg\}}$.

Fix $k, \ell \in K$ and put $h = k\ell^{-1} r(g)^{-1} g$. Note that $Kh = Kg$. If $x \in U_{k,\ell}$, then

$$x_{Kt} = (h \cdot x)_{Kt} = \Omega(Kt, h) x_{Kth} \qquad \text{for all } \ t \in G.$$

In particular, $x_{Kh^2} = \Omega(Kh,h)^{-1}\ell$ which means that the Kh^2 coordinate is completely determined for all $x \in U_{k,\ell}$. Thus if $Kh^2 \notin \{K, Kg\}$, we get that $U_{k,\ell} \subset K^{K\backslash G - \{K,Kg\}}$ has measure zero as wanted.

It remains to check what happens if $Kh^2 \in \{K, Kg\}$. Clearly, we cannot have that $Kh^2 = Kh$ since $h \notin K$, so we assume that $h^2 \in K$. Fix $t \in G$ such that $Kt \notin \{K, Kg\}$. Then also $Kth \notin \{K, Kg\}$ since $Kg = Kh = Kh^{-1}$. Note that $U_{k,\ell} \subset \{x \mid x_{Kt} = \Omega(Kt,h)x_{Kth}\}$. This is a null-set in $K^{K\backslash G-\{K,Kg\}}$ since both Kth and Kt do not belong to $\{K, Kg\}$. If $Kt = Kth$, then $\Omega(Kt,h) \neq e$, so that $U_{k,\ell}$ is even empty in this case.

We conclude that U_g is indeed a null-set. To finish the proof, we simply note that

$$\{x \in K^{K\backslash G} \mid \mathrm{Stab}_G(x) \neq \{e\}\} = \bigcup_{Kg \in K\backslash G} \{x \mid \mathrm{Stab}_G(x) \cap Kg \neq \{e\}\}$$
$$= \bigcup_{Kg \in K\backslash G} U_g.$$

Since $K\backslash G$ is countable, we conclude that the action $G \curvearrowright K^{K\backslash G}$ is free. □

The following folklore lemma is a key ingredient in our argument. It states that we can identify the action $K \curvearrowright K^{K\backslash G}$ with a translation action $K \curvearrowright K \times Y$. A complete proof can be found in [MRV13].

Lemma 3.12 ([MRV13, Lemma 10]). *Let K be a compact second countable group and let $K \curvearrowright (X, \mu)$ be a free pmp action on a standard probability space (X, μ). Denote by m the Haar measure on K. There exists a standard probability space (Y, η) and a Borel isomorphism*

$$\theta \colon K \times Y \to \{x \in X \mid \mathrm{Stab}_K(x) = \{e\}\},$$

such that $\theta_*(m \times \eta) = \mu$ and such that $\theta(kh, y) = k \cdot \theta(h, y)$ for all $k, h \in K$, $y \in Y$.

Using this identification, we can identify the Λ-invariant functions of $L^\infty(K^{K\backslash G})$ with $1_K \otimes L^\infty(Y)$. Denote by $p_Y \colon K \times Y \to Y$ the projection onto the Y-coordinate. In order to apply Proposition 3.9, we need to show that the map $\rho := p_Y \circ \theta^{-1}$ satisfies $\rho(g \cdot x) \neq \rho(x)$ for all $g \in \Gamma - \Lambda$ and almost all $x \in X$.

Fix $g \in \Gamma - \Lambda$ and put $X_0 = \{x \in X \mid \mathrm{Stab}_K(x) = \{e\}\}$. Given $x \in X_0$, write $\theta^{-1}(x) = (k, y)$ and $\theta^{-1}(g \cdot x) = (\tilde{k}, \tilde{y})$ with $k, \tilde{k} \in K$ and $y, \tilde{y} \in Y$. If $y = \tilde{y}$, then $k\tilde{k}^{-1}\pi(g^{-1}) \cdot x = x$. Since $g \notin \Lambda$, this implies that $\mathrm{stab}_G(x) \neq \{e\}$. So, we have shown that

$$\{x \in X_0 \mid \rho(g \cdot x) = \rho(x)\} \subset \{x \in X \mid \mathrm{stab}_G(x) \neq \{e\}\}.$$

Since the action $G \curvearrowright X$ is free, by Lemma 3.11, it follows that $\{x \in X \mid \rho(g \cdot x) = \rho(x)\}$ is a null-set. So, $\Gamma \curvearrowright X$ satisfies the assumption in Proposition 3.9 and we conclude that $A \subset M$ is a MASA.

Next, we show that $A \subset M$ is regular. Since A is a MASA, it is enough to show that the quasi-normalizer of A generates M, by [PS03, Theorem 2.7].

The *quasi-normalizer* $\mathcal{QN}_M(A)$ of A inside M is defined to be the set of elements $x \in M$ for which there exist finitely many elements $x_1, \dots, x_n, y_1, \dots, y_m \in M$ such that

$$xA \subset \sum_{i=1}^{n} Ax_i \quad \text{and} \quad Ax \subset \sum_{j=1}^{m} y_j A.$$

The fact that Λ is an almost normal subgroup of Γ easily implies that $u_g \in \mathcal{QN}_M(A)$, as shown in the following lemma.

Lemma 3.13. *For every $g \in \Gamma$, we have that $u_g \in \mathcal{QN}_M(A)$.*

Proof. Fix $g \in \Gamma$. Since Λ is almost normal, there exist finitely many elements $g_1, \dots, g_n \in \Gamma$ such that $g\Lambda \subset \bigcup_{i=1}^n \Lambda g_i$. Then

$$u_g A \subset \sum_{i=1}^{n} A u_{g_i} L^\infty(X)^\Lambda = \sum_{i=1}^{n} A L^\infty(X)^{g_i \Lambda g_i^{-1}} u_{g_i}.$$

By Lemma 3.14, we also have $(f_j)_{j=1}^m \subset L^\infty(X)$ such that

$$L^\infty(X)^{g_i \Lambda g_i^{-1} \cap \Lambda} \subset \sum_{j=1}^{m} L^\infty(X)^\Lambda f_j \quad \text{for all } \ i = 1, \dots, n.$$

It follows that we have a finite set of elements $x_1, \dots, x_N \in M$ such that $u_g A \subset \sum_{i=1}^N Ax_i$. Similarly, one finds a finite set of elements $y_1, \dots, y_k \in M$ such that $Au_g \subset \sum_{j=1}^k y_j A$. □

Lemma 3.14. *Let $\Gamma \curvearrowright (X, \mu)$ be a free pmp action of a countable group Γ on a standard probability space X. Assume that $\Lambda < \Gamma$ is a normal subgroup of finite index. Then $L^\infty(X)^\Gamma$ has finite index in $L^\infty(X)^\Lambda$, in the sense that there exist finitely many functions $(f_i)_{i=1}^n \subset L^\infty(X)^\Lambda$ such that*

$$L^\infty(X)^\Lambda \subset \sum_{i=1}^{n} L^\infty(X)^\Gamma f_i.$$

Proof. Since Λ is a normal subgroup of Γ, we have an action of Γ/Λ on $L^\infty(X)^\Lambda$. Write $L^\infty(X)^\Lambda = L^\infty(Y)$ for some standard probability space Y. Since Γ/Λ is finite, we can partition Y into finitely many Borel sets $Y_1, \dots, Y_n \subset Y$ such that each Y_i contains exactly one element from each orbit of the action $\Gamma/\Lambda \curvearrowright Y$.

We claim that the functions $f_i = \mathbb{1}_{Y_i} \in L^\infty(Y)$, $i = 1, \dots, n$, will do the job. Indeed, for any Borel subset $B \subset Y$, we have that

$$\mathbb{1}_B = \sum_{i=1}^n \mathbb{1}_{B\cap Y_i} = \sum_{i=1}^n \mathbb{1}_{\Gamma\cdot(B\cap Y_i)\cap Y_i} \in \sum_{i=1}^n L^\infty(Y)^\Gamma f_i.$$

Since $L^\infty(Y) \cong L^\infty(X)^\Lambda$ and $L^\infty(Y)^\Gamma \cong L^\infty(X)^\Gamma$, this finishes the proof. □

Lemma 3.15. *The subalgebra $A \subset M$ is regular.*

Proof. Since $A \subset M$ is a MASA, we have that $\mathcal{N}_M(A)'' = \mathcal{QN}_M(A)''$ by [PS03, Theorem 2.7]. It follows from Lemma 3.13 that $u_g \in \mathcal{N}_M(A)''$.

Using the identification $K^{\Lambda\backslash\Gamma} \cong K \times Y$ from Lemma 3.12 and using the fact that $L^\infty(K)$ is generated by the characters on K, we see that $L^\infty(K^{\Lambda\backslash\Gamma}) \cong L^\infty(K \times Y)$ is generated by the unitaries ω of the form

$$\omega(k, y) = \chi(k) f(y), \quad k \in K,\ y \in Y,$$

where χ is a character on K and f is a unitary in $L^\infty(Y)$. These unitaries normalize A since

$$\omega u_h \omega^* = \omega(h \cdot \omega^*) u_h = \chi(h) u_h \quad \text{for all} \ \ h \in \Lambda.$$

So, we have that $L^\infty(K^{\Lambda\backslash\Gamma}) \subset \mathcal{N}_M(A)''$. We conclude that $\mathcal{N}_M(A)'' = M$. □

We have now shown that $A \subset M$ is a Cartan subalgebra. Exactly as in the proof of (1), we see that A is not unitarily conjugate with $L^\infty(X)$. This finishes the proof of Theorem 3.3.

Chapter 4

A class of II_1 factors with exactly two group measure space decompositions

This chapter is based on my joint publication with Stefaan Vaes [KV15], in which we construct examples of II_1 factors with a prescribed number of group measure space decompositions. As stated in the introduction, we will here prove Theorem A.

Theorem A. (1) *For every integer $n \geq 0$, there exist II_1 factors M that have exactly 2^n group measure space Cartan subalgebras up to unitary conjugacy.*

(2) *For every integer $n \geq 1$, there exist II_1 factors M that have exactly n group measure space Cartan subalgebras up to conjugacy by an automorphism of M.*

The II_1 factors M in Theorem A are concretely constructed as follows. Let Γ be any torsion-free nonelementary hyperbolic group and let $\beta\colon \Gamma \curvearrowright (A_0, \tau_0)$ be any trace-preserving action on the amenable von Neumann algebra (A_0, τ_0) with $A_0 \neq \mathbb{C}1$ and $\operatorname{Ker}\beta \underset{\neq}{\subset} \{e\}$. We denote by $(A_0, \tau_0)^\Gamma$, or simply A_0^Γ, the infinite tensor product $\overline{\bigotimes}_\Gamma (A_0, \tau_0)$ as defined in Section 2.1.6. We then define $(A, \tau) = (A_0, \tau_0)^\Gamma$ and consider the action $\sigma\colon \Gamma \times \Gamma \curvearrowright (A, \tau)$ given by $\sigma_{(g,h)}(\pi_k(a)) = \pi_{gkh^{-1}}(\beta_h(a))$ for all $g, h, k \in \Gamma$ and $a \in A_0$, where $\pi_k\colon A_0 \to A$ denotes the embedding as the k'th tensor factor. Note that σ can be seen as the co-induced action of $\beta\colon \Gamma \curvearrowright A_0$ to $\Gamma \times \Gamma$, when viewing Γ as the diagonal subgroup of $\Gamma \times \Gamma$.

Our main result describes exactly all group measure space Cartan subalgebras of the crossed product $M = A_0^\Gamma \rtimes (\Gamma \times \Gamma)$.

Theorem 4.1. *Let $M = A_0^\Gamma \rtimes (\Gamma \times \Gamma)$ be as above. Up to unitary conjugacy, all group measure space Cartan subalgebras $B \subset M$ are of the form $B = B_0^\Gamma$ where $B_0 \subset A_0$ is a group measure space Cartan subalgebra of A_0 with the following two properties: $\beta_g(B_0) = B_0$ for all $g \in \Gamma$ and A_0 can be decomposed as $A_0 = B_0 \rtimes \Lambda_0$ with $\beta_g(\Lambda_0) = \Lambda_0$ for all $g \in \Gamma$.*

In Section 4.4, we actually prove a more general and more precise result, see Theorem 4.7. In Section 4.5, we give concrete examples and computations, thus proving Theorem A (see Theorem 4.24).

Our method relies on a technique of [PV09], using the so-called dual coaction that is associated to a group measure space decomposition. Given a II_1 factor M as in Theorem 4.1 and an arbitrary group measure space decomposition $M = B \rtimes \Lambda$, we can associate a *dual coaction*, i.e., a normal $*$-homomorphism $\Delta\colon M \to M \mathbin{\overline{\otimes}} M$ given by $\Delta(bv_s) = bv_s \otimes v_s$ for all $b \in B$, $s \in \Lambda$. By classifying all possible such embeddings $\Delta\colon M \to M \mathbin{\overline{\otimes}} M$ in terms of the initial structure of $M = A_0^\Gamma \rtimes (\Gamma \times \Gamma)$, we are able to relate the "mysterious" decomposition $M = B \rtimes \Lambda$ with the "original" decomposition $M = A_0^\Gamma \rtimes (\Gamma \times \Gamma)$. This allows us to classify all possible group measure space decompositions $M = B \rtimes \Lambda$.

Let us give a brief outline of the proof of Theorem 4.1. Given a II_1 factor M as in Theorem 4.1 and given the dual coaction $\Delta\colon M \to M \mathbin{\overline{\otimes}} M$ associated with an arbitrary group measure space decomposition $M = B \rtimes \Lambda$, Popa's key methods of malleability [Po03] and spectral gap rigidity [Po06b] for Bernoulli actions allow to prove that $\Delta(L(\Gamma \times \Gamma))$ can be unitarily conjugated into $M \mathbin{\overline{\otimes}} L(\Gamma \times \Gamma)$. An ultrapower technique of [Io11], in combination with the transfer-of-rigidity principle of [PV09], then shows that the "mysterious" group Λ must contain two commuting non-amenable subgroups Λ_1, Λ_2. Note here that the same combination of [Io11] and [PV09] was used in [CdSS15] to prove that if Γ_1, Γ_2 are nonelementary hyperbolic groups and $L(\Gamma_1 \times \Gamma_2) \cong L(\Lambda)$, then Λ must be a direct product of two non-amenable groups. Once we know that Λ contains two commuting non-amenable subgroups Λ_1, Λ_2, we use a combination of methods from [Io10] and [IPV10] to prove that $\Lambda_1\Lambda_2 \subset \Gamma \times \Gamma$. From that point on, it is not so hard any more to entirely unravel the structure of B and Λ. Throughout these arguments, we repeatedly use the crucial dichotomy theorem of [PV11, PV12] saying that hyperbolic groups Γ are relatively strongly solid: in arbitrary tracial crossed products $M = P \rtimes \Gamma$, if a von Neumann subalgebra $Q \subset M$ is amenable relative to P, then either Q embeds into P, or the normalizer of Q stays amenable relative to P.

4.1 Properties of the dual coaction

Let $M = B \rtimes \Lambda$ be any tracial crossed product von Neumann algebra and denote by $\{v_s\}_{s\in\Lambda}$ the canonical unitaries. To such a crossed product, we can associate a normal $*$-homomorphism $\Delta\colon M \to M \mathbin{\overline{\otimes}} M$ called the *dual coaction*, defined by $\Delta(bv_s) = bv_s \otimes v_s$ for all $b \in B$ and $s \in \Lambda$.

The following elementary lemma states that the only subalgebras $A \subset B \rtimes \Lambda$ that are invariant under Δ are the ones coming from the crossed product decomposition.

Lemma 4.2. *A von Neumann subalgebra $A \subset B \rtimes \Lambda$ satisfies $\Delta(A) \subset A \mathbin{\overline{\otimes}} A$ if and only if $A = B_0 \rtimes \Lambda_0$ for some von Neumann subalgebra $B_0 \subset B$ and some subgroup $\Lambda_0 < \Lambda$ that leaves B_0 globally invariant.*

Proof. Let $a \in A$ and write $a = \sum_{s\in\Lambda} a_s v_s$ with $a_s \in B$. Fix $s \in \Lambda$ such that $a_s \neq 0$ and define the normal linear functional ω on $B \rtimes \Lambda$ by $\omega(x) = \tau(xv_s^* a_s^*)$. Then $(\omega \otimes 1)\Delta(a) = \|a_s\|_2^2\, v_s$. Since $\Delta(a) \in A \mathbin{\overline{\otimes}} A$, it follows that $v_s \in A$. Similarly, we define a linear functional ρ on $B \rtimes \Lambda$ by $\rho(x) = \tau(xv_s^*)$. Then $(1 \otimes \rho)\Delta(a) = a_s v_s \in A$ and it follows that $a_s \in A$. Since this holds for all s, we conclude that $A = B_0 \rtimes \Lambda_0$ where $B_0 = A \cap B$ and $\Lambda_0 = \{s \in \Lambda \mid v_s \in A\}$. □

The proof of the next result is almost identical to the proof of [IPV10, Lemma 7.2(4)]. For the convenience of the reader, we provide all details.

Proposition 4.3. *Assume that (B, τ) is amenable. If $Q \subset M$ is a von Neumann subalgebra without amenable direct summand, then $\Delta(Q)$ is strongly non-amenable relative to $M \otimes 1$.*

Proof. We first prove that the $(M \mathbin{\overline{\otimes}} M)$-$M$-bimodule

$$_{M\overline{\otimes}M}\big(L^2(M \mathbin{\overline{\otimes}} M) \underset{M\otimes 1}{\otimes} L^2(M \mathbin{\overline{\otimes}} M)\big)_{\Delta(M)}$$

is weakly contained in the coarse $(M \mathbin{\overline{\otimes}} M)$-$M$-bimodule. Denoting by $\sigma\colon M \mathbin{\overline{\otimes}} M \to M \mathbin{\overline{\otimes}} M$ the flip automorphism, this bimodule is isomorphic with the $(M \mathbin{\overline{\otimes}} M)$-$M$-bimodule

$$_{M\overline{\otimes}M}L^2(M \mathbin{\overline{\otimes}} M \mathbin{\overline{\otimes}} M)_{(\mathrm{id}\otimes\sigma)(\Delta(M)\otimes 1)}\ .$$

So, it suffices to prove that the M-M-bimodule $_{M\otimes 1}L^2(M \mathbin{\overline{\otimes}} M)_{\Delta(M)}$ is weakly contained in the coarse M-M-bimodule. Noting that this M-M-bimodule is isomorphic with a multiple of the M-M-bimodule $_M\big(L^2(M) \otimes_B L^2(M)\big)_M$, the result follows from the amenability of B.

Assume now that $\Delta(Q)q$ is amenable relative to $M \otimes 1$ for some nonzero projection $q \in \Delta(Q)' \cap M \mathbin{\overline{\otimes}} M$. By the bimodule characterization of relative amenability (see Proposition 2.11), this means that $qL^2(M \mathbin{\overline{\otimes}} M)q$ is weakly contained in $qL^2(M \mathbin{\overline{\otimes}} M) \otimes_{M\otimes 1} L^2(M \mathbin{\overline{\otimes}} M)q$ as $q(M \mathbin{\overline{\otimes}} M)q$-$\Delta(Q)q$-bimodules.

Take $z \in Q$ such that $\Delta(z)$ is the support projection of $E_{\Delta(Q)}(q)$. Then z is a nonzero central projection in Q. We will prove that Qz is amenable and thereby reach a contradiction.

Clearly, Δ embeds the trivial Qz-bimodule into the $\Delta(Q)$-bimodule $L^2(\Delta(Qz))$. By Lemma 2.9 (2), it follows that the trivial Qz-bimodule is contained in

$$_{\Delta(Q)} qL^2(M \mathbin{\overline{\otimes}} M)q_{\Delta(Q)}.$$

By our relative amenability assumption and the fact that $q \leq \Delta(z)$, it follows that the trivial Qz-bimodule is weakly contained in

$$_{\Delta(Q)}(\Delta(z)L^2(M \mathbin{\overline{\otimes}} M) \otimes_{M\otimes 1} L^2(M \mathbin{\overline{\otimes}} M)\Delta(z))_{\Delta(Q)}.$$

By the first part of the proof, this bimodule is weakly contained in

$$_{\Delta(Q)}(\Delta(z)L^2(M \mathbin{\overline{\otimes}} M) \otimes L^2(M)z)_Q,$$

which is weakly contained in the coarse Qz-bimodule by Lemma 2.9 (1).

We conclude that the trivial Qz-bimodule is weakly contained in the coarse Qz-bimodule and hence Qz is amenable. □

4.2 Transfer of rigidity

Fix a trace-preserving action $\Lambda \curvearrowright (B,\tau)$ of a countable discrete group Λ and put $M = B \rtimes \Lambda$. We denote by $\{v_s\}_{s\in\Lambda}$ the canonical unitaries in $L(\Lambda) \subset M$. Whenever $\mathcal{G}$ is a family of subgroups of Λ, we say that a subset $F \subset \Lambda$ is *small relative to* $\mathcal{G}$ if F is contained in a finite union of subsets of the form $g\Sigma h$ where $g, h \in \Lambda$ and $\Sigma \in \mathcal{G}$ (see [BO08, Definition 15.1.1]).

Following the transfer of rigidity principle from [PV09, Section 3], we prove the following theorem.

Theorem 4.4. *Let $\Lambda \curvearrowright (B,\tau)$ be a trace-preserving action and put $M = B \rtimes \Lambda$. Let $\Delta\colon M \to M \mathbin{\overline{\otimes}} M$ be the dual coaction given by $\Delta(bv_s) = bv_s \otimes v_s$ for $b \in B, s \in \Lambda$. Let $\mathcal{G}$ be a family of subgroups of Λ. Let $P, Q \subset M$ be two von Neumann subalgebras satisfying*

(1) $\Delta(P) \prec_{M\overline{\otimes}M} M \mathbin{\overline{\otimes}} Q$,

(2) $P \not\prec_M B \rtimes \Sigma$ *for all* $\Sigma \in \mathcal{G}$.

Then there exists a finite set $x_1, \ldots, x_n \in M$ *and a* $\delta > 0$ *such that the following holds: whenever* $F \subset \Lambda$ *is small relative to* $\mathcal{G}$, *there exists an element* $s_F \in \Lambda - F$ *such that*

$$\sum_{i,k=1}^{n} \| E_Q\left(x_i v_{s_F} x_k^*\right) \|_2^2 \geq \delta \ .$$

Proof. Since $\Delta(P) \prec_{M \overline{\otimes} M} M \overline{\otimes} Q$, we can find a finite set $x_1, \ldots, x_n \in M$ and $\rho > 0$ such that

$$\sum_{i,k=1}^{n} \| E_{M \overline{\otimes} Q}\left((1 \otimes x_i)\Delta(w)(1 \otimes x_k^*)\right) \|_2^2 \geq \rho \quad \text{for all} \ \ w \in \mathcal{U}(P) \ .$$

Given a subset $F \subset \Lambda$, we denote by P_F the orthogonal projection of $L^2(M)$ onto the closed linear span of $\{bv_s \mid b \in B, s \in F\}$. Since $P \not\prec_M B \rtimes \Sigma$ for all $\Sigma \in \mathcal{G}$, it follows from [Va10, Lemma 2.4] that there exists a net of unitaries $(w_j)_{j \in J} \subset \mathcal{U}(P)$ such that $\|P_F(w_j)\|_2 \to 0$ for any set $F \subset \Lambda$ that is small relative to $\mathcal{G}$. For each $j \in J$, write $w_j = \sum_{s \in \Lambda} w_s^j v_s$ with $w_s^j \in B$ and compute

$$\begin{aligned}\sum_{i,k=1}^{n} \| E_{M \overline{\otimes} Q}\left((1 \otimes x_i)\Delta(w_j)(1 \otimes x_k^*)\right) \|_2^2 &= \sum_{i,k=1}^{n} \left\| \sum_{s \in \Lambda} w_s^j v_s \otimes E_Q(x_i v_s x_k^*) \right\|_2^2 \\ &= \sum_{i,k=1}^{n} \sum_{s \in \Lambda} \|w_s^j\|_2^2 \, \|E_Q(x_i v_s x_k^*)\|_2^2 \ .\end{aligned}$$

We now claim that the conclusion of the theorem holds with $\delta = \frac{\rho}{2}$. Indeed, assume for contradiction that there exists a subset $F \subset \Lambda$ that is small relative to $\mathcal{G}$ such that

$$\sum_{i,k=1}^{n} \| E_Q\left(x_i v_s x_k^*\right) \|_2^2 < \delta \quad \text{for all} \ \ s \in \Lambda - F \ .$$

Put $K = \max\{\|x_i\|^2 \|x_k^*\|_2^2 \mid i,k = 1, \ldots, n\}$ and choose $j_0 \in J$ such that $\|P_F(w_j)\|_2^2 = \sum_{s \in F} \|w_s^j\|_2^2 < \frac{\rho}{4Kn^2}$ for all $j \geq j_0$. We then get for $j \geq j_0$

$$\begin{aligned}\rho &\leq \sum_{s \in \Lambda} \sum_{i,k=1}^{n} \|w_s^j\|_2^2 \, \|E_Q(x_i v_s x_k^*)\|_2^2 \\ &\leq n^2 K \sum_{s \in F} \|w_s^j\|_2^2 + \sum_{s \in \Lambda - F} \|w_s^j\|_2^2 \, \delta < \frac{\rho}{4} + \frac{\rho}{2} \ ,\end{aligned}$$

which is a contradiction. ☐

4.3 Embeddings of group von Neumann algebras

Following [Io10, Section 4] and [IPV10, Section 3], we define the *height* h_Γ of an element in a group von Neumann algebra $L(\Gamma)$ as the absolute value of the largest Fourier coefficient, i.e.,

$$h_\Gamma(x) = \sup_{g\in\Gamma} |\tau(xu_g^*)| \quad \text{for } \ x \in L(\Gamma) \ .$$

Whenever $\mathcal{G} \subset L(\Gamma)$, we write

$$h_\Gamma(\mathcal{G}) = \inf\{h_\Gamma(x) \mid x \in \mathcal{G}\} \ .$$

In the following, we will view Γ as a subgroup of $\mathcal{U}(L(\Gamma))$ by identifying Γ with the canonical unitaries $\{u_g\}_{g\in\Gamma} \subset L(\Gamma)$. Note that $h_\Gamma(\Gamma) = 1$.

When Γ is an icc group and Λ is a countable group such that $L(\Lambda) = L(\Gamma)$ with $h_\Gamma(\Lambda) > 0$, it was proven in [IPV10, Theorem 3.1] that there exists a unitary $u \in L(\Gamma)$ such that $u\mathbb{T}\Lambda u^* = \mathbb{T}\Gamma$. We need the following generalization. For this, recall that a unitary representation is said to be *weakly mixing* if $\{0\}$ is the only finite-dimensional subrepresentation. Also recall the equivalent characterizations of weak mixing from Proposition 2.17.

Theorem 4.5. *Let Γ be a countable group and $p \in L(\Gamma)$ a projection. Assume that $\mathcal{G} \subset \mathcal{U}(pL(\Gamma)p)$ is a subgroup satisfying the following properties.*

(1) *The unitary representation $\{\operatorname{Ad} v\}_{v\in\mathcal{G}}$ on $L^2(pL(\Gamma)p \ominus \mathbb{C}p)$ is weakly mixing.*

(2) *If $g \in \Gamma$ and $g \neq e$, then $\mathcal{G}'' \not\prec L(C_\Gamma(g))$.*

(3) *We have $h_\Gamma(\mathcal{G}) > 0$.*

Then $p = 1$ and there exists a unitary $u \in L(\Gamma)$ such that $u\mathcal{G}u^* \subset \mathbb{T}\Gamma$.

Proof. Write $M = L(\Gamma)$ and denote by $\Delta \colon M \to M \mathbin{\overline{\otimes}} M \colon \Delta(u_g) = u_g \otimes u_g$ the comultiplication on $L(\Gamma)$. We first prove that the unitary representation on $L^2(\Delta(p)(M \mathbin{\overline{\otimes}} M)\Delta(p) \ominus \Delta(\mathbb{C}p))$ given by $\{\operatorname{Ad} \Delta(v)\}_{v\in\mathcal{G}}$ is weakly mixing. To prove this, assume that $\mathcal{H} \subset L^2(\Delta(p)(M \mathbin{\overline{\otimes}} M)\Delta(p))$ is a finite-dimensional subspace satisfying $\Delta(v)\mathcal{H}\Delta(v^*) = \mathcal{H}$ for all $v \in \mathcal{G}$. Writing $P = \mathcal{G}''$, it follows that the closed linear span of $\mathcal{H}\Delta(pMp)$ is a $\Delta(P)$-$\Delta(pMp)$-subbimodule of $L^2(\Delta(p)(M \mathbin{\overline{\otimes}} M)\Delta(p))$ that has finite right dimension. By [IPV10, Proposition 7.2] (using that $P \not\prec L(C_\Gamma(g))$ for $g \neq e$), we get that $\mathcal{H} \subset \Delta(L^2(pMp))$. Since the unitary representation $\{\operatorname{Ad} v\}_{v\in\mathcal{G}}$ on $L^2(pMp \ominus \mathbb{C}p)$ is weakly mixing, we conclude that $\mathcal{H} \subset \mathbb{C}\Delta(p)$.

Using the Fourier decomposition $v = \sum_{g\in\Gamma} (v)_g u_g$, we get for every $v \in \mathcal{G}$ that

$$\tau((v \otimes \Delta(v))(\Delta(v)^* \otimes v^*)) = \sum_{g\in\Gamma} |(v)_g|^4 \geq h_\Gamma(v)^4 \geq h_\Gamma(\mathcal{G})^4 \; .$$

Defining $X \in M \mathbin{\overline{\otimes}} M \mathbin{\overline{\otimes}} M$ as the element of minimal $\|\cdot\|_2$ in the weakly closed convex hull of $\{(v \otimes \Delta(v))(\Delta(v)^* \otimes v^*) \mid v \in \mathcal{G}\}$, we get that $\tau(X) \geq h_\Gamma(\mathcal{G})^4$, so that X is nonzero, and that $(v \otimes \Delta(v))X = X(\Delta(v) \otimes v)$ for all $v \in \mathcal{G}$. Also note that $(p \otimes \Delta(p))X = X = X(\Delta(p) \otimes p)$. By the weak mixing of both $\operatorname{Ad} v$ and $\operatorname{Ad}\Delta(v)$, it follows that XX^* is multiple of $p \otimes \Delta(p)$ and that X^*X is a multiple of $\Delta(p) \otimes p$. We may thus assume that

$$XX^* = p \otimes \Delta(p) \quad \text{and} \quad X^*X = \Delta(p) \otimes p \; .$$

Define $Y = (1 \otimes X)(X \otimes 1)$. Note that $Y \in M \mathbin{\overline{\otimes}} M \mathbin{\overline{\otimes}} M \mathbin{\overline{\otimes}} M$ is a partial isometry with $YY^* = p \otimes p \otimes \Delta(p)$ and $Y^*Y = \Delta(p) \otimes p \otimes p$. Also,

$$Y = (v \otimes v \otimes \Delta(v))Y(\Delta(v)^* \otimes v^* \otimes v^*) \quad \text{for all } \; v \in \mathcal{G} \; .$$

Since Y is nonzero, it follows that the unitary representation $\xi \mapsto (v \otimes v)\xi\Delta(v^*)$ of $\mathcal{G}$ on the Hilbert space $(p \otimes p)L^2(M \mathbin{\overline{\otimes}} M)\Delta(p)$ is not weakly mixing. We thus find a finite-dimensional irreducible representation $\omega \colon \mathcal{G} \to \mathcal{U}(\mathbb{C}^n)$ and a nonzero $Z \in M_{n,1}(\mathbb{C}) \otimes (p \otimes p)L^2(M \mathbin{\overline{\otimes}} M)\Delta(p)$ satisfying

$$(\omega(v) \otimes v \otimes v)Z = Z\Delta(v) \quad \text{for all } \; v \in \mathcal{G} \; .$$

By the weak mixing of $\operatorname{Ad} v$ and $\operatorname{Ad}\Delta(v)$ and the irreducibility of ω, it follows that ZZ^* is a multiple of $1 \otimes p \otimes p$ and that Z^*Z is a multiple of $\Delta(p)$. So, we may assume that $ZZ^* = 1 \otimes p \otimes p$ and that $Z^*Z = \Delta(p)$. It follows that $Z^*(M_n(\mathbb{C}) \otimes p \otimes p)Z$ is an n^2-dimensional globally $\{\operatorname{Ad}\Delta(v)\}_{v\in\mathcal{G}}$-invariant subspace of $\Delta(p)(M \mathbin{\overline{\otimes}} M)\Delta(p)$. Again by weak mixing, this implies that $n = 1$. But then, since $\tau(ZZ^*) = \tau(Z^*Z)$, we also get that $p = 1$. So, $Z \in M \mathbin{\overline{\otimes}} M$ is a unitary operator and $\omega \colon \mathcal{G} \to \mathbb{T}$ is a character satisfying $\omega(v)(v \otimes v)Z = Z\Delta(v)$ for all $v \in \mathcal{G}$.

Denoting by $\sigma \colon M \mathbin{\overline{\otimes}} M \to M \mathbin{\overline{\otimes}} M$ the flip map and using that $\sigma \circ \Delta = \Delta$, it follows that $Z\sigma(Z)^*$ commutes with all $v \otimes v$, $v \in \mathcal{G}$. By weak mixing, $Z\sigma(Z)^*$ is a multiple of 1. Using that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$, we similarly find that $(Z \otimes 1)(\Delta \otimes \mathrm{id})(Z)$ is a multiple of $(1 \otimes Z)(\mathrm{id} \otimes \Delta)(Z)$. By [IPV10, Theorem 3.3], there exists a unitary $u \in M$ such that $Z = (u^* \otimes u^*)\Delta(u)$. But then,

$$\Delta(uvu^*) = \omega(v)\, uvu^* \otimes uvu^* \quad \text{for all } \; v \in \mathcal{G} \; .$$

By [IPV10, Lemma 7.1], this means that $uvu^* \in \mathbb{T}\Gamma$ for every $v \in \mathcal{G}$. □

We end this section with some easy observations regarding the height h_Γ. Note that we can also define the height inside an amplification of a group von Neumann algebra $L(\Gamma)^n := M_n(\mathbb{C}) \otimes L(\Gamma)$ by the formula

$$h_\Gamma^n([x_{ij}]_{i,j=1}^n) = \max_{i,j=1,\dots,n} h_\Gamma(x_{ij}), \qquad x_{ij} \in L(\Gamma),$$

where we identify elements of $L(\Gamma)^n$ with $n \times n$-matrices over $L(\Gamma)$.

The following lemma shows that the property of having height bounded away from zero is preserved under conjugacy by a partial isometry.

Lemma 4.6. *Let $v \in M_{1,n}(\mathbb{C}) \otimes L(\Gamma)$ be a nonzero partial isometry and put $p = vv^* \in L(\Gamma)$. Given a bounded subset $\mathcal{G} \subset pL(\Gamma)p$, we have that $h_\Gamma(\mathcal{G}) > 0$ if and only if $h_\Gamma^n(v^*\mathcal{G}v) > 0$.*

Proof. Assume that there is a sequence $(x_k)_{k\in\mathbb{N}} \subset \mathcal{G}$ such that $h_\Gamma(x_k) \to 0$. Since $\mathcal{G}$ is bounded, we may assume that $\|x_k\| \leq 1$ for all k.

Write $v = (v_1, \dots, v_n)$ with $v_i \in L(\Gamma)$ and let $v_i = \sum_{g\in\Gamma} (v_i)_g u_g$ be the Fourier decompositions. We also write x_k in its Fourier decomposition $x_k = \sum_{g\in\Gamma} (x_k)_g u_g$. Then,

$$v^* x_k v = \sum_{i,j=1}^n v_i^* x_k v_j \otimes e_{ij} = \sum_{i,j=1}^n \sum_{s,g,h\in\Gamma} \overline{(v_i)_g} (x_k)_{gsh^{-1}} (v_j)_h u_s \otimes e_{ij}$$

and hence

$$h_\Gamma^n(v^* x_k v) = \max_{i,j=1,\dots,n} \sup_{s\in\Gamma} \Big| \sum_{g,h\in\Gamma} \overline{(v_i)_g} (x_k)_{gsh^{-1}} (v_j)_h \Big|.$$

Let $\varepsilon > 0$ be given and choose a finite set $F \subset \Gamma$ such that $\sum_{g\notin F} |(v_i)_g|^2 \leq \frac{\varepsilon^2}{9}$ for all $i, j = 1, \dots, n$. Then choose $N \in \mathbb{N}$ such that $h_\Gamma(x_k) < \frac{\varepsilon}{3|F|^2}$ for $k \geq N$. For any $s \in \Gamma$ and any $i, j \in \{1, \dots, n\}$, we have

$$\Big| \sum_{g,h\in\Gamma} \overline{(v_i)_g} (x_k)_{g^{-1}sh} (v_j)_h \Big| \leq \sum_{g,h\in F} h_\Gamma(x_k) + \Big| \sum_{g\notin F, h\in\Gamma} \overline{(v_i)_g} (x_k)_{g^{-1}sh} (v_j)_h \Big|$$
$$+ \Big| \sum_{g\in F, h\notin F} \overline{(v_i)_g} (x_k)_{g^{-1}sh} (v_j)_h \Big|. \tag{4.1}$$

We claim that each of the three terms occurring in this expression are smaller than $\frac{\varepsilon}{3}$ when $k \geq N$. This is clear for the first term, by our choice of N. For the second term, note that $\sum_{g\notin F, h\in\Gamma} \overline{(v_i)_g} (x_k)_{g^{-1}sh} (v_j)_h$ is a Fourier coefficient of the element $(1-P_F)(v_i^*)x_k v_j$, where $P_F \in B(\ell^2(\Gamma))$ denotes the orthogonal projection onto the linear span of $\{u_g \mid g \in F\}$. Therefore,

$$\Big| \sum_{g \notin F, h \in \Gamma} \overline{(v_i)_g} (x_k)_{g^{-1}sh} (v_j)_h \Big| \leq \|(1-P_F)(v_i^*)x_k v_j\|_2 \leq \frac{\varepsilon}{3}\|x_k\|\|v_j\| \leq \frac{\varepsilon}{3}.$$

Similarly, one shows that the third term in (4.1) is smaller than $\frac{\varepsilon}{3}$. From (4.1), it now follows that

$$h^n_\Gamma(v^* x_k v) = \max_{i,j=1,\dots,n} \sup_{s \in \Gamma} \Big| \sum_{g,h \in \Gamma} \overline{(v_i)_g} (x_k)_{g^{-1}sh} (v_j)_h \Big| \leq \varepsilon \quad \text{for} \ \ k \geq N.$$

We conclude that $h^n_\Gamma(v^* \mathcal{G} v) > 0$ implies $h_\Gamma(\mathcal{G}) > 0$. The other implication is shown analogously, using that $v(v^*\mathcal{G}v)v^* = p\mathcal{G}p = \mathcal{G}$. □

4.4 Proof of Theorem 4.1

Theorem 4.1 is an immediate consequence of the more general Theorem 4.7 that we prove in this section. In order to make our statements entirely explicit, we define a group measure space (gms) decomposition of a tracial von Neumann algebra (M,τ) to be any pair (B,Λ) where $B \subset M$ is a maximal abelian von Neumann subalgebra and $\Lambda \subset \mathcal{U}(M)$ is a subgroup normalizing B such that $M = (B \cup \Lambda)''$ and $E_B(v) = 0$ for all $v \in \Lambda \setminus \{1\}$. This of course amounts to writing $M = B \rtimes \Lambda$ for some free and trace-preserving action $\Lambda \curvearrowright (B,\tau)$.

We then say that two gms decompositions (B_i, Λ_i), $i = 0,1$, of M are

- identical if $B_0 = B_1$ and $\mathbb{T}\Lambda_0 = \mathbb{T}\Lambda_1$;
- unitarily conjugate if there exists a unitary $u \in \mathcal{U}(M)$ such that $uB_0u^* = B_1$ and $u\mathbb{T}\Lambda_0 u^* = \mathbb{T}\Lambda_1$;
- conjugate by an automorphism if there exists an automorphism $\theta \in \operatorname{Aut}(M)$ such that $\theta(B_0) = B_1$ and $\theta(\mathbb{T}\Lambda_0) = \mathbb{T}\Lambda_1$.

Recall from Section 2.6 the class $\mathcal{C}_{\text{rss}}$ consisting of all non-amenable relatively strongly solid groups.

Theorem 4.7. *Let Γ be a torsion-free group in the class $\mathcal{C}_{\text{rss}}$. Let (A_0, τ_0) be any amenable tracial von Neumann algebra with $A_0 \neq \mathbb{C}1$ and $\beta\colon \Gamma \curvearrowright (A_0,\tau_0)$ any trace-preserving action such that $\operatorname{Ker}\beta$ is a nontrivial subgroup of Γ. Define $(A,\tau) = (A_0,\tau_0)^\Gamma$ and denote by $\pi_k\colon A_0 \to A$ the embedding as the*

k'th tensor factor. Define the action $\sigma\colon \Gamma\times\Gamma \curvearrowright (A,\tau)$ given by $\sigma_{(g,h)}(\pi_k(a)) = \pi_{gkh^{-1}}(\beta_h(a))$ for all $g,k,h\in\Gamma$ and $a\in A_0$. Denote $M = A\rtimes(\Gamma\times\Gamma)$.

Up to unitary conjugacy, all gms decompositions of M are given as $M = B\rtimes\Lambda$ with $B = B_0^\Gamma$ and $\Lambda = \Lambda_0^{(\Gamma)}\rtimes(\Gamma\times\Gamma)$ where $A_0 = B_0\rtimes\Lambda_0$ is a gms decomposition of A_0 satisfying $\beta_g(B_0) = B_0$ and $\beta_g(\Lambda_0)=\Lambda_0$ for all $g\in\Gamma$.

Moreover, the gms decompositions of M associated with (B_0,Λ_0) and (B_1,Λ_1) are

1. *unitarily conjugate iff (B_0,Λ_0) is identical to (B_1,Λ_1);*
2. *conjugate by an automorphism of M iff there exists a trace-preserving automorphism $\theta_0\colon A_0\to A_0$ and an automorphism $\varphi\in\mathrm{Aut}(\Gamma)$ such that $\theta_0(B_0)=B_1$, $\theta_0(\mathbb{T}\Lambda_0)=\mathbb{T}\Lambda_1$ and $\theta_0\circ\beta_g = \beta_{\varphi(g)}\circ\theta_0$ for all $g\in\Gamma$.*

In Proposition 4.20 at the end of this section, we discuss when the Cartan subalgebras $B = B_0^\Gamma$ are unitarily conjugate and when they are conjugate by an automorphism of M.

First, let us show that the decompositions $M = B\rtimes\Lambda$ arising from β-invariant gms decompositions of A_0 as in Theorem 4.7 are indeed gms decompositions of M.

Proposition 4.8. *Let $M = A\rtimes(\Gamma\times\Gamma)$ be as in Theorem 4.7 and let $A_0 = B_0\rtimes\Lambda_0$ be a gms decomposition of A_0 satisfying $\beta_g(B_0)=B_0$ and $\beta_g(\Lambda_0)=\Lambda_0$ for all $g\in\Gamma$. Put $B = B_0^\Gamma$ and $\Lambda = \Lambda_0^{(\Gamma)}\rtimes(\Gamma\times\Gamma)$. Then $M = B\rtimes\Lambda$ is a gms decomposition of M.*

Proof. Assume that (B_0,Λ_0) is a gms decomposition of A_0 satisfying $\beta_g(B_0) = B_0$ and $\beta_g(\Lambda_0)=\Lambda_0$ for all $g\in\Gamma$. Then, $\{\beta_g\}_{g\in\Gamma}$ defines an action of Γ by automorphisms of the group Λ_0. We can co-induce this to the action of $\Gamma\times\Gamma$ by automorphisms of the direct sum group $\Lambda_0^{(\Gamma)}$ given by $(g,h)\cdot\pi_k(v) = \pi_{gkh^{-1}}(\beta_h(v))$ for all $g,h,k\in\Gamma$ and $v\in\Lambda_0$, where $\pi_k\colon\Lambda_0\to\Lambda_0^{(\Gamma)}$ denotes the embedding as the k'th direct summand. Putting $B := B_0^\Gamma$ and $\Lambda := \Lambda_0^{(\Gamma)}\rtimes(\Gamma\times\Gamma)$, we have found the crossed product decomposition $M = B\rtimes\Lambda$. It remains to check that the action $\sigma\colon\Lambda\curvearrowright B$ is free. So, we need to show that for all $b\in B$ and all $s\in\Lambda\setminus\{e\}$, we have that

$$bx = \sigma_s(x)b \ \text{ for all }\ x\in B \quad\Rightarrow\quad b = 0. \tag{4.2}$$

To prove this, let $s = (s_i)_{i\in\Gamma}(g,h)\in\Lambda\setminus\{e\}$, where $(s_i)_{i\in\Gamma}\in\Lambda_0^{(\Gamma)}$ and $(g,h)\in\Gamma\times\Gamma$. Since $\Lambda_0\curvearrowright B_0$ is free, clearly also $\Lambda_0^{(\Gamma)}\curvearrowright B_0^\Gamma$ is free. So, if $s\in\Lambda_0^{(\Gamma)}$

we are done. Assume therefore that $(g,h) \neq (e,e)$ and let $b \in B \setminus \{0\}$. Let $0 < \varepsilon < \frac{\|b\|_2}{2}$ and choose a finite set $\mathcal{F} \subset \Gamma$ and $b_0 \in B_0^{\mathcal{F}}$ such that $\|b - b_0\|_2 < \varepsilon$. Assume that $bx = \sigma_s(x)b$ for all $x \in B$. Then $\|b_0 x - \sigma_s(x) b_0\|_2 \leq 2\varepsilon \|x\|$ for all $x \in B$. Let $t \in \Gamma$ be such that $t \notin \mathcal{F} \cup g^{-1}\mathcal{F}h$ and such that $t \neq gth^{-1}$. We can do this because Γ is icc and either $g \neq e$ or $h \neq e$. Put $x = \pi_t(u) \in B$, where $u \in \mathcal{U}(B_0)$ is a unitary with $\tau(u) = 0$. Then

$$\|b_0 x - \sigma_s(x) b_0\|_2 = \|b_0\|_2 \|\pi_t(u) - \pi_{gth^{-1}}(s_{gth^{-1}} \cdot \beta_g(u))\|_2$$
$$= 2\|b_0\|_2 \geq 2(\|b\|_2 - \varepsilon) > 2\varepsilon,$$

a contradiction. Hence (4.2) holds and the action $\sigma \colon \Lambda \curvearrowright B$ is free. □

In the rest of this section, we prove Theorem 4.7. So, we fix a group Γ and an action $\sigma \colon \Gamma \times \Gamma \curvearrowright A$ as in the formulation of the theorem. We put $M = A \rtimes (\Gamma \times \Gamma)$.

We first prove the following lemma, which allows us to control commuting non-amenable subalgebras of M. The proof relies on Popa's spectral gap rigidity for Bernoulli actions, generalized to the setting of co-induced actions such as σ. We provide a full proof of these generalizations in Appendix A.

Lemma 4.9. *Let (N,τ) be a tracial factor and let $Q_1, Q_2 \subset N \overline{\otimes} M$ be commuting von Neumann subalgebras that are strongly non-amenable relative to $N \otimes 1$. Then $Q_1 \vee Q_2$ can be unitarily conjugated into $N \overline{\otimes} L(\Gamma \times \Gamma)$.*

Proof. Since A is amenable, we get that Q_1 and Q_2 are strongly non-amenable relative to $N \overline{\otimes} (A \rtimes L)$ whenever $L < \Gamma \times \Gamma$ is an amenable subgroup. For every $g \in \Gamma$, we denote by $\operatorname{Stab} g \subset \Gamma \times \Gamma$ the stabilizer of g under the left-right translation action $\Gamma \times \Gamma \curvearrowright \Gamma$. We also write $\operatorname{Stab}\{g,h\} = \operatorname{Stab} g \cap \operatorname{Stab} h$.

We start by proving that $Q_2 \not\prec N \overline{\otimes} (A \rtimes \operatorname{Stab} g)$ for all $g \in \Gamma$. Assume the contrary. Whenever $h \neq g$, the group $\operatorname{Stab}\{g,h\} \cong C_\Gamma(h^{-1}g)$ is amenable since Γ is torsion-free and in class $\mathcal{C}_{\text{rss}}$. By Proposition 2.16 it follows that, $Q_2 \not\prec N \overline{\otimes} (A \rtimes \operatorname{Stab}\{g,h\})$. Also by Proposition 2.16, along with [Va07, Remark 3.8], we can take projections $q \in Q_2$ and $p \in N \overline{\otimes} (A \rtimes \operatorname{Stab} g)$, a nonzero partial isometry $v \in q(N \overline{\otimes} M)p$ and a normal unital $*$-homomorphism

$$\theta \colon qQ_2q \to p(N \overline{\otimes} (A \rtimes \operatorname{Stab} g))p$$

such that $xv = v\theta(x)$ for all $x \in qQ_2q$ and such that, inside $N \overline{\otimes} (A \rtimes \operatorname{Stab} g)$, we have that $\theta(qQ_2q)$ is non-amenable relative to $N \overline{\otimes} A$ and we have that $\theta(qQ_2q) \not\prec N \overline{\otimes} (A \rtimes \operatorname{Stab}\{g,h\})$ whenever $h \neq g$.

Write $P := \theta(qQ_2q)' \cap p(N \mathbin{\overline{\otimes}} M)p$. By Lemma A.4, $P \subset p(N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} g))p$. In particular, $v^*v \in p(N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} g))p$ and we may assume that $v^*v = p$. Since $\operatorname{Stab} g \cong \Gamma$, we have $\operatorname{Stab} g \in \mathcal{C}_{\text{rss}}$ and Lemma 2.29 implies that $P \prec N \mathbin{\overline{\otimes}} A$. Conjugating with v and writing $e = vv^* \in (Q_2' \cap (N \mathbin{\overline{\otimes}} M))q$, we find that $e(Q_2' \cap (N \mathbin{\overline{\otimes}} M))e \prec N \mathbin{\overline{\otimes}} A$. Since $Q_1 \subset Q_2' \cap (N \mathbin{\overline{\otimes}} M)$, it follows that $Q_1 \prec N \mathbin{\overline{\otimes}} A$. By Proposition 2.16, this contradicts the strong non-amenability of Q_1 relative to $N \mathbin{\overline{\otimes}} A$. So, we have proved that $Q_2 \not\prec N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} g)$ for all $g \in \Gamma$.

Since Q_1 is strongly non-amenable relative to $N \mathbin{\overline{\otimes}} A$ and since $Q_2 \not\prec N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} g)$ for all $g \in \Gamma$, it follows from Theorem A.2 that $u^*Q_2u \subset N \mathbin{\overline{\otimes}} L(\Gamma \times \Gamma)$ for some unitary $u \in N \mathbin{\overline{\otimes}} M$. Since $u^*Q_2u \not\prec N \mathbin{\overline{\otimes}} L(\operatorname{Stab} g)$ for all $g \in \Gamma$, it follows from Lemma A.4 that also $u^*Q_1u \subset N \mathbin{\overline{\otimes}} L(\Gamma \times \Gamma)$. This concludes the proof of the lemma. □

We now also fix a gms decomposition $M = B \rtimes \Lambda$. We view Λ as a subgroup of $\mathcal{U}(M)$ and denote by $\Delta \colon M \to M \mathbin{\overline{\otimes}} M$ the associated dual coaction given by $\Delta(b) = b \otimes 1$ for all $b \in B$ and $\Delta(v) = v \otimes v$ for all $v \in \Lambda$.

Lemma 4.10. *Writing $Q_1 = L(\Gamma \times \{e\})$ and $Q_2 = L(\{e\} \times \Gamma)$, we have $\Delta(Q_2) \prec_{M \overline{\otimes} M} M \mathbin{\overline{\otimes}} Q_i$ for either $i = 1$ or $i = 2$.*

Proof. By Proposition 4.3, $\Delta(Q_1)$ and $\Delta(Q_2)$ are strongly non-amenable relative to $M \otimes 1$. So by Lemma 4.9, we can take a unitary $v \in M \mathbin{\overline{\otimes}} M$ such that

$$v^*\Delta(Q_1 \vee Q_2)v \subset M \mathbin{\overline{\otimes}} L(\Gamma \times \Gamma) .$$

We therefore have the two commuting subalgebras $v^*\Delta(Q_1)v$ and $v^*\Delta(Q_2)v$ inside $M \mathbin{\overline{\otimes}} L(\Gamma \times \Gamma)$. If $v^*\Delta(Q_1)v$ was amenable relative to both $M \mathbin{\overline{\otimes}} Q_1$ and $M \mathbin{\overline{\otimes}} Q_2$, then it would be amenable relative to $M \otimes 1$ by [PV11, Proposition 2.7], which is not the case. Hence $v^*\Delta(Q_1)v$ is non-amenable relative to either $M \mathbin{\overline{\otimes}} Q_1$ or $M \mathbin{\overline{\otimes}} Q_2$. Assuming that $v^*\Delta(Q_1)v$ is non-amenable relative to $M \mathbin{\overline{\otimes}} Q_1$, Lemma 2.29 implies that $\Delta(Q_2) \prec M \mathbin{\overline{\otimes}} Q_1$. □

In the following three lemmas, we prove that Λ contains two commuting non-amenable subgroups $\Lambda_1, \Lambda_2 < \Lambda$. The idea is as follows. Lemma 4.10 shows that essentially, $\Delta(Q_1) \subset M \mathbin{\overline{\otimes}} Q_i$ and $\Delta(Q_2) \subset M \mathbin{\overline{\otimes}} Q_j$, for $\{i, j\} = \{1, 2\}$. If we assume that these inclusions literally hold, then the same argument as in the proof of Lemma 4.2 gives two subgroups $\Lambda_1, \Lambda_2 < \Lambda$ such that $Q_1 \subset B \rtimes \Lambda_1$ and $Q_2 \subset B \rtimes \Lambda_2$. Then Λ_1 and Λ_2 are non-amenable and commute. Unfortunately, this simple argument completely breaks down if we merely have an intertwining as in Lemma 4.10 instead of a literal inclusion.

The method that we use to produce such commuting subgroups is taken from [Io11] and our proofs of Lemmas 4.11, 4.12 and 4.13 are very similar to the proof of [Io11, Theorem 3.1]. The same method was also used in [CdSS15, Theorem 3.3]. For completeness, we provide all details.

Combining Lemma 4.10 with our transfer of rigidity theorem (Theorem 4.4), we get the following.

Lemma 4.11. *Denote by $\mathcal{G}$ the family of all amenable subgroups of Λ. For either $i = 1$ or $i = 2$, there exists a finite set $x_1, \dots, x_n \in M$ and a $\delta > 0$ such that the following holds: whenever $F \subset \Lambda$ is small relative to $\mathcal{G}$, we can find an element $v_F \in \Lambda - F$ such that*

$$\sum_{k,j=1}^{n} \|E_{Q_i}(x_k v_F x_j^*)\|_2^2 \geq \delta \; .$$

Lemma 4.12. *There exists a decreasing sequence of non-amenable subgroups $\Lambda_n < \Lambda$ such that $Q_i \prec_M B \rtimes \left(\bigcup_n C_\Lambda(\Lambda_n)\right)$ for either $i = 1$ or $i = 2$, where $C_\Lambda(\Lambda_n)$ denotes the centralizer of Λ_n inside Λ.*

Proof. As in Lemma 4.11, we let $\mathcal{G}$ denote the family of all amenable subgroups of Λ. We denote by I the set of subsets of Λ that are small relative to $\mathcal{G}$. We order I by inclusion and choose a *cofinal* ultrafilter $\mathcal{U}$ on I, meaning that $\{S \in I \mid S \supset S_0\} \in \mathcal{U}$ for all $S_0 \in I$. Consider the ultrapower von Neumann algebra $M^{\mathcal{U}}$ and the ultrapower group $\Lambda^{\mathcal{U}}$. Every $v = (v_F)_{F \in I} \in \Lambda^{\mathcal{U}}$ can be viewed as a unitary in $M^{\mathcal{U}}$ and as explained at the end of Section 2.1.6, $\Lambda^{\mathcal{U}}$ and $B^{\mathcal{U}}$ are in crossed product position inside $M^{\mathcal{U}}$.

Assume without loss of generality that $i = 1$ in Lemma 4.11 and denote by $v = (v_F)_{F \in I}$ the element of $\Lambda^{\mathcal{U}}$ that we found in Lemma 4.11. Denote by $K \subset L^2(M^{\mathcal{U}})$ the closed linear span of MvM and by P_K the orthogonal projection from $L^2(M^{\mathcal{U}})$ onto K. Put $\Sigma = \Lambda \cap v\Lambda v^{-1}$. We claim that $Q_2 \prec_M B \rtimes \Sigma$.

Assume the contrary. This means that we can find a sequence of unitaries $a_n \in \mathcal{U}(Q_2)$ such that $\|E_{B \rtimes \Sigma}(x a_n y)\|_2 \to 0$ for any $x, y \in M$. We prove that $\langle a_n \xi a_n^*, \eta \rangle \to 0$ as $n \to \infty$ for all $\xi, \eta \in K$. For this, it suffices to prove that $\langle a_n x v x' a_n^*, y v y' \rangle \to 0$ for all $x, x', y, y' \in M$. First, note that for all $z \in M$, we have $E_M(v^* z v) = E_M(v^* E_{B \rtimes \Sigma}(z) v)$ by definition of the subgroup Σ. Hence

$$\begin{aligned} |\langle a_n x v x' a_n^*, y v y' \rangle| &= |\tau(E_M(v^* y^* a_n x v) x' a_n^* y'^*)| \\ &= |\tau(E_M(v^* E_{B \rtimes \Sigma}(y^* a_n x) v) x' a_n^* y'^*)| \\ &\leq \|x'\| \|y'\| \|E_{B \rtimes \Sigma}(y^* a_n x)\|_2 \to 0 \end{aligned}$$

as wanted.

Next, Lemma 4.11 provides a finite set $L \subset M$ such that

$$\sum_{x,y \in L} \|E_{Q_1^{\mathcal{U}}}(xvy^*)\|_2^2 \neq 0.$$

In particular, we can take $x, y \in L$ such that $E_{Q_1^{\mathcal{U}}}(xvy^*) \neq 0$. Put $\xi = P_K(E_{Q_1^{\mathcal{U}}}(xvy^*))$. We claim that $\xi \neq 0$. Since $E_{Q_1^{\mathcal{U}}}(xvy^*) \neq 0$, we get that $\|xvy^* - E_{Q_1^{\mathcal{U}}}(xvy^*)\|_2 < \|xvy^*\|_2$. Since $xvy^* \in K$, it follows that

$$\|xvy^* - \xi\|_2 = \|P_K(xvy^* - E_{Q_1^{\mathcal{U}}}(xvy^*))\|_2 < \|xvy^*\|_2.$$

Hence $\xi \neq 0$.

Since K is an M-bimodule and since Q_1 commutes with Q_2, we have that $a\xi = \xi a$ for all $a \in Q_2$. In particular, $\langle a_n \xi a_n^*, \xi\rangle = \|\xi\|_2^2 > 0$ in contradiction with the fact that $\langle a_n \xi a_n^*, \xi\rangle \to 0$. This proves that $Q_2 \prec_M B \rtimes \Sigma$.

It remains to show that there exists a decreasing sequence of subgroups $\Lambda_n < \Lambda$ such that for all n we have $\Lambda_n \notin \mathcal{G}$, and such that $\Sigma = \bigcup_n C_\Lambda(\Lambda_n)$. For every $\mathcal{T} \subset I$, we denote by $\Lambda_{\mathcal{T}}$ the subgroup of Λ generated by $\{v_F v_{F'}^{-1} \mid F, F' \in \mathcal{T}\}$. An element $w \in \Lambda$ belongs to Σ if and only if there exists a $\mathcal{T} \in \mathcal{U}$ such that w commutes with $\Lambda_{\mathcal{T}}$. Enumerating $\Sigma = \{w_1, w_2, \ldots\}$, choose $\mathcal{S}_n \in \mathcal{U}$ such that w_n commutes with $\Lambda_{\mathcal{S}_n}$. Then put $\mathcal{T}_n := \mathcal{S}_1 \cap \ldots \cap \mathcal{S}_n \in \mathcal{U}$. By construction, $\Sigma = \bigcup_n C_\Lambda(\Lambda_{\mathcal{T}_n})$. It remains to prove that $\Lambda_{\mathcal{T}} \notin \mathcal{G}$ for all $\mathcal{T} \in \mathcal{U}$.

Fix $\mathcal{T} \in \mathcal{U}$ and assume that $\Lambda_{\mathcal{T}} \in \mathcal{G}$. For fixed $F' \in \mathcal{T}$, we have that $\{v_F \mid F \in \mathcal{T}\} \subset \Lambda_{\mathcal{T}} v_{F'}$. So, $F_1 := \{v_F \mid F \in \mathcal{T}\}$ is small relative to $\mathcal{G}$. Define $\mathcal{T}' \subset I$ by $\mathcal{T}' = \{F \in I \mid F_1 \subset F\}$. Since $\mathcal{U}$ is a cofinal ultrafilter and $\mathcal{T} \in \mathcal{U}$, we get $\mathcal{T} \cap \mathcal{T}' \neq \emptyset$. So we can take $F \in \mathcal{T}$ with $F_1 \subset F$. Then, $v_F \in \Lambda - F \subset \Lambda - F_1$ but also $v_F \in F_1$. This being absurd, we have shown that $\Lambda_{\mathcal{T}} \notin \mathcal{G}$ for all $\mathcal{T} \in \mathcal{U}$. ☐

Lemma 4.13. *There exist two commuting non-amenable subgroups Λ_1 and Λ_2 inside Λ. Moreover, whenever $\Lambda_1, \Lambda_2 < \Lambda$ are commuting non-amenable subgroups, $L(\Lambda_1 \Lambda_2)$ can be unitarily conjugated into $L(\Gamma \times \Gamma)$.*

Proof. From Lemma 4.12, we find a decreasing sequence of non-amenable subgroups $\Lambda_n < \Lambda$ such that $Q_i \prec_M B \rtimes (\bigcup_n C_\Lambda(\Lambda_n))$ for either $i = 1$ or $i = 2$. Since Q_i has no amenable direct summand, we get that the group $\bigcup_n C_\Lambda(\Lambda_n)$ is non-amenable. It follows that $C_\Lambda(\Lambda_n)$ is non-amenable for some $n \in \mathbb{N}$. Then $\Lambda_1 := \Lambda_n$ and $\Lambda_2 := C_\Lambda(\Lambda_n)$ are commuting non-amenable subgroups of Λ.

Whenever $\Lambda_1, \Lambda_2 < \Lambda$ are commuting non-amenable subgroups, it follows from Lemma 4.9 applied to $N = \mathbb{C}1$ that $L(\Lambda_1) \vee L(\Lambda_2)$ can be unitarily conjugated into $L(\Gamma \times \Gamma)$. ☐

From now on, we fix commuting non-amenable subgroups $\Lambda_1, \Lambda_2 < \Lambda$. By Lemma 4.13, after a unitary conjugacy, we may assume that $L(\Lambda_1\Lambda_2) \subset L(\Gamma\times\Gamma)$.

Lemma 4.14. *If $N \subset L(\Gamma \times \Gamma)$ is an amenable von Neumann subalgebra such that the normalizer $\mathcal{N}_{L(\Gamma\times\Gamma)}(N)''$ contains $L(\Lambda_1\Lambda_2)$, then N is atomic. Also, $L(\Lambda_1\Lambda_2)' \cap L(\Gamma \times \Gamma)$ is atomic.*

Proof. Using [Va10, Proposition 2.6], we find a projection q in the center of the normalizer of N such that $Nq \prec^f L(\Gamma) \otimes 1$ and $N(1-q) \not\prec L(\Gamma) \otimes 1$.

Assume for contradiction that $q \neq 1$. Since $N(1-q) \not\prec L(\Gamma) \otimes 1$ and since $\Gamma \in \mathcal{C}_{\text{rss}}$, it follows that $L(\Lambda_i)(1-q)$ is amenable relative to $L(\Gamma) \otimes 1$ for both $i = 1, 2$. It then follows from [PV11, Proposition 2.7] that $L(\Lambda_1)(1-q)$ is non-amenable relative to $1 \otimes L(\Gamma)$, hence $L(\Lambda_2)(1-q) \prec 1 \otimes L(\Gamma)$ by Lemma 2.29. By Proposition 2.16, we get a nonzero projection $q_0 \leq 1-q$ that commutes with $L(\Lambda_2)$ such that $L(\Lambda_2)q_0$ is amenable relative to $1 \otimes L(\Gamma)$. But since $L(\Lambda_2)q_0$ is also amenable relative to $L(\Gamma) \otimes 1$, it follows from [PV11, Proposition 2.7] that $L(\Lambda_2)q_0$ is amenable relative to $\mathbb{C}1$, hence a contradiction.

We conclude that $q = 1$ so that $N \prec^f L(\Gamma) \otimes 1$. By symmetry, we also get that $N \prec^f 1 \otimes L(\Gamma)$. By [DHI16, Lemma 2.8], it follows that $N \prec^f (L(\Gamma) \otimes 1) \cap (1 \otimes L(\Gamma)) = \mathbb{C}1$ so that N is atomic.

To prove that $L(\Lambda_1\Lambda_2)' \cap L(\Gamma \times \Gamma)$ is atomic, it suffices to prove that every abelian von Neumann subalgebra $D \subset L(\Lambda_1\Lambda_2)' \cap L(\Gamma\times\Gamma)$ is atomic, by [BO08, Corollary F.14]. But then D is amenable and its normalizer contains $L(\Lambda_1\Lambda_2)$, so that D is indeed atomic. □

The proof of the following lemma is essentially contained in the proof of [OP03, Proposition 12]. It roughly states that after a unitary conjugacy, $L(\Lambda_1) \subset L(\Gamma)\otimes 1$ and $L(\Lambda_2) \subset 1\otimes L(\Gamma)$, up to amplifications and up to switching around Λ_1 and Λ_2.

Lemma 4.15. *For every minimal projection $e \in L(\Lambda_1\Lambda_2)' \cap L(\Gamma \times \Gamma)$, there exist projections $p \in M_n(\mathbb{C}) \otimes L(\Gamma)$, $q \in L(\Gamma) \otimes M_m(\mathbb{C})$ and a partial isometry $u \in M_{n,1}(\mathbb{C}) \otimes L(\Gamma \times \Gamma) \otimes M_{m,1}(\mathbb{C})$ such that $u^*u = e$, $uu^* = p \otimes q$ and such that either*

$$uL(\Lambda_1)u^* \subset p(M_n(\mathbb{C}) \otimes L(\Gamma))p \otimes q \quad \textit{and} \quad uL(\Lambda_2)u^* \subset p \otimes q(L(\Gamma) \otimes M_m(\mathbb{C}))q,$$

or

$$uL(\Lambda_1)u^* \subset p \otimes q(L(\Gamma) \otimes M_m(\mathbb{C}))q \quad \textit{and} \quad uL(\Lambda_2)u^* \subset p(M_n(\mathbb{C}) \otimes L(\Gamma))p \otimes q.$$

Proof. By [PV11, Proposition 2.7], $L(\Lambda_2)e$ is non-amenable relative to either $L(\Gamma)\otimes 1$ or $1\otimes L(\Gamma)$. Assume that $L(\Lambda_2)e$ is non-amenable relative to $L(\Gamma)\otimes 1$. By Lemma 2.29, $L(\Lambda_1)e \prec L(\Gamma)\otimes 1$. Take a projection $p \in M_n(\mathbb{C})\otimes L(\Gamma)$, a nonzero partial isometry $v \in (p\otimes 1)(M_{n,1}(\mathbb{C})\otimes L(\Gamma\times\Gamma))e$ and a unital normal $*$-homomorphism $\theta\colon L(\Lambda_1)\to p(M_n(\mathbb{C})\otimes L(\Gamma))p$ such that

$$(\theta(x)\otimes 1)v = vx \quad \text{for all } \ x\in L(\Lambda_1)\ .$$

Since $\Gamma\in\mathcal{C}_{\text{rss}}$ and Λ_1 is non-amenable, the relative commutant $\theta(L(\Lambda_1))'\cap p(M_n(\mathbb{C})\otimes L(\Gamma))p$ is atomic. Cutting with a minimal projection, we may assume that this relative commutant equals $\mathbb{C}p$.

Write $P := L(\Lambda_1)'\cap L(\Gamma\times\Gamma)$ and note that $v^*v, e\in P$ with $v^*v\le e$. Since $L(\Lambda_2)\subset P$, we have that $\mathcal{Z}(P)\subset L(\Lambda_1\Lambda_2)'\cap L(\Gamma\times\Gamma)$. It follows that $\mathcal{Z}(P)e = \mathbb{C}e$. So, ePe is a II_1 factor and we can take partial isometries $v_1,\dots,v_m\in ePe$ with $v_iv_i^*\le v^*v$ for all i and $\sum_{i=1}^m v_i^*v_i = e$. Define $u\in M_{n,1}(\mathbb{C})\otimes L(\Gamma\times\Gamma)\otimes M_{m,1}(\mathbb{C})$ given by $u=\sum_{i=1}^m vv_i\otimes e_{i1}$.

Since vPv^* commutes with $\theta(L(\Lambda_1))\otimes 1$, we have $vPv^*\subset p\otimes L(\Gamma)$ and we can define the normal $*$-homomorphism $\eta\colon v^*vPv^*v\to L(\Gamma)$ such that $vyv^* = p\otimes\eta(y)$ for all $y\in v^*vPv^*v$. By construction, $u^*u=e$ and $uu^*=p\otimes q$ where $q\in L(\Gamma)\otimes M_m(\mathbb{C})$ is the projection given by $q=\sum_{i=1}^m \eta(v_iv_i^*)\otimes e_{ii}$. Defining the $*$-homomorphism

$$\widetilde{\eta}\colon ePe\to q(L(\Gamma)\otimes M_m(\mathbb{C}))q\colon\quad \widetilde{\eta}(y)=\sum_{i,j=1}^m \eta(v_iyv_j^*)\otimes e_{ij}$$

and using that $L(\Lambda_2)e\subset ePe$, we get that

$$uL(\Lambda_1)u^* = \theta(L(\Lambda_1))\otimes q \quad\text{and}\quad uL(\Lambda_2)u^* = p\otimes\widetilde{\eta}(L(\Lambda_2)e)\ .$$

This concludes the proof of the lemma. □

Recall from Section 4.3 the notion of height of an element in a group von Neumann algebra (here, $L(\Gamma\times\Gamma)$), as well as the height of a subset of Ł$(\Gamma\times\Gamma)$. The proof of the following lemma is very similar to the proof of [Io10, Theorem 4.1].

Lemma 4.16. *For every projection $p\in L(\Lambda_1\Lambda_2)'\cap L(\Gamma\times\Gamma)$, we have that $h_{\Gamma\times\Gamma}(\Lambda_1\Lambda_2 p)>0$.*

Proof. It suffices to prove that $h_{\Gamma\times\Gamma}(\Lambda_1\Lambda_2 p)>0$ for all minimal projections $p\in L(\Lambda_1\Lambda_2)'\cap L(\Gamma\times\Gamma)$. Indeed, if $p\in L(\Lambda_1\Lambda_2)'\cap L(\Gamma\times\Gamma)$ is an arbitrary projection, then we find a minimal projection $p_0\le p$ since $L(\Lambda_1\Lambda_2)'\cap L(\Gamma\times\Gamma)$

is atomic. If $h_{\Gamma\times\Gamma}(\Lambda_1\Lambda_2 p_0) > 0$ then it follows from Lemma 4.6 that also $h_{\Gamma\times\Gamma}(\Lambda_1\Lambda_2 p) > 0$.

So, we fix a minimal projection $p \in L(\Lambda_1\Lambda_2)' \cap L(\Gamma\times\Gamma)$. Using the conjugacy of Lemma 4.15 along with Lemma 4.6, we see that the heights of $\Lambda_1 p$ and $\Lambda_2 p$ do not interact, so that it suffices to prove that $h_{\Gamma\times\Gamma}(\Lambda_i p) > 0$ for $i = 1, 2$. By symmetry, it is enough to prove this for $i = 1$.

Assume for contradiction that $h_{\Gamma\times\Gamma}(\Lambda_1 p) = 0$. Take a sequence $v_n \in \Lambda_1$ such that $h_{\Gamma\times\Gamma}(v_n p) \to 0$. For every finite subset $S \subset \Gamma\times\Gamma$, we denote by P_S the orthogonal projection of $L^2(M)$ onto the linear span of $L^2(A)u_g$, $g \in S$. We claim that for every sequence of unitaries $w_n \in L(\Gamma\times\Gamma)$, every $a \in M \ominus L(\Gamma\times\Gamma)$ and every finite subset $S \subset \Gamma\times\Gamma$, we have that

$$\lim_n \|P_S(pv_n a w_n)\|_2 = 0 \; .$$

Since $P_S(x) = \sum_{g\in S} E_A(xu_g^*)u_g$, it suffices to prove that $\|E_A(pv_n a w_n)\|_2 \to 0$ for all $a \in M \ominus L(\Gamma\times\Gamma)$. Every such a can be approximated by a linear combination of elements of the form $a_0 u_g$ with $a_0 \in A \ominus \mathbb{C}1$ and $g \in \Gamma\times\Gamma$. So, we may assume that $a \in A \ominus \mathbb{C}1$. Such an element a can be approximated by a linear combination of elementary tensors, so that we may assume that $a = \bigotimes_{i\in\mathcal{G}} a_i$ for some finite nonempty subset $\mathcal{G} \subset \Gamma$ and elements $a_i \in A_0 \ominus \mathbb{C}1$. Note that $\sigma_g(a) \perp \sigma_h(a)$ whenever $g, h \in \Gamma\times\Gamma$ and $g\cdot\mathcal{G} \neq h\cdot\mathcal{G}$ (where we use the left-right translation action of $\Gamma\times\Gamma$ on Γ). We also assume that $\|a\| \leq 1$.

Let $\varepsilon > 0$. By Lemma 4.15, we have that either $L(\Lambda_1)p \prec^f L(\Gamma)\otimes 1$ or $L(\Lambda_1)p \prec^f 1 \otimes L(\Gamma)$. So, we can take a finite subset $F_0 \subset \Gamma$ such that, writing $F = \Gamma\times F_0 \cup F_0 \times \Gamma$, we have $\|pv - P_F(pv)\|_2 \leq \varepsilon$ for all $v \in \Lambda_1$. Then,

$$\|E_A(pv_n a w_n) - E_A(P_F(pv_n) a w_n)\|_2 \leq \varepsilon$$

for all n, so that in order to prove the claim, it suffices to prove that $\|E_A(P_F(pv_n)aw_n)\|_2 \to 0$. Put $\kappa = 2|F_0||\mathcal{G}|^2$. Note that for every $h \in \Gamma\times\Gamma$, the set $\{g \in F \mid g\cdot\mathcal{G} = h\cdot\mathcal{G}\}$ contains at most κ elements. Using the Fourier decomposition for elements in $L(\Gamma\times\Gamma)$, we have

$$E_A(P_F(pv_n)aw_n) = \sum_{g\in F} (pv_n)_g \, (w_n)_{g^{-1}} \, \sigma_g(a) \; .$$

Thus, for all $h \in \Gamma\times\Gamma$, we have

$$|\langle E_A(P_F(pv_n)aw_n), \sigma_h(a)\rangle| \leq \kappa \, h_{\Gamma\times\Gamma}(pv_n) \; .$$

But then, using the Cauchy-Schwarz inequality, we get that

$$\|E_A(P_F(pv_n)aw_n)\|_2^2 \leq \sum_{h\in F} |\langle E_A(P_F(pv_n)aw_n)\,,\,(pv_n)_h\,(w_n)_{h^{-1}}\,\sigma_h(a)\rangle|$$

$$\leq \kappa\, h_{\Gamma\times\Gamma}(pv_n) \sum_{h\in\Gamma\times\Gamma} |(pv_n)_h|\,|(w_n)_{h^{-1}}|$$

$$\leq \kappa\, h_{\Gamma\times\Gamma}(pv_n)\,\|pv_n\|_2\,\|w_n\|_2 \leq \kappa\, h_{\Gamma\times\Gamma}(pv_n) \to 0\ .$$

So, the claim is proved.

Put $\delta = \|p\|_2/4$. Because $\Gamma \in \mathcal{C}_{\mathrm{rss}}$ and $B \subset M$ is a Cartan subalgebra, we have that $B \prec^f A$. By Lemma 2.14, we have $B \not\prec L(\Gamma\times\Gamma)$ and we can take a unitary $b \in \mathcal{U}(B)$ such that $\|E_{L(\Gamma\times\Gamma)}(b)\|_2 \leq \delta$. Since $B \prec^f A$, we can take a finite subset $S \subset \Gamma\times\Gamma$ such that $\|pd - P_S(pd)\|_2 \leq \delta$ for all $d \in \mathcal{U}(B)$ (using [Va10, Lemma 2.3]). For every n, we have that $v_n b v_n^* \in \mathcal{U}(B)$. Therefore,

$$\|pv_nbv_n^* - P_S(pv_nbv_n^*)\|_2 \leq \delta$$

for all n. Since $\|E_{L(\Gamma\times\Gamma)}(b)\|_2 \leq \delta$, writing $b_1 = b - E_{L(\Gamma\times\Gamma)}(b)$, we get

$$\|P_S(pv_nbv_n^*) - P_S(pv_nb_1v_n^*)\|_2 \leq \delta\ .$$

By the claim above, we can fix n large enough such that $\|P_S(pv_nb_1v_n^*)\|_2 \leq \delta$. It follows that

$$\|p\|_2 = \|pv_nbv_n^*\|_2 \leq 3\delta < \|p\|_2\ ,$$

which is absurd. So, we have shown that $h_{\Gamma\times\Gamma}(p\Lambda_1) > 0$ and the lemma is proved. □

Having height bounded away from zero as in Lemma 4.16 allows us to deduce that $\Lambda_1\Lambda_2$ sits as a subgroup of $\Gamma\times\Gamma$, up to unitary conjugacy.

Lemma 4.17. *There exists a unitary $u \in L(\Gamma\times\Gamma)$ such that $u\Lambda_1\Lambda_2u^* \subset \mathbb{T}(\Gamma\times\Gamma)$. Also, the unitary representation $\{\mathrm{Ad}\, v\}_{v\in\Lambda_1\Lambda_2}$ is weakly mixing on $L^2(M)\ominus\mathbb{C}1$.*

Proof. Write $\Lambda_0 = \Lambda_1\Lambda_2$. Denote the action of Λ on B by $\gamma_v(b) = vbv^*$ for all $v \in \Lambda$, $b \in B$. Define $K < \Lambda$ as the virtual centralizer of Λ_0 inside Λ, i.e., K consists of all $v \in \Lambda$ such that the set $\{wvw^{-1} \mid w \in \Lambda_0\}$ is finite. Equivalently, K consists of the elements that commute with a finite index subgroup of Λ_0. Define $B_0 \subset B$ as the von Neumann algebra generated by the unital $*$-algebra consisting of all $b \in B$ such that $\{\gamma_v(b) \mid v \in \Lambda_0\}$ spans a finite-dimensional subspace of B. Note that B_0 is globally invariant under γ_v, $v \in K$. Viewing

M as the crossed product $M = B \rtimes \Lambda$, we have by construction that the unitary representation $\{\operatorname{Ad} v\}_{v \in \Lambda_0}$ is weakly mixing on $L^2(M) \ominus L^2(B_0 \rtimes K)$ (see Lemma 2.23).

For every $g \in \Gamma$, define $\operatorname{Stab} g < \Gamma \times \Gamma$ to be the stabilizer of g under the left-right action $\Gamma \times \Gamma \curvearrowright \Gamma$. We have $L(\Lambda_0) \subset L(\Gamma \times \Gamma)$ and $L(\Lambda_0) \not\prec L(\operatorname{Stab} g)$ for all $g \in \Gamma$, since $\operatorname{Stab} g \cong \Gamma$ is in class $\mathcal{C}_{\text{rss}}$. Since $B_0 \rtimes K$ quasi-normalizes $L(\Lambda_0)$, Lemma A.4 yields $B_0 \rtimes K \subset L(\Gamma \times \Gamma)$. By definition of K, we can take a decreasing sequence of finite index subgroups $\Lambda_{0,n} < \Lambda_0$ such that $K = \bigcup_n C_\Lambda(\Lambda_{0,n})$. It follows from Lemma 4.14 that $L(K)$ is contained in a hyperfinite von Neumann algebra. So, K is amenable and thus also $B_0 \rtimes K$ is amenable. Since $B_0 \rtimes K$ is normalized by Λ_0, it follows from Lemma 4.14 that $B_0 \rtimes K$ is atomic. So, K is a finite group and B_0 is atomic. We can then take a minimal projection $p \in B_0 \rtimes K$ and finite index subgroups $\Lambda_3 < \Lambda_1$ and $\Lambda_4 < \Lambda_2$ such that p commutes with $\Lambda_3\Lambda_4$.

Lemmas 4.14, 4.15 and 4.16 apply to the commuting non-amenable subgroups $\Lambda_3, \Lambda_4 < \Lambda$. So, by Lemma 4.16, we get that $h_{\Gamma\times\Gamma}(p\Lambda_3\Lambda_4) > 0$. By construction, the unitary representation $\{\operatorname{Ad} v\}_{v \in \Lambda_3\Lambda_4}$ is weakly mixing on $pL(\Gamma \times \Gamma)p \ominus \mathbb{C}p$. For every $g \in \Gamma \times \Gamma$ with $g \neq e$, the centralizer $C_{\Gamma\times\Gamma}(g)$ is either amenable or of the form $\Gamma \times L$ or $L \times \Gamma$ with $L < \Gamma$ amenable, because Γ is torsion-free and in class $\mathcal{C}_{\text{rss}}$ (see end of Section 2.6). Therefore, $L(\Lambda_3\Lambda_4) \not\prec L(C_{\Gamma\times\Gamma}(g))$ for all $g \neq e$. It then first follows from Theorem 4.5 that $p = 1$, so that we could have taken $\Lambda_3 = \Lambda_1$ and $\Lambda_4 = \Lambda_2$, and then also that there exists a unitary $u \in L(\Gamma \times \Gamma)$ such that $u\Lambda_1\Lambda_2 u^* \subset \mathbb{T}(\Gamma \times \Gamma)$.

Since we also proved that $B_0 \rtimes K = \mathbb{C}1$, it follows as well that the unitary representation $\{\operatorname{Ad} v\}_{v \in \Lambda_1\Lambda_2}$ is weakly mixing on $L^2(M) \ominus \mathbb{C}1$. □

Lemma 4.18. *Whenever $\Lambda_2 \subset \mathbb{T}(\{e\} \times \Gamma)$ is a non-amenable subgroup, we have $M \cap \Lambda_2' = L(\Gamma) \otimes 1$.*

Proof. Define $\Gamma_2 < \Gamma$ such that $\mathbb{T}\Lambda_2 = \mathbb{T}(\{e\} \times \Gamma_2)$. Then Γ_2 is non-amenable and $M \cap \Lambda_2' = M \cap L(\{e\} \times \Gamma_2)'$. Since Γ is torsion-free and in class $\mathcal{C}_{\text{rss}}$, we even have that $\Gamma_2 < \Gamma$ is relatively icc. Hence $M \cap L(\{e\} \times \Gamma_2)' \subset A \rtimes (\Gamma \times \{e\})$. Since the action $\{e\} \times \Gamma_2 \curvearrowright A$ is weakly mixing, it follows that $M \cap L(\{e\} \times \Gamma_2)' \subset L(\Gamma \times \{e\})$. So, $M \cap \Lambda_2' \subset L(\Gamma) \otimes 1$ and the converse inclusion is obvious. □

Lemma 4.19. *There exist commuting subgroups $H_1, H_2 < \Lambda$ and a unitary $u \in M$ such that $\Lambda_i < H_i$ for $i = 1, 2$ and $u\mathbb{T}H_1H_2u^* = \mathbb{T}(\Gamma \times \Gamma)$.*

Proof. By Lemma 4.17, after a unitary conjugacy, we may assume that $\Lambda_1, \Lambda_2 < \Lambda$ are commuting non-amenable subgroups with $\Lambda_1\Lambda_2 \subset \mathbb{T}(\Gamma \times \Gamma)$. By Lemma 4.15, we have that $L(\Lambda_1) \prec L(\Gamma) \otimes 1$ and $L(\Lambda_2) \prec 1 \otimes L(\Gamma)$, after exchanging

Λ_1 and Λ_2 if needed. This means that $\Lambda_1 \subset \mathbb{T}(\Gamma \times F)$ and $\Lambda_2 \subset \mathbb{T}(F \times \Gamma)$ for some finite subset $F \subset \Gamma$, by Lemma 2.15. Since Γ is torsion-free, it follows that $\Lambda_1 \subset \mathbb{T}(\Gamma \times \{e\})$ and $\Lambda_2 \subset \mathbb{T}(\{e\} \times \Gamma)$.

Denote by $\{\gamma_v\}_{v\in\Lambda}$ the action of Λ on B. Define $H_1 < \Lambda$ as the virtual centralizer of Λ_2 inside Λ. So, H_1 consists of all $v \in \Lambda$ that commute with a finite index subgroup of Λ_2. Similarly, define B_1 as the von Neumann algebra generated by the $*$-algebra consisting of all $b \in B$ such that $\gamma_v(b) = b$ for all v in a finite index subgroup of Λ_2. Since finite index subgroups of Λ_2 are non-amenable, it follows from Lemma 4.18 that $B_1 \rtimes H_1 \subset L(\Gamma) \otimes 1$. We also find that

$$L(\Gamma) \otimes 1 \subset L(\Lambda_2)' \cap (B \rtimes \Lambda) \subset B_1 \rtimes H_1 \; .$$

So, $B_1 \rtimes H_1 = L(\Gamma) \otimes 1$. In particular, the subgroups $H_1, \Lambda_2 < \Lambda$ commute. Because $\Gamma \in \mathcal{C}_{\text{rss}}$ and $B_1 \subset L(\Gamma) \otimes 1$ is normalized by H_1, it follows that B_1 is atomic. Since $\Lambda_1 < H_1$, the unitaries $v \in \Lambda_1\Lambda_2$ normalize B_1. By Lemma 4.17, they induce a weakly mixing action on B_1. Since B_1 is atomic, this forces $B_1 = \mathbb{C}1$. We conclude that $L(H_1) = L(\Gamma) \otimes 1$.

We now apply Lemmas 4.14, 4.15 and 4.16 to the commuting non-amenable subgroups $H_1, \Lambda_2 < \Lambda$. We conclude that $h_\Gamma(H_1) > 0$. Since $L(H_1) = L(\Gamma) \otimes 1$, the group H_1 is icc. So, the action $\{\operatorname{Ad} v\}_{v\in H_1}$ on $L(\Gamma)$ is weakly mixing. Since for $g \neq e$, the group $C_\Gamma(g)$ is amenable, also $L(H_1) \not\prec L(C_\Gamma(g))$. So, by Theorem 4.5, there exists a unitary $u_1 \in L(\Gamma)$ such that $(u_1 \otimes 1)H_1(u_1^* \otimes 1) = \mathbb{T}(\Gamma \times \{e\})$.

Applying the same reasoning as above to the virtual centralizer of H_1 inside Λ, we find a subgroup $H_2 < \Lambda$, containing Λ_2 and commuting with H_1, and we find a unitary $u_2 \in L(\Gamma)$ such that $(1 \otimes u_2)H_2(1 \otimes u_2^*) \subset \mathbb{T}(\{e\} \rtimes \Gamma)$. So, we get that

$$(u_1 \otimes u_2)H_1H_2(u_1^* \otimes u_2^*) = \mathbb{T}(\Gamma \times \Gamma) \; .$$

□

Finally, we are ready to prove Theorem 4.7. As mentioned above, Theorem 4.1 is a direct consequence of Theorem 4.7.

Proof of Theorem 4.7. We already showed in Proposition 4.8 that any β-invariant gms decomposition $A_0 = B_0 \rtimes \Lambda_0$ of A_0 gives rise to a gms decomposition $M = B_0^\Gamma \rtimes (\Lambda_0^{(\Gamma)} \rtimes (\Gamma \times \Gamma))$ of M. We now show that all gms decompositions are of this form.

Assume that (B, Λ) is an arbitrary gms decomposition of M. By Lemma 4.19 and after a unitary conjugacy, we have $\Gamma \times \Gamma \subset \mathbb{T}\Lambda$. Denoting by $\Delta\colon M \to M \mathbin{\overline{\otimes}} M$ the dual coaction associated with (B, Λ) and given by $\Delta(b) = b \otimes 1$ for all

$b \in B$ and $\Delta(v) = v \otimes v$ for all $v \in \Lambda$, this means that $\Delta(u_{(g,h)})$ is a multiple of $u_{(g,h)} \otimes u_{(g,h)}$ for all $(g,h) \in \Gamma \times \Gamma$.

Denote $A_{0,e} := \pi_e(A_0) \subset A$ and observe that $A_{0,e}$ commutes with all $u_{(g,g)}$, $g \in \operatorname{Ker}\beta$. Then, $\Delta(A_{0,e})$ commutes with all $u_{(g,g)} \otimes u_{(g,g)}$, $g \in \operatorname{Ker}\beta$. Since Γ is a torsion-free group in $\mathcal{C}_{\text{rss}}$, the nontrivial normal subgroup $\operatorname{Ker}\beta < \Gamma$ must be non-amenable and thus relatively icc. It follows that the unitary representation $\{\operatorname{Ad} u_{(g,g)}\}_{g \in \operatorname{Ker}\beta}$ is weakly mixing on $L^2(M) \ominus L^2(A_{0,e})$. This implies that $\Delta(A_{0,e}) \subset A_{0,e} \overline{\otimes} A_{0,e}$.

By Lemma 4.2, we get a crossed product decomposition $A_0 = B_0 \rtimes \Lambda_0$ such that $\pi_e(B_0) = B \cap A_{0,e}$ and $\pi_e(\Lambda_0) = \Lambda \cap A_{0,e}$. For every $g \in \Gamma$, we have that $u_{(g,g)} \in \mathbb{T}\Lambda$. So, $u_{(g,g)}$ normalizes both B and $A_{0,e}$, so that $\beta_g(B_0) = B_0$. Also, $u_{(g,g)}$ normalizes both Λ and $A_{0,e}$, so that $\beta_g(\Lambda_0) = \Lambda_0$. For every $g \in \Gamma$, we have that $u_{(g,e)} \in \mathbb{T}\Lambda$ so that $u_{(g,e)}$ normalizes B and Λ. It follows that $\pi_g(B_0) \subset B$ and $\pi_g(\Lambda_0) \subset \Lambda$ for all $g \in \Gamma$. We conclude that

$$B_0^\Gamma \subset B \quad \text{and} \quad \Lambda_0^{(\Gamma)} \rtimes (\Gamma \times \Gamma) \subset \mathbb{T}\Lambda \; . \tag{4.3}$$

Since A_0 is generated by B_0 and Λ_0, we get that M is generated by B_0^Γ and $\Lambda_0^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$. Since M is also the crossed product of B and Λ, it follows from (4.3) that $B_0^\Gamma = B$ and $\mathbb{T}\Lambda_0^{(\Gamma)} \rtimes (\Gamma \times \Gamma) = \mathbb{T}\Lambda$. In particular, $B_0 \subset A_0$ must be maximal abelian. So, (B_0, Λ_0) is a gms decomposition of A_0 that is $\{\beta_g\}_{g \in \Gamma}$-invariant, while the gms decomposition (B, Λ) of M is unitarily conjugate to the gms decomposition associated with (B_0, Λ_0).

It remains to prove statements (1) and (2). Take $\{\beta_g\}_{g \in \Gamma}$-invariant gms decompositions (B_0, Λ_0) and (B_1, Λ_1) of A_0. Denote by (B, Λ) and (B', Λ') the associated gms decompositions of M.

To prove (1), assume that $u \in M$ is a unitary satisfying $uBu^* = B'$ and $u\mathbb{T}\Lambda u^* = \mathbb{T}\Lambda'$. It follows that for all $g \in \Gamma \times \Gamma$, we have $uu_gu^* \in \mathcal{U}(A)u_{\varphi(g)}$ where $\varphi \in \operatorname{Aut}(\Gamma \times \Gamma)$. Write $u = \sum_{h \in \Gamma \times \Gamma} a_h u_h$ with $a_h \in A$ for the Fourier decomposition of u. It follows that $\{\varphi(g)^{-1}hg \mid g \in \Gamma \times \Gamma\}$ is a finite set whenever $a_h \neq 0$. Since $\Gamma \times \Gamma$ is icc, it follows that a_h can only be nonzero for one $h \in \Gamma \times \Gamma$. So u is of the form $u = a_h u_h$. Since u_h normalizes both B and Λ, we may replace u with uu_h^* so that $u \in \mathcal{U}(A)$.

For each $g \in \Gamma$, we define $E_g \colon A \to A_0$ by $E_g(x) = \pi_g^{-1}(E_{\pi_g(A_0)}(x))$, $x \in A$. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence in Γ that tends to infinity, and let $b \in B_0$. Since $(\pi_{g_n}(b))_{n \in \mathbb{N}}$ is a asymptotically central in A, we get that

$$B_1 \ni E_{g_n}(u\pi_{g_n}(b)u^*) \to b \; ,$$

hence $B_0 \subset B_1$. By symmetry, it follows that $B_0 = B_1$. Similarly, we see that $\mathbb{T}\Lambda_0 = \mathbb{T}\Lambda_1$ so we conclude that (B_0, Λ_0) and (B_1, Λ_1) are identical gms decompositions of A_0.

To prove (2), assume that $\theta \in \text{Aut}(M)$ is an automorphism satisfying $\theta(B) = B'$ and $\theta(\mathbb{T}\Lambda) = \mathbb{T}\Lambda'$. Define the commuting subgroups $\Lambda_1, \Lambda_2 < \Lambda'$ such that $\theta(\mathbb{T}(\Gamma \times \{e\})) = \mathbb{T}\Lambda_1$ and $\theta(\mathbb{T}(\{e\} \times \Gamma)) = \mathbb{T}\Lambda_2$. Applying Lemma 4.19 to the gms decomposition (B', Λ') of M and these commuting subgroups $\Lambda_1, \Lambda_2 < \Lambda'$, we find commuting subgroups $H_1, H_2 < \Lambda'$ and a unitary $u \in M$ such that $\Lambda_i < H_i$ for $i = 1, 2$ and $u\mathbb{T}H_1H_2u^* = \mathbb{T}(\Gamma \times \Gamma)$. Since $\Gamma \times \{e\}$ and $\{e\} \times \Gamma$ are each other's centralizer inside Λ and since $\theta(\mathbb{T}\Lambda) = \mathbb{T}\Lambda'$, we must have that $\Lambda_i = H_i$ for $i = 1, 2$.

Writing $\theta_1 = \text{Ad}\, u \circ \theta$, we have proved that $\theta_1(\mathbb{T}(\Gamma \times \Gamma)) = \mathbb{T}(\Gamma \times \Gamma)$. This equality induces an automorphism of $\Gamma \times \Gamma$. Since Γ is a torsion-free group in $\mathcal{C}_{\text{rss}}$, all automorphisms of $\Gamma \times \Gamma$ are either of the form $(g, h) \mapsto (\varphi_1(g), \varphi_2(h))$ or of the form $(g, h) \mapsto (\varphi_1(h), \varphi_2(g))$ for some automorphisms $\varphi_i \in \text{Aut}(\Gamma)$. Indeed, $\varphi(\Gamma \times \{e\})$ and $\varphi(\{e\} \times \Gamma)$ are commuting non-amenable subgroups of $\Gamma \times \Gamma$. Since Γ is in $\mathcal{C}_{\text{rss}}$, it follows that $\varphi(\Gamma \times \{e\}) \prec L(\Gamma) \otimes 1$ or $\varphi(\Gamma \times \{e\}) \prec 1 \otimes L(\Gamma)$. In the first case, we get by Lemma 2.15 that $\varphi(\Gamma \times \{e\}) \subset \mathbb{T}(\Gamma \times F)$ for some finite subset $F \subset \Gamma$. Since Γ is torsion-free, it follows that $\varphi(\Gamma \times \{e\}) \subset \mathbb{T}(\Gamma \times \{e\})$. In the second case, we get instead that $\varphi(\Gamma \times \{e\}) \subset \mathbb{T}(\{e\} \times \Gamma)$. Reasoning similarly with $\varphi(\{e\} \times \Gamma)$, we see that φ has the desired form.

The formulas $\zeta(u_{(g,h)}) = u_{(h,g)}$ and $\zeta(\pi_k(a)) = \pi_{k^{-1}}(\beta_k(a))$ for all $g, h, k \in \Gamma$ and $a \in A_0$ define an automorphism $\zeta \in \text{Aut}(M)$ satisfying $\zeta(B') = B'$ and $\zeta(\Lambda') = \Lambda'$. So composing θ with ζ if necessary, we may assume that we have $\varphi_1, \varphi_2 \in \text{Aut}(\Gamma)$ such that $\theta_1(u_{(g,h)}) \in \mathbb{T}u_{(\varphi_1(g), \varphi_2(h))}$ for all $g, h \in \Gamma$. We still have that the gms decompositions $(\theta_1(B), \theta_1(\Lambda))$ and (B', Λ') of M are unitarily conjugate.

Because $u_{(g,g)}$ commutes with $\pi_e(A_0)$ for all $g \in \text{Ker}\,\beta$, the unitary representation on $L^2(M) \ominus \mathbb{C}1$ given by $\{\text{Ad}\, u_{(\varphi_1(g), \varphi_2(g))}\}_{g \in \text{Ker}\,\beta}$ is not weakly mixing. There thus exists a $k \in \Gamma$ such that $\varphi_1(g)k = k\varphi_2(g)$ for all g in a finite index subgroup of $\text{Ker}\,\beta$. So after replacing θ_1 by $(\text{Ad}\, u_{(e,k)}) \circ \theta_1$, we may assume that $\varphi_1(g) = \varphi_2(g)$ for all g in a finite index subgroup of $\text{Ker}\,\beta$. Let $K < \text{Ker}\,\beta$ denote this finite index subgroup, i.e., $K = \{g \in \text{Ker}\,\beta \mid \varphi_1(g) = \varphi_2(g)\}$.

We now show that in fact, $K = \text{Ker}\,\beta$. Assume that this is not the case. Then there exists $g \in \text{Ker}\,\beta$ such that $\varphi_1(g) \neq \varphi_2(g)$. Since $\text{Ker}\,\beta < \Gamma$ is relatively icc and since $K < \text{Ker}\,\beta$ has finite index, we have that the set

$$\{\varphi_1(h)(\varphi_1(g)\varphi_2(g)^{-1})\varphi_1(h)^{-1} \mid h \in K\} = \{\varphi_1(hgh^{-1})\varphi_2(hg^{-1}h^{-1}) \mid h \in K\}$$

is infinite. On the other hand, for $h, k \in K$ we have that

$$\begin{aligned}
\varphi_1(hgh^{-1})\varphi_2(hg^{-1}h^{-1}) = \varphi_1(kgk^{-1})\varphi_2(kg^{-1}k^{-1}) \quad &\Leftrightarrow \\
\varphi_1(kg^{-1}k^{-1}hgh^{-1}) = \varphi_2(kg^{-1}k^{-1}hgh^{-1}) \quad &\Leftrightarrow \\
\varphi_1(g^{-1}k^{-1}hg) = \varphi_2(g^{-1}k^{-1}hg) \quad &\Leftrightarrow \\
g^{-1}k^{-1}hg \in K \quad &\Leftrightarrow \\
g^{-1}hgK = g^{-1}kgK. &
\end{aligned}$$

Since $K < \operatorname{Ker}\beta$ has finite index, we have reached a contradiction. We conclude that $K = \operatorname{Ker}\beta$, i.e., $\varphi_1(g) = \varphi_2(g)$ for all $g \in \operatorname{Ker}\beta$. Since $\operatorname{Ker}\beta$ is a normal subgroup of Γ, it follows that $\varphi_1(k)\varphi_2(k)^{-1}$ commutes with $\varphi_1(g)$ for all $k \in \Gamma$ and $g \in \operatorname{Ker}\beta$. Using again that $\operatorname{Ker}\beta$ is relatively icc, it follows that $\varphi_1 = \varphi_2$ and we denote this automorphism by φ.

Taking the commutant of the unitaries $u_{(g,g)}$, $g \in \operatorname{Ker}\beta$, it follows that $\theta_1(\pi_e(A_0)) = \pi_e(A_0)$. We define the automorphism $\theta_0 \in \operatorname{Aut}(A_0)$ such that $\theta_1 \circ \pi_e = \pi_e \circ \theta_0$. Since $\theta_1(u_{(g,g)}) \in \mathbb{T}u_{(\varphi(g),\varphi(g))}$ and since $\pi_e \circ \beta_g = \operatorname{Ad} u_{(g,g)} \circ \pi_e$, we get that $\theta_0 \circ \beta_g = \beta_{\varphi(g)} \circ \theta_0$ for all $g \in \Gamma$. It follows that $(\theta_0(B_0), \theta_0(\Lambda_0))$ is a $\{\beta_g\}_{g\in\Gamma}$-invariant gms decomposition of A_0. The associated gms decomposition of M is $(\theta_1(B), \theta_1(\Lambda))$. This gms decomposition of M is unitarily conjugate with the gms decomposition (B', Λ'). It then follows from (1) that $\theta_0(B_0) = B_1$ and $\theta_0(\mathbb{T}\Lambda_0) = \mathbb{T}\Lambda_1$. ☐

We finally prove a criterion for when the possible gms Cartan subalgebras of M are conjugate.

Proposition 4.20. *Under the same hypotheses and with the same notations as in Theorem 4.7, if (B_0, Λ_0) and (B_1, Λ_1) are $\{\beta_g\}_{g\in\Gamma}$-invariant gms decompositions of A_0, then the associated Cartan subalgebras of M given by B_0^Γ and B_1^Γ are*

(1) *unitarily conjugate iff $B_0 = B_1$;*

(2) *conjugate by an automorphism of M iff there exists a trace-preserving automorphism $\theta_0 \colon A_0 \to A_0$ and an automorphism $\varphi \in \operatorname{Aut}(\Gamma)$ such that $\theta_0(B_0) = B_1$ and $\theta_0 \circ \beta_g = \beta_{\varphi(g)} \circ \theta_0$ for all $g \in \Gamma$.*

Proof. To prove (1), it suffices to prove that $B_0^\Gamma \not\prec B_1^\Gamma$ if $B_0 \neq B_1$. Take a unitary $u \in \mathcal{U}(B_0)$ such that $u \notin B_1$. Then $\|E_{B_1}(u)\|_2 < 1$. Let $\{g_1, g_2, \ldots\}$

be an enumeration of Γ and define the sequence of unitaries $(w_n) \subset \mathcal{U}(B_0^\Gamma)$ by $w_n = \pi_{g_{n+1}}(u)\,\pi_{g_{n+2}}(u)\,\cdots\,\pi_{g_{2n}}(u)$. We will show that

$$\|E_{B_1^\Gamma}(xw_ny)\|_2 \to 0 \quad \text{for all } \ x, y \in M, \tag{4.4}$$

so that $B_0^\Gamma \not\prec B_1^\Gamma$. It is enough to prove this for $x = au_{(g,h)}$ and $y = bu_{(s,t)}$ with $g, h, s, t \in \Gamma$ and $a, b \in A$. Moreover, we may assume that $a, b \in \pi_F(A_0^F)$ for some finite subset $F \subset \Gamma$. If $(g, h) \neq (s, t)^{-1}$, we have that $\|E_{B_1^\Gamma}(xw_ny)\|_2 = 0$ for all n, so we assume that $(g, h) = (s, t)^{-1}$. Then

$$\|E_{B_1^\Gamma}(xw_ny)\|_2 = \|E_{B_1^\Gamma}(a\sigma_{(g,h)}(w_n)\sigma_{(g,h)}(b))\|_2.$$

Choose n_0 so large that g_n does not belong to the finite set $F \cup g^{-1}Fh$ for $n \geq n_0$. For $n \geq n_0$, we then have that

$$\begin{aligned}\|E_{B_1^\Gamma}(a\sigma_{(g,h)}(w_n)\sigma_{(g,h)}(b))\|_2 &= \|E_{B_1^\Gamma}(a\sigma_{(g,h)}(b))\|_2\|E_{B_1^\Gamma}(\sigma_{(g,h)}(w_n))\|_2 \\ &\leq \|a\|\|b\|\|E_{B_1}(\beta_h(u))\|_2^n \to 0,\end{aligned}$$

since $\|E_{B_1}(\beta_h(u))\|_2 < 1$. So, (4.4) holds and this finishes the proof of (1).

To prove (2), denote by (B, Λ) and (B', Λ') the gms decompositions of M associated with (B_0, Λ_0) and (B_1, Λ_1). Assume that $\theta \in \mathrm{Aut}(M)$ satisfies $\theta(B) = B'$. Then, $(B', \theta(\Lambda))$ is a gms decomposition of M. By Theorem 4.7, $(B', \theta(\Lambda))$ is unitarily conjugate with the gms decomposition associated with a $\{\beta_g\}_{g\in\Gamma}$-invariant gms decomposition (B_2, Λ_2) of A_0. By (1), we must have $B_2 = B_1$. So the gms decompositions associated with (B_0, Λ_0) and (B_1, Λ_2) are conjugate by an automorphism of M. By Theorem 4.7(2) there exists an automorphism $\theta_0 \in \mathrm{Aut}(A_0)$ as in (2). □

4.5 Examples of II_1 factors with a prescribed number of group measure space decompositions

For every amenable tracial von Neumann algebra (A_0, τ_0) and for every trace-preserving action of $\Gamma = \mathbb{F}_\infty$ on (A_0, τ_0) with nontrivial kernel, Theorem 4.7 gives a complete description of all gms decompositions of the II_1 factor $M = A_0^\Gamma \rtimes (\Gamma \times \Gamma)$ in terms of the Γ-invariant gms decompositions of A_0.

In this section, we construct a family of examples where these Γ-invariant gms decompositions of A_0 can be explicitly determined. In particular, this gives a proof of Theorem A. We will construct A_0 of the form $A_0 = L^\infty(K) \rtimes H_1$ where H_1 is a countable abelian group and $H_1 \hookrightarrow K$ is an embedding of H_1 as a dense subgroup of the compact second countable group K. Note that we can equally view K as $\widehat{H_2}$ where H_2 is a countable abelian group and the embedding $H_1 \hookrightarrow \widehat{H_2}$ is given by a bicharacter $\Omega\colon H_1 \times H_2 \to \mathbb{T}$ that is non-degenerate: if $g \in H_1$ and $\Omega(g,h) = 1$ for all $h \in H_2$, then $g = e$; if $h \in H_2$ and $\Omega(g,h) = 1$ for all $g \in H_1$, then $h = e$.

We consider the crossed product $L^\infty(\widehat{H_2}) \rtimes H_1$ associated with the left translation action of H_1 on $\widehat{H_2}$. We can then view $L^\infty(\widehat{H_2}) \rtimes H_1$ as being generated by the group von Neumann algebras $L(H_1)$ and $L(H_2) \cong L^\infty(\widehat{H_2})$, with canonical unitaries $\{u_g\}_{g \in H_1}$ and $\{u_h\}_{h \in H_2}$ satisfying $u_g u_h = \Omega(g,h) u_h u_g$ for all $g \in H_1$, $h \in H_2$.

Given a subset $S \subset H_1$, we define $S^\perp < H_2$ to be the orthogonal complement with respect to Ω, i.e., the set of elements $h \in H_2$ such that $\Omega(g,h) = 1$ for all $g \in S$. Similarly, we define $T^\perp < H_1$ when $T \subset H_2$. We call a direct sum decomposition $H_1 = S_1 \oplus T_1$ *admissible* if the closures of S_1, T_1 in $\widehat{H_2}$ give a direct sum decomposition of $\widehat{H_2}$.

Lemma 4.21. *$H_1 = S_1 \oplus T_1$ is admissible if and only if there exists a direct sum decomposition $H_2 = S_2 \oplus T_2$ satisfying $S_1 = S_2^\perp$, $S_2 = S_1^\perp$, $T_2 = T_1^\perp$ and $T_1 = T_2^\perp$.*

Proof. Assume that $H_1 = S_1 \oplus T_1$ is admissible. Since $H_2 = \widehat{\widehat{H_2}}$ and since $\widehat{H_2} = \overline{S_1} \oplus \overline{T_1}$, we have that $H_2 = S_2 \oplus T_2$ where $S_2 := \widehat{\overline{T_1}}$ and $T_2 := \widehat{\overline{S_1}}$. Note that

$$S_2 = \{h_2 \in H_2 \mid \varphi(h_2) = 1 \text{ for all } \varphi \in \overline{S_1}\}$$

$$= \{h_2 \in H_2 \mid \Omega(s_1, h_2) = 1 \text{ for all } s_1 \in S_1\} = S_1^\perp,$$

and similarly $T_2 = T_1^\perp$. Moreover, since $\widehat{H_2} = \overline{S_1} \oplus \overline{T_1}$ we have that $S_2^\perp = (S_1^\perp)^\perp = \overline{S_1} \cap H_1 = S_1$ and similarly $T_2^\perp = T_1$.

For the converse, assume instead that $H_2 = S_2 \oplus T_2$ with $S_1 = S_2^\perp$, $S_2 = S_1^\perp$, $T_2 = T_1^\perp$ and $T_1 = T_2^\perp$. If $\varphi \in \overline{S_1} \cap \overline{T_1}$, we have that $\varphi(h_2) = 1$ for all $h_2 \in S_1^\perp = S_2$ and all $h_2 \in T_1^\perp = T_2$. Since $H_2 = S_2 \oplus T_2$, this implies that $\varphi = 1$. So, $\widehat{H_2} = \overline{S_1} \oplus \overline{T_1}$ and thus the decomposition $H_1 = S_1 \oplus T_1$ is admissible. □

Proposition 4.22. *Let L_1, L_2 be torsion-free abelian groups and $L_1 \hookrightarrow \widehat{L_2}$ a dense embedding. Put $\Gamma_0 = \mathrm{SL}(3,\mathbb{Z})$ and $H_i = L_i^3$. Consider the natural action of Γ_0 on the direct sum embedding $H_1 \hookrightarrow \widehat{H_2}$, defining the trace-preserving action $\{\beta_g\}_{g \in \Gamma_0}$ of Γ_0 on $A_0 = L^\infty(\widehat{H_2}) \rtimes H_1$.*

Whenever $L_1 = P_1 \oplus Q_1$ *is an admissible direct sum decomposition with corresponding* $L_2 = P_2 \oplus Q_2$*, put* $S_i = P_i^3$*,* $T_i = Q_i^3$ *and define* $B_0 = L(S_1) \vee L(S_2)$*,* $\Lambda_0 = T_1 T_2$*.*

Then (B_0, Λ_0) *is a* $\{\beta_g\}_{g \in \Gamma_0}$*-invariant gms decomposition of* A_0*. Every* $\{\beta_g\}_{g \in \Gamma_0}$*-invariant gms decomposition of* A_0 *is of this form for a unique admissible direct sum decomposition* $L_1 = P_1 \oplus Q_1$*.*

Proof. Let $L_1 = P_1 \oplus Q_1$ be an admissible direct sum decomposition, and let $B_0 = L(S_1) \vee L(S_2)$ and $\Lambda_0 = T_1 T_2$ be as above. We clearly have a crossed product decomposition $A_0 = B_0 \rtimes \Lambda_0$ satisfying $\beta_g(B_0) = B_0$, $\beta_g(\Lambda_0) = \Lambda_0$ for all $g \in \Gamma_0$. Using the relation $u_g u_h = \Omega(g,h) u_h u_g$ for $g \in H_1$, $h \in H_2$ and the fact that $S_1 = S_2^{\perp}$, $S_2 = S_1^{\perp}$, we also get that $B_0 \subset A_0$ is maximal abelian. So, (B_0, Λ_0) is a $\{\beta_g\}_{g \in \Gamma_0}$-invariant gms decomposition of A_0.

Conversely, let (B_0, Λ_0) be an arbitrary $\{\beta_g\}_{g \in \Gamma_0}$-invariant gms decomposition of A_0. Define the subgroup $\Gamma_1 < \Gamma_0$ as

$$\Gamma_1 = \Gamma_0 \cap \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

We also put $H_1^{(1)} = L_1 \oplus 0 \oplus 0$. Because L_1 is torsion-free, the following holds.

- $a \cdot g = g$ for all $a \in \Gamma_1$ and $g \in H_1^{(1)}$.
- $\Gamma_1 \cdot g$ is infinite for all $g \in H_1 \setminus H_1^{(1)}$.
- $\Gamma_1^T \cdot h$ is infinite for all $h \in H_2 \setminus \{0\}$, where Γ_1^T denotes the transpose of Γ_1.

From these observations, it follows that $L(H_1^{(1)})$ is equal to the algebra of Γ_1-invariant elements in A_0 and that $L(H_1^{(1)})$ is also equal to the algebra of elements in A_0 that are fixed by some finite index subgroup of Γ_1. Since both B_0 and Λ_0 are globally Γ_0-invariant, it follows that $L(H_1^{(1)}) = B_0^{(1)} \rtimes \Lambda_0^{(1)}$ where $\Lambda_0^{(1)} < \Lambda_0$ denotes the subgroup of elements that are fixed by a finite index subgroup of Γ_1 and $B_0^{(1)} \subset B_0$ denotes the von Neumann subalgebra generated by elements that are fixed by a finite index subgroup of Γ_1.

We similarly consider $H_1^{(2)} = 0 \oplus L_1 \oplus 0$ and $H_1^{(3)} = 0 \oplus 0 \oplus L_1$. We conclude that $L(H_1^{(i)}) = B_0^{(i)} \rtimes \Lambda_0^{(i)}$ for all $i = 1, 2, 3$. The subgroups $H_1^{(1)}, H_1^{(2)}$ and $H_1^{(3)}$ generate H_1 and H_1 is abelian. So, "everything" commutes and we conclude that $L(H_1) = B_1 \rtimes \Lambda_1$ for some von Neumann subalgebra $B_1 \subset B_0$ and subgroup $\Lambda_1 < \Lambda_0$. A similar reasoning applies to $L^\infty(\widehat{H_2}) = L(H_2)$ and we get that $L(H_2) = B_2 \rtimes \Lambda_2$ for $B_2 \subset B_0$ and $\Lambda_2 < \Lambda_0$.

Since $L(H_1)L(H_2)$ is $\|\cdot\|_2$-dense in A_0 and $L(H_1)\cap L(H_2)=\mathbb{C}1$, we get that $\Lambda_1\Lambda_2=\Lambda_0=\Lambda_2\Lambda_1$ and $\Lambda_1\cap\Lambda_2=\{e\}$. It then follows that for all $b_i\in B_i$ and $s_i\in\Lambda_i$, $i=1,2$, we have that $E_{B_0}(b_1v_{s_1}b_2v_{s_2})$ equals zero unless $s_1=e$ and $s_2=e$, in which case, we get b_1b_2. We conclude that B_1B_2 is $\|\cdot\|_2$-dense in B_0.

For every $x\in L(H_i)$ and $g\in H_i$, we denote by $(x)_g=\tau(xu_g^*)$ the g-th Fourier coefficient of x. Comparing Fourier decompositions, we get for all $x_i\in L(H_i)$ that

$$x_1x_2=x_2x_1 \;\text{ iff }\; \Omega(g,h)=1 \;\text{ whenever }\; g\in H_1, h\in H_2, (x_1)_g\neq 0, (x_2)_h\neq 0\;. \tag{4.5}$$

Since $B_i\subset L(H_i)$ and since B_1, B_2 commute, we obtain from (4.5) subgroups $S_i\subset H_i$ such that $\Omega(g,h)=1$ for all $g\in S_1$, $h\in S_2$ and such that $B_i\subset L(S_i)$. Since B_1B_2 is dense in B_0, it follows that $B_0\subset L(S_1)\vee L(S_2)$. Since $L(S_1)\vee L(S_2)$ is abelian and B_0 is maximal abelian, we conclude that $B_0=L(S_1)\vee L(S_2)$. Thus, $B_i=L(S_i)$ for $i=1,2$. When $g\in S_2^\perp$, the unitary u_g commutes with $L(S_2)$, but also with $L(S_1)$ because $L(S_1)=B_1\subset L(H_1)$ and $L(H_1)$ is abelian. Since B_0 is maximal abelian, we get that $g\in S_1$. So, $S_1=S_2^\perp$ and similarly $S_2=S_1^\perp$.

The next step of the proof is to show that Λ_0 is abelian, i.e., that Λ_1 and Λ_2 are commuting subgroups of Λ_0. Put $T_i=H_i/S_i$. Since $S_1=S_2^\perp$ and $S_2=S_1^\perp$, we have the canonical dense embeddings $T_1\hookrightarrow\widehat{S_2}$ and $T_2\hookrightarrow\widehat{S_1}$. Viewing $L^\infty(\widehat{S_1}\times\widehat{S_2})=L(S_1)\vee L(S_2)$ as a Cartan subalgebra of A_0, the associated equivalence relation is given by the orbits of the action

$$T_1\times T_2\curvearrowright\widehat{S_1}\times\widehat{S_2}: \quad (g,h)\cdot(y,z)=(h\cdot y, g\cdot z)\;,$$

where the actions on the right, namely $T_1\curvearrowright\widehat{S_2}$ and $T_2\curvearrowright\widehat{S_1}$, are given by translation. Indeed, the automorphisms induced by the normalizing unitaries $\{u_g\}_{g\in H_1}$ and $\{u_h\}_{h\in H_2}$ of $L^\infty(\widehat{S_1}\times\widehat{S_2})$ are exactly given by the above action. But viewing $B_0=L(S_1)\vee L(S_2)$, the same equivalence relation is given by the orbits of the action $\Lambda_0\curvearrowright\widehat{S_1}\times\widehat{S_2}$. So, we have an orbit equivalence $\Phi\colon\widehat{S_1}\times\widehat{S_2}\to\widehat{S_1}\times\widehat{S_2}$ satisfying $\Phi((T_1\times T_2)\cdot x)=\Lambda_0\cdot\Phi(x)$ for $x\in\widehat{S_1}\times\widehat{S_2}$. To such an orbit equivalence, we can associate a 1-cocycle $\omega\colon(T_1\times T_2)\times(\widehat{S_1}\times\widehat{S_2})\to\Lambda_0$ defined by the formula

$$\Phi(t\cdot x)=\omega(t,x)\cdot\Phi(x)\quad\text{for }\; t\in T_1\times T_2,\; x\in\widehat{S_1}\times\widehat{S_2}.$$

The 1-cocycle relation for ω states that

$$\omega(t_1t_2,x)=\omega(t_1,t_2\cdot x)\omega(t_2,x)\quad\text{for }\; t_1,t_2\in T_1\times T_2,\;\; x\in\widehat{S_1}\times\widehat{S_2}.$$

By construction (see also [Si55, Lemma 2.2 and Corollary 2.3]), for all $g\in H_1$, $h\in H_2$ and $s\in\Lambda_0$, the support of the Fourier coefficient $E_{B_0}(v_s^*u_gu_h)$ is the projection in $L^\infty(\widehat{S_1} \times \widehat{S_2})$ given by the set

$$\{(y,z) \in \widehat{S_1} \times \widehat{S_2} \mid \omega((gS_1, hS_2),(y,z)) = s\} \ .$$

Since $L(H_1) = B_1 \rtimes \Lambda_1$, we get for all $g \in T_1$ that the projection given by the set

$$\{(y,z) \in \widehat{S_1} \times \widehat{S_2} \mid \omega((g,e),(y,z)) = s\}$$

belongs to B_1 for all $s \in \Lambda_0$ and is zero for $s \notin \Lambda_1$. This means that the map $(y,z) \mapsto \omega((g,e),(y,z))$ only depends on the first variable and takes values in Λ_1 a.e. Reasoning similarly for $h \in T_2$, we find $\omega_i \colon T_i \times \widehat{S}_i \to \Lambda_i$ such that

$$\omega((g,e),(y,z)) = \omega_1(g,y) \quad \text{and} \quad \omega((e,h),(y,z)) = \omega_2(h,z) \quad \text{a.e.}$$

Writing $(g,h) = (g,e)(e,h)$ and $(g,h) = (e,h)(g,e)$, the 1-cocycle relation for ω implies that

$$\omega_1(g, h \cdot y)\, \omega_2(h,z) = \omega_2(h, g \cdot z)\, \omega_1(g,y) \tag{4.6}$$

for all $g \in T_1$, $h \in T_2$ and a.e. $y \in \widehat{S_1}$, $z \in \widehat{S_2}$.

Define the subgroup $G_1 < \Lambda_1$ by

$$G_1 = \{s \in \Lambda_1 \mid \forall t \in \Lambda_2, tst^{-1} \in \Lambda_1\} \ .$$

Similarly, define $G_2 < \Lambda_2$. Note that G_1 and G_2 are normal subgroups of Λ_0 since $\Lambda_0 = \Lambda_1\Lambda_2$. Since $\Lambda_1 \cap \Lambda_2 = \{e\}$, we also have that G_1 and G_2 commute. Rewriting (4.6) as

$$\omega_1(g,y)\, \omega_2(h,z) = \omega_2(h, g \cdot z)\, \omega_1(g, h^{-1} \cdot y) \ ,$$

we find that for all $g \in T_1$, $h \in T_2$ and a.e. $y, y' \in \widehat{S_1}$, $z \in \widehat{S_2}$,

$$\omega_2(h,z)^{-1}\, \omega_1(g,y')^{-1}\, \omega_1(g,y)\, \omega_2(h,z) \in \Lambda_1 \ .$$

Since $L(H_2) = B_2 \rtimes \Lambda_2$, the essential range of ω_2 equals Λ_2. It thus follows that

$$\omega_1(g,y')^{-1}\, \omega_1(g,y) \in G_1$$

for all $g \in T_1$ and a.e. $y, y' \in \widehat{S_1}$. For every $g \in T_1$, we choose $\delta_1(g) \in \Lambda_1$ such that $\omega_1(g,y) = \delta_1(g)$ on a non-negligible set of $y \in \widehat{S_1}$. We conclude that $\omega_1(g,y) = \delta_1(g)\, \mu_1(g,y)$ with $\mu_1(g,y) \in G_1$ a.e. We similarly decompose $\omega_2(h,z) = \delta_2(h)\, \mu_2(h,z)$.

With these decompositions of ω_1 and ω_2 and using that G_1, G_2 are commuting normal subgroups of Λ_0, it follows from (4.6) that for all $g \in T_1$, $h \in T_2$, the commutator $\delta_2(h)^{-1}\delta_1(g)^{-1}\delta_2(h)\delta_1(g)$ belongs to G_1G_2, so that it can be uniquely written as $\eta_1(g,h)\eta_2(g,h)^{-1}$ with $\eta_i(g,h) \in G_i$. It then follows from (4.6) that

$$\mu_1(g, h\cdot y) = \delta_2(h)\,\eta_1(g,h)\,\mu_1(g,y)\,\delta_2(h)^{-1}\ , \tag{4.7}$$

$$\mu_2(h, g\cdot z) = \delta_1(g)\,\eta_2(g,h)\,\mu_2(h,z)\,\delta_1(g)^{-1}\ ,$$

almost everywhere. Since $S_1 < H_1$ is torsion-free, $\widehat{S_1}$ has no finite quotients and thus no proper closed finite index subgroups. It follows that finite index subgroups of T_2 act ergodically on $\widehat{S_1}$. We claim that for every $g \in T_1$, the map $y \mapsto \mu_1(g,y)$ is essentially constant. To prove this claim, fix $g \in T_1$ and denote $\xi\colon \widehat{S_1} \to G_1 : \xi(y) = \mu_1(g,y)$. For every $h \in T_2$, define the permutation

$$\rho_h\colon G_1 \to G_1 : \quad \rho_h(s) = \delta_2(h)\,\eta_1(g,h)\,s\,\delta_2(h)^{-1}\ .$$

So, (4.7) says that $\xi(h\cdot y) = \rho_h(\xi(y))$ for all $h \in T_2$ and a.e. $y \in \widehat{S_1}$. Defining $V_1 \subset G_1$ as the essential range of ξ, it follows that $\{\rho_h\}_{h\in T_2}$ is an action of T_2 on V_1. The push forward via ξ of the Haar measure on $\widehat{S_1}$ is a $\{\rho_h\}_{h\in T_2}$-invariant probability measure on the countable set V_1 and has full support. It follows that all orbits of the action $\{\rho_h\}_{h\in T_2}$ on V_1 are finite. Choosing $s \in V_1$, the set $\xi^{-1}(\{s\}) \subset \widehat{S_1}$ is non-negligible and globally invariant under a finite index subgroup of T_2. It follows that $\xi(y) = s$ for a.e. $y \in \widehat{S_1}$, thus proving the claim.

Similarly, for every $h \in T_2$, the map $z \mapsto \mu_2(h,z)$ is essentially constant. So we have proved that $\omega_1(g,y) = \delta_1(g)$ and $\omega_2(h,z) = \delta_2(h)$ a.e. But then, (4.6) implies that Λ_1 and Λ_2 commute, so that Λ_0 is an abelian group.

Since A_0 is a factor, $B_0^{\Lambda_0} = \mathbb{C}1$ and thus $L(\Lambda_0) \subset A_0$ is maximal abelian. Since $L(\Lambda_0) = L(\Lambda_1) \vee L(\Lambda_2)$ with $L(\Lambda_i) \subset L(H_i)$, the same reasoning as with $B_i \subset L(H_i)$, using (4.5), gives us subgroups $T_i \subset H_i$ such that $L(\Lambda_i) = L(T_i)$ and $T_1 = T_2^{\perp}$, $T_2 = T_1^{\perp}$. Since $L(H_i) = B_i \rtimes \Lambda_i$ with $B_i = L(S_i)$ and $L(\Lambda_i) = L(T_i)$, we get that $H_i = S_i \oplus T_i$.

So far, we have proved that $B_0 = L(S_1) \vee L(S_2)$ and $L(\Lambda_0) = L(T_1) \vee L(T_2)$. In any crossed product $B_0 \rtimes \Lambda_0$ by a faithful action, the only unitaries in $L(\Lambda_0)$ that normalize B_0 are the multiples of the canonical unitaries $\{v_s\}_{s\in\Lambda}$. Therefore, $\mathbb{T}T_1T_2 = \mathbb{T}\Lambda_0$. We have thus proved that the gms decomposition (B_0, Λ_0) is identical to the gms decomposition $(L(S_1) \vee L(S_2), T_1T_2)$.

Since Λ_0, H_1 and H_2 are globally $\{\beta_g\}_{g\in\Gamma_0}$-invariant, it follows that T_i is a globally $\mathrm{SL}(3,\mathbb{Z})$-invariant subgroup of H_i. Thus, $T_i = Q_i^3$ for some subgroup $Q_i < L_i$. Since B_0, H_1 and H_2 are globally Γ_0-invariant, it follows in the same way that $S_i = P_i^3$ for some subgroups $P_i < L_i$. Then, $L_i = P_i \oplus Q_i$ and P_1, P_2, as well as Q_1, Q_2, are each other's orthogonal complement under Ω. So, $L_1 = P_1 \oplus Q_1$ is an admissible direct sum decomposition. □

We now combine Proposition 4.22 with Theorem 4.7 and Proposition 4.20. We fix once and for all $\Gamma = \mathbb{F}_\infty$, $\Gamma_0 = \mathrm{SL}(3,\mathbb{Z})$ and a surjective homomorphism $\beta\colon \Gamma \twoheadrightarrow \Gamma_0$ so that the automorphism $g \mapsto (g^{-1})^T$ of Γ_0 lifts to an automorphism of Γ. An obvious way to do this is by enumerating $\Gamma_0 = \{g_0, g_1, \ldots\}$ and defining $\beta\colon \Gamma \to \Gamma_0$ by $\beta(s_i) = g_i$ for $i \geq 0$, where $(s_i)_{i\in\mathbb{N}}$ are free generators of Γ. Note that $\operatorname{Ker}\beta$ is automatically nontrivial.

We also fix countable abelian torsion-free groups L_1, L_2 and a dense embedding $L_1 \hookrightarrow \widehat{L_2}$. Put $H_i = L_i^3$ and let Γ act on $H_1 \hookrightarrow \widehat{H_2}$ through β. Then define $A_0 = L^\infty(\widehat{H_2}) \rtimes H_1$ together with the natural action $\beta\colon \Gamma \curvearrowright (A_0, \tau_0)$. Put $(A,\tau) = (A_0,\tau_0)^\Gamma$ with the action $\Gamma \times \Gamma \curvearrowright (A,\tau)$ given by $(g,h)\cdot \pi_k(a) = \pi_{gkh^{-1}}(\beta_h(a))$ for all $g,h,k \in \Gamma$, $a \in A_0$. Write $M = A \rtimes (\Gamma \times \Gamma)$.

We call an automorphism of L_1 admissible if it extends to a continuous automorphism of $\widehat{L_2}$. We call an isomorphism $\theta\colon L_1 \to L_2$ admissible if it extends to a continuous isomorphism $\widehat{L_2} \to \widehat{L_1}$.

Theorem 4.23. *Whenever $L_1 = P_1 \oplus Q_1$ is an admissible direct sum decomposition with corresponding $L_2 = P_2 \oplus Q_2$, we define $B(P_1,Q_1) := (L(P_1^3) \vee L(P_2^3))^\Gamma$ and $\Lambda(P_1,Q_1) = (Q_1^3 \oplus Q_2^3)^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$.*

- *Every $B(P_1,Q_1), \Lambda(P_1,Q_1)$ gives a gms decomposition of M.*
- *Every gms decomposition of M is unitarily conjugate with a $B(P_1,Q_1)$, $\Lambda(P_1,Q_1)$ for a unique admissible direct sum decomposition $L_1 = P_1 \oplus Q_1$.*
- *Let $L_1 = P_1 \oplus Q_1$ and $L_1 = P_1' \oplus Q_1'$ be two admissible direct sum decompositions with associated gms decompositions (B,Λ) and (B',Λ').*
  - *(B,Λ) and (B',Λ') are conjugate by an automorphism of M if and only if there exists an admissible automorphism $\theta\colon L_1 \to L_1$ with $\theta(P_1) = P_1'$, $\theta(Q_1) = Q_1'$, or an admissible isomorphism $\theta\colon L_1 \to L_2$ with $\theta(P_1) = P_2'$, $\theta(Q_1) = Q_2'$.*
  - *The Cartan subalgebras B and B' are unitarily conjugate if and only if $P_1 = P_1'$.*
  - *The Cartan subalgebras B and B' are conjugate by an automorphism of M if and only if there exists an admissible automorphism $\theta\colon L_1 \to L_1$ with $\theta(P_1) = P_1'$ or an admissible isomorphism $\theta\colon L_1 \to L_2$ with $\theta(P_1) = P_2'$.*

Proof. Because of Proposition 4.22, Theorem 4.7 and Proposition 4.20, it only remains to describe all automorphisms $\psi\colon A_0 \to A_0$ that normalize the action $\beta\colon \Gamma \curvearrowright A_0$. This action β is defined through the quotient homomorphism $\Gamma \twoheadrightarrow \Gamma_0$. Every automorphism of $\Gamma_0 = \mathrm{SL}(3,\mathbb{Z})$ is, up to an inner automorphism, either the identity or $g \mapsto (g^{-1})^T$ (see [HR51]). So, we only need to describe all automorphisms $\psi\colon A_0 \to A_0$ satisfying either $\psi \circ \beta_g = \beta_g \circ \psi$ for all $g \in \Gamma_0$, or $\psi \circ \beta_g = \beta_{(g^{-1})^T} \circ \psi$.

In the first case, reasoning as in the first paragraphs of the proof of Proposition 4.22, we get that $\psi(L(H_i)) = L(H_i)$ for $i = 1, 2$. So, for every $g \in H_1$, $\psi(u_g)$ is a unitary in $L(H_1)$ that normalizes $L(H_2)$. This forces $\psi(u_g) \in \mathbb{T}H_1$ and we conclude that $\psi(\mathbb{T}H_1) = \mathbb{T}H_1$. Similarly, $\psi(\mathbb{T}H_2) = \mathbb{T}H_2$. In the second case, we obtain in the same way that $\psi(\mathbb{T}H_1) = \mathbb{T}H_2$ and $\psi(\mathbb{T}H_2) = \mathbb{T}H_1$. The further analysis is analogous in both cases and we only give the details of the first case.

We find automorphisms $\theta_i\colon H_i \to H_i$ such that $\psi(u_g) \in \mathbb{T}u_{\theta_i(g)}$ for all $i = 1, 2$ and $g \in H_i$. Since θ_1 commutes with the action of $\mathrm{SL}(3, \mathbb{Z})$ on H_1, we get that $\theta_1 = \theta^3$ for some automorphism $\theta\colon L_1 \to L_1$. Similarly, we get an automorphism $\eta\colon L_2 \to L_2$ such that $\theta_2 = \eta^3$. Because ψ is an automorphism of A_0, it follows that $\Omega(g, h) = \Omega(\theta(g), \eta(h))$ for $g \in L_1$, $h \in L_2$. This means that the dual automorphism $\widehat{\eta}\colon \widehat{L_2} \to \widehat{L_2}$ extends θ and hence $\theta\colon L_1 \to L_1$ is an admissible automorphism. It follows that ψ maps the gms decomposition associated with $L_1 = P_1 \oplus Q_1$ to the gms decomposition associated with $L_1 = \theta(P_1) \oplus \theta(Q_1)$. This concludes the proof of the theorem. □

The following concrete examples provide a proof of Theorem A.

Theorem 4.24. *Consider the following two embeddings $\pi_i\colon \mathbb{Z}^n \hookrightarrow \mathbb{T}^{2n}$, for $n \geq 1$.*

- $\pi_1(k) = (\alpha_1^{k_1}, \alpha_2^{k_1}, \dots, \alpha_{2n-1}^{k_n}, \alpha_{2n}^{k_n})$ *for rationally independent irrational angles $\alpha_j \in \mathbb{T}$.*
- $\pi_2(k) = (\alpha^{k_1}, \beta^{k_1}, \dots, \alpha^{k_n}, \beta^{k_n})$ *for rationally independent irrational angles $\alpha, \beta \in \mathbb{T}$.*

Applying Theorem 4.23 to the embeddings π_1 and π_2, we obtain

- *a II_1 factor M that has exactly 2^n gms decompositions up to unitary conjugacy, and with the associated 2^n Cartan subalgebras not conjugate by an automorphism of M ;*
- *a II_1 factor M that has exactly $n+1$ gms decompositions up to conjugacy by an automorphism of M, and with the associated $n+1$ Cartan subalgebras not conjugate by an automorphism of M.*

Proof. Whenever $\mathcal{F} \subset \{1, \dots, n\}$, we have the direct sum decomposition $\mathbb{Z}^n = P(\mathcal{F}) \oplus P(\mathcal{F}^c)$ where $P(\mathcal{F}) = \{x \in \mathbb{Z}^n \mid \forall i \notin \mathcal{F}, x_i = 0\}$.

In the case of π_1, these are exactly all the admissible direct sum decompositions of $\mathbb{Z}^n$. Also, the only admissible automorphisms of $\mathbb{Z}^n$ are the ones of the form $(x_1, \dots, x_n) \mapsto (\varepsilon_1 x_1, \dots, \varepsilon_n x_n)$ with $\varepsilon_i = \pm 1$. Since $\mathbb{Z}^n \ncong \mathbb{Z}^{2n}$, there are no isomorphisms "exchanging L_1 and L_2". So, we get 2^n gms Cartan subalgebras and these are all not conjugate by an automorphism.

In the case of π_2, all direct sum decompositions and all automorphisms of $\mathbb{Z}^n$ are admissible. For every direct sum decomposition $\mathbb{Z}^n = P_1 \oplus Q_1$, there exists a unique $k \in \{0, \dots, n\}$ and an automorphism $\theta \in \mathrm{GL}(n, \mathbb{Z})$ such that $\theta(P_1) = P(\{1, \dots, k\})$ and $\theta(Q_1) = P(\{k+1, \dots, n\})$. Again, there are no isomorphisms exchanging L_1 and L_2. So, the $n+1$ direct sum decompositions $\mathbb{Z}^n = P(\{1, \dots, k\}) \oplus P(\{k+1, \dots, n\})$, $0 \leq k \leq n$, exactly give the possible gms decompositions of M up to conjugacy by an automorphism of M. When $k \neq k'$, there is no isomorphism $\theta \in \mathrm{GL}(n, \mathbb{Z})$ with $\theta(P(\{1, \dots, k\})) = P(\{1, \dots, k'\})$. Therefore, the $n+1$ associated Cartan subalgebras are not conjugate by an automorphism. □

Remark 4.25. In this remark, we show that the number of gms decompositions (up to unitary conjugacy) of the II_1 factors produced by Theorem 4.23 is always either infinite or a power of 2.

When L_1, L_2 are torsion-free abelian groups and $L_1 \hookrightarrow \widehat{L_2}$ is a dense embedding, then the set of admissible homomorphisms $L_1 \to L_1$ is a ring $\mathcal{R}$ that is torsion-free as an additive group. The admissible direct sum decompositions of L_1 are in bijective correspondence with the idempotents of $\mathcal{R}$. Indeed, any idempotent $\varphi \in \mathcal{R}$ gives rise to an admissible direct sum decomposition $L_1 = P \oplus Q$ by letting $P = \varphi(L_1)$ and $Q = (1 - \varphi)(L_1)$. Conversely, any admissible direct sum decomposition $L_1 = P \oplus Q$ gives rise to an idempotent admissible homomorphism $\varphi \colon L_1 \to L_1$ given by $\varphi(p, q) = (p, 0)$ for $p \in P$, $q \in Q$. Given $\varphi \in \mathcal{R}$, we will use the notation $\varphi^\perp := 1 - \varphi \in \mathcal{R}$.

As a torsion-free ring, $\mathcal{R}$ either has infinitely many idempotents, or finitely many that are all central. To see this, assume that $p \in \mathcal{R}$ is a non-central idempotent. Take $x \in \mathcal{R}$ such that $px \neq xp$. Note that $px = pxp + pxp^\perp$ and $xp = pxp + p^\perp xp$. Since $px \neq xp$, we have that either $pxp^\perp \neq 0$ or $p^\perp xp \neq 0$. Assume without loss of generality that $pxp^\perp \neq 0$. For any $n \in \mathbb{N}$, we put $p_n = p + npxp^\perp \in \mathcal{R}$. It is easy to check that p_n is an idempotent for all n. Since $\mathcal{R}$ is torsion-free, all p_n are different so that $\mathcal{R}$ contains infinitely many idempotents.

So, if $\mathcal{R}$ has finitely many idempotents then they are all central. In that case, there exists a central idempotent that is minimal with respect to the order relation $p \leq q \Leftrightarrow pq = p$. Let $\{p_i\}_{i=1}^n$ be a maximal family of minimal central idempotents. By maximality, we have that $\sum_{i=1}^n p_i = 1$. Given any idempotent $p \in \mathcal{R}$, we therefore have $p = \sum_{i=1}^{n} pp_i$ and $pp_i \in \{0, p_i\}$ for all i by minimality of p_i. It follows that there are 2^n idempotents in $\mathcal{R}$, namely $\{\sum_{i=1}^{n} \varepsilon_i p_i \mid \varepsilon_i \in \{0,1\} \ \forall i\}$. We conclude that the number of idempotents in $\mathcal{R}$ is either infinite or a power of 2.

Remark 4.26. Still in the context of Theorem 4.23, we call a subgroup $P_1 < L_1$ admissible if $L_1 \cap \overline{P_1} = P_1$, where $\overline{P_1}$ denotes the closure of P_1 inside $\widehat{L_2}$. Note that $P_1 < L_1$ is admissible if and only if there exists a subgroup $P_2 < L_2$ such that $P_2 = P_1^{\perp}$ and $P_1 = P_2^{\perp}$. Whenever $P_1 < L_1$ is an admissible subgroup, we define $B(P_1) := (L(P_1^3) \vee L(P_2^3))^{\Gamma}$. It is easy to check that all $B(P_1)$ are Cartan subalgebras of M and that $B(P_1)$ is unitarily conjugate with $B(P_1')$ if and only if $P_1 = P_1'$.

Also note that an admissible subgroup $P_1 < L_1$ cannot necessarily be complemented into an admissible direct sum decomposition $L_1 = P_1 \oplus Q_1$. In such a case, $B(P_1)$ is a Cartan subalgebra of M that is not of group measure space type. It is highly plausible that these $B(P_1)$ describe all Cartan subalgebras of M up to unitary conjugacy. We could however not prove this because all our techniques make use of the dual coaction associated with a gms decomposition of M.

Chapter 5

Thin II_1 factors with no Cartan subalgebras

In this chapter, which is based on my joint article [KV16] with Stefaan Vaes, we prove Theorem B from the introduction, in which we find examples of s-thin II_1 factors that have no Cartan subalgebras. Recall that the s-thin approximation property, as stated in the introduction, was introduced by Popa in search of an intrinsic characterization for a II_1 factor to have a Cartan subalgebra. In [Po16], Popa showed that a II_1 factor M is s-thin if and only if M admits an s-MASA. We will use this as the definition of s-thinness.

Definition 5.1. Let A be a MASA in a II_1 factor M. Then $A \subset M$ is called an *s-MASA* if the A-bimodule ${}_A L^2(M)_A$ is cyclic, i.e., if there exists a vector $\xi \in L^2(M)$ such that $A\xi A$ spans a dense subspace of $L^2(M)$.

We say that M is *s-thin* if M contains an s-MASA.

When a II_1 factor is s-thin, we can think of it as being "thin" relative to its abelian subalgebras. This is closely related to the notion of being *thin* defined in [GP98], where the thinness is measured relative to the hyperfinite subalgebras of a given II_1 factor.

Note that a MASA $A \subset M$ is an s-MASA if and only if the abelian von Neumann subalgebra $A \vee JAJ \subset B(L^2(M))$ is a MASA, where J denotes the canonical involution on $L^2(M)$ (see for instance [AP16, Theorem 3.1.4]).

Using the Feldman-Moore theorem, Theorem 2.4, It is easy to see that a separable Cartan subalgebra is also an s-MASA (see [FM75, Proposition 2.9]) and therefore, any separable II_1 factor that admits a Cartan subalgebra is also s-thin. The converse implication was left as a question by Popa in [Po16], and was the motivation for our work. The following is our main theorem.

Theorem 5.2. *There exist non-amenable s-thin II_1 factors that are strongly solid.*

Recall from Section 2.6 the notion of strong solidity, which is a stronger property than absence of Cartan subalgebras.

Our examples of II_1 factors are given by Shlyakhtenko's construction of A-valued semicircular systems (see [Sh97] and Section 5.1 below), generalizing Voiculescu's free Gaussian functor [Vo83]. The data for this construction consists of a tracial von Neumann algebra (A, τ) and a symmetric A-bimodule ${}_AH_A$, where the symmetry is given by an anti-unitary operator $J\colon H \to H$ satisfying $J^2 = 1$ and $J(a \cdot \xi \cdot b) = b^* \cdot J\xi \cdot a^*$. The construction produces a tracial von Neumann algebra M containing A such that ${}_AL^2(M)_A$ can be identified with the full Fock space

$$L^2(A) \oplus \bigoplus_{n\geq 1} \big(\underbrace{H \otimes_A \cdots \otimes_A H}_{n \text{ times}}\big) .$$

We refer to Section 5.1 for further details on this construction.

In the same way as the free Gaussian functor transforms direct sums of real Hilbert spaces into free products of von Neumann algebras, the construction of [Sh97] transforms direct sums of A-bimodules into free products that are amalgamated over A. Therefore, the deformation/rigidity results and methods for amalgamated free products introduced in [IPP05, Io12], and in particular Popa's s-malleable deformation obtained by "doubling and rotating" the A-bimodule, can be applied and yield the following result, proved in Corollaries 5.14 and 5.22 below (see Theorem 5.21 for the most general statement).

Theorem 5.3. *Let (A, τ) be a tracial von Neumann algebra and let M be the von Neumann algebra associated with a symmetric A-bimodule ${}_AH_A$. Assume that ${}_AH_A$ is weakly mixing (Definition 2.24) and that the left action of A on H is faithful. Then, M has no Cartan subalgebra. If moreover ${}_AH_A$ is mixing and A is amenable, then M is strongly solid.*

In the particular case where A is diffuse abelian and the bimodule ${}_AH_A$ is weakly mixing, we have that $A \subset M$ is a singular MASA. As explained in the introduction, interesting examples arise by taking $A = L^\infty(K, \mu)$ where K is a second countable compact group with Haar probability measure μ. Any symmetric probability measure ν on K gives rise to a symmetric A-bimodule H_ν (see Section 5.5), and when ν satisfies certain special properties, the von

Neumann algebra M_ν associated with H_ν is a strongly solid II_1 factor that contains A as an s-MASA (see Proposition 5.47 for the precise statement). In Theorem 5.49, we construct a compact group K and a probability measure ν on K satisfying these "special" properties. As a consequence, we obtain the first examples of s-thin II_1 factors that have no Cartan subalgebras, proving our main theorem (Theorem 5.2).

As we explain in Remark 5.11, the so-called free Bogoljubov crossed products $L(\mathbb{F}_\infty) \rtimes G$ associated with an (infinite-dimensional) orthogonal representation of a countable group G can be written as the von Neumann algebra associated with a symmetric A-bimodule where $A = L(G)$. Therefore, our Theorem 5.3 is a generalization of similar results proved in [Ho12b] for free Bogoljubov crossed products. Although free Bogoljubov crossed products $M = L(\mathbb{F}_\infty) \rtimes G$ with G abelian provide examples of MASAs $L(G) \subset M$ with interesting properties (see [HS09, Ho12a]), $L(G) \subset M$ can never be an s-MASA (see Remark 5.48).

The point of view of A-valued semicircular systems is more flexible and even offers advantages in the study of free Bogoljubov crossed products $M = L(\mathbb{F}_\infty) \rtimes G$. Indeed, in Corollary 5.24, we prove that these II_1 factors M never have a Cartan subalgebra, while in [Ho12b], this could only be proved for special classes of orthogonal representations.

In Theorem 5.19, we prove several maximal amenability results for the inclusion $A \subset M$ associated with a symmetric A-bimodule (H, J). The study of maximal amenable von Neumann subalgebras was originally motivated by a question of Kadison from the 1960s: is any self-adjoint element in a II_1 factor M contained in a hyperfinite II_1 factor? This question was answered in the negative by Popa in [Po83]. He showed that the von Neumann subalgebra of a free group factor $L(\mathbb{F}_n)$, $n \geq 2$, generated by one of the free generators of $\mathbb{F}_n$, is maximal amenable. This provided the first example of a maximal amenable subalgebra that is also abelian. Using the same approach, known as Popa's asymptotic orthogonality, many more maximal amenability results have been obtained since [Po83]. Recently, new methods for proving maximal amenability results were developed in [BC14, BH16], based on the study of centralizers of states. By combining the methods of [Po83, BH16], we prove in Theorem 5.19 among other things that $A \subset M$ is maximal amenable whenever A is amenable and H is weakly mixing, where M is the von Neumann algebra associated with a symmetric A-bimodule (H, J). Again, these results generalize [Ho12a, Ho12b] where the same was proved for free Bogoljubov crossed products.

Finally, we have added a section about property Gamma, Section 5.6, which did not appear in [KV16]. The definition of property Gamma goes back to the work of Murray and von Neumann, [MvN43], and was originally used to distinguish the hyperfinite II_1 factor R from the free group factors $L(\mathbb{F}_n)$,

$n \geq 2$. A separable II_1 factor M is said to have *property Gamma* if there exists a nontrivial central sequence in M, i.e., a bounded sequence $a_n \in M$ with $\|a_n x - x a_n\|_2 \to 0$ for all $x \in M$, such that a_n is not asymptotically scalar. When M is the von Neumann algebra associated with some symmetric A-bimodule (H, J) that is weakly mixing, we show that all central sequences of M asymptotically lie in A (see Theorem 5.55), and this allows us to characterize when M has property Gamma. Similar results were proved in [Ho12b] for free Bogoljubov crossed products and our proof uses the same methods. In particular, when $M = M_\nu$ arises from a symmetric probability measure ν on a compact group K as in Section 5.5, we will characterize when M_ν has property Gamma in terms of the measure ν.

5.1 Shlyakhtenko's A-valued semicircular systems

We first recall Voiculescu's free Gaussian functor from the category of real Hilbert spaces to the category of tracial von Neumann algebras. Let $H_\mathbb{R}$ be a real Hilbert space and let H be its complexification. The *full Fock space* of H is defined as

$$\mathcal{F}(H) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n} .$$

The unit vector Ω is called the *vacuum vector*. Given a vector $\xi \in H$, we define the *left creation operator* $\ell(\xi) \in B(\mathcal{F}(H))$ by

$$\ell(\xi)(\Omega) = \xi \quad \text{and} \quad \ell(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n .$$

Put

$$\Gamma(H_\mathbb{R})'' := \{\ell(\xi) + \ell(\xi)^* \mid \xi \in H_\mathbb{R}\}'' .$$

In the literature, the notation $\Gamma(H_\mathbb{R})$ is used to denote the C*-algebra generated by the operators $\ell(\xi) + \ell(\xi)^*$ for $\xi \in H_\mathbb{R}$.

The von Neumann algebra $\Gamma(H_\mathbb{R})''$ is equipped with the faithful trace given by $\tau(\cdot) = \langle \cdot\, \Omega, \Omega \rangle$. In [Vo83], it is proved that the operator $X = \ell(\xi) + \ell(\xi)^*$ has a semicircular distribution with respect to the trace τ, in the sense that

$$\tau(X^n) = \frac{2}{\pi R^2} \int_{-R}^{R} t^n \sqrt{R^2 - t^2}\, \mathrm{d}t \quad \text{for all } n \in \mathbb{N},$$

where $R = \|X\|$, and it is proved that $\Gamma(H_\mathbb{R})'' \cong L(\mathbb{F}_{\dim H_\mathbb{R}})$. By the functoriality of the construction, any orthogonal transformation u of $H_\mathbb{R}$ gives rise to an automorphism α_u of $\Gamma(H_\mathbb{R})''$ satisfying $\alpha_u(\ell(\xi) + \ell(\xi)^*) = \ell(u\xi) + \ell(u\xi)^*$ for all $\xi \in H_\mathbb{R}$. So, every orthogonal representation $\pi \colon G \to O(H_\mathbb{R})$ of a countable group G gives rise to the *free Bogoljubov action* $\sigma_\pi \colon G \curvearrowright \Gamma(H_\mathbb{R})''$ given by $\sigma_\pi(g) = \alpha_{\pi(g)}$ for all $g \in G$.

In [Sh97], Shlyakhtenko introduced a generalization of Voiculescu's free Gaussian functor, this time being a functor from the category of symmetric A-bimodules (where A is any von Neumann algebra) to the category of von Neumann algebras containing A. We will here repeat this construction in the case where A is a tracial von Neumann algebra.

Definition 5.4. Let (A, τ) be a tracial von Neumann algebra. A *symmetric* A-bimodule (H, J) is an A-bimodule ${}_A H_A$ equipped with an anti-unitary operator $J \colon H \to H$ such that $J^2 = 1$ and

$$J(a \cdot \xi \cdot b) = b^* \cdot J\xi \cdot a^*, \qquad \forall\, a, b \in A\ .$$

Note that the complexification of a real Hilbert space can be seen as a symmetric $\mathbb{C}$-bimodule, where the symmetry J is given by complex conjugation.

Let (A, τ) be a tracial von Neumann algebra and let (H, J) be a symmetric A-bimodule. We denote by $H^{\otimes_A^n}$ the n-fold Connes tensor product $H \otimes_A H \otimes_A \cdots \otimes_A H$. The full Fock space of the A-bimodule ${}_A H_A$ is defined by

$$\mathcal{F}_A(H) = L^2(A) \oplus \bigoplus_{n=1}^{\infty} H^{\otimes_A^n}\ . \tag{5.1}$$

We denote by $\mathcal{H}$ the set of left and right A-bounded vectors in H, as defined in Section 2.2. Since A is a tracial von Neumann algebra, $\mathcal{H}$ is dense in H. Given a right bounded vector $\xi \in H$, we define the left creation operator $\ell(\xi)$ on $\mathcal{F}_A(H)$ analogously to the case where $A = \mathbb{C}$ by

$$\ell(\xi)(a) = \xi a, \quad a \in A\ ,$$

$$\ell(\xi)(\xi_1 \otimes_A \ldots \otimes_A \xi_n) = \xi \otimes_A \xi_1 \otimes_A \ldots \otimes_A \xi_n, \quad \xi_i \in \mathcal{H}\ .$$

Note that $a\ell(\xi) = \ell(a\xi)$ and $\ell(\xi)a = \ell(\xi a)$ for $a \in A$ and that the adjoint map $\ell(\xi)^*$ satisfies

$$\ell(\xi)^*(a) = 0 \quad \text{for all } \ a \in L^2(A)\ ,$$

$$\ell(\xi)^*(\xi_1 \otimes_A \ldots \otimes_A \xi_n) = \langle \xi, \xi_1 \rangle_A\, \xi_2 \otimes_A \ldots \otimes_A \xi_n \quad \text{for } \ \xi_i \in \mathcal{H}\ .$$

We also have a right creation operator $r(\xi)$ on $\mathcal{F}_A(H)$ defined by

$$r(\xi)(a) = a\xi, \quad a \in A\ ,$$

$$r(\xi)(\xi_1 \otimes_A \ldots \otimes_A \xi_n) = \xi_1 \otimes_A \ldots \otimes_A \xi_n \otimes_A \xi, \quad \xi_i \in \mathcal{H}\ .$$

Definition 5.5. Given a tracial von Neumann algebra (A, τ) and a symmetric A-bimodule (H, J), we consider the full Fock space $\mathcal{F}_A(H)$ given by (5.1) and define

$$\Gamma(H, J, A, \tau)'' := A \vee \{\ell(\xi) + \ell(\xi)^* \mid \xi \in \mathcal{H},\ J\xi = \xi\}'' \subset B(\mathcal{F}_A(H)) ,$$

where $A \subset B(\mathcal{F}_A(H))$ is given by the left action on $\mathcal{F}_A(H)$. We also have

$$\Gamma(H, J, A, \tau)'' = A \vee \{\ell(\xi) + \ell(J\xi)^* \mid \xi \in \mathcal{H}\}'' .$$

We denote by Ω the vacuum vector in $\mathcal{F}_A(H)$ given by $\Omega = 1_A \in L^2(A)$. We define τ as the vector state on $M = \Gamma(H, J, A, \tau)''$ given by the vacuum vector Ω. Whenever $n \geq 1$ and $\xi_1, \ldots, \xi_n \in \mathcal{H}$, we define the Wick product as in [HR10, Lemma 3.2] by

$$W(\xi_1, \ldots, \xi_n) = \sum_{i=0}^{n} \ell(\xi_1) \cdots \ell(\xi_i) \ell(J\xi_{i+1})^* \cdots \ell(J\xi_n)^* . \tag{5.2}$$

As in [HR10, Lemma 3.2], we get the following lemma.

Lemma 5.6. *For $n \geq 1$ and $\xi_1, \ldots, \xi_n \in \mathcal{H}$, we have that $W(\xi_1, \ldots, \xi_n) \in M$ and*

$$W(\xi_1, \ldots, \xi_n)\Omega = \xi_1 \otimes_A \cdots \otimes_A \xi_n .$$

Moreover, the set of Wick products $W(\xi_1, \ldots, \xi_n)$ with $n \geq 1$ and $\xi_i \in \mathcal{H}$, together with A, spans an SOT-dense $*$-subalgebra of M.

Proof. Since $\ell(\xi)^*\Omega = 0$ for any $\xi \in \mathcal{H}$, it is clear that $W(\xi_1, \ldots, \xi_n)\Omega = \xi_1 \otimes_A \otimes \cdots \otimes_A \xi_n$. We prove by induction on n that $W(\xi_1, \ldots, \xi_n) \in M$.

For $n = 1$, we have $W(\xi_1) = \ell(\xi_1) + \ell(J\xi_1)^* \in M$. Next, assume that $W(\xi_1, \ldots, \xi_k) \in M$ for all $k \leq n$. A direct computation shows that

$$W(\xi_0)W(\xi_1, \ldots, \xi_n) = W(\xi_0, \xi_1, \ldots, \xi_n) + \langle J\xi_0, \xi_1\rangle_A W(\xi_2, \ldots, \xi_n).$$

So, by induction hypothesis

$$W(\xi_0, \ldots, \xi_n) = W(\xi_0)W(\xi_1, \ldots, \xi_n) - \langle J\xi_0, \xi_1\rangle_A W(\xi_2, \ldots, \xi_n) \in M.$$

To prove the final statement, let M_0 be the linear span of $\{W(\xi_1, \ldots, \xi_n) \mid n \geq 1, \xi_i \in \mathcal{H}\} \cup A$. Since M is generated by M_0, it is enough to show that M_0 is a $*$-subalgebra. We have that

$$W(\xi_1, \ldots, \xi_n)^* = W(J\xi_n, \ldots, J\xi_1),$$

so that M_0 is closed under taking the adjoint. Moreover, assuming that $m \leq n$, we have that

$$W(\xi_1, \dots, \xi_n) W(\eta_1, \dots, \eta_m) = \sum_{i=0}^{m} W(\xi_1, \dots, \xi_{n-i} a_i, \eta_{1+i}, \dots, \eta_m) \in M_0,$$

where $a_i = \ell(J\xi_{n-i+1})^* \cdots \ell(J\xi_n)^* \ell(\eta_1) \cdots \ell(\eta_i) \in A$ for $i = 1, \dots, m$ and $a_0 = 1$. So, M_0 is an SOT-dense $*$-subalgebra of M. □

Proposition 5.7 ([Sh97])**.** *The state $\tau(\cdot) = \langle \cdot\, \Omega, \Omega \rangle$ defined above is a faithful trace on M.*

Proof. Define $\mathcal{J} \colon \mathcal{F}_A(H) \to \mathcal{F}_A(H)$ by $\mathcal{J}(a) = a^*$ for $a \in A$ and

$$\mathcal{J}(\xi_1 \otimes_A \cdots \otimes_A \xi_n) = J\xi_n \otimes_A \cdots \otimes_A J\xi_1$$

for $\xi_1, \dots, \xi_n \in \mathcal{H}$. Then $\mathcal{J}$ is an anti-unitary map satisfying $\mathcal{J}^2 = 1$. One easily checks that $\mathcal{J}\ell(\xi)\mathcal{J} = r(J\xi)$ for all $\xi \in \mathcal{H}$ and that $\mathcal{J} a \mathcal{J}$ is just right multiplication by a^* on $\mathcal{F}_A(H)$. This implies that $\mathcal{J} M \mathcal{J}$ commutes with M. Indeed, for $\xi, \eta \in \mathcal{H}$ with $J\xi = \xi$ and $J\eta = \eta$, we have $\langle \xi, a\eta \rangle_A = {}_A\langle \xi a, \eta \rangle$ since

$$\langle Jr(\xi a)^* r(\eta) Jx, y \rangle = \langle r(\xi a) y^*, r(\eta) x^* \rangle = \langle y^* \xi a, x^* \eta \rangle = \langle J(x^* \eta), J(y^* \xi a) \rangle$$

$$= \langle \eta x, a^* \xi y \rangle = \langle \ell(\xi)^* \ell(a\eta) x, y \rangle \;,$$

for all $x, y \in A$. It follows that

$$(\ell(\xi)^* r(\eta) + \ell(\xi) r(\eta)^*)(a) = \langle \xi, a\eta \rangle_A = {}_A\langle \xi a, \eta \rangle = (r(\eta)^* \ell(\xi) + r(\eta) \ell(\xi)^*)(a),$$

for all $a \in A$. Since $\ell(\xi)$ and $r(\eta)^*$ clearly commute when restricted to $\mathcal{F}_A(H) \ominus L^2(A)$, it follows that $\ell(\xi) + \ell(\xi)^*$ commutes with $r(\eta) + r(\eta)^*$. We conclude that M commutes with $\mathcal{J} M \mathcal{J}$.

Next, we show that $\mathcal{J}(x\Omega) = x^*\Omega$ for all $x \in M$. This clearly holds for $x \in A$ so it suffices to prove it for x of the form $x = W(\xi_1, \dots, \xi_n)$ with $\xi_i \in \mathcal{H}$. We have

$$\mathcal{J}(W(\xi_1, \dots, \xi_n)\Omega) = \mathcal{J}(\xi_1 \otimes_A \cdots \otimes_A \xi_n) = J\xi_n \otimes_A \cdots \otimes_A J\xi_1$$

$$= W(J\xi_n, \dots, J\xi_1)\Omega = W(\xi_1, \dots, \xi_n)^* \Omega \,.$$

We now get that

$$\tau(xy) = \langle xy\Omega, \Omega \rangle = \langle x\mathcal{J}(y^*\Omega), \Omega \rangle = \langle x\mathcal{J} y^* \mathcal{J}\Omega, \Omega \rangle = \langle \mathcal{J} y^* \mathcal{J} x\Omega, \Omega \rangle$$

$$= \langle x\Omega, \mathcal{J} y \mathcal{J}\Omega \rangle = \langle x\Omega, y^*\Omega \rangle = \langle yx\Omega, \Omega \rangle = \tau(yx) \;,$$

for all $x, y \in M$ and hence τ is a trace.

It follows from Lemma 5.6 that $\Omega \in \mathcal{F}_A(H)$ is a cyclic vector for both M and $\mathcal{J} M \mathcal{J}$. Hence Ω is also separating for M and it follows that τ is faithful. $\square$

By construction, we have that $L^2(M) \cong \mathcal{F}_A(H)$ as A-bimodules.

In [Sh97], Shlyakhtenko used the terminology *A-valued semicircular system* for the family $\{\ell(\xi) + \ell(\xi)^* \mid \xi \in \mathcal{H}, J\xi = \xi\}$, as an analogue to the free Gaussian functor case, where the operator $\ell(\xi) + \ell(\xi)^*$ has a semicircular distribution with respect to τ.

Remark 5.8. In the original article [Sh97], Shlyakhtenko used a seemingly different setup to define the von Neumann algebras $\Gamma(H, J, A, \tau)''$. Instead of using a symmetric A-bimodule (H, J) as input, he used a system of completely positive maps on A. More precisely, to any von Neumann algebra A and any set of linear maps $\varphi_{ij} \colon A \to A$, $i, j \in I$ such that $a \mapsto (\varphi_{ij}(a))_{i,j \in I} \colon A \to A \mathbin{\overline{\otimes}} B(\ell^2(I))$ is normal and completely positive, Shlyakhtenko associated a von Neumann algebra $\Phi(A, \varphi)$. Let us briefly explain how a symmetric A-bimodule (H, J) gives rise to such a system of maps $(\varphi_{ij})_{i,j\in I}$ in the case where A is tracial.

Let $(\xi_i)_{i\in I}$ be a maximal family of symmetric unit vectors such that the A-subbimodules $\overline{A\xi_i A}$ are all pairwise orthogonal. We then obtain the desired system of maps $(\varphi_{ij})_{i,j\in I}$ by letting $\varphi_{ij} \colon A \to A$ be the linear map given by $\varphi_{ij}(a) = \ell(\xi_i)^* a \ell(\xi_j) = \langle \xi_i, a\xi_j \rangle_A$. It is easy to check that our construction of $\Gamma(H, J, A, \tau)''$ coincides with Shlyakhtenko's construction of $\Phi(A, \varphi)$. Moreover, [Sh97, Lemma 2.2] shows that any such system $(\varphi_{ij})_{i,j\in I}$ gives rise to an A-bimodule H spanned by vectors $\xi_i \in H$ satisfying $\langle \xi_i, a\xi_j \rangle_A = \varphi_{ij}(a)$. Note however that Shlyakhtenko's construction also works when A is non-tracial. In this case, the associated von Neumann algebra $\Phi(A, \varphi)$ is of type III.

Example 5.9. (1) When $H = L^2(A)$ is the trivial A-bimodule with $J(a) = a^*$, we simply get

$$\Gamma(H, J, A, \tau)'' = A \mathbin{\overline{\otimes}} L^\infty[0, 1] \,.$$

Indeed, A commutes with $\ell(1_A) + \ell(1_A)^*$ and they together generate $\Gamma(H, J, A, \tau)''$. It is not hard to see that $\ell(1_A)$ has the same $*$-distribution as a free creation operator from the free Gaussian functor. Hence, the operator $\ell(1_A) + \ell(1_A)^*$ has a semicircular $*$-distribution and in particular, it generates a von Neumann algebra isomorphic with $L^\infty[0, 1]$. It is also easy to see that $\ell(1_A) + \ell(1_A)^*$ is independent from A with respect to τ, so that $\{\ell(1_A) + \ell(1_A)^*\}''$ and A are in tensor product position with each other.

From this example, we see that $\Gamma(H, J, A, \tau)''$ is not always a factor.

(2) When $H = L^2(A) \otimes L^2(A)$ is the coarse A-bimodule with $J(a \otimes b) = b^* \otimes a^*$, we get (see [Sh97, Example 3.3])

$$\Gamma(H, J, A, \tau)'' = (A, \tau) * L^\infty[0, 1] .$$

This example shows that the construction of $\Gamma(H, J, A, \tau)''$ may depend on the trace on A. Indeed, if $A = \mathbb{C}^2$ we can consider the trace τ_δ for any $\delta \in (0, 1)$ given by $\tau_\delta(a, b) = \delta a + (1 - \delta)b$, $a, b \in \mathbb{C}$. By [Dy92, Lemma 1.6], we have that $L(\mathbb{Z}) * (A, \tau_\delta) = L(\mathbb{F}_{1+2\delta(1-\delta)})$, the interpolated free group factor. It is wide open whether the interpolated free group factors are all isomorphic. So at least, there is no obvious isomorphism between $\Gamma(H, J, A, \tau_{\delta_1})''$ and $\Gamma(H, J, A, \tau_{\delta_2})''$ for $\delta_1 \neq \delta_2$. In Example 5.12, we shall actually see that even the factoriality of $\Gamma(H, J, A, \tau)''$ may depend on the choice of the trace τ. For a general factoriality criterion for $\Gamma(H, J, A, \tau)''$, see Theorem 5.21.

Note that the construction of $\Gamma(H, J, A, \tau)''$ is functorial in the following sense. If $U \in \mathcal{U}(H)$ is a unitary operator that is A-bimodular and commutes with J, then U defines a trace-preserving automorphism of $M = \Gamma(H, J, A, \tau)''$ in the following way. Since U is A-bimodular, we can define a unitary U^n on $H^{\otimes_A^n}$ by $U^n(\xi_1 \otimes_A \cdots \otimes_A \xi_n) = U\xi_1 \otimes_A \cdots \otimes_A U\xi_n$. The direct sum of these unitaries (and the identity on $L^2(A)$) then gives an A-bimodular unitary operator on $\mathcal{F}_A(H)$, which we will still denote by U. Note that $U\ell(\xi)U^* = \ell(U\xi)$ for all $\xi \in \mathcal{H}$. Since U commutes with J, it follows that $UMU^* = M$ so that $\operatorname{Ad} U$ defines an automorphism of M.

Recall that for Voiculescu's free Gaussian functor, we have that the direct sum of Hilbert spaces translates into the free product of von Neumann algebras, in the sense that $\Gamma(H_1 \oplus H_2)'' = \Gamma(H_1)'' * \Gamma(H_2)''$. In the setting of A-bimodules in general, we instead get the amalgamated free product over A as stated in the following proposition.

Proposition 5.10 ([Sh97, Proposition 2.17]). *Let (H_1, J_1) and (H_2, J_2) be symmetric A-bimodules. Put $H = H_1 \oplus H_2$ and $J = J_1 \oplus J_2$. Then*

$$\Gamma(H, J, A, \tau)'' \cong \Gamma(H_1, J_1, A, \tau)'' *_A \Gamma(H_2, J_2, A, \tau)'' ,$$

with respect to the unique trace-preserving conditional expectation onto A.

Remark 5.11. As we recalled in the beginning of this section, to every orthogonal representation $\pi : G \to O(K_\mathbb{R})$ of a countable group G on a real Hilbert space $K_\mathbb{R}$ is associated the free Bogoljubov action $\sigma_\pi : G \curvearrowright \Gamma(K_\mathbb{R})''$. Write $A = L(G)$ and equip A with its canonical tracial state τ. Denote by K

the complexification of $K_{\mathbb{R}}$ and define the symmetric A-bimodule ${}_A H_A$ given by

$$\begin{aligned} H = \ell^2(G) \otimes K \quad \text{with} \quad & u_g \cdot (\delta_h \otimes \xi) \cdot u_k = \delta_{ghk} \otimes \pi(g)\xi \\ \text{and} \quad & J(\delta_h \otimes \xi) = \delta_{h^{-1}} \otimes \pi(h^{-1})\overline{\xi} \end{aligned} \tag{5.3}$$

where $(\delta_g)_{g\in G}$ denotes the canonical orthonormal basis of $\ell^2(G)$. We then have a canonical trace-preserving isomorphism

$$\Gamma(K_{\mathbb{R}})'' \rtimes^{\sigma_\pi} G \cong \Gamma(H, J, A, \tau)''$$

that maps $L(G)$ onto A identically and $W(\xi_1, \dots, \xi_n)$ with $\xi_i \in K_{\mathbb{R}}$ onto $W(\delta_e \otimes \xi_1, \dots, \delta_e \otimes \xi_n)$.

Example 5.12. This final example illustrates that even the factoriality of $\Gamma(H, J, A, \tau)''$ may depend on the choice of τ. Take $A = \mathbb{C}^2$, $\alpha \in \mathrm{Aut}(A)$ the flip automorphism and $H = \mathbb{C}^2$ with A-bimodule structure given by $a\cdot\xi\cdot b = \alpha(a)\xi b$. Define $J\colon H \to H$ by $J(a) = \alpha(a)^*$. For every $0 < \delta < 1$, denote by τ_δ the trace on A given by $\tau_\delta(a, b) = \delta a + (1-\delta)b$. By symmetry, it suffices to consider the case $0 < \delta \leq 1/2$. For any δ, when A is equipped with the trace τ_δ, the n-fold tensor power $H^{\otimes_A^n}$ can be identified with $\mathbb{C}^2$ with the bimodule structure given by

$$a \cdot \xi \cdot b = \begin{cases} a\xi b & \text{if } n \text{ is even,} \\ \alpha(a)\xi b & \text{if } n \text{ is odd.} \end{cases}$$

Denote by ℓ_δ the left creation operator $\ell(1_A)$ with respect to the trace τ_δ. When we represent ℓ_δ on the Hilbert space $\bigoplus_{n\geq 0} \mathbb{C}^2$ via the identification above, we get that

$$\ell_\delta = \ell\,\lambda(D^{-1/2}) \,,$$

where ℓ denotes the shift operator on $\bigoplus_{n\geq 0} \mathbb{C}^2$, λ denotes the left A-action and $D = (\delta, 1-\delta)$ is the Radon-Nikodym derivative between τ_δ and the usual inner product on $\mathbb{C}^2$.

Put

$$M_\delta := \Gamma(H, J, A, \tau_\delta)'' = \lambda(A) \vee \{\ell_\delta + \ell_\delta^*\}''.$$

We still denote by τ_δ the canonical trace on M_δ. Note that $M = \lambda(A) \vee \{S_\delta\}''$, where $S_\delta = \ell\lambda(\Delta^{-1/4}) + \ell^*\lambda(\Delta^{1/4})$ and $\Delta = D\alpha(D)^{-1} = (\delta/(1-\delta), (1-\delta)/\delta)$. Indeed, we have that

$$S_\delta \lambda(\alpha(D^{-1/4})D^{-1/4}) = \ell_\delta + \ell_\delta^*.$$

Note also that $S_\delta = S_\delta^*$. Denoting by $e = (1, 0)$ and $f = (0, 1)$ the minimal projections in A, we have that $S_\delta e = f S_\delta$.

Next, we calculate the possible eigenvalues of S_δ. Given $\xi = \bigoplus_{n\geq 0}\xi_n \in \bigoplus_{n\geq 0}\mathbb{C}^2$, we have that

$$S_\delta(\xi) = \bigoplus_{n=1}^{\infty}\alpha^{n-1}(\Delta^{-1/4})\xi_{n-1} + \bigoplus_{n=0}^{\infty}\alpha^{n+1}(\Delta^{1/4})\xi_{n+1}. \tag{5.4}$$

Assume that $\xi \in \ker(S_\delta)$. Using (5.4) coordinate-wise, we get

$$\alpha(\Delta^{1/4})\xi_1 = 0,$$

$$\alpha^{n-1}(\Delta^{-1/4})\xi_{n-1} + \alpha^{n+1}(\Delta^{1/4})\xi_{n+1} = 0 \quad \text{for } n \geq 1.$$

It follows that $\xi_n = 0$ for all n odd and that $\xi_{2k} = (-1)^k\Delta^{-k/2}\xi_0$ for all $k \geq 0$. Writing $\xi_0 = (\xi_{01}, \xi_{02}) \in \mathbb{C}^2$, we thus get

$$\|\xi\|^2 = \sum_{k=0}^{\infty}\|\Delta^{-k/2}\xi_0\|^2 = |\xi_{01}|^2\sum_{n=0}^{\infty}\left(\frac{1-\delta}{\delta}\right)^k + |\xi_{02}|^2\sum_{n=0}^{\infty}\left(\frac{\delta}{1-\delta}\right)^k.$$

If $\delta = 1/2$, both of the sums above are divergent and hence $\xi_{01} = \xi_{02} = 0$. So, $\ker S_{1/2} = \{0\}$ in this case. If $\delta < \frac{1}{2}$, we get that the first sum in the expression above diverges while the second sum converges. This forces $\xi_{01} = 0$ and we get that $\ker S_\delta$ has dimension 1.

Next, assume that ξ is an eigenvector for S_δ, with eigenvalue $\lambda \neq 0$. Again, using (5.4) coordinate-wise yields

$$\Delta^{-1/4}\xi_1 = \lambda\xi_0,$$

$$\Delta^{-1/4}\xi_{n-1} + \Delta^{1/4}\xi_{n+1} = \lambda\xi_n \quad \text{for } n \text{ odd},$$

$$\Delta^{1/4}\xi_{n-1} + \Delta^{-1/4}\xi_{n+1} = \lambda\xi_n \quad \text{for } n \text{ even}.$$

It follows that $\xi_n = a_n\xi_0$ where $a_n \in \mathbb{C}^2$ is defined recursively by

$$a_n = \Delta^{(-1)^{n+1}/4}\lambda a_{n-1} - \Delta^{(-1)^{n+1}/2}a_{n-2}, \quad n \geq 2.$$

Assume without loss of generality that the first coordinate of ξ_0 is nonzero. Letting $b_n \in \mathbb{C}$ denote the first coordinate of a_n, we get that b_n satisfies the following recurrence relation

$$b_n = \lambda\left(\frac{\delta}{1-\delta}\right)^{(-1)^{n+1}/4} b_{n-1} - \left(\frac{\delta}{1-\delta}\right)^{(-1)^{n+1}/2} b_{n-2}.$$

One can show that this recurrence relation is not stable so that in particular, b_n does not converge to zero. Thus

$$\|\xi\|^2 = \sum_{n=0}^{\infty}\|a_n\xi_0\|^2 \geq \sum_{n=0}^{\infty}|b_n\xi_{01}|^2 = \infty,$$

a contradiction. We conclude that S_δ has no eigenvalue different from zero, for all $0 < \delta \leq \frac{1}{2}$. This means that S_δ is diffuse on the orthogonal complement of $\ker S_\delta$.

We have now shown the following: When $\delta = 1/2$, the operator S_δ is non-singular and diffuse; When $0 < \delta < 1/2$, the kernel of S_δ has dimension 1 and S_δ is diffuse on its orthogonal complement. We denote by z_δ the projection onto the kernel of S_δ. Using the description of $\ker S_\delta$ from the computations above, we get that z_δ is a minimal and central projection in M_δ with $\tau_\delta(z_\delta) = 1 - 2\delta$. We conclude that there is a trace-preserving $*$-isomorphism

$$(M_\delta, \tau_\delta) \cong \underbrace{M_2(\mathbb{C}) \otimes B}_{\delta(\mathrm{Tr} \otimes \tau_0)} \oplus \underbrace{\mathbb{C}}_{1-2\delta} \tag{5.5}$$

where (B, τ_0) is a diffuse abelian von Neumann algebra with normal faithful tracial state τ_0 and where we emphasized the choice of trace at the right hand side. Under the isomorphism (5.5), we have that

$$e \mapsto (e_{11} \otimes 1) \oplus 0\ , \quad f \mapsto (e_{22} \otimes 1) \oplus 1\ , \quad S_\delta \mapsto ((e_{12} + e_{21}) \otimes b) \oplus 0\ , \quad z_\delta \mapsto 0 \oplus 1$$

where $b \in B$ is a positive non-singular element generating B.

Next, taking $H \oplus H$ and $J \oplus J$, it follows from Proposition 5.10 that

$$\mathcal{M}_\delta := \Gamma(H \oplus H, J \oplus J, A, \tau_\delta)'' = M_\delta *_A M_\delta \ ,$$

where we used at the right hand side the amalgamated free product with respect to the unique τ_δ-preserving conditional expectations, which we will denote by E_A. We denote with superscripts $^{(1)}$ and $^{(2)}$ the elements of M_δ viewed in the first, resp. second copy of M_δ in the amalgamated free product. Note that $f^{(1)} = f^{(2)}$ and that, denoting this projection as f, we get that $fM_\delta^{(1)}f$ and $fM_\delta^{(2)}f$ are free inside $f\mathcal{M}_\delta f$. Indeed, this follows from the fact that $M_\delta^{(1)}$ and $M_\delta^{(2)}$ are free with respect to E_A and that $E_A(x) = \tau_\delta(x)f$ for any $x \in fM_\delta^{(i)}f$, $i = 1, 2$. It now follows from [Vo86] that the projection $z := z_\delta^{(1)} \wedge z_\delta^{(2)}$ is nonzero if and only if $\delta < 1/3$. We also have that z is a minimal central projection in $\mathcal{M}_\delta$.

By [IPP05, Theorem 1.2.1], using that $B^{(i)}$ is diffuse, we have that

$$(B^{(i)})' \cap (1 - z_\delta^{(i)})\mathcal{M}_\delta(1 - z_\delta^{(i)}) \subset M_\delta^{(i)}(1 - z_\delta^{(i)}), \qquad i = 1, 2. \tag{5.6}$$

Also note that $(1 - z_\delta^{(1)}) \vee (1 - z_\delta^{(2)}) = 1 - z$. We claim that

$$(B^{(1)} \vee B^{(2)})' \cap \mathcal{M}_\delta(1 - z) = A(1 - z). \tag{5.7}$$

Indeed, let $x \in (B^{(1)} \vee B^{(2)})' \cap \mathcal{M}_\delta(1-z)$. By (5.6), we have that $x_i := x(1-z_\delta^{(i)}) \in M_\delta^{(i)}(1-z_\delta^{(i)})$ for both $i = 1, 2$. By assumption, x commutes with $b_1 b_2$ for any unitaries $b_1 \in \mathcal{U}(B^{(1)})$ and $b_2 \in \mathcal{U}(B^{(2)})$. It follows that $b_1^* x_1 b_1 = b_2 x_2 b_2^* \in A(1-z)$. Thus $x_i \in A(1-z)$ for both $i = 1, 2$ and since $(1-z_\delta^{(1)}) \vee (1-z_\delta^{(2)}) = 1-z$, it follows that $x \in A(1-z)$.

By (5.7), $\mathcal{Z}(\mathcal{M}_\delta)(1-z) \subset A(1-z)$. Since $aS_\delta = S_\delta \alpha(a)$ for $a \in A$, this implies that $\mathcal{Z}(\mathcal{M}_\delta)(1-z) = \mathbb{C}(1-z)$. Hence

$$\mathcal{Z}(\mathcal{M}_\delta) = \mathbb{C}z + \mathbb{C}(1-z).$$

We conclude that $\Gamma(H \oplus H, J \oplus J, A, \tau_\delta)''$ is a factor if and only if $1/3 \leq \delta \leq 2/3$.

5.2 Normalizers and (relative) strong solidity

The main result of this section is the following relative strong solidity theorem for A-valued semicircular systems. In the special case of free Bogoljubov crossed products (see Remark 5.11), this result was proven in [Ho12b, Theorem B]. As explained in the introduction, the A-valued semicircular systems fit perfectly into Popa's deformation/rigidity theory. The proof of Theorem 5.13 therefore follows closely [IPP05, HS09, HR10, Io12, Ho12b], using in the same way Popa's s-malleable deformation given by "doubling and rotating" the initial A-bimodule ${}_A H_A$ (see below).

Theorem 5.13. *Let (A, τ) be a tracial von Neumann algebra and (H, J) a symmetric A-bimodule. Put $M = \Gamma(H, J, A, \tau)''$. Let $q \in M$ be a projection and $B \subset qMq$ a von Neumann subalgebra. If B is amenable relative to A, then at least one of the following statements holds: $B \prec_M A$ or $\mathcal{N}_M(B)''$ stays amenable relative to A.*

As a consequence of Theorem 5.13, we get the following strong solidity theorem.

Corollary 5.14. *Let (A, τ) be a tracial von Neumann algebra and (H, J) a symmetric A-bimodule. Put $M = \Gamma(H, J, A, \tau)''$. Assume that ${}_A H_A$ is mixing. If $B \subset M$ is a diffuse von Neumann subalgebra that is amenable relative to A, then $\mathcal{N}_M(B)''$ stays amenable relative to A.*

So if A is amenable and ${}_A H_A$ is mixing, we get that M is strongly solid.

Proof. Let $P := \mathcal{N}_M(B)''$. Since $B \vee (B' \cap M) \subset P$, we have $P' \cap M = \mathcal{Z}(P)$. Therefore, there exists a largest projection $z \in \mathcal{Z}(P)$ such that $Pz \prec^f A$. In particular, Pz is amenable relative to A. It remains to prove that also

$P(1-z)$ is amenable relative to A. Note that by maximality of z, we have that $P(1-z) \not\prec A$.

Put $z_0 = 1 - z$. Since the bimodule ${}_A H_A$ is mixing, the inclusion $A \subset M$ is mixing in the sense of [Po03, Proof of Theorem 3.1] and [Io12, Definition 9.2]. Since $\mathcal{N}_{z_0 M z_0}(Bz_0)'' = Pz_0$, since Bz_0 is diffuse and since $Pz_0 \not\prec_M A$, it follows from [Io12, Lemma 9.4] that $Bz_0 \not\prec_M A$. It then follows from Theorem 5.13 that Pz_0 is amenable relative to A. □

To prove Theorem 5.13, we fix a tracial von Neumann algebra (A, τ) and a symmetric A-bimodule (H, J). Put $M = \Gamma(H, J, A, \tau)''$ as in Definition 5.5. Recall that $L^2(M) = \mathcal{F}_A(H) = L^2(A) \oplus \bigoplus_{n=1}^\infty H^{\otimes_A^n}$.

We construct as follows an s-malleable deformation of M in the sense of [Po03]. Put

$$\mathcal{M} = \Gamma(H \oplus H, J \oplus J, A, \tau)'' .$$

By Proposition 5.10, we have $\mathcal{M} = M *_A M$. We denote by π_1 and π_2 the two canonical embeddings of M into $\mathcal{M}$. When no embedding is explicitly mentioned, we will always consider $M \subset \mathcal{M}$ via the embedding π_1.

Let $U_t \in \mathcal{U}(H \oplus H)$, $t \in \mathbb{R}$, be the rotation with angle t, i.e.,

$$U_t(\xi, \eta) = (\cos(t)\xi - \sin(t)\eta, \sin(t)\xi + \cos(t)\eta) \quad \text{for} \;\; \xi, \eta \in H \; .$$

Since the construction of $\Gamma(H, J, A, \tau)''$ is functorial, this gives rise to an automorphism $\theta_t := \operatorname{Ad} U_t \in \operatorname{Aut}(\mathcal{M})$. Note that $\theta_{\pi/2} \circ \pi_1 = \pi_2$.

Define $\beta \in \mathcal{U}(H)$ by $\beta(\xi, \eta) = (\xi, -\eta)$ for $\xi, \eta \in H$. Again by functoriality, we have that β defines an automorphism of $\mathcal{M}$. Now, β satisfies $\beta(x) = x$ for all $x \in \pi_1(M)$, $\beta^2 = \mathrm{id}$ and $\beta \circ \theta_t = \theta_{-t} \circ \beta$ for all t. Hence $(\mathcal{M}, (\theta_t)_{t \in \mathbb{R}})$ is an s-malleable deformation of M.

The following two lemmas are the key ingredients in the proof of Theorem 5.13.

Lemma 5.15. *Let $q \in M$ be a projection and $P \subset qMq$ a von Neumann subalgebra. If $\theta_t(P) \prec_{\mathcal{M}} \pi_i(M)$ for some $i \in \{1, 2\}$ and some $t \in (0, \frac{\pi}{2})$, then $P \prec_M A$.*

Lemma 5.16. *Let $q \in M$ be a projection and $P \subset qMq$ a von Neumann subalgebra. If $\theta_t(P)$ is amenable relative to A inside $\mathcal{M}$ for all $t \in (0, \frac{\pi}{2})$, then P is amenable relative to A inside M.*

Before proving Lemma 5.15 and Lemma 5.16, we first show how Theorem 5.13 follows from these two lemmas.

Proof of Theorem 5.13. Put $P = \mathcal{N}_{qMq}(B)''$. We apply [Va13, Theorem A] to the subalgebra $\theta_t(B) \subset M *_A M$ for a fixed $t \in (0, \frac{\pi}{2})$. Note that $\theta_t(B)$ is normalized by $\theta_t(P)$. So, we get that one of the following holds:

(1) $\theta_t(B) \prec_{\mathcal{M}} A$.

(2) $\theta_t(P) \prec_{\mathcal{M}} \pi_i(M)$ for some $i \in \{1, 2\}$.

(3) $\theta_t(P)$ is amenable relative to A inside $\mathcal{M}$.

If 1 or 2 holds, it follows by Lemma 5.15 that $B \prec_M A$. So, if we assume that $B \not\prec_M A$, we get that $\theta_t(P)$ is amenable relative to A inside $\mathcal{M}$ for all $t \in (0, \frac{\pi}{2})$. It then follows from Lemma 5.16 that $P = \mathcal{N}_{qMq}(B)''$ is amenable relative to A inside M. □

Proof of Lemma 5.15

We now turn to the proof of Lemma 5.15. We first give a sketch of the proof. For each $k \in \mathbb{N}$, we let $p_k \in B(L^2 M)$ denote the projection onto $H^{\otimes_A^k}$. Given a von Neumann subalgebra $P \subset qMq$, we first show that if $\theta_t(P) \prec_{\mathcal{M}} \pi_i(M)$ for some $i \in \{1, 2\}$ and some $t \in (0, \frac{\pi}{2})$, then P has "bounded tensor length", in the sense that there exists $k \in \mathbb{N}$ and $\delta > 0$ such that $\| \sum_{i=0}^k p_i(a)\|_2 \geq \delta$ for all $a \in \mathcal{U}(P)$ (see Lemma 5.18). Next, we reason exactly as in the proof of [Po03, Theorem 4.1]. Since θ_t converges uniformly to id on the unit ball of $p_i(M)$ for any fixed $i \in \mathbb{N}$, we get a $t \in (0, \frac{\pi}{2})$ and a nonzero partial isometry $v \in \mathcal{M}$ such that $\theta_t(a)v = va$ for all $a \in \mathcal{U}(P)$. Using the automorphism β, we can even obtain $t = \pi/2$, i.e., $\pi_2(a)v = v\pi_1(a)$ for all $a \in \mathcal{U}(P)$. Using results of [IPP05] on amalgamated free product von Neumann algebras, this implies that $P \prec_M A$.

For simplicity, we put $M_i = \pi_i(M) \subset \mathcal{M}$ for $i \in \{1, 2\}$. Note that

$$L^2(M_1) = L^2(A) \oplus \bigoplus_{k=1}^{\infty} (H \oplus 0)^{\otimes_A^k}, \quad L^2(M_2) = L^2(A) \oplus \bigoplus_{k=1}^{\infty} (0 \oplus H)^{\otimes_A^k} ,$$

as subspaces of $L^2(\mathcal{M}) = \mathcal{F}_A(H \oplus H)$. Denote by $e_{M_i} \in B(L^2(\mathcal{M}))$ the projection onto $L^2(M_i)$.

Lemma 5.17. *If $\mu_n \in L^2(M_1)$ is a bounded net of vectors such that $\lim_{n\to\infty} \|p_k(\mu_n)\| = 0$ for all $k \geq 0$, then for all $i = 1, 2$, $0 < t < \frac{\pi}{2}$, integers $a, b, c, d \geq 0$ and vectors $\xi_i, \eta_i, \gamma_i, \rho_i \in \mathcal{H} \oplus \mathcal{H}$, we have*

$$\|e_{M_i}(\ell(\xi_1)\cdots\ell(\xi_a)\ell(\eta_b)^*\cdots\ell(\eta_1)^* r(\gamma_c)\cdots r(\gamma_1) r(\rho_1)^*\cdots r(\rho_d)^* U_t \mu_n)\| \to 0 ,$$

as $n \to \infty$.

Proof. Fix $t \in (0, \frac{\pi}{2})$ and define $\delta_1 = \cos t$ and $\delta_2 = \sin t$. Define the operator $Z_i \in B(L^2\mathcal{M})$ for $i = 1, 2$ by

$$Z_i = \bigoplus_{e \geq b+d} \delta_i^{e-b-d}(U_t^{\otimes_A^b} \otimes_A 1^{\otimes_A^{(e-b-d)}} \otimes_A U_t^{\otimes_A^d}) \ .$$

Denote $p_{\geq\kappa} = \sum_{i=\kappa}^{\infty} p_i$ and $p_{<\kappa} = \sum_{i=0}^{k-1} p_i$. When $\kappa \geq b+d$, we have $\|Z_i p_{\geq\kappa}\| = \delta_i^{\kappa-b-d}$. Since $\lim_n \|p_{<\kappa}(\mu_n)\| = 0$ for every κ, we get that $\lim_n \|Z_i(\mu_n)\| = 0$. So, it suffices to prove that

$$\begin{aligned} & e_{M_i}(\ell(\xi_1)\cdots\ell(\xi_a)\ell(\eta_b)^*\cdots\ell(\eta_1)^* r(\gamma_c)\cdots r(\gamma_1) r(\rho_1)^*\cdots r(\rho_d)^* U_t p_{\geq b+d}(\mu)) \\ &\qquad = \ell(q_i\xi_1)\cdots\ell(q_i\xi_a)\ell(\eta_b)^*\cdots\ell(\eta_1)^* r(q_i\gamma_c)\cdots r(q_i\gamma_1) r(\rho_1)^*\cdots r(\rho_d)^* Z_i(\mu) \end{aligned}$$

for all $\mu \in L^2(M_1)$, where q_1 and q_2 denote the orthogonal projections of $H \oplus H$ onto $H \oplus 0$ and $0 \oplus H$, respectively. It is sufficient to check this formula for $\mu = \mu_1 \otimes_A \cdots \otimes_A \mu_e$ with $\mu_i \in \mathcal{H} \oplus 0$ and $e \geq b + d$, where it follows by a direct computation. Indeed, let $x = \ell(\eta_b)^* \cdots \ell(\eta_1)^*(U_t\mu_1 \otimes_A \cdots \otimes_A U_t\mu_b) \in A$ and $y = r(\rho_1)^* \cdots r(\rho_d)^*(U_t\mu_{e-d+1} \otimes_A \cdots \otimes_A U_t\mu_e) \in A$. Then

$$\begin{aligned} & e_{M_i}(\ell(\xi_1)\cdots\ell(\xi_a)\ell(\eta_b)^*\cdots\ell(\eta_1)^* r(\gamma_c)\cdots r(\gamma_1) r(\rho_1)^*\cdots r(\rho_d)^* U_t(\mu)) \\ &\qquad = e_{M_i}(\xi_1 \otimes \cdots \otimes \xi_a \otimes x \cdot U_t\mu_{b+1} \otimes \cdots \otimes U_t\mu_{e-d} \cdot y \otimes \gamma_1 \otimes \cdots \otimes \gamma_c) \\ &\qquad = q_i\xi_1 \otimes \cdots \otimes q_i\xi_a \otimes x\delta_i\mu_{b+1} \otimes \cdots \otimes \delta_i\mu_{e-d} y \otimes q_i\gamma_1 \otimes \cdots \otimes q_i\gamma_c \\ &\qquad = \ell(q_i\xi_1)\cdots\ell(q_i\xi_a)\ell(\eta_b)^*\cdots\ell(\eta_1)^* r(q_i\gamma_c)\cdots r(q_i\gamma_1) r(\rho_1)^*\cdots r(\rho_d)^* Z_i(\mu). \end{aligned}$$

□

Lemma 5.18. *If $a_n \in M$ is a bounded net with $\lim_n \|p_k(a_n)\|_2 = 0$ for all $k \geq 0$, then*

$$\lim_{n\to\infty} \|E_{M_i}(x\theta_t(a_n)y)\|_2 = 0 \ ,$$

for all $i \in \{1, 2\}$, $0 < t < \frac{\pi}{2}$ and $x, y \in \mathcal{M}$.

Proof. It suffices to take $x = W(\xi_1, \ldots, \xi_k)$ and $y = W(\eta_1, \ldots, \eta_m)$ with $\xi_i, \eta_i \in \mathcal{H} \oplus \mathcal{H}$ (as defined in Section 5.1), since these elements span a $\|\cdot\|_2$-dense subspace of $\mathcal{M} \ominus A$. Then,

$$\begin{aligned} E_{M_i}(x\theta_t(a_n)y) &= e_{M_i}(xJy^*JU_t(a_n\Omega)) \\ &= \sum_{s=0}^{k}\sum_{r=0}^{m} e_{M_i}(\ell(\xi_1)\cdots\ell(\xi_s)\ell(J\xi_{s+1})^*\cdots\ell(J\xi_k)^* \\ &\qquad r(\eta_m)\cdots r(\eta_{r+1}) r(J\eta_r)^*\cdots r(J\eta_1)^* U_t(a_n\Omega)) \ , \end{aligned}$$

and the result now follows from Lemma 5.17. □

We are now ready to finish the proof of Lemma 5.15.

Proof of Lemma 5.15. Assume that $\theta_t(P) \prec M_i$ for some $i \in \{1,2\}$ and $t \in (0, \frac{\pi}{2})$. Without loss of generality, we assume that $i = 1$. By Lemma 5.18, we get a $\delta > 0$ and $\kappa > 0$ such that $\| \sum_{j=0}^{\kappa} p_j(a)\|_2^2 \geq 2\delta$ for all $a \in \mathcal{U}(P)$. Note that $\langle U_t(p_i(a)), p_j(a)\rangle = 0$ if $i \neq j$ and that $\langle U_t(p_i(a)), p_i(a)\rangle = \cos(t)^i \|p_i(a)\|_2^2$. Choose $t_0 \in (0, \frac{\pi}{2})$ such that $\cos(t_0)^i \geq 1/2$ for all $i = 0, \dots, \kappa$. Note that we may choose t_0 of the form $t_0 = \pi/2^n$. For all $a \in \mathcal{U}(P)$, we then have

$$\tau(\theta_{t_0}(a)a^*) = \langle U_{t_0}(a), a\rangle = \sum_{i,j=0}^{\infty} \langle U_{t_0}(p_i(a)), p_j(a)\rangle = \sum_{i=0}^{\infty} \cos(t_0)^i \|p_i(a)\|_2^2$$

$$\geq \sum_{i=0}^{\kappa} \cos(t_0)^i \|p_i(a)\|_2^2 \geq \frac{1}{2} 2\delta = \delta \ .$$

Let v be the unique element of minimal 2-norm in the weakly closed convex hull of $\{\theta_{t_0}(a)a^* \mid a \in \mathcal{U}(P)\}$. Then $v \in \mathcal{M}$ and $\theta_{t_0}(a)v = va$ for all $a \in \mathcal{U}(P)$. Moreover, $v \neq 0$ since $\tau(v) \geq \delta$.

Put $w_1 = \theta_{t_0}(v\beta(v^*))$. Then w_1 satisfies $w_1 a = \theta_{2t_0}(a)w_1$ for all $a \in \mathcal{U}(P)$. However, we do not know yet that w_1 is nonzero. Assuming that $P \not\prec_M A$, we have from Proposition 5.10 and [IPP05, Theorem 1.2.1] that $P' \cap q\mathcal{M}q \subset qMq$, hence $v^*v \in qMq$. Thus

$$w_1^* w_1 = \theta_{t_0}(\beta(v)v^*v\beta(v^*)) = \theta_{t_0}(\beta(vv^*)) \neq 0 \ .$$

By iterating this process, we obtain $w = w_{n-1} \neq 0$ such that $wa = \theta_{\pi/2}(a)w$, i.e., $w\pi_1(a) = \pi_2(a)w$ for all $a \in P$. This means that $P \prec_{\mathcal{M}} M_2$. As in [Ho07, Claim 5.3], this is incompatible with our assumption $P \not\prec_M A$. So it follows that $P \prec_M A$ and the lemma is proved. □

Proof of Lemma 5.16

Proof. Let $P \subset qMq$ and assume that $\theta_t(P)$ is amenable relative to A in $\mathcal{M}$ for all $t \in (0, \frac{\pi}{2})$. As in the proof of [Io12, Theorem 5.1] (and [Va13, Theorem 3.4]), we let I be the set of all quadruples $i = (X, Y, \delta, t)$ where $X \subset \mathcal{M}$ and $Y \subset \mathcal{U}(P)$ are finite subsets, $\delta \in (0,1)$ and $t \in (0, \frac{\pi}{2})$. Then I is a directed set when equipped with the ordering $(X, Y, \delta, t) \leq (X', Y', \delta', t')$ if and only if $X \subset X'$, $Y \subset Y'$, $\delta' \leq \delta$ and $t' \leq t$.

By [OP07, Theorem 2.1], we can for each $i = (X, Y, \delta, t) \in I$ choose a vector $\xi_i \in \theta_t(q)L^2(\mathcal{M}) \otimes_A L^2(\mathcal{M})\theta_t(q)$ such that $\|\xi_i\|_2 \leq 1$ and

$$|\langle x\xi_i, \xi_i\rangle - \tau(x\theta_t(q))| \leq \delta \quad \text{for all } x \in X \text{ or } x = (\theta_t(y) - y)^*(\theta_t(y) - y),\ y \in Y,$$

$$\|\theta_t(y)\xi_i - \xi_i\theta_t(y)\|_2 \leq \delta \quad \text{for all } y \in Y\ .$$

Moreover, we may assume that ξ_i satisfies $\langle x\xi_i, \xi_i\rangle = \langle \xi_i x, \xi_i\rangle$ for all $x \in \mathcal{M}$ and $i \in I$.

We now prove that the qMq-P-bimodule ${}_{qMq}L^2(qMq)_P$ is weakly contained in ${}_{qMq}(qL^2(\mathcal{M}) \otimes_A L^2(\mathcal{M})q)_P$. By Lemma 2.8, it suffices to show that

$$\begin{aligned} &\lim_i \langle x\xi_i, \xi_i\rangle = \tau(x) \quad \text{for every } \ x \in q\mathcal{M}q\ , \\ &\lim_i \|y\xi_i - \xi_i y\|_2 = 0 \quad \text{for every } \ y \in P\ . \end{aligned} \tag{5.8}$$

Let $y \in \mathcal{U}(P)$ and $\varepsilon > 0$ be given. Choose $t > 0$ small enough so that $\|\theta_t(y) - y\|_2 \leq \frac{\varepsilon}{\sqrt{18}}$. We have

$$\|y\xi_i - \xi_i y\|_2 \leq \|(y - \theta_t(y))\xi_i\|_2 + \|\theta_t(y)\xi_i - \xi_i\theta_t(y)\|_2 + \|\xi_i(\theta_t(y) - y)\|_2$$

for any $i \in I$. Moreover, for $i \geq (\{0\}, \{y\}, \frac{\varepsilon^2}{18}, t)$ we have

$$\begin{aligned} \|(y - \theta_t(y))\xi_i\|_2^2 &= \langle(\theta_t(y) - y)^*(\theta_t(y) - y)\xi_i, \xi_i\rangle \\ &\leq \|(\theta_t(y) - y)\theta_t(q)\|_2^2 + \frac{\varepsilon^2}{18} \leq \frac{\varepsilon^2}{9}. \end{aligned}$$

Similarly, we get that $\|\xi_i(\theta_t(y) - y)\|_2 \leq \frac{\varepsilon}{3}$. Thus, we conclude that $\|y\xi_i - \xi_i y\|_2 \leq \varepsilon$ for $i \geq (\{0\}, \{y\}, \frac{\varepsilon^2}{18}, t)$ and so the second assertion of (5.8) holds true. The first assertion is proved similarly, using that $\|\theta_t(q) - q\|_2 \to 0$ as $t \to 0$.

By Proposition 5.10, we have $\mathcal{M} = M_1 *_A M_2$. Under our identification $M = M_1$, we claim that ${}_M L^2(\mathcal{M})_A \cong {}_M(L^2(M) \otimes_A \mathcal{K})_A$ for some A-bimodule ${}_A\mathcal{K}_A$. To see this, note that

$$E_{M_1}(x_1 \cdots x_{2m+1}) = 0,$$

whenever $m \geq 1$, $x_1, x_{2m+1} \in M_1$, $x_{2j} \in M_2 \ominus A$ and $x_{2j+1} \in M_1 \ominus A$ for all $j = 1, \ldots, m-1$. Also, $L^2(\mathcal{M} \ominus M_1)$ is the closed linear span of such elements. It follows that

$$L^2(\mathcal{M} \ominus M_1) \cong \bigoplus_{m \geq 1} L^2(M_1) \otimes_A L^2(M_2 \ominus A) \otimes_A \cdots \otimes_A L^2(M_2 \ominus A) \otimes_A L^2(M_1),$$

where the tensor product on the right hand side should be understood as an alternating tensor product between $L^2(M_2 \ominus A)$ and $L^2(M_1 \ominus A)$, with $2m+1$ tensor factors in total. So, letting

$$K = \bigoplus_{m \geq 1} L^2(M_2 \ominus A) \otimes_A \cdots \otimes_A L^2(M_2 \ominus A) \otimes_A L^2(M_1) \oplus L^2(A),$$

it follows that ${}_M L^2(\mathcal{M})_A \cong {}_M(L^2(M) \otimes_A \mathcal{K})_A$ as claimed.

We conclude that the bimodule ${}_{qMq}L^2(qMq)_P$ is weakly contained in ${}_{qMq}(qL^2(M) \otimes_A (\mathcal{K} \otimes_A L^2(\mathcal{M})q))_P$. It then follows from Proposition 2.11 that P is amenable relative to A inside M. This finishes the proof of Lemma 5.16. □

5.3 Maximal amenability

Fix a tracial von Neumann algebra (A, τ) and a symmetric Hilbert A-bimodule ${}_A H_A$ with symmetry $J \colon H \to H$. Denote by $M = \Gamma(H, J, A, \tau)''$ the associated von Neumann algebra with faithful normal tracial state τ. We prove the following maximal amenability property by combining Popa's asymptotic orthogonality [Po83] with the method of [BH16]. In the special case of free Bogoljubov crossed products (see Remark 5.11), part 3 of Theorem 5.19 was proved in [Ho12b, Theorem D].

Theorem 5.19. *Assume that ${}_A H_A$ is weakly mixing. Then the following properties hold.*

1. $\mathcal{Z}(M) = \{a \in \mathcal{Z}(A) \mid a\xi = \xi a \text{ for all } \xi \in H\}$.
2. *If $B \subset M$ is a von Neumann subalgebra that is amenable relative to A inside M and if the bimodule ${}_{B \cap A}H_A$ is left weakly mixing, then $B \subset A$.*
3. *A von Neumann subalgebra of M that properly contains A is not amenable relative to A inside M. If the A-bimodule ${}_A H_A$ is faithful[1], then M has no amenable direct summand. If A is amenable, then $A \subset M$ is a maximal amenable subalgebra.*

Proof. As above, identify

$$L^2(M) = L^2(A) \oplus \bigoplus_{n \geq 1} H^{\otimes_A^n}$$

[1] A P-Q-bimodule ${}_P H_Q$ is called faithful if the $*$-homomorphisms $P \to B(H)$ and $Q^{\text{op}} \to B(H)$ are faithful.

and denote by $\mathcal{H} \subset H$ the subspace of vectors that are both left and right bounded.

1. Since ${}_A H_A$ is weakly mixing, it follows from Proposition 2.26 that the n-fold tensor products $H^{\otimes_A^n}$ (with $n \geq 1$) have no A-central vectors. Therefore, $A' \cap M = \mathcal{Z}(A)$. Looking at the commutator of $a \in \mathcal{Z}(A)$ and $\ell(\xi) + \ell(J\xi)^*$, the conclusion follows.

2. Since B is amenable relative to A inside M, we can fix a B-central state $\omega \in \langle M, e_A\rangle^*$ such that $\omega|_M = \tau$.

Claim I. For every $\xi \in \mathcal{H}$ and every $\varepsilon > 0$, there exists a projection $p \in A$ such that $\tau(1-p) < \varepsilon$ and such that

$$\omega(\ell(\xi p)\ell(\xi p)^*) < \varepsilon \; .$$

To prove this claim, fix $\xi \in \mathcal{H}$ and $\varepsilon > 0$. Define $a = \sqrt{\langle \xi, \xi\rangle_A}$ and denote by $q \in A$ the support projection of a. Take a projection $q_1 \in qAq$ that commutes with a, such that $\tau(q - q_1) < \varepsilon/2$ and such that aq_1 is invertible in $q_1 A q_1$. Denote by $b \in q_1 A q_1$ this inverse and define $\eta = \xi b$. By construction, $\ell(\eta)^*\ell(\eta) = q_1$ and $\xi q_1 = \eta a$.

Pick a positive integer N such that $2^{-N} < \varepsilon/(2\|a\|^2)$. Put $\kappa = 2^N$. Then pick $\delta > 0$ such that $\delta < \varepsilon/(\kappa 2\|a\|^2)$. We start by constructing unitary operators $v_1, \ldots, v_\kappa \in \mathcal{U}(A \cap B)$ and a projection $q_2 \in q_1 A q_1$ such that $\tau(q_1 - q_2) < \varepsilon/2$ and such that the vectors $\eta_i = v_i \eta$ satisfy

$$\|q_2 \langle \eta_i, \eta_j\rangle_A \, q_2\| < \delta \quad \text{whenever} \;\; i \neq j \tag{5.9}$$

(and where we indeed use the operator norm at the left hand side of (5.9)).

We put $e_0 = q_1$ and $v_1 = 1$. Denoting by $(a_i)_{i \in I}$ the net of unitaries in $B \cap A$ witnessing the left weak mixing of ${}_{B\,\cap\,A} H_A$, we get that $\lim_i \|\langle \eta, a_i \eta\rangle_A\|_2 = 0$. Let $r_i \in q_1 A q_1$ be the spectral projection of $\langle \eta, a_i\eta\rangle_A^* \langle \eta, a_i \eta\rangle_A$ associated with the interval $[0, \delta^2]$. Then $(r_i)_{i\in I}$ is a net of projections such that

$$\tau(q_1 - r_i) \leq \delta^{-2}\tau(\langle \eta, a_i\eta\rangle_A^*\langle \eta, a_i\eta\rangle_A) \to 0$$

and

$$\|\langle \eta, a_i \eta\rangle_A \, r_i\|^2 = \|r_i \langle \eta, a_i\eta\rangle_A^* \langle \eta, a_i \eta\rangle_A r_i\| < \delta^2 \quad \text{for every } i.$$

Take i large enough such that $\tau(q_1 - r_i) < \varepsilon/4$ and define $e_1 := r_i$ and $v_2 := a_i$. We have now constructed v_1, v_2. Inductively, we double the length of the sequence, until we arrive at $v_1, \ldots, v_\kappa$. After k steps, we have constructed the projections $e_1 \geq \cdots \geq e_k$ and unitaries $v_1, \ldots, v_{2^k}$ in $\mathcal{U}(B \cap A)$ such that $\tau(e_{j-1} - e_j) < 2^{-j-1}\varepsilon$ and such that the vectors $\eta_i = v_i \eta$ satisfy

$$\|e_k \, \langle \eta_i, \eta_j\rangle_A \, e_k\| < \delta \quad \text{whenever} \;\; i \neq j \; .$$

As in the first step, we can pick a unitary $a \in \mathcal{U}(B \cap A)$ and a projection $e_{k+1} \in e_k A e_k$ such that $\tau(e_k - e_{k+1}) < 2^{-k-2}\varepsilon$ and such that

$$\|e_{k+1} \langle \eta_i, a\eta_j \rangle_A \, e_{k+1}\| < \delta$$

for all $i, j \in \{1, \ldots, 2^k\}$. It now suffices to put $v_{2^k+i} = av_i$ for all $i = 1, \ldots, 2^k$. We have doubled our sequence. We continue for N steps and put $q_2 = e_N$. So, (5.9) is proved.

Put $\mu_i = \eta_i q_2 = v_i \eta q_2$. Define $P_i = \ell(\mu_i)\ell(\mu_i)^*$ and note that $P_i = v_i P_1 v_i^*$. Also note that P_i is a projection since η was chosen such that P_1 is a projection. By construction, $\|P_i P_j\| < \delta$ whenever $i \neq j$. Writing $P = \sum_{i=1}^{\kappa} P_i$ it follows that $\|P^2 - P\| < \kappa^2\delta$. Since P is a positive operator, we conclude that $\|P\| < 1 + \kappa^2\delta$. Since ω is B-central and $v_i \in B$ for all i, we get that

$$\kappa\, \omega(P_1) = \sum_{i=1}^{\kappa} \omega(P_i) = \omega(P) \leq \|P\| < 1 + \kappa^2\delta \ .$$

Therefore, $\omega(P_1) < \kappa^{-1} + \kappa\delta < \|a\|^{-2}\varepsilon$.

Since q_1 and a commute, the right support of $(q_1 - q_2)a$ is smaller than q_1, hence it is a projection of the form $q_1 - p_0$ where $p_0 \in q_1 A q_1$. Since the left and right supports of an element have the same trace and since the left support of $(q_1 - q_2)a$ is smaller than $q_1 - q_2$, we have that $\tau(q_1 - p_0) \leq \tau(q_1 - q_2) < \varepsilon/2$. By construction, $q_1 a p_0 = q_2 a p_0$. Since $p_0 \leq q_1$ and $\eta = \eta q_1$, it follows that

$$\xi p_0 = \xi q_1 p_0 = \eta a p_0 = \eta q_1 a p_0 = \eta q_2 a p_0 \ .$$

Define the projection $p \in A$ given by $p = (1 - q) + p_0$. Since $\xi(1 - q) = 0$, we still have $\xi p = \eta q_2 a p_0$. Because $1 - p = (q - q_1) + (q_1 - p_0)$, we get that $\tau(1 - p) < \varepsilon$. Finally,

$$\begin{aligned} \omega(\ell(\xi p)\ell(\xi p)^*) &= \omega(\ell(\eta q_2)\, a p_0 a^* \, \ell(\eta q_2)^*) \leq \|a\|^2 \, \omega(\ell(\eta q_2)\ell(\eta q_2)^*) \\ &= \|a\|^2 \, \omega(P_1) < \varepsilon \ . \end{aligned}$$

So, we have proven Claim I.

Claim II. For every $\xi \in \mathcal{H}$ and every $\varepsilon > 0$, there exists a projection $p \in A$ such that $\tau(1 - p) < \varepsilon$ and such that $\omega(\ell(\xi p)\ell(\xi p)^*) = 0$.

For every integer $k \geq 1$, Claim I gives a projection $p_k \in A$ with $\tau(1 - p_k) < 2^{-k}\varepsilon$ and $\omega(\ell(\xi p_k)\ell(\xi p_k)^*) < 1/k$. Defining $p = \bigwedge_k p_k$, we get that $\tau(1 - p) < \varepsilon$ and, for every $k \geq 1$,

$$\omega(\ell(\xi p)\ell(\xi p)^*) = \omega(\ell(\xi) p \ell(\xi)^*) \leq \omega(\ell(\xi) p_k \ell(\xi)^*) = \omega(\ell(\xi p_k)\ell(\xi p_k)^*) < 1/k \ .$$

So, $\omega(\ell(\xi p)\ell(\xi p)^*) = 0$ and claim II is proved.

We can now conclude the proof of 2. Denote by $E_A \colon M \to A$ and $E_B \colon M \to B$ the unique trace-preserving conditional expectations. It is sufficient to prove that $E_B \circ E_A = E_B$. So we have to prove that $E_B(x) = 0$ for all $x \in M \ominus A$. Using the Wick products defined in (5.2), it suffices to prove that $E_B(W(\xi_1, \dots, \xi_k)) = 0$ for all $k \geq 1$ and all $\xi_1, \dots, \xi_k \in \mathcal{H}$.

Since ω is B-central and $\omega|_M = \tau$, there is a unique conditional expectation $\Phi \colon \langle M, e_A \rangle \to B$ such that $\Phi|_M = E_B$ and $\omega = \tau \circ \Phi$. Indeed, given $T \in \langle M, e_A \rangle_+$, we define a positive linear functional ω_T on B by $\omega_T(x) = \omega(Tx)$. For $b \in B_+$, we have

$$|\omega_T(b)|^2 = |\omega(b^{1/2} T b^{1/2})|^2 \leq \omega(b^{1/2} T^2 b^{1/2}) \omega(b) \leq \|T\|^2 \omega(b)^2 = \|T\|^2 \tau(b)^2.$$

By the Radon-Nikodym Theorem, there exists a unique element $\Phi(T) \in B_+$ such that $\omega(Tb) = \tau(\Phi(T)b)$ for $b \in B$. Extending Φ to all of $\langle M, e_A \rangle$ gives a conditional expectation $\Phi \colon \langle M, e_A \rangle \to B$ such that $\Phi|_M = E_B$ and $\omega = \tau \circ \Phi$.

We first consider $k \geq 2$ and $\xi_1, \dots, \xi_k \in \mathcal{H}$. By Claim II, we can take sequences of projections $p_n, q_n \in A$ such that $p_n \to 1$ and $q_n \to 1$ strongly and

$$\Phi(\ell(\xi_1 p_n)\ell(\xi_1 p_n)^*) = 0 = \Phi(\ell(J(\xi_k) q_n)\ell(J(\xi_k) q_n)^*)$$

for all n. Then also $\Phi(\ell(\xi_1 p_n)T) = 0 = \Phi(T\ell(J(\xi_k) q_n)^*)$ for all n and all $T \in \langle M, e_A \rangle$, by the Cauchy-Schwarz inequality. We conclude that

$$E_B(W(\xi_1 p_n, \xi_2, \dots, \xi_{k-1}, q_n \xi_k)) = \Phi(W(\xi_1 p_n, \xi_2, \dots, \xi_{k-1}, q_n \xi_k)) = 0$$

for all n. Since E_B is normal, it follows that $E_B(W(\xi_1, \dots, \xi_k)) = 0$.

We next consider the case $k = 1$. So it remains to prove that $E_B(\ell(\xi) + \ell(J\xi)^*) = 0$ for all $\xi \in \mathcal{H}$. For this, it is sufficient to prove that $\Phi(\ell(\xi)) = 0$ for all $\xi \in \mathcal{H}$. By Claim II and reasoning as above, we find a sequence of projections $p_n \in A$ such that $p_n \to 1$ strongly and $\Phi(\ell(\xi p_n)T) = 0$ for all n and all $T \in \langle M, e_A \rangle$. In particular, we can take $T = 1$ and get that $\Phi(\ell(\xi)p_n) = 0$ for all n. Write $e_n = 1 - p_n$. Using the Cauchy-Schwarz inequality, we get

$$\Phi(\ell(\xi))^* \Phi(\ell(\xi)) = \Phi(\ell(\xi)e_n)^* \Phi(\ell(\xi)e_n) \leq \|\ell(\xi)\|^2 \, \Phi(e_n) = \|\ell(\xi)\|^2 \, E_B(e_n) \, .$$

Since $E_B(e_n) \to 0$ strongly, we conclude that $\Phi(\ell(\xi)) = 0$. This concludes the proof of 2.

3. It follows from 2 that a von Neumann subalgebra of M properly containing A is not amenable relative to A and thus, not amenable itself. Whenever $H \neq \{0\}$, we have $A \neq M$ and we conclude that M is not amenable. By 1, any

direct summand of M is given as the von Neumann algebra associated with the symmetric weakly mixing Az-bimodule Hz where $z \in \mathcal{Z}(A)$ is a nonzero central projection satisfying $\xi z = z\xi$ for all $\xi \in H$. If ${}_A H_A$ is faithful, we have $Hz \neq \{0\}$ and it follows that this direct summand is not amenable. The final statement is an immediate consequence of 2. □

5.4 Absence of Cartan subalgebras

In this section, we give a complete description of the structure of the von Neumann algebra $M = \Gamma(H, J, A, \tau)''$ associated with an arbitrary symmetric A-bimodule (H, J).[2] We describe the trivial direct summands of M and then prove that the remaining direct summand never has a Cartan subalgebra and describe its center (see Theorem 5.21). In all interesting cases, there are no trivial direct summands and this allows us to prove absence of Cartan subalgebras whenever H is a weakly mixing A-bimodule (Corollary 5.22), when A is a II_1 factor and H is not the trivial bimodule nor the bimodule given by a period 2 automorphism of A (Corollary 5.23), and finally for arbitrary free Bogoljubov crossed products (Corollary 5.24). This last result improves [Ho12b, Corollary C].

To prove our general structure theorem, we need the following terminology. Fix a separable tracial von Neumann algebra (A, τ). We say that an A-bimodule H is given by a partial automorphism if one of the following two equivalent conditions holds.

- The commutant of the right A-action on H equals the left A-action, and vice versa.
- There exists a projection $e \in B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} A$, a central projection $z \in \mathcal{Z}(A)$ and a normal surjective $*$-isomorphism $\alpha\colon Az \to e(B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} A)e$ such that ${}_A H_A \cong e(\ell^2(\mathbb{N}) \otimes L^2(A))$ with the bimodule structure given by $a \cdot \xi \cdot b = \alpha(a)\xi b$.

If (H, J) is a symmetric A-bimodule that is given by a partial automorphism, then $\Gamma(H, J, A, \tau)''$ is "trivial", in the sense that it is essentially equal to A. We can compute this using the methods of Example 5.12.

Proposition 5.20. *Let (A, τ) be a separable tracial von Neumann algebra and (H, J) a symmetric A-bimodule such that H is given by a partial automorphism of A. Put $M = \Gamma(H, J, A, \tau)''$. Then M contains a subalgebra N of index 2 (in*

[2] For simplicity, we will in this section restrict ourselves to the case where A is a separable von Neumann algebra.

the sense that N is fixed by an order 2 automorphism of M) such that N is a direct sum of a corner of A and a corner of $A \mathbin{\overline{\otimes}} L^\infty[0,1]$.

Proof. Let $\sigma\colon \mathbb{Z}/2\mathbb{Z} \curvearrowright H$ denote the action given by $\sigma(1)(\xi) = -\xi$ for $\xi \in H$. Clearly σ commutes with J and with the left and right A-actions. By functoriality, we get an action $\sigma\colon \mathbb{Z}/2\mathbb{Z} \curvearrowright M$ satisfying $\sigma(1)W(\xi_1, \dots, \xi_n) = (-1)^n W(\xi_1, \dots, \xi_n)$ for $\xi_1, \dots, \xi_n \in \mathcal{H}$. Let

$$N = A \vee \{W(\xi_1, \dots, \xi_n) \mid n \text{ even}, \xi_1, \dots, \xi_n \in \mathcal{H}\}'' \subset M,$$

and note that N equals the subalgebra of M that is fixed under σ. So, N is an index 2 subalgebra of M.

Let $\alpha\colon Az \to p(B(\ell^2\mathbb{N})\overline{\otimes}A)p$ be an isomorphism such that $H = p(\ell^2(\mathbb{N})\otimes L^2(A))$ with A-bimodular actions given by $a\cdot\xi\cdot b = \alpha(a)\xi b$. We then have that $H \otimes_A \overline{H} = L^2(p(B(\ell^2\mathbb{N}) \mathbin{\overline{\otimes}} A)p)$ with A-bimodular actions given by $a \cdot \xi \cdot b = \alpha(a)\xi\alpha(b)$. So, α gives an isomorphism between $H \otimes_A \overline{H}$ and the trivial Az-bimodule. Since H is symmetric, we have that $H \cong \overline{H}$ and hence $H \otimes_A H$ is isomorphic with the trivial Az-bimodule. Let $\Phi\colon H \otimes_A H \to L^2(A)z$ denote this isomorphism.

Let $\xi_0 = \Phi^{-1}(z) \in H \otimes_A H$. Since ξ_0 is a left and right bounded vector in $L^2(M)$, there is a unique element $S \in M$ such that $S\Omega = \xi_0$. Note that $S \in N$, S is self-adjoint and S commutes with A. We claim that $N = A \vee \{S\}''$. Indeed, since $\overline{A\xi_0 A} = H \otimes_A H$, we have that $W(\xi_1, \xi_2) \in A \vee \{S\}''$ for any $\xi_1, \xi_2 \in \mathcal{H}$. Inductively, it follows that $W(\xi_1, \dots, \xi_{2k}) \in A \vee \{S\}''$ for any $k \in \mathbb{N}$ and $\xi_i \in \mathcal{H}$, and thus $N \subset A \vee \{S\}''$. The reverse inclusion is trivial.

Write $\{S\}'' = L^\infty(X, \mu)$, where μ is the spectral measure of S. By decomposing μ into its continuous and atomic parts, we get that $\{S\}'' \cong L^\infty[0,1] \oplus \bigoplus_{i\in I} \mathbb{C}$, where I is a set indexing the atoms of μ. Let $z_0 \in \{S\}''$ denote the projection onto the continuous part and let $(z_i)_{i\in I}$ denote the projections onto the atoms. Then

$$N = A \vee \{S\}'' \cong L^\infty[0,1] \mathbin{\overline{\otimes}} Az_0 \oplus \bigoplus_{i\in I} Az_i.$$

□

Fix a symmetric A-bimodule (H, J) and denote $M = \Gamma(H, J, A, \tau)''$. Then, M has two trivial direct summands. First denote by $z_0 \in \mathcal{Z}(A)$ the largest projection such that $z_0 H = \{0\}$. Then, $z_0 \in \mathcal{Z}(M)$ and $Mz_0 = Az_0$. Next, there is a largest projection $z_1 \in \mathcal{Z}(A)(1 - z_0)$ such that $z_1 H = H z_1$ and such that the A-bimodule Hz_1 is given by a partial automorphism of A (see Lemma 5.40 for details). Again $z_1 \in \mathcal{Z}(M)$ and Mz_1 is essentially equal to a corner of A, up to amplifications and an index 2 extension (see Proposition 5.20).

Writing $z_2 = 1 - (z_0 + z_1)$, we thus get that

$$M = Az_0 \oplus \Gamma(Hz_1, J, Az_1, \tau)'' \oplus \Gamma(Hz_2, J, Az_2, \tau)''$$

and only the third direct summand is "interesting and nontrivial". By Lemma 5.40, the symmetric Az_2-bimodule Hz_2 is *completely nontrivial* in the following sense: the left action of Az_2 on H is faithful and there are no nonzero projections $e, f \in \mathcal{Z}(A)z_2$ such that $eH = Hf$ and such that eH is given by a partial automorphism of Az_2. So it suffices to describe the structure of the von Neumann algebra associated with an arbitrary completely nontrivial symmetric A-bimodule.

We denote by $\dim_{-A}(K)$ the right A-dimension of a right Hilbert A-module K. Recall that the value of $\dim_{-A}(K)$ depends on the choice of the trace τ. We similarly define $\dim_{A-}(K)$ for a left Hilbert A-module K. As in (5.10), for every A-bimodule H, there is a unique element Δ_H^ℓ in the extended positive part of $\mathcal{Z}(A)$ characterized by $\tau(\Delta_H^\ell e) = \dim_{-A}(eH)$ for every projection $e \in \mathcal{Z}(A)$.

Theorem 5.21. *Let (A, τ) be a separable tracial von Neumann algebra and (H, J) a completely nontrivial symmetric A-bimodule. Write $M = \Gamma(H, J, A, \tau)''$. There is a canonical central projection $q \in \mathcal{Z}(M)$ (which most of the time is zero) such that the following holds.*

(a) No direct summand of $M(1-q)$ is amenable relative to $A(1-q)$.

(b) No direct summand of $M(1-q)$ admits a Cartan subalgebra.

(c) $Mq = Aq$ and the support of $E_A(1-q)$ equals 1.

(d) Defining $C := \{a \in \mathcal{Z}(A) \mid a\xi = \xi a \text{ for all } \xi \in H\}$, we get that $\mathcal{Z}(M) = \mathcal{Z}(A)q + C(1-q)$.

Moreover, we have that $E_A(q) = Z(\Delta_H^\ell)$, where $Z\colon (0, +\infty) \to \mathbb{R}$ is the positive function given by $Z(t) = 1 - t$ when $t \in (0, 1)$ and $Z(t) = 0$ when $t \geq 1$.

Corollary 5.22. *Let (A, τ) be a separable tracial von Neumann algebra and (H, J) a symmetric A-bimodule. Put $M = \Gamma(H, J, A, \tau)''$. If ${}_AH_A$ is weakly mixing and faithful, then no direct summand of M has a Cartan subalgebra and $\mathcal{Z}(M) = \{a \in \mathcal{Z}(A) \mid a\xi = \xi a \text{ for all } \xi \in H\}$.*

Proof. Let $z \in \mathcal{Z}(A)$ be a nonzero central projection. Since $zH \neq \{0\}$ and zH is still left weakly mixing as an A-bimodule, we have that $\dim_{-A}(zH) = +\infty$ and that zH is not given by a partial automorphism of A. So the conclusions follow from Theorem 5.21. □

When A is a II_1 factor, the results of Theorem 5.21 can be formulated more easily as follows.

Corollary 5.23. *Let A be a separable II_1 factor with its unique tracial state τ and let (H, J) be a symmetric A-bimodule. Denote $M = \Gamma(H, J, A, \tau)''$. Unless H is zero or H is the trivial A-bimodule or H is the symmetric A-bimodule associated with a period 2 outer automorphism of A, the following holds: M is a factor, M is not amenable relative to A, and M has no Cartan subalgebra.*

Proof. Since A is a II_1 factor, the only symmetric A-bimodules given by a partial automorphism of A are the trivial A-bimodule and the A-bimodule given by $\alpha \in \mathrm{Aut}(A)$ with $\alpha \circ \alpha$ being inner. Indeed, assume that $\alpha\colon A \to e(B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} A)e$ is a $*$-isomorphism and $H = e(\ell^2(\mathbb{N}) \otimes L^2(A))$ with A-bimodular actions given by $a \cdot \xi \cdot b = \alpha(a)\xi b$. By [AP16, Proposition 12.1.5] and since H is symmetric, we have that

$$(\mathrm{Tr} \otimes \tau)(e) = \dim_{-A}(H) = \dim_{A-}(H) = \frac{1}{(\mathrm{Tr} \otimes \tau)(e)},$$

and hence $(\mathrm{Tr} \otimes \tau)(e) = 1$. It follows that $H = L^2(A)$ and that $\alpha\colon A \to A$ is a $*$-isomorphism. Since H is symmetric, we must have that α^2 is an inner automorphism of A.

When a symmetric A-bimodule H is not given by a partial automorphism of A, we have that $\dim_{-A}(H) > 1$. So, $Z(\Delta_H^\ell) = 0$, where $Z\colon (0, \infty) \to \mathbb{R}$ denotes the function defined in Theorem 5.21. The conclusion now follows from Theorem 5.21. □

We finally deduce that free Bogoljubov crossed products never have a Cartan subalgebra. In [Ho12b, Corollary C], this was proven under extra assumptions on the underlying orthogonal representation.

Corollary 5.24. *Let G be an arbitrary countable group and $\pi\colon G \to \mathcal{O}(K_\mathbb{R})$ an orthogonal representation of G with $\dim(K_\mathbb{R}) \geq 2$. Denote by $\sigma_\pi\colon G \curvearrowright \Gamma(K_\mathbb{R})'' \cong L(\mathbb{F}_{\dim K_\mathbb{R}})$ the associated free Bogoljubov action with crossed product $M := \Gamma(K_\mathbb{R})'' \rtimes^{\sigma_\pi} G$ (see Remark 5.11). Then no direct summand of M has a Cartan subalgebra. Also, M is a factor if and only if $\pi(g) \neq 1$ for every $g \in G \setminus \{e\}$ that has a finite conjugacy class.*

Proof. Write $A = L(G)$ with its canonical tracial state τ. By Remark 5.11, we can view $M = \Gamma(H, J, A, \tau)''$ where the symmetric A-bimodule (H, J) is given by (5.3). Denote by K the complexification of $K_\mathbb{R}$. Observe that $H \cong \ell^2(G) \otimes K$ with bimodule structure $a \cdot \xi \cdot b = \alpha(a)\xi b$, where $\alpha\colon L(G) \to L(G) \mathbin{\overline{\otimes}} B(K)$ is given by $\alpha(u_g) = u_g \otimes \pi(g)$ for all $g \in G$. Since $(\tau \otimes \mathrm{id})\alpha(a) = \tau(a)1$ for all $a \in L(G)$, it follows that $\Delta^\ell_H = \dim(K_\mathbb{R})\,1$.

The left and right actions of A on H are faithful. Since $H \otimes_A \overline{H}$ can be identified with the bimodule associated with the representation $\pi \otimes \overline{\pi}$, the center valued dimension of $H \otimes_A \overline{H}$ as a left A-module equals $\dim(K_\mathbb{R})^2\,1$. It follows from Lemma 5.39 below that H is completely nontrivial. Since $\dim K_\mathbb{R} > 1$, it follows from Theorem 5.21 that no direct summand of M has a Cartan and that $\mathcal{Z}(M) = \{a \in \mathcal{Z}(A) \mid a\xi = \xi a \text{ for all } \xi \in H\}$.

Let $a \in \mathcal{Z}(A)$ be such that $a\xi = \xi a$ for all $\xi \in H$. By writing $a = \sum_{g\in\Gamma} a_g u_g$ for the Fourier decomposition of a and using that a commutes with $\delta_e \otimes \eta$ for all $\eta \in K$, we get

$$\sum_{g\in G} a_g \delta_g \otimes \eta = \sum_{g\in G} a_g \delta_g \otimes \pi(g)\eta, \quad \eta \in K.$$

It follows that $\pi(g) = 1$ for any $g \in G$ that appears with nonzero coefficient in the Fourier decomposition of a. But since $a \in \mathcal{Z}(A)$, the only group elements that appear in the Fourier decomposition are the ones with finite conjugacy class. It follows that M is a factor whenever $\pi(g) \neq 1$ for every $g \in G \setminus \{e\}$ with a finite conjugacy class. Conversely, assume that $\pi(g) = 1$ for some $g \in G \setminus \{e\}$ with a finite conjugacy class $F = \{hgh^{-1} \mid h \in G\}$. Then the element $a = \sum_{g\in F} u_g$ is a nontrivial central element of M. We conclude that M is a factor if and only if $\pi(g) \neq 1$ for every $g \in G \setminus \{e\}$ that has a finite conjugacy class. □

Before proving Theorem 5.21, we will first introduce some general notions for bimodules and prove some technical lemmas.

5.4.1 Preliminaries on bimodules

Let (A, τ) be a tracial von Neumann algebra and denote by $\widehat{\mathcal{Z}(A)}$ the *extended positive part* of $\mathcal{Z}(A)$, i.e., when we identify $\mathcal{Z}(A) = L^\infty(X, \mu)$, then $\widehat{\mathcal{Z}(A)}$ consists of all measurable functions $f \colon X \to [0, +\infty]$ up to identification of functions that are equal almost everywhere.

Whenever (B, τ) and (A, τ) are tracial von Neumann algebras and H is a B-A-bimodule, we denote by $\Delta^\ell_H \in \widehat{\mathcal{Z}(B)}$ the unique element in the extended positive part of $\mathcal{Z}(B)$ characterized by

$$\tau(\Delta^\ell_H e) = \dim_{-A}(eH) \quad \text{for all projections } e \in \mathcal{Z}(B)\,. \tag{5.10}$$

Writing $H \cong p(\ell^2(I) \otimes L^2(A))$ with the bimodule action given by $b \cdot \xi \cdot a = \alpha(b)\xi a$ where $\alpha\colon B \to p(B(\ell^2(I)) \mathbin{\overline{\otimes}} A)p$ is a normal $*$-homomorphism, we get that $\tau(\Delta_H^\ell \,\cdot\,) = (\mathrm{Tr} \otimes \tau)\alpha(\,\cdot\,)$ and this also allows to construct Δ_H^ℓ.

By symmetry, we can also define $\Delta_H^r \in \widehat{\mathcal{Z}(A)}$ characterized by the formula $\tau(\Delta_H^r e) = \dim_{B-}(He)$ for every projection $e \in \mathcal{Z}(A)$.

Recall that a right Hilbert A-module K is finitely generated if and only if $K \cong p(\mathbb{C}^n \otimes L^2(A))$ for some $n \in \mathbb{N}$ and some projection $p \in M_n(\mathbb{C}) \otimes A$. Any finitely generated right A-module K admits a *Pimsner-Popa basis*, i.e., right bounded elements $\xi_1, \dots, \xi_n \in K$ such that

$$\xi = \sum_{i=1}^n \xi_i \, \langle \xi_i, \xi \rangle_A \tag{5.11}$$

for all right bounded elements $\xi \in K$. Indeed, identifying K with $p(\mathbb{C}^n \otimes L^2(A))$, this amounts to letting $\xi_i = p(e_i \otimes 1_A)$ where $\{e_i\}_{i=1}^n$ denotes the canonical basis of $\mathbb{C}^n$. We denote by $t_K \in K \otimes_A \overline{K}$ the associated vector given by

$$t_K := \sum_{i=1}^n \xi_i \otimes_A \overline{\xi_i} \; . \tag{5.12}$$

Lemma 5.25. *Let K be an A-bimodule that is finitely generated as a right A-module. Choose a Pimsner-Popa basis $\{\xi_i\}_{i=1}^n \subset K$ and let $t_K \in K \otimes_A \overline{K}$ be the associated vector defined in (5.12). Then t_K is an A-central vector and $\langle t_K, t_K \rangle_A = \Delta_K^\ell$. If Δ_K^ℓ is bounded, i.e., $\Delta_K^\ell \in \mathcal{Z}(B)$, then ξ_i is also left bounded for each i.*

Proof. Giving a Pimsner-Popa basis $\{\xi_k\}_{k=1}^n$ for the right Hilbert A-module K is the same as defining a right A-linear unitary operator $\theta\colon e(\mathbb{C}^n \otimes L^2(A)) \to K$ for some projection $e \in A^n := M_n(\mathbb{C}) \otimes A$, with $\xi_k = \theta(e(e_k \otimes 1))$. Define the faithful normal $*$-homomorphism $\alpha\colon A \to eA^n e$ such that $\theta(\alpha(a)\xi) = a\theta(\xi)$ for all $a \in A$ and $\xi \in e(\mathbb{C}^n \otimes L^2(A))$.

For $a \in A$, we now have that

$$a t_K = \sum_{i,j=1}^n \xi_j \cdot \alpha(a)_{ji} \otimes_A \overline{\xi_i} = \sum_{i,j=1}^n \xi_j \otimes_A \overline{\xi_i \cdot \alpha(a^*)_{ij}} = t_K a,$$

so that t_K is A-central. Moreover, for $x, y \in A$,

$$\begin{aligned}\langle \ell(t_K)^* \ell(t_K) x, y \rangle_2 &= \sum_{i,j=1}^{n} \langle \xi_i \otimes_A \overline{\xi_i} \cdot x, \xi_j \otimes_A \overline{\xi_j} \cdot y \rangle \\ &= \sum_{i,j} \langle \overline{\alpha(x^*)(e_i \otimes 1)}, \langle \xi_i, \xi_j \rangle_A \cdot \overline{\alpha(y^*)(e_j \otimes 1)} \rangle \\ &= \sum_{i} \langle \alpha(y^*)(e_i \otimes 1), \alpha(x^*)(e_i \otimes 1) \rangle \\ &= (\operatorname{Tr} \otimes \tau)(\alpha(xy^*)) = \langle \Delta_K^\ell x, y \rangle_2,\end{aligned}$$

which means that $\langle t_K, t_K \rangle_A = \Delta_K^\ell$.

Finally, assume that Δ_K^ℓ is bounded. Define the vector $\xi \in \overline{\mathbb{C}^n} \otimes K$ given by

$$\xi = \sum_{k=1}^{n} \overline{e_k} \otimes \xi_k \ .$$

Then, $b\xi = \xi\alpha(b)$ for all $b \in A$ and, in particular, $\xi \in (\overline{\mathbb{C}^n} \otimes K)e$.

Define the normal positive functional $\omega \colon A \to \mathbb{C} \colon \omega(a) = \langle a\xi, \xi \rangle$. Then ω is A-central, so we find $\Delta \in L^1(\mathcal{Z}(A))^+$ such that $\omega(a) = \tau(a\Delta)$ for all $a \in A$. But for all projections $q \in A$, we have

$$\tau(q\Delta) = \omega(q) = \langle q\xi, \xi \rangle = \langle \xi\alpha(q), \xi \rangle = (\operatorname{Tr} \otimes \tau)(\alpha(q)) = \dim_{-A}(qK) \ .$$

This means that $\Delta = \Delta_K^\ell$. Since Δ_K^ℓ is bounded, we get that

$$\sum_{i=1}^{n} \|a\xi_i\|^2 = \|a\xi\|^2 = \tau(a^* a \Delta_K^\ell) \leq \|\Delta_K^\ell\| \, \|a\|_2^2 \qquad \text{for all } \ a \in A,$$

which implies that the vectors $\xi_k \in H$ are left A-bounded. □

Recall that the A-dimension of a left or a right A-module depends on the choice of trace on A. Therefore, a better way to keep track of the dimension of an A-module in the case where A is not a factor, is to consider the *center valued dimension*, which is defined as follows. Let L be a right A-module. Then we may identify L with $p(\ell^2(I) \otimes L^2(A))$ for some projection $p \in B(\ell^2(I)) \mathbin{\overline{\otimes}} A$. The *center valued right A-dimension* of L is defined by

$$\operatorname{zdim}_{-A}(L) = (\operatorname{Tr} \otimes E_{\mathcal{Z}(A)})(p) \in \widehat{\mathcal{Z}(A)},$$

where $E_{\mathcal{Z}(A)}$ denotes the conditional expectation of A onto its center $\mathcal{Z}(A)$. Similarly, if L is a left A-module, we can define the *center valued left A-dimension* denoted by $\mathrm{zdim}_{A-}(L)$. We have that L is finitely generated as a right (resp. left) Hilbert A-module if and only if $\mathrm{zdim}_{-A}(L)$ (resp. $\mathrm{zdim}_{A-}(L)$) is bounded (see [AP16, Proposition 9.3.2]).

Note that the definition of the center valued dimension is independent of the trace τ on A since $E_{\mathcal{Z}(A)}$ is intrinsic. Moreover, the center valued dimension is a complete invariant for left/right A-modules: two left/right A-modules are isomorphic if and only if they have the same center valued left/right A-dimension (see [AP16, Section 9.3]).

We will also need the notion of essentially finite index for inclusions of tracial von Neumann algebras. Given a tracial von Neumann algebra (Q, τ) and a von Neumann subalgebra $P \subset Q$, the *Jones index* of the inclusion $P \subset Q$ is defined to be $[Q : P] := \dim_{-P} L^2(Q)$. Note that the value of $[Q : P]$ is not canonical since it depends on τ. Therefore, when we speak about a *finite index inclusion*, it will always be with respect to a certain trace. In [Va07, Appendix A], Vaes introduced a broader notion of a finite index inclusion, which is independent of the choice of trace.

Definition 5.26 ([Va07, Definition A.2])**.** A von Neumann subalgebra P of a tracial von Neumann algebra (Q, τ) is said to be of *essentially finite index* if there exist projections $q \in P' \cap Q$ arbitrarily close to 1 such that $Pq \subset qQq$ has finite Jones index.

Proposition 5.27 ([Va07, Proposition A.2])**.** *Let P be a von Neumann subalgebra of (Q, τ). The following are equivalent.*

(1) *P is of essentially finite index in Q.*

(2) *$qQq \prec_{qQq} Pq$ for every nonzero projection $q \in P' \cap Q$.*

(3) *There exists a projection $q \in P' \cap Q$ arbitrarily close to 1 and there exist finitely many elements $y_1, \ldots, y_n \in Qq$ such that*

$$xq = \sum_{i=1}^{n} y_i E_P(y_i^* x) \quad \text{for all } x \in Q.$$

5.4.2 Relative diffuse subalgebras

In this subsection, we will introduce the notion of a relative diffuse subalgebra and we will provide a structural characterization of being relative diffuse. Our goal is to prove Lemma 5.38, in which the setup is as follows: $B \subset (M, \tau)$ is a von Neumann subalgebra of a separable tracial von Neumann algebra M, $S \in M$ is a self-adjoint operator that commutes with B, and $D = \{S\}'' \vee B \subset M$. For simplicity, we will only consider this specific setup.

We start by considering the abelian case. Let $A \subset D$ be an inclusion of abelian von Neumann algebras and write $A = L^\infty(Y,\nu)$ and $D = L^\infty(X,\mu)$. Such an inclusion $\iota\colon L^\infty(Y,\nu) \to L^\infty(X,\mu)$ amounts to a surjective Borel map $\pi\colon (X,\mu) \to (Y,\nu)$ satisfying $\nu = \pi_*\mu$ and $\iota(f) = f \circ \pi$ for all $f \in L^\infty(Y,\nu)$ (see for instance [AP16, Theorem 3.3.4]).

A Borel map $\pi\colon (X,\mu) \to (Y,\nu)$ satisfying $\pi_*\mu = \nu$ is called a *factor map*. It is a classical result in measure theory that such a factor map $\pi\colon (X,\mu) \to (Y,\nu)$ gives rise to a disintegration of μ, as stated in the following theorem. For a proof, we refer to [Bo07, Theorem 10.4.14].

Theorem 5.28 (The Disintegration Theorem)**.** *Let (X,μ) and (Y,ν) be standard probability spaces, and let $\pi\colon X \to Y$ be a factor map. There exists a disintegration of μ with respect to π, i.e., a family of probability measures $\{\mu_y\}_{y\in Y}$ on X such that*

(1) μ_y is supported on the fiber $\pi^{-1}(\{y\})$ for all $y \in Y$;

(2) the map $y \mapsto \mu_y$ is a Borel map in the following sense: for each Borel set $B \subset X$, the map $y \mapsto \mu_y(B)$ is Borel;

(3) we have that $\mu = \int_Y \mu_y \, d\nu(y)$. More precisely, for any measurable function $f\colon X \to [0,\infty]$,

$$\int_X f(x)\, d\mu(x) = \int_Y \int_{\pi^{-1}(\{y\})} f(x)\, d\mu_y(x)\, d\nu(y).$$

Moreover, the disintegration of μ is unique in the following sense: if $\{\mu'_y\}_{y\in Y}$ is another disintegration of μ with respect to π, then $\mu_y = \mu'_y$ ν-almost everywhere.

Definition 5.29. Let (X,μ) and (Y,ν) be standard probability spaces and let $\pi\colon (X,\mu) \to (Y,\nu)$ be a factor map. Let $\{\mu_y\}_{y\in Y}$ be the associated disintegration of μ. We say that the factor map π is *diffuse* if μ_y is non-atomic for almost all $y \in Y$.

If $A = L^\infty(Y,\nu) \subset L^\infty(X,\mu) = D$ is an inclusion of abelian von Neumann algebras, then we say that D is *diffuse relative to* A (or that $A \subset D$ is relatively diffuse) if the associated factor map $\pi\colon (X,\mu) \to (Y,\nu)$ is diffuse.

Example 5.30. Let (Y,ν) be a standard probability space and let $(X,\mu) = ([0,1] \times Y, \lambda \times \nu)$, where λ denotes the Lebesgue measure on $[0,1]$. We then have a factor map $p_Y\colon [0,1]\times Y \to Y$ given by $p_Y(x,y) = y$. Note that this is exactly the factor map associated with the canonical inclusion $L^\infty(Y,\nu) \subset L^\infty[0,1] \mathbin{\overline{\otimes}} L^\infty(Y,\nu)$. The disintegration of μ associated with the factor map p_Y is trivial, in the sense that $\mu_y = \lambda \times \delta_y$ for all y. Therefore, p_Y is called the *trivial* diffuse factor map.

It is a folklore result that any diffuse factor map $\pi\colon (X,\mu) \to (Y,\nu)$ is trivial, in the sense that there is a measure space isomorphism $\theta\colon (X,\mu) \to ([0,1]\times Y, \lambda\times\nu)$ such that $p_Y \circ \theta = \pi$. As a corollary, we get that any relatively diffuse inclusion $A \subset D$ of abelian von Neumann algebras is isomorphic with the canonical inclusion $A \subset L^\infty[0,1] \mathbin{\overline{\otimes}} A$. Since I was unable to find a reference for these facts, I will here provide a full proof.

Recall that any non-atomic standard probability space (X,μ) is isomorphic with $([0,1],\lambda)$, where λ denotes the Lebesgue measure. So, if $\pi\colon (X,\mu) \to (Y,\nu)$ is a diffuse factor map, we get a family of measure space isomorphisms $\theta_y\colon (X,\mu_y) \to ([0,1],\lambda)$. In the following, we will show that these isomorphisms θ_y can be chosen in a uniform way so that $\theta\colon (X,\mu) \to ([0,1]\times Y, \lambda\times\nu)$ given by $\theta(x) = (\theta_{\pi(x)}(x), \pi(x))$ defines an isomorphism with the trivial factor map.

Lemma 5.31 ([Ke95, Theorem 17.41])**.** *Let μ be any non-atomic probability measure on $[0,1]$ and let λ denote the Lebesgue measure. Define*

$$g_\mu\colon [0,1] \to [0,1] : \quad g_\mu(x) = \mu([0,x]),$$

and let $N_\mu \subset [0,1]$ be the set of y for which $g_\mu^{-1}(\{y\})$ contains more than one point. Then N_μ is countable, in particular $\lambda(N_\mu) = 0$, and letting $W_\mu = g_\mu^{-1}(N_\mu)$, we have that

$$g_\mu\colon [0,1]\setminus W_\mu \to [0,1]\setminus N_\mu$$

is continuous, bijective and satisfies $(g_\mu)_*(\mu) = \lambda$.

Proof. Because μ has no atoms, we have that g_μ is continuous. Moreover, g_μ is increasing and satisfies $g_\mu(0) = 0$ and $g_\mu(1) = 1$. It follows that g_μ is surjective. So, for each $y \in [0,1]$, the preimage $g_\mu^{-1}(\{y\})$ is a closed interval, and all of these intervals are pairwise disjoint. There can be only countably many disjoint closed intervals with a positive length inside $[0,1]$. So, it follows that N_μ is countable.

Since $g_\mu\colon [0,1] \to [0,1]$ is surjective, also the restriction $g_\mu\colon [0,1]\setminus W_\mu \to [0,1]\setminus N_\mu$ is surjective. To show that it is injective, assume that $x_1 < x_2$ and $g_\mu(x_1) = g_\mu(x_2)$. Then $\mu([x_1,x_2]) = 0$ and $g_\mu(x) = g_\mu(x_1)$ for all $x \in [x_1,x_2]$. It follows that $g_\mu(x_1) \in N_\mu$ and hence $x_1 \in W_\mu$.

Finally, when $y \in [0,1] \setminus N_\mu$, write $y = g_\mu(x)$ for some $x \in [0,1] \setminus W_\mu$. Note that $g_\mu^{-1}([0,y]) = [0,x]$. Thus,

$$((g_\mu)_*\mu)([0,y]) = \mu([0,x]) = g_\mu(x) = y = \lambda([0,y]).$$

So, $(g_\mu)_*\mu = \lambda$. □

Proposition 5.32. *Let (X,μ) and (Y,ν) be standard probability spaces and let $\pi\colon X \to Y$ be a diffuse factor map. Then π is isomorphic with the trivial diffuse factor map $p_Y\colon ([0,1] \times Y, \lambda \times \nu) \to (Y,\nu)$.*

Proof. Define $Z \subset X \times Y$ by $Z = \{(x,\pi(x)) \mid x \in X\}$. Then Z is a Borel set, being the graph of the Borel map π, and it is isomorphic with X. Let $\{\mu_y\}_{y\in Y}$ be the disintegration of μ with respect to π and define the probability measure γ on $X \times Y$ by

$$\gamma(A) = \int_Y \mu_y(A_y)\,\mathrm{d}\nu(y), \qquad A \subset X \times Y \text{ Borel},$$

where A_y denotes the fiber $A_y = \{x \in X \mid (x,y) \in A\}$. Note that $\gamma(Z) = 1$.

We may assume that $X = [0,1]$. By Lemma 5.31, we have continuous bijections $g_{\mu_y}\colon X \setminus W_{\mu_y} \to X \setminus N_{\mu_y}$ such that $(g_{\mu_y})_*\mu_y = \lambda$. Moreover, g_{μ_y} is given by $g_{\mu_y}(x) = \mu_y([0,x])$. Note that $(x,y) \mapsto \mu_y([0,x])$ is a Borel map by [Ke95, Theorem 17.25].

Define the Borel map $\theta\colon X \times Y \to X \times Y$ given by $\theta(x,y) = (\mu_y([0,x]),y)$. With the notation N_{μ_y} from Lemma 5.31, put

$$N = \{(a,y) \in X \times Y \mid a \in N_{\mu_y}\} = \{(a,y) \in X \times Y \mid \lambda(g_{\mu_y}^{-1}(\{a\})) > 0\}.$$

We claim that N is a Borel set. Indeed, define the set

$$V = \{(y,a,x) \in Y \times X \times X \mid g_{\mu_y}(x) = a\}.$$

Note that V is Borel, being the graph of the Borel map $(x,y) \mapsto g_{\mu_y}(x)$. By [Ke95, Theorem 17.25], the map $(y,a) \mapsto \lambda(V_{(y,a)})$ is Borel. It follows that

$$N = \{(a,y) \in X \times Y \mid \lambda(V_{(y,a)}) > 0\}$$

is indeed a Borel set.

Put $W = \theta^{-1}(N)$. Then $W = \{(x,y) \in X \times Y \mid x \in W_{\mu_y}\}$, so that

$$\gamma(W) = \int_Y \mu_y(W_{\mu_y})\,\mathrm{d}\nu(y) = 0.$$

Now, $\theta\colon (X \times Y) \setminus W \to (X \times Y) \setminus N$ is a bijection because this holds fiber wise. Moreover, $\theta_*\gamma = \lambda \times \nu$. So, under the isomorphism $X \cong Z \subset X \times Y$, we have that $\theta\colon X \to [0,1] \times Y$ is a measure space isomorphism satisfying $\pi = p_Y \circ \theta$. □

Proposition 5.33. *Let $A \subset D$ be an inclusion of separable abelian von Neumann algebras. The following are equivalent.*

(i) D is diffuse relative to A.

(ii) There is an isomorphism $\Phi\colon D \to L^\infty[0,1]\,\overline{\otimes}\,A$ satisfying $\Phi(a) = 1 \otimes a$ for all $a \in A$.

(iii) There exists a unitary $u \in D$ such that $E_A(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$ and such that $\{u\}'' \vee A = D$.

(iv) $D \not\prec_D A$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows directly from Proposition 5.32.

(ii) $\Rightarrow$ (iii): Assume that $D \cong L^\infty[0,1]\,\overline{\otimes}\,A$. Pick a Haar unitary $u_0 \in L^\infty[0,1]$, i.e., $\tau(u_0^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. Letting $u = u_0 \otimes 1_A \in \mathcal{U}(D)$, we get the desired unitary.

(iii) $\Rightarrow$ (ii): Let $u \in \mathcal{U}(D)$ be a unitary as in the statement of (iii). Note that u generates a diffuse von Neumann subalgebra of D that is in tensor product position with A. So, $D = \{u\}'' \vee A \cong L^\infty[0,1]\,\overline{\otimes}\,A$ and this isomorphism is the identity on A.

(ii) $\Rightarrow$ (iv): This is a consequence of Popa's intertwining-by-bimodules theorem, Theorem 2.12.

(iv) $\Rightarrow$ (i): Write $A = L^\infty(Y, \nu)$ and $D = L^\infty(X, \mu)$, and let $\mu = \int_Y \mu_y \,\mathrm{d}\nu(y)$ be the disintegration of μ associated with the inclusion $A \subset D$. If D is not diffuse relative to A, then we find a set $Y_0 \subset Y$ of positive measure such that μ_y has an atom for all $y \in Y_0$. Assume first that there is a Borel map $f\colon Y_0 \to X$ such that $f(y)$ is an atom of μ_y for all $y \in Y_0$. Since f is injective, the image $X_0 := \{f(y) \mid y \in Y_0\}$ is a Borel set, by [Ke95, Corollary 15.2]. Then $p := \mathbb{1}_{X_0} \in L^\infty(X, \mu)$ is a nonzero projection such that

$$pL^\infty(X, \mu) = L^\infty(X_0, \mu) \cong L^\infty(Y_0, \nu),$$

via the isomorphism $\pi^*\colon L^\infty(Y_0, \nu) \to L^\infty(X_0, \mu)$ induced by π. In particular, $D \prec_D A$.

It remains to show that the Borel map f from above does indeed exist. By [Ke95, Theorem 17.25], we have that the map $x \mapsto \mu_{\pi(x)}(\{x\})$ is a Borel map from X to $\mathbb{R}$. In particular, the set

$$X_0 := \{x \in X \mid \mu_{\pi(x)}(\{x\}) > 0\}$$

is Borel. Now, the map $\pi\colon X_0 \to Y_0$ is a surjective countable-to-one Borel map and hence it has a Borel cross-section $f\colon Y_0 \to X_0$ by [Bo07, Theorem 6.9.6]. This gives us the desired map f. □

We now turn to the non-commutative case, in the setting of Lemma 5.38. Let M be a separable tracial von Neumann algebra, let $B \subset M$ be a von Neumann subalgebra and let $S \in M$ be a self-adjoint operator commuting with B. Put $D = B \vee \{S\}''$.

We first show that the subalgebras B and $\mathcal{Z}(B) \vee \{S\}''$ of D form a so-called commuting square, which is a notion due to Popa [Po83]. Two von Neumann subalgebras Q_1, Q_2 of a tracial von Neumann algebra (M, τ) are said to form a *commuting square* if $E_{Q_1} \circ E_{Q_2} = E_{Q_2} \circ E_{Q_1}$, where E_{Q_i} denotes the unique τ-preserving conditional expectation of M onto Q_i. In that case, we have that $E_{Q_1} \circ E_{Q_2} = E_{Q_1 \cap Q_2}$.

Lemma 5.34. *The von Neumann subalgebras B and $\mathcal{Z}(B) \vee \{S\}''$ form a commuting square inside D, with $B \cap (\mathcal{Z}(B) \vee \{S\}'') = \mathcal{Z}(B)$.*

Proof. For any $a \in \mathcal{Z}(B) \vee \{S\}''$, we have that $b \mapsto \tau(ba)$ is a trace on B. Since $E_{\mathcal{Z}(B)}$ is preserved by any trace on B, it follows that $\tau(ba) = \tau(E_{\mathcal{Z}(B)}(b)a)$ for all $b \in B$ and all $a \in \mathcal{Z}(B) \vee \{S\}''$. This means that $E_{\mathcal{Z}(B)\vee\{S\}''}(b) = E_{\mathcal{Z}(B)}(b)$ for all $b \in B$. Thus, $E_{\mathcal{Z}(B)\vee\{S\}''} \circ E_B = E_{\mathcal{Z}(B)}$.

Letting e_B, $e_{\mathcal{Z}(B)}$ and $e_{\mathcal{Z}(B)\vee\{S\}''}$ denote the associated Jones projections in $B(L^2(M))$, we now have that $e_{\mathcal{Z}(B)\vee\{S\}''}\, e_B = e_{\mathcal{Z}(B)}$. Taking the adjoint, also $e_B e_{\mathcal{Z}(B)\vee\{S\}''} = e_{\mathcal{Z}(B)}$. Restricting to M yields $E_B \circ E_{\mathcal{Z}(B)\vee\{S\}''} = E_{\mathcal{Z}(B)}$. □

Definition 5.35. Let $B \subset D$ be as above. We say that D is diffuse relative to B if the abelian inclusion $\mathcal{Z}(B) \subset \mathcal{Z}(B) \vee \{S\}''$ is relatively diffuse.

Because of the commuting square property, we get the following characterizations of relative diffuseness, motivating Definition 5.35 above.

Proposition 5.36. *Let $B \subset D$ be as above. The following are equivalent.*

(i) D is diffuse relative to B.

(ii) $D \not\prec_D B$.

(iii) The D-bimodule $L^2(D) \otimes_B L^2(D)$ contains no nonzero D-central vectors.

(iv) For any B-D-bimodule K, the D-bimodule $L^2(D) \otimes_B K$ contains no nonzero D-central vectors.

Proof. Note that $D \not\prec_D B$ exactly means that ${}_D L^2(D)_B$ is left weakly mixing. By Proposition 2.26, we get that (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

(i) $\Rightarrow$ (ii): Assume that D is diffuse relative to B. By Proposition 5.33, this means that $\{S\}'' \vee \mathcal{Z}(B) \cong L^\infty[0,1] \mathbin{\overline{\otimes}} \mathcal{Z}(B)$ via an isomorphism that is the identity on $\mathcal{Z}(B)$. Let $u_n \in L^\infty[0,1]$ be a sequence of unitaries tending to 0 weakly. Then $w_n := u_n \otimes 1_B \in D$ is a sequence of unitaries such that $\|E_{\mathcal{Z}(B)}(xw_ny)\|_2 \to 0$ for all $x, y \in \{S\}'' \vee \mathcal{Z}(B)$. Since B and $\{S\}'' \vee \mathcal{Z}(B)$ form a commuting square, it follows that $\|E_B(xw_ny)\|_2 \to 0$ for all $x, y \in D$. Indeed, it is enough to show this for $x, y \in \{S\}''$. In that case, we have that $xw_ny \in \mathcal{Z}(B) \vee \{S\}''$ and hence

$$\|E_B(xw_ny)\|_2 = \|(E_B \circ E_{\mathcal{Z}(B)\vee\{S\}''})(xw_ny)\|_2 = \|E_{\mathcal{Z}(B)}(xw_ny)\|_2 \to 0.$$

By Popa's intertwining-by-bimodules theorem, Theorem 2.12, this means that $D \not\prec_D B$.

(ii) $\Rightarrow$ (i): Assume that D is not diffuse relative to B, i.e., $D_0 := \mathcal{Z}(B) \vee \{S\}''$ is not diffuse relative to $\mathcal{Z}(B)$. By Proposition 5.33, this means that $D_0 \prec_{D_0} \mathcal{Z}(B)$. In fact, we proved the following stronger property in the proof of Proposition 5.33: there exist nonzero projections $z \in \mathcal{Z}(B)$ and $p \in D_0 z$ such that the linear map $\alpha\colon D_0 p \to \mathcal{Z}(B)z$ given by $\alpha(x) = E_{\mathcal{Z}(B)z}(x)E_{\mathcal{Z}(B)z}(p)^{-1}$ for $x \in D_0 p$ is a $*$-isomorphism. Since D_0 and B form a commuting square inside $D = D_0 \vee B$, with $B \cap D_0 = \mathcal{Z}(B)$, it follows that the linear map $\widetilde{\alpha}\colon Dp \to Bz$ given by $\widetilde{\alpha}(x) = E_{Bz}(x)E_{Bz}(p)^{-1}$ is a $*$-isomorphism extending α. Also note that $\widetilde{\alpha}(xb) = \widetilde{\alpha}(x)b$ for $x \in D$ and $b \in B$. In particular, the D-B-bimodule $L^2(D)p$ has right B-dimension equal to $\tau(z)$, and thus $D \prec_D B$. □

We finish this subsection with two lemmas that will be needed in the proof of Theorem 5.21.

Lemma 5.37. *Let $\Phi\colon D \to B$ be a conditional expectation of the form $\Phi(x) = E_B(xa)$ for some positive element $a \in \mathcal{Z}(B) \vee \{S\}''$. If D is diffuse relative to B, then there exists a unitary $u \in \mathcal{U}(\mathcal{Z}(B) \vee \{S\}'')$ such that $\Phi(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$.*

Proof. Assume first that $B = L^\infty(Y,\nu)$ and $D = L^\infty(X,\mu)$ are abelian. Let $\{\mu_y\}_{y\in Y}$ be the disintegration of μ associated with the factor map $\pi\colon X \to Y$ coming from the inclusion $B \subset D$. Note that Φ is given by

$$\Phi(d)(y) = \int_X d(x)a(x)\,\mathrm{d}\mu_y(x), \quad d \in D.$$

Since Φ is a conditional expectation, we have that $\Phi(1) = 1$ and hence $\int_X a(x)\,\mathrm{d}\mu_y(x) = 1$ for ν-almost all $y \in Y$. It follows that $\pi\colon (X, a\mu) \to (Y,\nu)$

is a factor map and that $y \mapsto a\mu_y$ is the associated disintegration of $a\mu$. By Proposition 5.32, we get a measure space isomorphism $\theta\colon (X, a\mu) \to ([0,1] \times Y, \lambda \times \nu)$ such that $p_Y \circ \theta = \pi$. This induces a $*$-isomorphism $\widetilde{\theta}\colon D \to L^\infty[0,1]\overline{\otimes}B$ satisfying $\widetilde{\theta}(\Phi(f)) = E_B(\widetilde{\theta}(f))$ for $f \in D$, where $E_B\colon L^\infty[0,1]\overline{\otimes}B \to B$ denotes the usual trace-preserving conditional expectation. Let $w \in L^\infty[0,1]$ be a Haar unitary and put $u = \widetilde{\theta}^{-1}(w \otimes 1_B) \in D$. Then,

$$\widetilde{\theta} \circ \Phi(u^k) = E_B(w^k \otimes 1_B) = \tau(w^k)1_B = 0, \qquad k \in \mathbb{Z} \setminus \{0\}.$$

This finishes the proof in the abelian case.

If B is not abelian, notice that since B and $D_0 := \mathcal{Z}(B) \vee \{S\}''$ form a commuting square inside D, we have that $\Phi(x) = E_B(xa) = E_{\mathcal{Z}(B)}(xa)$ for all $x \in D_0$. By applying the first part of the proof to the abelian inclusion $\mathcal{Z}(B) \subset D_0$, we find a unitary $u \in D_0$ such that $\Phi(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. □

Lemma 5.38. *Let (A, τ) be a separable tracial von Neumann algebra and (H, J) a symmetric A-bimodule. Write $M = \Gamma(H, J, A, \tau)''$. Let $p \in A$ be a projection and $B \subset pAp$ a von Neumann subalgebra such that $B' \cap pAp = \mathcal{Z}(B)$. Let $K \subset pH$ be a B-A-subbimodule that is finitely generated as a right Hilbert A-module. Assume that Δ_K^ℓ is bounded and satisfies $\Delta_K^\ell \geq p$, as B-A-bimodule.*

Let $(\xi_k)_{k=1}^n$ be a Pimsner-Popa basis for K as a right A-module. Then the vectors ξ_k are also left A-bounded and using the notation of (5.2), we define $S \in pMp$ given by

$$S := \sum_{k=1}^{n} W(\xi_k, J(\xi_k)) \ . \tag{5.13}$$

Then $S \in B' \cap pMp$, S is self-adjoint and the von Neumann algebra $D := \{S\}'' \vee B$ is diffuse relative to B.

Proof. Giving a Pimsner-Popa basis $(\xi_k)_{k=1}^n$ for the right Hilbert A-module K is the same as defining a right A-linear unitary operator $\theta\colon e(\mathbb{C}^n \otimes L^2(A)) \to K$ for some projection $e \in A^n := M_n(\mathbb{C}) \otimes A$, with $\xi_k = \theta(e(e_k \otimes 1))$. Define the faithful normal $*$-homomorphism $\alpha\colon B \to eA^n e$ such that $\theta(\alpha(b)\xi) = b\theta(\xi)$ for all $b \in B$ and $\xi \in e(\mathbb{C}^n \otimes L^2(A))$. View $\overline{\mathbb{C}^n} \otimes K$ as a B-A^n-subbimodule of $\overline{\mathbb{C}^n} \otimes pH$. Define the vector $\xi \in \overline{\mathbb{C}^n} \otimes K$ given by

$$\xi = \sum_{k=1}^{n} \overline{e_k} \otimes \xi_k \ .$$

Then, $b\xi = \xi\alpha(b)$ for all $b \in B$ and, in particular, $\xi \in (\overline{\mathbb{C}^n} \otimes K)e$.

Define the normal positive functional $\omega\colon pAp \to \mathbb{C}\colon \omega(a) = \langle a\xi, \xi\rangle$. Since ω is B-central and $B' \cap pAp = \mathcal{Z}(B)$, we find $\Delta \in L^1(\mathcal{Z}(B))^+$ such that $\omega(a) = \tau(a\Delta)$

for all $a \in pAp$. As in the proof of Lemma 5.25, we have that $\Delta = \Delta_K^\ell$ and that the vectors $\xi_k \in H$ are left A-bounded.

So, the vectors ξ_k are both left and right A-bounded and thus the operator S given by (5.13) is a well-defined element of pMp. Since

$$S = \sum_{k=1}^{n} \bigl(\ell(\xi_k)\ell(J(\xi_k)) + \ell(\xi_k)\ell(\xi_k)^* + \ell(J(\xi_k))^*\ell(\xi_k)^*\bigr) \; ,$$

we get that $S = S^*$. From this formula, we also get that S commutes with B. Put $S_1 := \Delta + S$. Since $\Delta \in \mathcal{Z}(B)$, it suffices to prove that $\{S_1\}'' \vee B$ is diffuse relative to B.

Write $A_1 = pAp$ and $A_2 = eA^n e$. Equip A_1 and A_2 with the non-normalized traces given by restricting τ to A_1 and $\mathrm{Tr} \otimes \tau$ to A_2. View ξ as a vector in the A_1-A_2-bimodule $L := (\overline{\mathbb{C}^n} \otimes pH)e$ and note that

$$\langle \xi, \xi \rangle_{A_2} = e \;\; , \;\; {}_{A_1}\langle \xi, \xi \rangle = \Delta \; .$$

Write $L' := e(\mathbb{C}^n \otimes Hp)$, view L' as an A_2-A_1-bimodule and note that the anti-unitary operator

$$J_1 \colon L \to L' \colon \quad J_1\bigl(\sum_{k=1}^{n} \overline{e_k} \otimes \mu_k\bigr) = \sum_{k=1}^{n} e_k \otimes J(\mu_k)$$

satisfies $J_1(a\mu b) = b^* J_1(\mu) a^*$ for all $\mu \in L$, $a \in A_1$ and $b \in A_2$. Define $\xi' \in L'$ given by $\xi' = J_1(\xi)\Delta^{-1/2}$. Then ξ' satisfies the following properties.

$$\langle \xi', \xi' \rangle_{A_1} = p \;\; , \;\; {}_{A_2}\langle \xi', \xi' \rangle = \alpha(\Delta^{-1}) \;\text{ and }\; \alpha(b)\xi' = \xi' b \;\; \forall b \in B \; .$$

Define the Hilbert spaces

$$L_{\mathrm{even}} = L^2(A_1) \oplus \bigoplus_{m=1}^{\infty} \bigl(L \otimes_{A_2} L'\bigr)^{\otimes_{A_1}^m} \; ,$$

$$L_{\mathrm{odd}} = L' \otimes_{A_1} L_{\mathrm{even}} = \bigoplus_{m=0}^{\infty} \Bigl(L' \otimes_{A_1} \bigl(L \otimes_{A_2} L'\bigr)^{\otimes_{A_1}^m}\Bigr) \; .$$

Note that L_{even} is an A_1-bimodule, while L_{odd} is an A_2-A_1-bimodule. Then,

$$W := \ell(\xi')\Delta^{1/2} + \ell(\xi)^* \tag{5.14}$$

is a well-defined bounded operator from L_{even} to L_{odd} and $W^*W \in B(L_{\mathrm{even}})$.

Using the natural isometry $L \otimes_{A_2} L' \hookrightarrow p(H \otimes_A H)p$, we define the isometry $V\colon L_{\text{even}} \to pL^2(M)p$ given as the direct sum of the compositions of

$$\big(L \otimes_{A_2} L'\big)^{\otimes^m_{A_1}} \hookrightarrow \big(p(H \otimes_A H)p\big)^{\otimes^m_{A_1}} \hookrightarrow p\big(H^{\otimes^{2m}_A}\big)p \; .$$

Then V is A_1-bimodular and

$$V\,W^*W = S_1\,V \; . \tag{5.15}$$

To compute the $*$-distribution of $B \cup \{S_1\}$ with respect to the trace τ, it is thus sufficient to compute the $*$-distribution of $B \cup \{W^*W\}$ acting on L_{even} and with respect to the vector functional implemented by $p \in L^2(A_1) \subset L_{\text{even}}$.

Define the closed subspaces $L^0_{\text{even}} \subset L_{\text{even}}$ and $L^0_{\text{odd}} \subset L_{\text{odd}}$ given as the closed linear span

$$L^0_{\text{even}} = \overline{\text{span}}\{L^2(B), (\xi \otimes_{A_2} \xi')^{\otimes^m_{A_1}} B \mid m \geq 1\} \; ,$$

$$L^0_{\text{odd}} = \overline{\text{span}}\{(\xi' \otimes_{A_1} (\xi \otimes_{A_2} \xi')^{\otimes^m_{A_1}})B \mid m \geq 0\} \; .$$

Since $\xi \otimes_{A_2} \xi'$ is a B-central vector and since $\langle \xi, \xi \rangle_{A_2} = e$ and $\langle \xi', \xi' \rangle_{A_1} = p$, we find that $W(L^0_{\text{even}}) \subset L^0_{\text{odd}}$ and $W^*(L^0_{\text{odd}}) \subset L^0_{\text{even}}$. So to compute the $*$-distribution of $B \cup \{W^*W\}$, we may restrict B and W^*W to L^0_{even}.

Consider the full Fock space $\mathcal{F}(\mathbb{C}^2)$ of the 2-dimensional Hilbert space $\mathbb{C}^2$, with creation operators $\ell_1 = \ell(e_1)$ and $\ell_2 = \ell(e_2)$ given by the standard basis vectors $e_1, e_2 \in \mathbb{C}^2$. Denote by η the vector state on $B(\mathcal{F}(\mathbb{C}^2))$ implemented by the vacuum vector $\Omega \in \mathcal{F}(\mathbb{C}^2)$. For every $\lambda \geq 1$, consider the operator $X(\lambda) \in B(\mathcal{F}(\mathbb{C}^2))$ given by $X(\lambda) = \sqrt{\lambda}\ell_2 + \ell_1^*$. We find that $X(\lambda)^* X(\lambda) = \lambda y^* y$ with $y = \ell_2 + \lambda^{-1/2}\ell_1^*$. It then follows from [Sh96, Lemma 4.3 and discussion after Definition 4.1] that the spectral measure of $X(\lambda)^* X(\lambda)$ has no atoms. Also for every $\lambda \geq 1$, η is a faithful state on $\{X(\lambda)^* X(\lambda)\}''$.

Identify $\mathcal{Z}(B) = L^\infty(Z, \mu)$ for some standard probability space (Z, μ). View Δ as a bounded function from Z to $[1, +\infty)$ and define $Y \in B(\mathcal{F}(\mathbb{C}^2)) \mathbin{\overline{\otimes}} L^\infty(Z, \mu)$ given by $Y(z) = X(\Delta(z))$. We can view Y as an element of $B(\mathcal{F}(\mathbb{C}^2)) \mathbin{\overline{\otimes}} B$ acting on the Hilbert space $\mathcal{F}(\mathbb{C}^2) \otimes L^2(B)$. Also, $\eta \otimes \tau$ is faithful on $(1 \otimes B \cup \{Y^*Y\})''$. Define the isometry

$$U\colon L^0_{\text{even}} \to \mathcal{F}(\mathbb{C}^2) \otimes L^2(B)\colon \quad U\big((\xi \otimes_{A_2} \xi')^{\otimes^m_{A_1}} b\big) = (e_1 \otimes e_2)^{\otimes m} \otimes b \; .$$

By construction, $UW^*W = Y^*YU$ and U is B-bimodular. It follows that the $*$-distribution of $B \cup \{S_1\}$ with respect to τ equals the $*$-distribution of $1 \otimes B \cup \{Y^*Y\}$ with respect to $\eta \otimes \tau$. So there is a unique normal $*$-isomorphism

$$\Psi\colon (1 \otimes B \cup \{Y^*Y\})'' \to (B \cup \{S_1\})''$$

satisfying $\Psi(1 \otimes b) = b$ for all $b \in B$ and $\Psi(Y^*Y) = S_1$. Also, $\tau \circ \Psi = \eta \otimes \tau$. Since for all $z \in Z$, the spectral measure of $Y(z)^*Y(z)$ has no atoms, we have that $1 \otimes B \vee \{Y^*Y\}''$ is diffuse relative to $1 \otimes B$. Hence, $B \vee \{S_1\}''$ is diffuse relative to B. □

5.4.3 Technical lemmas

Recall from the beginning of this section the notion of an A-bimodule given by a partial automorphism of A.

Lemma 5.39. *Let (A, τ) be a separable tracial von Neumann algebra and T an A-bimodule with left support e. Denote $\Sigma := \operatorname{zdim}_{A-}(T \otimes_A \overline{T})$. Then, the support of Σ equals e and $\Sigma \geq e$. Defining $e_1 = \mathbb{1}_{\{1\}}(\Sigma)$, the following holds.*

(1) *Denoting by $f_1 \in \mathcal{Z}(A)$ the right support of $e_1 T$, we have that $e_1 T = T f_1$ and that the A-bimodule $e_1 T$ is given by a partial automorphism of A.*

(2) *When $e_2 \in \mathcal{Z}(A)e$ and $f_2 \in \mathcal{Z}(A)$ are projections such that $e_2 T = T f_2$ and such that the A-bimodule $e_2 T$ is given by a partial automorphism of A, then $e_2 \leq e_1$.*

(3) *If $e_3 \in \mathcal{Z}(A)e$ is a projection such that $e_3 T$ is finitely generated as a right Hilbert A-module, then the left support of $e_3 T \otimes_A \overline{T} \cap (t_{e_3 T} A)^\perp$ equals $e_3(1 - e_1)$.*

Proof. Choose a projection $p \in B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} A$ and a normal unital $*$-homomorphism $\alpha \colon A \to p(B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} A)p$ such that $T \cong p(\ell^2(\mathbb{N}) \otimes L^2(A))$ with the A-bimodule structure given by $a \cdot \xi \cdot b = \alpha(a)\xi b$. Note that e equals the support of α. Also note that $T \otimes_A \overline{T} \cong L^2(p(B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} A)p)$ with the A-bimodule structure given by $a \cdot \xi \cdot b = \alpha(a)\xi\alpha(b)$.

Define $e_0 = \mathbb{1}_{(0,1]}(\Sigma)$ and denote by $f_0 \in \mathcal{Z}(A)$ the right support of $e_0 T$. Note that $(1 \otimes f_0)p$ is the central support of $\alpha(e_0)$ inside $p(B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} A)p$. By construction, $\operatorname{zdim}_{A-}(e_0 T \otimes_A \overline{T}) = e_0 \Sigma \leq e_0$. It follows that $e_0 T \otimes_A \overline{T}$ is finitely generated as a left A-module and hence the commutant of the left A-action on $e_0 T \otimes_A \overline{T}$ is a finite von Neumann algebra. A fortiori, $p(B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} A)p(1 \otimes f_0)$ is a finite von Neumann algebra. We can thus choose a sequence of projections $q_n \in \mathcal{Z}(A)f_0$ such that $q_n \to f_0$ and $p(1 \otimes q_n)$ has finite trace for all n. Denote by $p_n \in \mathcal{Z}(A)e_0$ the support of the homomorphism that maps $a \in Ae_0$ to $\alpha(a)(1 \otimes q_n)$. It follows that $p_n \to e_0$.

Since the closure of $\alpha(Ae_0)(1 \otimes q_n)$ inside $L^2(p(B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} A)p)$ has zdim_{A-} equal to p_n, we conclude that $\Sigma p_n \geq p_n$ for all n and thus $\Sigma e_0 \geq e_0$. From the definition of e_0, it then follows that $\Sigma e_0 = e_0$ and $e_0 = e_1$ (as defined in the formulation of the lemma), as well as $\Sigma \geq e$ and $f_0 = f_1$. Since $p_n\Sigma = p_n$ for all n, it also follows that $\alpha(Ap_n)(1 \otimes q_n)$ is dense in $\alpha(p_n)L^2(B(\ell^2(\mathbb{N}))\overline{\otimes} A)p$ for all n, because the orthogonal complement has dimension zero. This means that $\alpha(e_1) = (1 \otimes f_1)p$ and that $\alpha\colon Ae_1 \to p(B(\ell^2(\mathbb{N}))\overline{\otimes} A)p(1 \otimes f_1)$ is a surjective $*$-isomorphism. So, $e_1T = Tf_1$ and this A-bimodule is given by a partial automorphism of A.

The first statement of the lemma is now proved. Take $e_2 \in \mathcal{Z}(A)e$ and $f_2 \in \mathcal{Z}(A)$ as in the second statement of the lemma. It follows that $e_2T \otimes_A \overline{T} = e_2T \otimes_A \overline{e_2T} \cong e_2L^2(A)$ so that $\mathrm{zdim}_{A-}(e_2T \otimes_A \overline{T}) = e_2$. Hence, $e_2\Sigma = e_2$, meaning that $e_2 \leq e_1$.

Finally take $e_3 \in \mathcal{Z}(A)$ as in the last statement of the lemma. We have $(\mathrm{Tr}\otimes\tau)\alpha(e_3) = \dim_{-A}(e_3T) < \infty$. Under the above isomorphism between $T \otimes_A \overline{T}$ and $L^2(p(B(\ell^2(\mathbb{N}))\overline{\otimes} A)p)$, the vector t_{e_3T} corresponds to $\alpha(e_3)$. So we have to determine the left support z of $\alpha(e_3)pL^2(B(\ell^2(\mathbb{N}))\overline{\otimes} A)p \cap \alpha(Ae_3)^\perp$. A projection $e_4 \in \mathcal{Z}(A)e_3$ is orthogonal to z if and only if $\alpha(Ae_4)$ is dense in $\alpha(e_4)pL^2(B(\ell^2(\mathbb{N}))\overline{\otimes} A)p$. This holds if and only if there exists a projection $f_4 \in \mathcal{Z}(A)$ such that $\alpha(e_4) = (1 \otimes f_4)p$ and $\alpha(Ae_4) = p(B(\ell^2(\mathbb{N}))\overline{\otimes} A)p(1 \otimes f_4)$. Since this is equivalent with $e_4 \leq e_1$, we have proved that $z = e_3(1 - e_1)$. □

Lemma 5.40. *Let (A, τ) be a separable tracial von Neumann algebra and (H, J) a symmetric A-bimodule with left (and thus also, right) support $e \in \mathcal{Z}(A)$. There is a unique projection $e_1 \in \mathcal{Z}(A)$ such that $e_1H = He_1$, the A-bimodule e_1H is given by a partial automorphism of A and the $A(e - e_1)$-bimodule $(1 - e_1)H$ is completely nontrivial.*

Proof. By Lemma 5.39, we find projections $e_1, f_1 \in \mathcal{Z}(A)e$ such that $e_1H = Hf_1$, the A-bimodule e_1H is given by a partial automorphism of A and writing $e_2 := e - e_1$, $f_2 = e - f_1$, the Ae_2-Af_2-bimodule $e_2H = Hf_2$ is completely nontrivial. Since $H \cong \overline{H}$, we must have $e_1 = f_1$ and $e_2 = f_2$. The uniqueness of e_1 also follows from Lemma 5.39. □

Lemma 5.41. *Let (A, τ) be a separable tracial von Neumann algebra and T an A-bimodule with left support $e \in \mathcal{Z}(A)$ and right support $f \in \mathcal{Z}(A)$. If $\Delta^\ell_T \leq e$ and $\Delta^r_T \leq f$, then $\Delta^\ell_T = e$, $\Delta^r_T = f$ and T is given by a partial automorphism of A.*

Proof. Let $e_0 \in \mathcal{Z}(A)e$ be the maximal projection with the following properties: the right support $f_0 \in \mathcal{Z}(A)f$ of e_0T satisfies $e_0T = Tf_0$, the A-bimodule e_0T is given by a partial automorphism of A and $\Delta^\ell_T = e_0$, $\Delta^r_T = f_0$. We have to prove that $e_0 = e$.

Assume that e_0 is strictly smaller than e. Since $e_0 T = T f_0$, also f_0 is strictly smaller than f. Denote $e_1 = e - e_0$ and $f_1 = f - f_0$. Note that $e_1 T = T f_1$. Since $\dim_{-A}(T) = \tau(\Delta^\ell_T) \leq \tau(e) \leq 1$ and similarly $\dim_{A-}(T) \leq 1$, it follows from [PSV15, Proposition 2.3] that there exists a nonzero A-subbimodule $K \subset e_1 T$ with the following properties: K is finitely generated, both as a left Hilbert A-module and as a right Hilbert A-module, and denoting by $e_2 \in \mathcal{Z}(A)e_1$ and $f_2 \in \mathcal{Z}(A)f_1$ the left, resp. right, support of K, there is a surjective $*$-isomorphism $\alpha\colon \mathcal{Z}(A)f_2 \to \mathcal{Z}(A)e_2$ such that $\xi a = \alpha(a)\xi$ for all $\xi \in K$, $a \in \mathcal{Z}(A)f_2$.

Denote by D the Radon-Nikodym derivative between $\tau \circ \alpha$ and τ, so that $\tau(b) = \tau(\alpha(b)D)$ for all $b \in \mathcal{Z}(A)f_2$. By a direct computation, we get that

$$\Delta^\ell_K = D\,\alpha(\mathrm{zdim}_{-A}(K)) \quad \text{and} \quad \alpha(\Delta^r_K) = D^{-1}\,\mathrm{zdim}_{A-}(K)\;.$$

In particular, we get that

$$\Delta^\ell_K\,\alpha(\Delta^r_K) = \mathrm{zdim}_{A-}(K)\,\alpha(\mathrm{zdim}_{-A}(K))\;. \tag{5.16}$$

By Lemma 5.39 and the computation in the proof of [PSV15, Lemma 2.2], we have

$$\mathrm{zdim}_{A-}(K)\,\alpha(\mathrm{zdim}_{-A}(K)) = \mathrm{zdim}_{A-}(K \otimes_A \overline{K}) \geq e_2\;. \tag{5.17}$$

Since $\Delta^\ell_K \leq e_2$ and $\Delta^r_K \leq f_2$, in combination with (5.16), it follows that $\Delta^\ell_K = e_2$ and $\Delta^r_K = f_2$. From (5.17), we then also get that $\mathrm{zdim}_{A-}(K \otimes_A \overline{K}) = e_2$. By Lemma 5.39, K is given by a partial automorphism of A.

Since $e_2 \geq \Delta^\ell_{e_2 T} = \Delta^\ell_K + \Delta^\ell_{e_2 T \ominus K} = e_2 + \Delta^\ell_{e_2 T \ominus K}$, we conclude that $e_2 T \ominus K = \{0\}$. So, $e_2 T = K$ and $e_2 T$ is given by a partial automorphism of A. This then contradicts the maximality of e_0. ☐

Lemma 5.42. *Let (A,τ) be a separable tracial von Neumann algebra and (H,J) a symmetric A-bimodule. Write $M = \Gamma(H,J,A,\tau)''$.*

Let $p \in A$ be a projection and $B \subset pAp$ a von Neumann subalgebra such that $B' \cap pAp = \mathcal{Z}(B)$ and such that $\mathcal{N}_{pAp}(B)''$ has essentially finite index in pAp. Let $K_1 \subset pH$ be a B-A-subbimodule satisfying the following three properties.

1. *K_1 is a direct sum of B-A-subbimodules of finite right A-dimension.*
2. *The left action of B on K_1 is faithful.*
3. *The A-bimodule $\overline{AK_1}$ is left weakly mixing.*

Then there exists a diffuse abelian von Neumann subalgebra $D \subset B' \cap pMp$ that is in tensor product position with respect to B. More precisely, there exists a unitary $u \in B' \cap pMp$ such that $E_B(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$.

Proof. We claim that for every $\varepsilon > 0$, there exists a projection $z \in \mathcal{Z}(B)$ with $\tau(p - z) < \varepsilon$ and a B-A-subbimodule $L \subset zH$ such that L is finitely generated as a right Hilbert A-module and such that Δ^ℓ_L is bounded and satisfies $\Delta^\ell_L \geq z$. To prove this claim, denote $K := \overline{AK_1}$ and let $(K_i)_{i\in I}$ be a maximal family of mutually orthogonal nonzero B-A-subbimodules of pK that are finitely generated as a right A-module. Denote by R the closed linear span of all K_i. Whenever $u \in \mathcal{N}_{pAp}(B)$ and $i \in I$, also uK_i is a B-A-subbimodule of pK that is finitely generated as a right A-module. By the maximality of the family $(K_i)_{i\in I}$, we get that $uK_i \subset R$. So, $uR = R$ for all $u \in \mathcal{N}_{pAp}(B)$. Writing $P := \mathcal{N}_{pAp}(B)''$, we conclude that R is a P-A-subbimodule of pK.

Since $P \subset pAp$ is essentially of finite index and since ${}_AK_A$ is left weakly mixing, Lemma 5.44 says that for every projection $q \in P$, the right A-module qR is either $\{0\}$ or of infinite right A-dimension. By the assumptions of the lemma and the maximality of the family $(K_i)_{i\in I}$, the left B-action on R is faithful. So $qR \neq \{0\}$ and thus $\dim_{-A}(qR) = \infty$ for every nonzero projection $q \in B$. This means that for every nonzero projection $q \in B$,

$$\sum_{i\in I} \tau(q\Delta^\ell_{K_i}) = \sum_{i\in I} \dim_{-A}(qK_i) = \dim_{-A}(qR) = \infty\ .$$

So we can find a projection $z \in \mathcal{Z}(B)$ and a finite subset $I_0 \subset I$ such that $\tau(p - z) < \varepsilon$ and such that the operator $\Delta := \sum_{i\in I_0} \Delta^\ell_{K_i} z$ is bounded and satisfies $\Delta \geq z$. Defining $L = \sum_{i\in I_0} zK_i$, the claim is proved.

Combining the claim with Lemma 5.38, we find for every $\varepsilon > 0$, a projection $z \in \mathcal{Z}(B)$ with $\tau(p - z) < \varepsilon$ and a unitary $u \in (Bz)' \cap zMz$ such that $E_B(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. So, we find projections $z_n \in \mathcal{Z}(B)$ and unitaries $u_n \in (Bz_n)' \cap z_nMz_n$ such that $E_B(u_n^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$ and such that $\bigvee_n z_n = p$. We can then choose projections $z_n' \in \mathcal{Z}(B)$ with $z_n' \leq z_n$ and $\sum_n z_n' = p$. Defining $u = \sum_n u_n z_n'$, we have found a unitary in $B' \cap pMp$ satisfying $E_B(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. So, the lemma is proved. □

Above we also needed the following two lemmas.

Lemma 5.43. *Let (N, τ) be a separable tracial von Neumann algebra and $B \subset N$ an abelian von Neumann subalgebra. Assume that $D \subset B' \cap N$ is a diffuse abelian von Neumann subalgebra that is in tensor product position with respect to B. Then there is no nonzero projection $q \in B' \cap N$ satisfying $q(B' \cap N)q = Bq$.*

Proof. Put $P = B' \cap N$ and assume that $q \in P$ is a nonzero projection such that $qPq = Bq$. Note that $B \subset \mathcal{Z}(P)$ because B is abelian. Take a nonzero projection $z \in \mathcal{Z}(P)$ such that $z = \sum_{i=1}^n v_i v_i^*$ where $v_1, \ldots, v_n$ are partial isometries in Pq. Note that $zq \neq 0$ and write $p = zq$. Then,

$$Pp = zPq = \text{span}\{v_i qPq \mid i = 1 \ldots, n\} = \text{span}\{v_i B \mid i = 1, \ldots, n\} \,.$$

So, $L^2(P)p$ is finitely generated as a right Hilbert B-module. Define $Q = B \vee D$ and denote by $e \in Q$ the support projection of $E_Q(p)$. Then $\xi \mapsto \xi p$ is an injective right B-linear map from $L^2(Q)e$ to $L^2(P)p$. So also $L^2(Q)e$ is finitely generated as a right Hilbert B-module. Since $Q \cong B \mathbin{\overline{\otimes}} D$ with D diffuse and since e is a nonzero projection in $Q \cong B \mathbin{\overline{\otimes}} D$, this is absurd. □

Lemma 5.44. *Let (A, τ) be a separable tracial von Neumann algebra and ${}_A K_A$ an A-bimodule that is left weakly mixing. Let $p \in A$ be a projection and $P \subset pAp$ a von Neumann subalgebra that is essentially of finite index (see Definition 5.26). If $L \subset pK$ is a P-A-subbimodule and $q \in P$ is a projection such that $qL \neq \{0\}$, then the right A-dimension of qL is infinite.*

Proof. Assume for contradiction that $q \in P$ is a projection such that qL is nonzero and such that qL has finite right A-dimension.

By Proposition 5.27, since $P \subset pAp$ is essentially of finite index, there exist projections $p_1 \in P' \cap pAp$ that lie arbitrarily close to p such that Ap_1 is finitely generated as a right Pp_1 module. There also exist central projections $z \in \mathcal{Z}(P)$ that lie arbitrarily close to p such that Pzq is finitely generated as a right qPq-module. Indeed, take a projection $z \in \mathcal{Z}(P)$ arbitrarily close to the central support of q such that $z = \sum_{i=1}^n v_i v_i^*$ with partial isometries $v_1, \ldots, v_n \in Pq$. Then $Pzq \subset \sum_{i=1}^n v_i(qPq)$ as wanted.

Take such p_1 and z with $p_1 zqL \neq \{0\}$. Then $Ap_1 zq$ is finitely generated as a right qPq-module. Therefore, the closed linear span of $Ap_1 zqL$ is a nonzero A-subbimodule of K having finite right A-dimension. This contradicts the left weak mixing of ${}_A H_A$. □

5.4.4 Proof of Theorem 5.21

Proof of Theorem 5.21. Let $K \subset H$ be the maximal left weakly mixing A-subbimodule of H, i.e., the orthogonal complement of the span of all A-subbimodules of H having finite right A-dimension. Denote by $z_0 \in \mathcal{Z}(A)$ the support of the left A-action on K. In the first part of the proof, assuming $z_0 \neq 0$, we show that

(1) $\mathcal{Z}(M)z_0 \subset \mathcal{Z}(A)z_0$,

(2) every M-central state ω on $\langle M, e_A \rangle$ that is normal on M satisfies $\omega(z_0) = 0$.

Note that $K \subset z_0 H$. Denote by $\mathcal{K} \subset K$ the dense subspace of vectors that are both left and right bounded. Define the von Neumann subalgebra $N \subset z_0 M z_0$ given by

$$N := \left(Az_0 \cup \{W(\xi, J(\mu)) \mid \xi, \mu \in \mathcal{K}\}\right)'' , \tag{5.18}$$

where we used the notation of (5.2). Then, the linear span of Az_0 and elements of the form $W(\xi_1, J(\mu_1), \ldots, \xi_k, J(\mu_k))$, $k \geq 1$, $\xi_i, \mu_i \in \mathcal{K}$, is a dense $*$-subalgebra of N.

Whenever $K_1, \ldots, K_n \subset H$ are A-subbimodules, we denote by concatenation $K_1 \cdots K_n$ the A-subbimodule of $L^2(M)$ given by

$$K_1 \cdots K_n := K_1 \otimes_A \cdots \otimes_A K_n \subset H \otimes_A \cdots \otimes_A H \subset L^2(M) .$$

In the same way, we write powers of A-subbimodules and when $K_i \subset H^{k_i}$ are A-subbimodules, then $K_1 \cdots K_n \subset H^{k_1 + \cdots + k_n}$ is a well-defined A-subbimodule.

Using this notation, note that $L^2(N)$ is the direct sum of $L^2(Az_0)$ and the spaces $L_n := (K\, J(K))^n$, $n \geq 1$. Since K is a left weakly mixing A-bimodule, it follows that $N \cap (Az_0)' = \mathcal{Z}(A) z_0$.

We claim that

(3) $N \not\prec_N Az_0$, meaning that the N-A-bimodule $L^2(N)$ is left weakly mixing.

Since $N \cap (Az_0)' = \mathcal{Z}(A)z_0$, to prove this claim, it suffices to show that $\dim_{-A}(L^2(N)e) = +\infty$ for every nonzero projection $e \in \mathcal{Z}(A)z_0$. Since the left action of Az_0 on K is faithful and K is left weakly mixing, we get that $\dim_{-A}(K\, J(K)\, e) = +\infty$. So certainly $\dim_{-A}(L^2(N)e) = +\infty$ and the claim follows.

Proof of (1). Define the A-subbimodule $R \subset L^2(M)$ given as

$$R := \left(H \ominus \overline{(K + J(K))} \right) \oplus \bigoplus_{n=0}^{\infty} (H \ominus K)\, H^n\, (H \ominus J(K)) .$$

Since K is left weakly mixing and $J(K)$ is right weakly mixing, all A-central vectors in $L^2(M)$ belong to $L^2(A) + R$. Next note that left and right multiplication by elements of N induces an N-bimodular unitary operator

$$L^2(N) \otimes_A R \otimes_A L^2(N) \to \overline{NRN} \subset L^2(z_0 M z_0) .$$

Since the N-A-bimodule $L^2(N)$ is left weakly mixing, it follows that $\overline{NRN}$ has no nonzero N-central vectors. Every element $x \in \mathcal{Z}(M)z_0$ defines a vector in

$L^2(z_0 M z_0)$ that is both A-central and N-central. By A-centrality, we conclude that $x \in A z_0 + z_0 R z_0$. In particular, $x \in L^2(N) + \overline{NRN}$. Since x is N-central and $\overline{NRN}$ has no nonzero N-central vectors, we get that $x \in L^2(N)$ and thus, $x \in \mathcal{Z}(A) z_0$.

Proof of (2). Denote $L_{\text{even}} := L^2(N)$ and define L_{odd} as the direct sum of the A-bimodules $(K\, J(K))^n\, K$, $n \geq 0$. Note that both L_{even} and L_{odd} are N-A-bimodules. The same argument as in the proof of Theorem 5.19, using the left weak mixing of K, shows that the von Neumann algebras $B(L_{\text{even}}) \cap (A^{\text{op}})'$ and $B(L_{\text{odd}}) \cap (A^{\text{op}})'$ admit no N-central states that are normal on N. Note that we have the following decomposition of $L^2(z_0 M)$ as an N-A-bimodule:

$$L^2(z_0 M) = \Big(L_{\text{even}} \otimes_A \Big(L^2(A) \oplus \bigoplus_{n \geq 0} (H \ominus K)\, H^n\Big)\Big) \oplus$$

$$\Big(L_{\text{odd}} \otimes_A \Big(L^2(A) \oplus \bigoplus_{n \geq 0} (H \ominus J(K))\, H^n\Big)\Big) .$$

This decomposition induces $*$-homomorphisms from $B(L_{\text{even}}) \cap (A^{\text{op}})'$ and $B(L_{\text{odd}}) \cap (A^{\text{op}})'$ to $B(z_0 L^2(M)) \cap (A^{\text{op}})' = z_0 \langle M, e_A \rangle z_0$. So, $z_0 \langle M, e_A \rangle z_0$ admits no N-central state that is normal on N. A fortiori, (2) holds.

Next we define the projection $z_1 \in \mathcal{Z}(A)(1 - z_0)$ given by

$$z_1 = \mathbb{1}_{(1,+\infty]}\big(\Delta^{\ell}_{(1-z_0)H}\big) . \tag{5.19}$$

We also write $z = z_0 + z_1$ and $z_2 = 1 - z$.

Denote by $e' \in \mathcal{Z}(A) z_1$ the maximal projection with the following properties: the right support $f \in \mathcal{Z}(A)$ of $e'H$ satisfies $e'H = zHf$ and the A-bimodule $e'H$ is given by a partial automorphism of A. Define $e = z_1 - e'$.

By the definition of z_0, we get that the A-bimodule $(1 - z_0)H$ is a sum of A-bimodules that are finitely generated as a right Hilbert A-module. It then follows from the definition of z_1 that we can choose a projection $e_1 \in \mathcal{Z}(A) z_1$ that lies arbitrarily close to z_1 and for which there exists an A-subbimodule $L_1 \subset z_1 H$ with the following properties:

- the left support of L_1 equals e_1,
- L_1 is finitely generated as a right Hilbert A-module,
- $\Delta^{\ell}_{L_1}$ is bounded and satisfies $\Delta^{\ell}_{L_1} \geq \delta_1 e_1$ for some real number $\delta_1 > 1$.

Denote by e_2 the left support of $e_1(H \ominus L_1)$. Making e_1 slightly smaller, but still arbitrarily close to z_1, we may assume that e_2 is the left support of an

A-subbimodule $L_2 \subset e_1(H \ominus L_1)$ with the following properties: L_2 is finitely generated as a right Hilbert A-module and $\Delta^\ell_{L_2}$ is bounded. By construction, $e_2 \leq e_1$. Since $e_2 L_1$ and L_2 are orthogonal and have the same left support e_2, it follows that for nonzero projections $s \in \mathcal{Z}(A)e_2$, the A-bimodule sH is not given by a partial automorphism of A. This means that $e_2 \leq e$ and thus, $e_2 \leq ee_1$. Define $L = L_1 + L_2$. Using notation (5.12), it follows from Lemma 5.39 (1) and (3) that the left support of $e_2 L\, J(L) e_2 \cap (t_{e_2 L} A)^\perp$ equals e_2. A fortiori, the left support of $e_2 L\, Hz \cap (t_{e_2 L} A)^\perp$ equals e_2.

We put $e_3 = ee_1 - e_2$. Since e_2 is the left support of $e_1(H \ominus L_1)$, we get that $e_3 H = e_3 L_1 = e_3 L$. Since $e_3 \leq e$, applying Lemma 5.39 to the A-bimodule zH, we conclude that the left support of $e_3 L\, Hz \cap (t_{e_3 H} A)^\perp$ equals e_3. Summarizing, L has the following properties:

- the left support of L equals e_1,
- L is finitely generated as a right Hilbert A-module,
- Δ^ℓ_L is bounded and satisfies $\Delta^\ell_L \geq \delta e_1$ for some real number $\delta > 1$,
- the left support of $L\, Hz \cap (t_L A)^\perp$ equals ee_1.

Denote by $s \in \mathcal{Z}(A)$ the left support of $L\, H(z_0 + e_1) \cap (t_L A)^\perp$. Since e_1 could be chosen arbitrarily close to z_1, it follows that s lies arbitrarily close to e.

We next prove that

(4) $\mathcal{Z}(M)s \subset \mathcal{Z}(A)s$,

(5) every M-central state ω on $\langle M, e_A \rangle$ that is normal on M satisfies $\omega(s) = 0$.

Write $\Delta := \Delta^\ell_L$, choose a Pimsner-Popa basis $(\xi_i)_{i=1}^n$ for the right Hilbert A-module L and put

$$t := t_L = \sum_{i=1}^n \xi_i \otimes_A J(\xi_i) \; .$$

Since Δ is bounded, the vectors $\xi_i \in H$ are both left and right bounded by Lemma 5.25.

Denoting by P_T the orthogonal projection onto a Hilbert subspace $T \subset H$, the main properties of t, used throughout the proof, are:

$$\langle t, t \rangle_A = {}_A\langle t, t \rangle = \Delta \;\; , \;\; \ell(\xi)^* t = J(P_L(\xi)) \;\text{ and }\; r(\xi)^* t = P_L(J(\xi)) \;\; ,$$

for all left and right bounded vectors $\xi \in \mathcal{H}$.

Since the vectors ξ_i are both left and right bounded, we can define the self-adjoint element $S_1 \in e_1Me_1$ given by

$$S_1 := \sum_{i=1}^{n} W(\xi_i, J(\xi_i)) \ .$$

By Lemma 5.38, we have that $S_1 \in e_1Me_1 \cap (Ae_1)'$ and that the von Neumann algebra $D := Ae_1 \vee \{S_1\}''$ is a subalgebra of $e_1Me_1 \cap (Ae_1)'$ that is diffuse relative to Ae_1. Using Lemma 5.37, we fix a unitary $u \in \mathcal{U}(D)$ satisfying $E_{Ae_1}(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$.

Defining

$$S_k := \sum_{i_1,\ldots,i_k=1}^{n} W(\xi_{i_1}, J(\xi_{i_1}), \ldots, \xi_{i_k}, J(\xi_{i_k})) \ ,$$

and denoting by $\Omega \in L^2(M)$ the vacuum vector, we get that

$$t_k := S_k\Omega = \underbrace{t \otimes_A \cdots \otimes_A t}_{k \text{ times}} \ . \tag{5.20}$$

By induction, we see that $S_k \in D$ for all $k \geq 1$. With the convention that $S_0 = e_1$, the elements $\{S_k\}_{k\geq 0}$, form a Pimsner-Popa basis for the right A-module $L^2(D)_A$. More precisely, we have that $\langle S_k, S_\ell\rangle_A = E_A(S_kS_\ell) = 0$ for $k \neq \ell$, and every element $x \in D$ can be written as a $\|\cdot\|_2$-converging sum $x = \sum_{k=0}^{\infty} S_kx_k$ with $x_k = \Delta^{-k}E_A(xS_k)$.

Proof of (4). We start by proving that an element $x \in \mathcal{Z}(M)e_1$ must belong to D. Define $T_0 \subset H^2$ as the closure of tA. Note that $\ell(t)\ell(t)^*\Delta^{-1}$ is the orthogonal projection of H^2 onto T_0. Then define $T_2 := H^2 \ominus T_0$ and $T_3 := H^3 \ominus (T_0H + HT_0)$. Observe that $L^2(e_1Me_1 \ominus D)$ is spanned by the D-subbimodules

$$\overline{DHD} \ \ , \ \ \overline{DT_2D} \ \ , \ \ \overline{DT_3D} \ \ , \ \ \overline{DT_2H^nT_2D} \ \text{ with } n \geq 0 \ . \tag{5.21}$$

We will prove that each of the D-bimodules in (5.21) is contained in a multiple of a D-bimodule of the form $L^2(D) \otimes_A K$ for some A-D-bimodule K.

For the first one, $\overline{DHD}$, fix a left and right bounded vector $\mu \in H$ with $\|\mu\| \leq 1$. Using the notation t_k introduced in (5.20), one checks that for $k \geq 1$,

$$S_kW(\mu)\Omega = t_k \otimes_A \mu + t_{k-1} \otimes_A P_L(\mu) \quad \text{and}$$

$$W(\mu)S_k\Omega = \mu \otimes_A t_k + P_{J(L)}(\mu) \otimes_A t_{k-1} \ .$$

When $\mu, \eta \in H$ are left and right bounded vectors, we have $\langle t_k \otimes_A \mu, \eta \otimes_A t_l \rangle = 0$ if $k \neq l$, while

$$\begin{aligned}\langle t_k \otimes_A \mu, \eta \otimes_A t_k \rangle &= \langle \ell(\eta)^*(t_k \otimes_A \mu), t_k \rangle \\ &= \langle J(P_L(\eta)) \otimes_A t_{k-1} \otimes_A \mu, t_k \rangle \\ &= \langle J(P_L(\eta)) \otimes_A t_{k-1}, r(\mu)^* t_k \rangle \\ &= \langle J(P_L(\eta)) \otimes_A t_{k-1}, t_{k-1} \otimes_A P_L(J(\mu)) \rangle \ .\end{aligned}$$

It follows by induction that $\langle t_k \otimes_A \mu, \eta \otimes_A t_k \rangle = \langle (P_L J)^{2k} \mu, \eta \rangle$ for any $k \geq 0$. Thus, for $a, b \in A$,

$$\langle S_k a W(\mu) b S_l, W(\mu) \rangle = \begin{cases} \langle a\mu b, \mu \rangle & \text{if } k = l = 0, \\ \langle a((P_L J)^{2k} + (J P_L)^{2k}) \mu b, \mu \rangle & \text{if } k = l \geq 1, \\ \langle a (P_L J)^{2(k-1)} P_L \mu b, \mu \rangle & \text{if } k = l + 1 \geq 1, \\ \langle a J (P_L J)^{2k+1} \mu b, \mu \rangle & \text{if } k = l - 1 \geq 0. \end{cases}$$

We next claim that

$$\begin{aligned}\xi := e_1 \otimes_A \mu \otimes_A e_1 + \sum_{k=1}^{\infty} \Big(\Delta^{-k} S_k \otimes_A ((P_L J)^{2k} + (J P_L)^{2k})(\mu) \otimes_A \Delta^{-k} S_k \Big) \\ + \sum_{k=0}^{\infty} \Big(\Delta^{-k-1} S_{k+1} \otimes_A (P_L J)^{2k} P_L(\mu) \otimes_A \Delta^{-k} S_k \\ + \Delta^{-k} S_k \otimes_A J(P_L J)^{2k+1}(\mu) \otimes_A \Delta^{-k-1} S_{k+1} \Big)\end{aligned}$$

is a well-defined element in $L^2(D) \otimes_A H \otimes_A L^2(D)$. This follows because $E_A(S_k^2) = \langle t_k, t_k \rangle_A = \Delta^k$ and thus

$$\|\Delta^{-k} S_k\|_2^2 = \tau(\Delta^{-2k} S_k^2) = \tau(\Delta^{-k}) \leq \delta^{-k} \ ,$$

where $\delta > 1$.

By construction,

$$\langle S_k a W(\mu) b S_l, W(\mu) \rangle = \langle S_k a \, \xi \, b S_l, \ \xi_0 \rangle \ ,$$

where $\xi_0 = e_1 \otimes_A \mu \otimes_A e_1$. This means that the D-bimodule $\overline{D\mu D}$ is contained in $L^2(D) \otimes_A H \otimes_A L^2(D)$. Since this holds for any $\mu \in \mathcal{H}$, we have that the D-bimodule $\overline{DHD}$ is contained in a multiple of $L^2(D) \otimes_A H \otimes_A L^2(D)$.

Next, we do similar computations for the remaining D-bimodules occurring in (5.21). Let $K \subset H^n$ be one of the A-bimodules T_2, T_3 or $T_2 H^k T_2$, $k \geq 1$, and fix a left and right bounded vector $\mu \in K$. Then $\ell(t)^*\mu = 0$ and $r(t)^*\mu = 0$, so it follows that

$$S_k W(\mu)\Omega = t_k \otimes_A \mu + t_{k-1} \otimes_A (P_L \otimes 1^{\otimes(n-1)})(\mu) \quad \text{and}$$

$$W(\mu) S_k \Omega = \mu \otimes_A t_k + (1^{\otimes(n-1)} \otimes P_{J(L)})(\mu) \otimes_A t_{k-1} \; ,$$

for $k \geq 1$. Note that

$$\langle (P_L \otimes 1^{\otimes(n-1)})(\mu), t \rangle = \langle \mu, (P_L \otimes 1^{\otimes(n-1)})(t) \rangle = \langle \mu, t \rangle = 0,$$

and similarly $\langle (1^{\otimes(n-1)} \otimes P_{J(L)})(\mu), t \rangle = 0$. It follows that $S_k W(\mu)$ is orthogonal to $W(\mu) S_\ell$ whenever $k, \ell \geq 1$ except when $k = \ell = 1$. In the remaining cases, we have that

$$\langle S_1 W(\mu), W(\mu) \rangle = \langle (P_L \otimes 1^{\otimes(n-1)})(\mu), \mu \rangle,$$

$$\langle W(\mu) S_1, W(\mu) \rangle = \langle (1^{\otimes(n-1)} \otimes P_{J(L)})(\mu), \mu \rangle,$$

$$\langle S_1 W(\mu) S_1, W(\mu) \rangle = \langle (P_L \otimes 1^{\otimes(n-2)} \otimes P_{J(L)})(\mu), \mu \rangle.$$

Define $\xi \in L^2(D) \otimes_A K \otimes_A L^2(D)$ by

$$\begin{aligned} \xi = e_1 \otimes_A \mu \otimes_A e_1 + S_1 \Delta^{-1} \otimes_A (P_L \otimes 1^{\otimes(n-1)})(\mu) \otimes_A e_1 \\ + e_1 \otimes_A (1^{\otimes(n-1)} \otimes P_{J(L)})(\mu) \otimes_A S_1 \Delta^{-1} \\ + S_1 \Delta^{-1} \otimes_A (P_L \otimes 1^{\otimes(n-2)} \otimes P_{J(L)})(\mu) \otimes_A S_1 \Delta^{-1}. \end{aligned}$$

By construction,

$$\langle S_k a W(\mu) S_\ell b, W(\mu) \rangle = \langle S_k a \, \xi \, S_\ell b, \xi_0 \rangle \quad \text{for} \;\; k, \ell \geq 0, \;\; a, b \in A,$$

where $\xi_0 = e_1 \otimes_A \mu \otimes_A e_1$. This means that the D-bimodule $\overline{DKD}$ is contained in a multiple of $L^2(D) \otimes_A K \otimes_A L^2(D)$.

We have thus proved that all D-bimodules in (5.21) are contained in a multiple of a bimodule of the form $L^2(D) \otimes_A K$ for some A-D-bimodule K. Since D is diffuse relative to A, it follows from Proposition 5.36 that $L^2(e_1 M e_1 \ominus D)$ has no D-central vectors. In particular, $\mathcal{Z}(M) e_1 \subset D$.

We are now ready to prove (4). Fix $x \in \mathcal{Z}(M)$. We have to prove that $xs \in A$. Because of (1) and the previous paragraphs, we can uniquely decompose $x(z_0 + e_1)$

as the $\|\cdot\|_2$-convergent sum

$$x(z_0+e_1) = a_0 + \sum_{k=1}^{\infty} S_k a_k \tag{5.22}$$

with $a_0 \in A(z_0+e_1)$ and $a_k \in Ae_1$ for all $k \geq 1$. Note that $a_0 = E_A(x)(z_0+e_1)$ and $a_k = \Delta^{-k} E_A(S_k x)$ for all $k \geq 1$.

Let now $\eta \in L\,H(z_0+e_1) \cap (tA)^\perp$ be an arbitrary left and right bounded vector. Note that

$$\eta = \sum_{i=1}^{n} \xi_i \otimes_A J(\eta_i) \tag{5.23}$$

where the vectors $\eta_i \in (z_0+e_1)H$ are both left and right bounded. Define

$$W(\eta) := \sum_{i=1}^{n} W(\xi_i, J(\eta_i))$$

and note that $W(\eta) \in sM(z_0+e_1) \subset e_1 M(z_0+e_1)$.

Using that $W(\eta)$ commutes with x and using the decomposition of $x(z_0+e_1)$ in (5.22), we find that

$$\begin{aligned} W(\eta)x\Omega &= W(\eta)(z_0+e_1)x\Omega = W(\eta)a_0\Omega + \sum_{k=1}^{\infty} W(\eta)S_k a_k \Omega \\ &= \eta a_0 + \sum_{k=1}^{\infty} \Big(\eta \otimes_A t_k a_k + (1 \otimes P_{J(L)})(\eta) \otimes_A t_{k-1} a_k \Big) \ , \\ xW(\eta)\Omega &= xe_1 W(\eta)\Omega = a_0 e_1 W(\eta)\Omega + \sum_{k=1}^{\infty} a_k S_k W(\eta)\Omega \\ &= a_0 \eta + \sum_{k=1}^{\infty} \Big(a_k t_k \otimes_A \eta + a_k t_{k-1} \otimes_A (P_L \otimes 1)(\eta) \Big) \\ &= (a_0+a_1)\eta + \sum_{k=1}^{\infty} (a_k + a_{k+1}) t_k \otimes_A \eta \ . \end{aligned}$$

In this last expression for $xW(\eta)\Omega$, all terms except $(a_0+a_1)\eta$ are orthogonal to $W(\eta)x\Omega$. We conclude that $(a_k+a_{k+1})t_k \otimes_A \eta = 0$ and thus ${}_A\langle \eta, \eta \rangle (a_k + a_{k+1}) = 0$ for all $k \geq 1$ and for all choices of η. Since the left support of $L\,H(z_0+e_1) \cap (tA)^\perp$ equals s, it follows that $(a_k + a_{k+1})s = 0$ for all $k \geq 1$. This means that $a_k s = (-1)^{k-1} a_1 s$ for all $k \geq 1$.

Since

$$+\infty > \|x\|_2^2 \geq \sum_{k=1}^{\infty} \|S_k a_k s\|_2^2 = \sum_{k=1}^{\infty} \tau(s a_1^* \Delta^k a_1 s) \geq \sum_{k=1}^{\infty} \delta^k \|a_1 s\|_2^2 \; ,$$

it follows that $a_1 s = 0$. So, $a_k s = 0$ for all $k \geq 1$. From (5.22), it follows that $xs \in A$, so that (4) is proved.

Proof of (5). Fix an M-central state ω on $\langle M, e_A\rangle$ that is normal on M. We have to prove that $\omega(s) = 0$. Recall that we defined $T_0 \subset H^2$ as the closure of tA. Consider the following orthogonal decomposition of $e_1 L^2(M)$ as an A-bimodule:

$$e_1 L^2(M) = V_0 \oplus V_1 \oplus V_2 \qquad \text{where} \qquad V_0 := \bigoplus_{n=0}^{\infty} T_0 H^n \;\; ,$$

$$V_1 := L^2(Ae_1) \oplus \bigoplus_{n=0}^{\infty} (e_1 H \ominus L) H^n \;\; , \quad V_2 := L \oplus \bigoplus_{n=0}^{\infty} (LH \ominus T_0) H^n \;\; .$$

Denote by $Q_i \in e_1 \langle M, e_A\rangle e_1$ the projections onto V_i, for $i = 0, 1, 2$. So, $e_1 = Q_0 + Q_1 + Q_2$. Also note that the projections Q_i commute with A. We prove below that $\omega(sQ_0) = \omega(Q_1) = \omega(Q_2) = 0$. Once these statements are proved, (5) follows.

To prove that $\omega(Q_1) = 0$, note that for all $\mu \in V_1$ and all $k \geq 1$, we have that $S_k \mu = t_k \otimes_A \mu$ and thus, $S_k \mu$ is orthogonal to V_1. So, for all $\mu, \mu' \in V_1$ and $d \in D$, we get that

$$\langle d\mu, \mu' \rangle = \langle E_A(d)\mu, \mu' \rangle \; .$$

Above we introduced the unitary element $u \in \mathcal{U}(D)$ satisfying $E_A(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. It follows that the subspaces $u^k V_1$ are all orthogonal. So, the projections $u^k Q_1 u^{-k}$ are all orthogonal. By M-centrality, ω takes the same value on each of these projections. So, $\omega(Q_1) = 0$.

To prove that $\omega(Q_2) = 0$, we argue similarly. For all $\mu \in V_2$ and all $k \geq 2$, we have that $S_k \mu = t_k \otimes_A \mu + t_{k-1} \otimes_A \mu$ and thus, $S_k \mu$ is orthogonal to V_2. On the other hand, $S_1 \mu = t \otimes_A \mu + \mu$ and here, only $t \otimes_A \mu$ is orthogonal to V_2. It follows that for all $\mu, \mu' \in V_2$ and $d \in D$,

$$\langle d\mu, \mu' \rangle = \langle \Phi(d)\mu, \mu' \rangle \; ,$$

where $\Phi \colon D \to D$ is the linear map given by $\Phi(d) = E_A(d) + \Delta^{-1} E_A(dS_1)$. Notice that $\Phi(d) = E_A(dx)$ where $x = e_1 + \Delta^{-1} S_1 \in \mathcal{Z}(D)$ and that $\Phi(a) = a$ for $a \in Ae_1$. Moreover, x is a positive element since

$$\Delta + S_1 = \sum_{i=1}^{n} X_i X_i^* \geq 0 \qquad \text{where} \qquad X_i = \ell(\xi_i) + \ell(J\xi_i)^*.$$

Since D is diffuse relative to Ae_1, Lemma 5.37 now gives us a unitary $v \in \mathcal{U}(D)$ such that $\Phi(v^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. It follows that the subspaces $v^k V_2$ are all orthogonal. As in the previous paragraph, we get that $\omega(Q_2) = 0$.

It remains to prove that $\omega(sQ_0) = 0$. Fix $\eta \in L\,H(z_0 + e_1) \ominus T_0$ as in (5.23) and define

$$\eta' = \sum_{i=1}^{n} \eta_i \otimes_A J(\xi_i)\ .$$

Note that $\eta' \in (z_0 + e_1)H\,J(L) \ominus T_0$. From (2), we already know that $\omega(z_0) = 0$. Since $e_1\eta' \in V_1 + V_2$, we also know that $\omega(\ell(e_1\eta')\ell(e_1\eta')^*) = 0$. Both together imply that $\omega(\ell(\eta')\ell(\eta')^*) = 0$.

For all $n \geq 0$ and $\mu \in H^n$, we have that

$$W(\eta)(\eta' \otimes_A t \otimes_A \mu) = \eta \otimes_A \eta' \otimes_A t \otimes_A \mu + \sum_{i=1}^{n} \ell(\xi_i)\ell(\eta_i)^*(\eta' \otimes_A t \otimes_A \mu) + \langle \eta', \eta' \rangle_A\,(t \otimes_A \mu).$$

Since

$$\ell(t)^* \sum_{i=1}^{n} \ell(\xi_i)\ell(\eta_i)^*\eta' = \sum_{i=1}^{n} \ell(J(\xi_i))^*\ell(\eta_i)^*\eta' = \ell(\eta')^*\eta' = \langle \eta', \eta' \rangle_A$$

and since the projection Q_0 is given by $Q_0 = \Delta^{-1}\ell(t)\ell(t)^*$, we get that

$$Q_0 W(\eta)(\eta' \otimes_A t \otimes_A \mu) = \langle \eta', \eta' \rangle_A\,\Delta^{-1}\,(t \otimes_A t \otimes_A \mu) + \langle \eta', \eta' \rangle_A\,(t \otimes_A \mu)$$

for all $n \geq 0$ and all $\mu \in H^n$. This means that

$$Q_0 W(\eta)\ell(\eta' \otimes_A t) = \langle \eta', \eta' \rangle_A \left(\Delta^{-1}\,\ell(t \otimes_A t) + \ell(t)\right) = \ell(t)\,\langle \eta', \eta' \rangle_A\,(1 + \Delta^{-1}\ell(t)).$$

Because

$$\|\Delta^{-1}\ell(t)\|^2 = \|\Delta^{-2}\ell(t)^*\ell(t)\| = \|\Delta^{-1}\| \leq \delta^{-1} < 1\ ,$$

the operator $R := 1 + \Delta^{-1}\ell(t)$ is invertible. Also note that $\ell(t)\ell(t)^* \leq \|\Delta\| 1$ so that

$$\ell(\eta' \otimes_A t)\ell(\eta' \otimes_A t)^* \leq \|\Delta\|\,\ell(\eta')\ell(\eta')^*\ .$$

So, we find $\varepsilon > 0$ and $\kappa > 0$ such that

$$\begin{aligned} \varepsilon\,\ell(t)\,(\langle \eta', \eta' \rangle_A)^2\,\ell(t)^* &\leq \ell(t)\,\langle \eta', \eta' \rangle_A\,RR^*\,\langle \eta', \eta' \rangle_A\,\ell(t)^* \\ &= Q_0 W(\eta)\ell(\eta' \otimes_A t)\ell(\eta' \otimes_A t)^* W(\eta)^* Q_0 \qquad (5.24) \\ &\leq \kappa Q_0 W(\eta)\ell(\eta')\ell(\eta')^* W(\eta)^* Q_0\ . \end{aligned}$$

We already proved that $\omega(\ell(\eta')\ell(\eta')^*) = 0$. Since ω is M-central, also

$$\omega(W(\eta)\ell(\eta')\ell(\eta')^*W(\eta)^*) = 0 \; .$$

Because $e_1 = Q_0 + Q_1 + Q_2$ and $\omega(Q_1) = \omega(Q_2) = 0$, the Cauchy-Schwarz inequality implies that $\omega(Y) = \omega(Q_0 Y) = \omega(Y Q_0)$ for all $Y \in e_1 \langle M, e_A \rangle e_1$. Therefore,

$$\omega\big(Q_0 W(\eta)\ell(\eta')\ell(\eta')^*W(\eta)^* Q_0\big) = \omega(W(\eta)\ell(\eta')\ell(\eta')^*W(\eta)^*) = 0 \; .$$

It then follows from (5.24) that

$$\omega\big((\langle \eta', \eta' \rangle_A)^2 \, \Delta \, Q_0\big) = 0$$

for all bounded vectors $\eta' \in (z_0 + e_1)H\,J(L) \ominus T_0$. By the Cauchy-Schwarz inequality and the normality of ω restricted to M, we get that $\omega(a_i Q_0) \to \omega(a Q_0)$ whenever $a_i \in A$ is a bounded sequence such that $\|a_i - a\|_2 \to 0$. Since the right support of the A-bimodule $(z_0 + e_1)H\,J(L) \ominus T_0$ equals s, it follows that $\omega(s Q_0) = 0$. Since we already proved that $\omega(Q_1) = \omega(Q_2) = 0$, it follows that (5) holds.

Since s lies arbitrarily close to e, it follows from (1)-(2) and (4)-(5) that

(6) $\mathcal{Z}(M)(z_0 + e) \subset \mathcal{Z}(A)(z_0 + e)$,

(7) every M-central state ω on $\langle M, e_A \rangle$ that is normal on M satisfies $\omega(z_0 + e) = 0$.

Recall that $z = z_0 + z_1$ and $z_2 = 1 - (z_0 + z_1)$. Note that $\Delta^\ell_{z_2 H} \leq z_2$. We claim that $z_2 H z_2 = \{0\}$. Denote by $e_0 \in \mathcal{Z}(A) z_2$ the left support of $z_2 H z_2$. Note that by symmetry, e_0 also is the right support of $z_2 H z_2$. By Lemma 5.41, we get that $\Delta^\ell_{e_0 H e_0} = e_0$ and that $e_0 H e_0$ is given by a partial automorphism of A. Since

$$\Delta^\ell_{e_0 H} = \Delta^\ell_{e_0 H e_0} + \Delta^\ell_{e_0 H(1-e_0)} = e_0 + \Delta^\ell_{e_0 H(1-e_0)}$$

and since $\Delta^\ell_{e_0 H} \leq e_0$, we get that $e_0 H(1 - e_0) = \{0\}$. We conclude that $e_0 H = H e_0 = e_0 H e_0$ and that this A-bimodule is given by a partial automorphism of A. Since H is assumed to be completely nontrivial, we get that $e_0 = 0$ and the claim is proved.

Recall that $e \in \mathcal{Z}(A) z_1$ was defined as $e = z_1 - e'$ where $e' \in \mathcal{Z}(A) z_1$ has the following properties: denoting by $f \in \mathcal{Z}(A)$ the right support of $e'H$, we have that $e'H = zHf$ and that the A-bimodule $e'H$ is given by a partial automorphism of A. We claim that $f \leq z$. To prove this claim, denote $f_1 := f z_2$. If $f_1 \neq 0$, we find a nonzero projection $e'' \in \mathcal{Z}(A)e'$ such that $e''H = zHf_1$ and such that this A-bimodule is given by a partial automorphism of A. Above, we have proved that $z_2 H z_2 = \{0\}$. A fortiori, $z_2 H f_1 = \{0\}$, meaning that $H f_1 = z H f_1$. But then, $e'' H = H f_1$, contradicting the complete non-triviality of H. So, we have proved that $f \leq z$.

We next claim that $f \leq z_0 + e$. To prove this claim, assume that $f' := fe'$ is nonzero. Then, $f'H = fe'H = fzHf \subset Hz$ because $f \leq z$. Applying the symmetry J, it follows that $Hf' = zHf'$ and thus $e''H = Hf'$ for some nonzero projection $e'' \in \mathcal{Z}(A)e'$, again contradicting the complete non-triviality of H. So, we have proved that $f \leq z_0 + e$.

Since $e'H$ is given by a partial automorphism of A, we can take projections $e'' \in \mathcal{Z}(A)e'$ arbitrarily close to e' such that $e''H$ is finitely generated as a right Hilbert A-module and $\Delta^\ell_{e''H}$ is bounded. Denote by $f' \in \mathcal{Z}(A)f$ the right support of $e''H$. Since the right A-action equals the commutant of the left A-action on $e'H$, we can for each $a \in \mathcal{Z}(A)e''$ find a unique element $\alpha(a) \in \mathcal{Z}(A)f'$ such that $a\xi = \xi\alpha(a)$. This gives rise to a $*$-isomorphism $\alpha \colon \mathcal{Z}(A)e'' \to \mathcal{Z}(A)f'$ satisfying $a\xi = \xi\alpha(a)$ for all $a \in \mathcal{Z}(A)e''$. Let $(\gamma_i)_{i=1}^n$ be a Pimsner-Popa basis of the right A-module $e''H$ and define

$$R_i = \ell(\gamma_i) + \ell(J(\gamma_i))^* \quad \text{and} \quad R = \sum_{i=1}^n R_i R_i^* = \Delta^\ell_{e''H} + \sum_{i=1}^n W(\gamma_i, J(\gamma_i)) \; .$$

Note that $R_i \in e''Mf'$ and $R \in e''Me''$. Since $\Delta^\ell_{e''H} = e''\Delta^\ell_H \geq e''$, it follows from Lemma 5.38 that the support projection of R equals e''.

Let $x \in \mathcal{Z}(M)$ and using (6), take $a \in \mathcal{Z}(A)(z_0 + e)$ such that $(z_0 + e)x = a$. Since $f' \leq z_0 + e$, we have $f'x = af'$ and thus

$$xR = \sum_{i=1}^n R_i x R_i^* = \sum_{i=1}^n R_i \, af' \, R_i^* = \alpha^{-1}(af')R \; .$$

Since the support projection of R equals e'', we have proved that $\mathcal{Z}(M)e'' \subset \mathcal{Z}(A)e''$. Since e'' lies arbitrarily close to e', together with (6), it follows that

(8) $\quad \mathcal{Z}(M)z \subset \mathcal{Z}(A)z$.

Next, we will show that

(9) $\quad$ every M-central state ω on $\langle M, e_A \rangle$ that is normal on M satisfies $\omega(z) = 0$.

Using (7), it is enough to show that $\omega(e') = 0$. We do this similarly to the argument above. With $R_i \in e''Mf'$ and $R \in e''Me''$ as above, we get by M-centrality of ω that

$$\omega(R) = \sum_{i=1}^n \omega(R_i^* R_i) = 0,$$

since $R_i^* R_i \in f'Mf'$ and $\omega(f') \leq \omega(z_0 + e) = 0$. Since e'' is the support projection of R, it follows that $\omega(e'') = 0$ and hence $\omega(e') = 0$, as e'' was chosen arbitrarily close to e'. This finishes the proof of (9).

To prove the first two statements of the theorem, it remains to see what happens under the projection z_2.

Denote $\Delta_2 := \Delta^\ell_{z_2 H}$. By the definition of z_2, we have that $\Delta_2 \leq z_2$. Let $(\mu_i)_{i\in I}$ be a (possibly infinite) Pimsner-Popa basis for the right A-module $z_2 H$. Since Δ_2 is bounded, we may choose the vectors μ_i to be left and right bounded. For the same reason,

$$s := \sum_{i\in I} \mu_i \otimes_A J(\mu_i)$$

is a well-defined bounded A-central vector in $z_2 H\, H z_2$ and the infinite sums

$$G_n = \sum_{i_1,\ldots,i_n} W(\mu_{i_1}, J(\mu_{i_1}), \ldots, \mu_{i_n}, J(\mu_{i_n}))$$

are well-defined bounded operators in $z_2 M z_2 \cap (A z_2)'$ satisfying

$$G_n \Omega = s_n := \underbrace{s \otimes_A \cdots \otimes_A s}_{n \text{ times}} .$$

By convention, we put $G_0 = z_2$. From the definition of G_n, we obtain the recurrence relation

$$G_1 G_n = G_{n+1} + G_n + \Delta_2\, G_{n-1} \tag{5.25}$$

for all $n \geq 1$, and thus $G_{n+1} = (G_1 - 1) G_n - \Delta_2\, G_{n-1}$ for all $n \geq 1$.

Denote by $q \in z_2 M z_2$ the projection onto the kernel of $G_1 + \Delta_2$. Although the sum defining G_1 is infinite, the computations in the proof of Lemma 5.38 remain valid and it follows that the kernel of $(G_1 + \Delta_2)\, \mathbb{1}_{\{1\}}(\Delta_2)$ is reduced to zero. So, $q \leq \mathbb{1}_{(0,1)}(\Delta_2)$.

With the convention that $s_0 = z_2 \Omega$, we claim that

$$q\Omega = \sum_{k=0}^{\infty} (-1)^k (z_2 - \Delta_2) s_k = \sum_{k=0}^{\infty} (-1)^k s_k (z_2 - \Delta_2) . \tag{5.26}$$

Because

$$\sum_{k=0}^{\infty} \|(z_2 - \Delta_2) s_k\|_2^2 = \sum_{k=0}^{\infty} \tau\big(\langle s_k, s_k\rangle_A\, (z_2 - \Delta_2)^2\big)$$

$$= \sum_{k=0}^{\infty} \tau\big(\Delta_2^k (z_2 - \Delta_2)^2\big) = \tau(z_2 - \Delta_2) < \infty ,$$

the right hand side of (5.26) is a well-defined element $p \in L^2(z_2 M z_2)$ satisfying, with $\| \cdot \|_2$-convergence,

$$p = \sum_{k=0}^{\infty} (-1)^k (z_2 - \Delta_2)\, G_k \; .$$

Note that $p = p^*$. Using the recurrence relation (5.25), it follows that $(G_1 + \Delta_2)p = 0$ and thus $p = qp$. Taking the adjoint, also $p = pq$.

On the other hand, because $(G_1 + \Delta_2)q = 0$, we have $G_1 q = -\Delta_2 q$. Using the recurrence relation (5.25), it follows that $G_k q = (-1)^k \Delta_2^k q$ for all $k \geq 0$. It then follows that

$$pq = \sum_{k=0}^{\infty} (z_2 - \Delta_2)\Delta_2^k \, q = \mathbb{1}_{(0,1)}(\Delta_2)\, q = q \; .$$

We already proved that $pq = p$, so that $p = q$ and (5.26) is proved.

From (5.26), we get for all $\xi \in \mathcal{H}$ that

$$(\ell(\xi) + \ell(J(\xi))^*)\, q\, \Omega = (\ell(\xi z_2) + \ell(J(\xi z_2))^*)\, q\, \Omega = 0 \; .$$

So, for all $x \in M$, we have that $xq = E_A(x)q$. Taking the adjoint, also $qx = qE_A(x)$ for all $x \in M$. Since q commutes with A, it follows that $q \in \mathcal{Z}(M)$ and $Mq = Aq$. From (5.26), we also get that $E_A(q) = z_2 - \Delta_2$ and thus $E_A(q) = Z(\Delta_H^\ell)$ where $Z\colon (0, +\infty) \to \mathbb{R}$ is defined as in the formulation of the theorem. So, $E_A(1 - q) = z + \Delta_2$ and this operator has support equal to 1. Statement (c) of the theorem is now proved.

We next prove that

(10) $\mathcal{Z}(M)(z_2 - q) \subset \mathcal{Z}(A)(z_2 - q)$.

Take $x \in \mathcal{Z}(M)$ and write

$$x z_2 \Omega = \sum_{n=0}^{\infty} \zeta_n \quad \text{with} \;\; \zeta_n \in z_2 H^n \; .$$

Using (8), take $a \in \mathcal{Z}(A)z$ such that $xz = a$. Also write $a_0 = E_A(xz_2)$ and note that $\zeta_0 = a_0 \Omega$.

Since $z_2 \mathcal{H} z_2 = 0$, we have $z_2 \mathcal{H} = z_2 \mathcal{H} z$ and we get, for every $\xi \in \mathcal{H}$, that

$$\begin{aligned}\sum_{n=0}^{\infty} \big(\ell(\xi)^* + \ell(J(\xi))\big)\, \zeta_n &= \big(\ell(\xi)^* + \ell(J(\xi))\big)\, x z_2 \Omega \\ &= x\, \big(\ell(\xi)^* + \ell(J(\xi))\big)\, z_2 \Omega \\ &= x\, J(z_2 \xi) = xz\, J(z_2\xi) = a\, J(z_2 \xi) \; .\end{aligned}$$

Comparing the components in H^n for all $n \geq 0$, we find that

$$\ell(\xi)^* \zeta_1 = 0 \quad , \quad \ell(\xi)^* \zeta_2 = a\, J(\xi) - J(\xi)\, a_0 \quad , \quad \ell(\xi)^* \zeta_{n+1} = -J(\xi) \otimes_A \zeta_{n-1}$$

for all $\xi \in z_2\mathcal{H}$ and all $n \geq 2$. Since $\zeta_n \in z_2 H^n$ for all n, it first follows that $\zeta_1 = 0$ and then inductively, that $\zeta_n = 0$ for all odd n.

Next, we get that $\zeta_2 = s_a - s a_0$, where

$$s_a := \sum_{i \in I} \mu_i \otimes_A aJ(\mu_i)$$

is a well-defined A-central vector in $z_2 H^2 z_2$.

Before continuing the proof, we give another expression for s_a. For all $\mu, \mu' \in z_2\mathcal{H} = z_2\mathcal{H}z$, we have that $W(J(\mu), \mu') \in zMz$. Since $xz = a$ and $x \in \mathcal{Z}(M)$, it follows that a commutes with $W(J(\mu), \mu')$. This means that

$$a\, J(\mu) \otimes_A \mu' = J(\mu) \otimes_A \mu'\, a \quad \text{for all} \;\; \mu, \mu' \in z_2\mathcal{H} \; .$$

It follows that $a\, J(\mu) \otimes_A s = J(\mu) \otimes_A s_a$ for all $\mu \in z_2\mathcal{H}$. Defining the normal completely positive map $\varphi \colon Az \to Az_2$ given by

$$\varphi(b) = \sum_{i \in I} \langle J(\mu_i), b\, J(\mu_i) \rangle_A \quad \text{for all} \;\; b \in Az \; ,$$

we get that $\varphi(a)\, s = \Delta_2\, s_a$. Since $\varphi(z) = \Delta_2$, there is a unique normal completely positive map $\psi \colon Az \to Az_2$ such that $\psi(b)\Delta_2 = \varphi(b)$ for all $b \in Az$. We conclude that $s_a = \psi(a)\, s = s\, \psi(a)$.

Writing $a_1 = \psi(a) - a_0$, we get that $\zeta_2 = s\, a_1$. We then conclude that $\zeta_{2n} = (-1)^{n+1}\, s_n\, a_1$ for all $n \geq 1$. Define the spectral projection $r = \mathbb{1}_{\{1\}}(\Delta_2)$. Since

$$\langle \zeta_{2n}, \zeta_{2n} \rangle_A = a_1^* \, \langle s_n, s_n \rangle_A \, a_1 = a_1^* \, \Delta_2^n \, a_1 \; ,$$

we get that $\|\zeta_{2n} r\| = \|a_1 r\|_2$ for all n. Since $\sum_n \|\zeta_{2n} r\|^2 < \infty$, we conclude that $a_1 r = 0$ and thus $xr \in A$.

Using (5.26), it follows that $x(z_2 - \Delta_2) = q a_1 + a_2$ for some element $a_2 \in A$. Since $xr \in A$, it follows that $x(z_2 - q) \in A(z_2 - q)$. Since the support of $E_A(z_2 - q)$ equals z_2, it follows that (10) holds.

Using (8) and (10), to conclude the proof of statement (d), it suffices to prove that for any $a \in \mathcal{Z}(A)$, we have $a(1-q) \in \mathcal{Z}(M)$ if and only if $a \in C$, where C is defined in the formulation of the theorem. This follows immediately by expressing the commutation with $\ell(\xi) + \ell(J(\xi))^*$ for all $\xi \in \mathcal{H}$ and using that $\bigl(\ell(\xi) + \ell(J(\xi))^*\bigr)\, q = 0$, as shown above.

Let ω be an M-central state on $\langle M, e_A\rangle$ that is normal on M. To conclude the proof of statement (a), we have to show that $\omega(1-q) = 0$. By (9), we already know that $\omega(z) = 0$. With $\mu_i \in z_2\mathcal{H} = z_2\mathcal{H}z$ as above, define $y_i := \ell(\mu_i) + \ell(J(\mu_i))^*$. Note that $y_i \in z_2Mz$ and that $G_1 + \Delta_2 = \sum_i y_i y_i^*$. By M-centrality and normality of ω on M, and because $y_i^* y_i \in zMz$, we get that $\omega(G_1 + \Delta_2) = 0$. So, $\omega(z_2 - q) = 0$ since $z_2 - q$ is the support projection of $G_1 + \Delta_2$. Since we already know that $\omega(z) = 0$, we conclude that $\omega(1-q) = 0$.

It remains to prove statement (b). Assume that $s \in \mathcal{Z}(M)(1-q)$ is a nonzero projection and that $B \subset Ms$ is a Cartan subalgebra. Since $\mathcal{N}_{Ms}(B)'' = Ms$, a combination of statement (a) and Theorem 5.13 implies that $B \prec_M A(1-q)$. The A-subbimodule $z_2H = z_2Hz$ of $L^2(M)$ has finite right A-dimension equal to $\tau(\Delta_2)$ and realizes a full intertwining of $A(z_2-q)$ into Az. It then follows that $B \prec_M Az$.

By Theorem 2.12, we can take projections $q_1 \in B$, $p \in Az$, a faithful normal unital $*$-homomorphism $\theta \colon Bq_1 \to pAp$ and a nonzero partial isometry $v \in q_1Mp$ such that $bv = v\theta(b)$ for all $b \in Bq_1$. Since $B \subset Ms$ is maximal abelian, we may assume that $vv^* = q_1$. By [Io11, Lemma 1.5], we may assume that $B_0 := \theta(Bq_1)$ is a maximal abelian subalgebra of pAp. Write $q_2 = v^*v$ and note that $q_2 \in B_0' \cap pMp$. We may assume that the support projection of $E_A(q_2)$ equals p.

Since $z = z_0 + z_1$, at least one of the projections pz_0, pz_1 is nonzero. Since we can cut down everything with the projections z_0 and z_1, we may assume that either $p \leq z_0$ or $p \leq z_1$.

Proof in the case where $p \leq z_0$. Recall that we denoted by $K \subset H$ the largest A-subbimodule that is left weakly mixing and that z_0 is the left support of K. First assume that the B_0-A-bimodule pK is left weakly mixing. Define the orthogonal decomposition of the pAp-bimodule $pL^2(M)p$ given by $pL^2(M)p = U_1 \oplus U_2$ with

$$U_1 = \bigoplus_{n=0}^{\infty} pKH^n p \quad \text{and} \quad U_2 = L^2(pAp) \oplus \bigoplus_{n=0}^{\infty} p(H \ominus K)H^n p \; .$$

We claim that $v^*\mathcal{N}_{q_1Mq_1}(Bq_1)v \subset U_2$. To prove this claim, take $u \in \mathcal{N}_{q_1Mq_1}(Bq_1)$ and write $u^*bu = \alpha(b)$ for all $b \in Bq_1$. Put $x = v^*uv$ and denote by y the orthogonal projection of x onto U_1. Since U_1 is a pAp-subbimodule of $pL^2(M)p$, we get that y is a right pAp-bounded vector in U_1 and that $\theta(b)y = y\theta(\alpha(b))$ for all $b \in Bq$. Since the B_0-A-bimodule pK is left weakly mixing, also U_1 is left weakly mixing as a B_0-pAp-bimodule. So, we can take a sequence of unitaries $b_n \in \mathcal{U}(Bq_1)$ such that $\lim_n \|\langle \theta(b_n)y, y\rangle_{pAp}\|_2 = 0$. But,

$$\langle \theta(b_n)y, y\rangle_{pAp} = \langle y\theta(\alpha(b_n)), y\rangle_{pAp} = \theta(\alpha(b_n)^*)\, \langle y, y\rangle_{pAp} \; .$$

Since $\theta(\alpha(b_n))$ is a unitary in B_0, we have $\|\theta(\alpha(b_n)^*)\,\langle y,y\rangle_{pAp}\|_2 = \|\langle y,y\rangle_{pAp}\|_2$ for all n. We conclude that $y=0$ and thus $v^*uv \in U_2$. Since the linear span of $\mathcal{N}_{q_1Mq_1}(Bq_1)$ is $\|\cdot\|_2$-dense in q_1Mq_1, we get that $q_2Mq_2 \subset U_2$.

Again consider the von Neumann subalgebra $N \subset z_0Mz_0$ introduced in (5.18). Since

$$P_{pL^2(N)p}(U_2) \subset L^2(pAp)\;,$$

we get that $E_{pNp}(q_2Mq_2) \subset pAp$. Denote by $N_0 \subset pNp$ the von Neumann algebra generated by the subspace $E_{pNp}(q_2Mq_2)$. So, $N_0 \subset pAp$. In particular, $E_N(q_2) \in A$, so that $E_N(q_2) = E_A(q_2)$ and thus, $E_N(q_2)$ has support p. By [Io11, Lemma 1.6] combined with Proposition 5.27, the inclusion $N_0 \subset pNp$ is essentially of finite index in the sense of Definition 5.26. A fortiori, $pAp \subset pNp$ is essentially of finite index. This contradicts the left weak mixing of the N-A-bimodule $L^2(N)$ that we obtained in (3).

Next assume that the B_0-A-bimodule pK is not left weakly mixing and take a nonzero B_0-A-subbimodule $K_1 \subset pK$ that is finitely generated as a right Hilbert A-module. Denote by $z_0' \in \mathcal{Z}(B_0)$ the support projection of the left action of B_0 on K_1. Since $K_1 \neq \{0\}$, also $z_0' \neq 0$. Since the support of $E_A(q_2)$ equals p, we get that $E_A(q_2z_0') = E_A(q_2)z_0' \neq 0$. So, $q_2z_0' \neq 0$ and we can cut down everything by z_0' and assume that the left B_0 action on K_1 is faithful.

Put $P = \mathcal{N}_{pAp}(B_0)''$. Whenever $u \in \mathcal{N}_{q_1Mq_1}(Bq_1)$ with $ubu^* = \alpha(b)$ for all $b \in Bq_1$, we have $E_A(v^*uv)\theta(b) = \theta(\alpha(b))E_A(v^*uv)$ for all $b \in Bq_1$. Since $B_0 \subset pAp$ is maximal abelian, it follows that $E_A(v^*uv) \in P$. So, $E_A(q_2Mq_2) \subset P$. From [Io11, Lemma 1.6] combined with Proposition 5.27, we conclude that the inclusion $P \subset pAp$ is essentially of finite index in the sense of Definition 5.26. So, all conditions of Lemma 5.42 are satisfied and we can choose a diffuse abelian von Neumann subalgebra $D \subset B_0' \cap pMp$ that is in tensor product position with respect to B_0. Since $Bq_1 \subset q_1Mq_1$ is maximal abelian, also $B_0q_2 \subset q_2Mq_2$ is maximal abelian. So, $q_2(B_0' \cap pMp)q_2 = B_0q_2$, contradicting Lemma 5.43.

Proof in the case where $p \leq z_1$. As proven above, we can find projections $e_1 \in \mathcal{Z}(A)z_1$ that lie arbitrarily close to z_1 and for which there exists an A-subbimodule $L \subset z_1H$ with the following properties: the left support of L equals e_1, L is finitely generated as a right Hilbert A-module, Δ_L^ℓ is bounded and $\Delta_L^\ell \geq e_1$. Taking e_1 close enough to z_1 and cutting down with e_1, we may assume that $p \leq e_1$. By Lemma 5.38, we can choose a diffuse abelian von Neumann subalgebra $D \subset (Ae_1)' \cap e_1Me_1$ that is in tensor product position with respect to Ae_1. Then $Dp \subset B_0' \cap pMp$ and Dp is in tensor product position with respect to B_0. Since Dp is diffuse abelian and $q_2 \in B_0' \cap pMp$ is a projection satisfying $q_2(B_0' \cap pMp)q_2 = B_0q_2$, this again contradicts Lemma 5.43. □

5.5 Compact groups, free subsets, c_0 probability measures and the proof of Theorem B

For every second countable compact group K with Haar probability measure μ and for every symmetric probability measure ν on K, we consider $A = L^\infty(K, \mu)$, the A-bimodule $H_\nu = L^2(K \times K, \mu \times \nu)$ with A-actions given by

$$(F \cdot \xi \cdot G)(x, y) = F(xy)\, \xi(x, y)\, G(x) \quad \text{for all } F, G \in A \text{ and } \xi \in H_\nu \;, \tag{5.27}$$

and the symmetry $J_\nu \colon H_\nu \to H_\nu$ given by

$$(J_\nu \xi)(x, y) = \overline{\xi(xy, y^{-1})} \quad \text{for all } \; x, y \in K \; . \tag{5.28}$$

We put $M_\nu = \Gamma(H_\nu, J_\nu, A, \mu)''$.

In Proposition 5.47 below, we characterize when the bimodule H_ν is mixing (so that M_ν becomes strongly solid by Corollary 5.14) and when $A \subset M_\nu$ is an s-MASA. For the latter, the crucial property will be that the support S of ν is of the form $S = F \cup F^{-1}$ where $F \subset K$ is a closed subset that is *free* in the following sense.

Definition 5.45. A subset F of a group G is called *free* if

$$g_1^{\varepsilon_1} \cdots g_n^{\varepsilon_n} \neq e$$

for all nontrivial *reduced words*, i.e., for all $n \geq 1$ and all $g_1, \ldots, g_n \in F$, $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$ satisfying $\varepsilon_i = \varepsilon_{i+1}$ whenever $1 \leq i \leq n-1$ and $g_i = g_{i+1}$.

On the other hand, the mixing property of H_ν will follow from the following c_0 condition on the measure ν.

Whenever K is a compact group, we denote by $\lambda \colon K \to \mathcal{U}(L^2(K))$ the left regular representation. For every probability measure ν on K and every unitary representation $\pi \colon K \to \mathcal{U}(H)$, we denote

$$\pi(\nu) = \int_K \pi(x) \, \mathrm{d}\nu(x) \in B(H) \; .$$

Definition 5.46. A probability measure ν on a compact group K is said to be c_0 if the operator $\lambda(\nu) \in B(L^2(K))$ is compact.

We denote by $\mathrm{Irr}(K)$ the set of equivalence classes of the irreducible representations of K, and we denote by $\epsilon \in \mathrm{Irr}(K)$ the equivalence class of the trivial representation. Since the regular representation of K decomposes as the direct sum of all irreducible representations $\pi \in \mathrm{Irr}(K)$, each appearing with multiplicity equal to its dimension, we get that a probability measure ν is c_0 if and only if

$$\lim_{\pi \in \mathrm{Irr}(K), \pi \to \infty} \|\pi(\nu)\| = 0 ,$$

i.e., if and only if the map $\mathrm{Irr}(K) \to \mathbb{R} : \pi \mapsto \|\pi(\nu)\|$ is c_0. In particular, when K is an abelian compact group, a probability measure ν on K is c_0 if and only if the Fourier transform of ν is a c_0 function on $\widehat{K}$.

Proposition 5.47. *Let K be a second countable compact group with Haar probability measure μ. Put $A = L^\infty(K, \mu)$. Let ν be a symmetric probability measure on K without atoms. Define the A-bimodule H_ν with symmetry J_ν by* (5.27) *and* (5.28)*. Denote by $M_\nu = \Gamma(H_\nu, J_\nu, A, \mu)''$ the associated tracial von Neumann algebra. Let S be the support of ν, i.e., the smallest closed subset of K with $\nu(S) = 1$.*

(1) *The bimodule H_ν is weakly mixing, $A \subset M_\nu$ is a singular MASA, M_ν has no Cartan subalgebra and $A \subset M_\nu$ is a maximal amenable subalgebra.*

(2) *The von Neumann algebra M_ν has no amenable direct summand. The center $\mathcal{Z}(M_\nu)$ of M_ν equals $L^\infty(K/K_0)$ where $K_0 \subset K$ is the closure of the subgroup generated by S. So if S topologically generates K, then M_ν is a non-amenable II_1 factor.*

(3) *If S is of the form $S = F \cup F^{-1}$ where $F \subset K$ is a closed subset that is free in the sense of Definition 5.45, then $A \subset M_\nu$ is an s-MASA.*

(4) *If ν is c_0 in the sense of Definition 5.46, then the bimodule H_ν is mixing and thus M_ν is strongly solid.*

Proof. 1. Note that

$$H_\nu^{\otimes_A^n} \cong L^2(K \times \underbrace{K \times \cdots \times K}_{n \text{ times}}, \mu \times \underbrace{\nu \times \cdots \times \nu}_{n \text{ times}}) \tag{5.29}$$

with the A-bimodule structure given by

$$(F \cdot \xi \cdot G)(x, y_1, \ldots, y_n) = F(xy_1 \cdots y_n)\, \xi(x, y_1, \ldots, y_n)\, G(x) .$$

Indeed, we can define an A-bimodular isometry $\Phi\colon H_\nu^{\otimes_A^n} \to L^2(K \times K^n, \mu \times \nu^n)$ by

$$\Phi(\xi_1 \otimes_A \cdots \otimes_A \xi_n)(x, y_1, \ldots, y_n) = \prod_{i=1}^{n} \xi_i(xy_1 \cdots y_{n-i}, y_{n-i+1}).$$

Define $D \subset K \times K$ given by $D = \{(y, y^{-1}) \mid y \in K\}$. Since ν has no atoms, we have $(\nu \times \nu)(D) = 0$. It then follows that $H_\nu \otimes_A H_\nu$ has no nonzero A-central vectors. By Proposition 2.26, the A-bimodule H_ν is weakly mixing, so also $L^2(M_\nu) \ominus L^2(A)$ is a weakly mixing A-bimodule. Let $a_n \in \mathcal{U}(A)$ be a sequence of unitaries such that $\|\langle a_n \xi, \eta \rangle_A\|_2 \to 0$ for all left and right bounded vectors $\xi, \eta \in L^2(M_\nu) \ominus L^2(A)$. For $x \in \mathcal{N}_{M_\nu}(A)$, we have that

$$\|\langle a_n x, x \rangle_A\|_2 = \|\langle x^* a_n x \Omega, \Omega \rangle_A\|_2 = \|x^* a_n x\|_2 = 1.$$

This implies that $x \in A$ by the choice of $(a_n)_{n \in \mathbb{N}}$. So, we have that $\mathcal{N}_{M_\nu}(A) \subset A$. Hence $A \subset M_\nu$ is a MASA and this MASA is singular. By Theorem 5.21, M_ν has no Cartan subalgebra. By Theorem 5.19, we get that $A \subset M_\nu$ is a maximal amenable subalgebra.

2. Since H_ν is weakly mixing, we get from Theorem 5.19 that M_ν has no amenable direct summand and that $\mathcal{Z}(M_\nu)$ consists of all $a \in A$ satisfying $a \cdot \xi = \xi \cdot a$ for all $\xi \in H_\nu$. It is then clear that $L^\infty(K/K_0) \subset \mathcal{Z}(M_\nu)$. To prove the converse, fix $a \in A$ with $a \cdot \xi = \xi \cdot a$ for all $\xi \in H_\nu$. We find in particular that $a(xy) = a(x)$ for $\mu \times \nu$-a.e. $(x, y) \in K \times K$. Let $\mathcal{U}_n$ be a decreasing sequence of basic neighborhoods of e in K. Define the functions b_n given by

$$b_n(y) = \mu(\mathcal{U}_n)^{-1} \int_{\mathcal{U}_n} a(xy)\, d\mu(x)\ .$$

For every fixed n, the functions b_n still satisfy $b_n(xy) = b(x)$ for $\mu \times \nu$-a.e. $(x, y) \in K \times K$. But the functions b_n are continuous. It follows that $b_n(xy) = b_n(x)$ for all $x \in K$ and all $y \in S$. So, $b_n \in C(K/K_0)$. Since $\lim_n \|b_n - a\|_1 = 0$, we get that $a \in L^\infty(K/K_0)$.

3. Denote by $W_n \subset (F \cup F^{-1})^n$ the subset of reduced words of length n. Since ν has no atoms, we find that $\nu^n(W_n) = 1$. Denote by $\pi_n \colon K^n \to K$ the multiplication map and put $S_n := \pi_n(W_n)$. Since F is free, the subsets $S_n \subset K$ are disjoint. By freeness of F, we also have that the restriction of π_n to W_n is injective. Define the probability measures $\nu_n := (\pi_n)_*(\nu^n)$ and then $\eta = \frac{1}{2}\delta_0 + \sum_{n=1}^\infty 2^{-n-1}\nu_n$. Using (5.29), we get that $H^{\otimes_A^n} \cong L^2(K \times S_n, \mu \times \nu_n)$ for all $n \geq 1$. Since the $S_n \subset K$ are disjoint, it follows that ${}_A L^2(M_\nu)_A$ is isomorphic with the A-bimodule

$$L^2(K \times K, \mu \times \eta) \quad \text{with} \quad (F \cdot \xi \cdot G)(x, y) = F(xy)\, \xi(x, y)\, G(x)\ .$$

So, ${}_A L^2(M_\nu)_A$ is a cyclic bimodule and $A \subset M_\nu$ is an s-MASA.

4. Define $\xi_0 \in H_\nu$ by $\xi_0(x, y) = 1$ for all $x, y \in K$ and note that ξ_0 is a cyclic vector for ${}_A(H_\nu)_A$. Denote by $\varphi \colon A \to A$ the completely positive map given by $\varphi(a) = \langle \xi_0, a\xi_0 \rangle_A$. Note that as a function in $L^\infty(K, \mu)$, we have that

$\varphi(a)(x) = \int_K a(xy)\,\mathrm{d}\nu(y)$ for $x \in K$. To prove that H_ν is mixing, it is sufficient to prove that $\lim_n \|\varphi(a_n)\|_2 = 0$ whenever (a_n) is a bounded sequence in A that converges weakly to 0. Denoting by $\rho\colon K \to L^2(K)$ the right regular representation, we get that $\varphi(a) = \rho(\nu)(a)$ for all $a \in A \subset L^2(K)$. Since $\rho(\nu)$ is a compact operator, we indeed get that $\lim_n \|\rho(\nu)(a_n)\|_2 = 0$. So, H_ν is a mixing A-bimodule. By Corollary 5.14, M_ν is strongly solid. □

Remark 5.48. In the special case where K is abelian, we identify $L^\infty(K,\mu) = L(G)$, with $G := \widehat{K}$ being a countable abelian group. Then the symmetric $L^\infty(K,\mu)$-bimodule H_ν given by (5.27) and (5.28) is isomorphic with the symmetric $L(G)$-bimodule associated, as in Remark 5.11, with the cyclic orthogonal representation of G with spectral measure ν. In particular, as in Remark 5.11, the von Neumann algebras $M_\nu = \Gamma(H_\nu, J_\nu, L^\infty(K),\mu)''$ can also be realized as a free Bogoljubov crossed product by the countable abelian group G. In this way, Proposition 5.47 generalizes the results of [HS09, Ho12a]. Note however that for a free Bogoljubov crossed product $M = \Gamma(K_\mathbb{R})'' \rtimes G$ with G abelian, the subalgebra $L(G) \subset M$ is *never* an s-MASA. Indeed, if we let $\pi\colon G \to \mathcal{O}(K_\mathbb{R})$ be an orthogonal representation, $A = L(G)$ and $H = \ell^2(G) \otimes K$ the associated A-bimodule, then $H \otimes_A H \cong \ell^2(G) \otimes K \otimes K$ is the bimodule associated with $\pi \otimes \pi$. Let $F \in B(H \otimes_A H)$ denote the operator that flips the two copies of K and view F as an operator on $L^2(M)$. Then F commutes with the left and right A-actions but $F \notin A \vee JAJ$. So, $A \vee JAJ \subset B(L^2M)$ is not maximal abelian and hence A is not an s-MASA. This shows that our more general construction is essential to prove Theorem B.

For non-abelian compact groups K, we can still view $K = \widehat{G}$, but G is no longer a countable group, rather a discrete Kac algebra. It is then still possible to identify the II_1 factors M in Proposition 5.47 with a crossed product $\Gamma(K_\mathbb{R})'' \rtimes G$, where the discrete Kac algebra action of G on $\Gamma(K_\mathbb{R})''$ is the free Bogoljubov action associated in [Va02] with an orthogonal co-representation of the quantum group G.

The main result of this section says that in certain sufficiently non-abelian compact groups K, one can find "large" free subsets $F \subset K$, where "large" means that F carries a non-atomic probability measure that is c_0. We conjecture that the compact Lie groups $\mathrm{SO}(n)$, $n \geq 3$, admit free subsets carrying a c_0 probability measure. For our purposes, it is however sufficient to prove that these exist in more ad hoc groups.

For every prime number p, denote by Γ_p the finite group $\Gamma_p = \mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$. The following is the main result of this section. Recall that the support of a probability measure ν on a compact space K is defined as the smallest closed subset $S \subset K$ with $\nu(S) = 1$.

Theorem 5.49. *There exists a sequence of prime numbers p_n tending to infinity, a closed free subset $F \subset K := \prod_{n=1}^{\infty} \Gamma_{p_n}$ topologically generating K and a symmetric, non-atomic, c_0 probability measure ν on K whose support equals $F \cup F^{-1}$.*

We then immediately get:

Proof of Theorem B. Take K and ν as in Theorem 5.49. Denote by M_ν the associated von Neumann algebra with abelian subalgebra $A \subset M_\nu$ as in Proposition 5.47. By Proposition 5.47, we get that M_ν is a non-amenable, strongly solid II_1 factor and that $A \subset M_\nu$ is an s-MASA. □

Before proving Theorem 5.49, we need some preparation.

The Alon-Roichman theorem [AR92] asserts that the Cayley graph given by a random and independent choice of $k \geq c(\varepsilon) \log |G|$ elements in a finite group G has expected second eigenvalue at most ε, with the normalization chosen so that the largest eigenvalue is 1. In [LR04, Theorem 2], a simple proof of that result was given. The same proofs yields the following result. For completeness, we provide the argument.

Whenever G is a group, $\pi\colon G \to \mathcal{U}(H)$ is a unitary representation and $g_1, \ldots, g_k \in G$, we write

$$\pi(g_1, \ldots, g_k) := \frac{1}{k} \sum_{j=1}^{k} \pi(g_j) \ . \tag{5.30}$$

Lemma 5.50 ([LR04])**.** *Let G_n be a sequence of finite groups and k_n a sequence of positive integers such that $k_n / \log |G_n| \to \infty$. For every $\varepsilon > 0$ and for a uniform and independent choice of k_n elements $g_1, \ldots, g_{k_n} \in G_n$, we have that*

$$\lim_{n \to \infty} \mathrm{P}\Big(\; \|\pi(g_1, \ldots, g_{k_n})\| \leq \varepsilon \;\; \textit{for all} \;\; \pi \in \mathrm{Irr}(G_n) \setminus \{\epsilon\} \; \Big) = 1 \ .$$

Proof. Fix a finite group G and a positive integer k. Let $g_1, \ldots, g_k$ be a uniform and independent choice of elements of G. Denote by $\lambda_0\colon G \to \mathcal{U}(\ell^2(G) \ominus \mathbb{C}1)$ the regular representation restricted to $\ell^2(G) \ominus \mathbb{C}1$. Put $d = |G| - 1$. Both

$$T(g_1, \ldots, g_k) = \frac{1}{k} \sum_{j=1}^{k} \frac{\lambda_0(g_j) + \lambda_0(g_j)^*}{2} \quad \text{and}$$

$$S(g_1, \ldots, g_k) = \frac{1}{k} \sum_{j=1}^{k} \frac{i\lambda_0(g_j) - i\lambda_0(g_j)^*}{2}$$

are sums of k independent self-adjoint $d \times d$ matrices of norm at most 1 and having expectation 0. We apply [AW01, Theorem 19] to the independent random variables

$$X_j = \frac{2 + \lambda_0(g_j) + \lambda_0(g_j)^*}{4} \; ,$$

satisfying $0 \leq X_j \leq 1$ and having expectation $1/2$. We conclude that for every $0 \leq \varepsilon \leq 1/2$,

$$\begin{aligned} \mathrm{P}\big(\|T(g_1,\ldots,g_k)\| \leq \varepsilon \big) &= \mathrm{P}\Big((1-\varepsilon)\frac{1}{2} \leq \frac{1}{k}\sum_{j=1}^k X_j \leq (1+\varepsilon)\frac{1}{2} \Big) \\ &\geq 1 - 2d \, \exp\big(-k \frac{\varepsilon^2}{4 \log 2}\big) \; . \end{aligned}$$

The same estimate holds for $S(g_1,\ldots,g_k)$. Since $\lambda_0(g_1,\ldots,g_k) = T(g_1,\ldots,g_k) - iS(g_1,\ldots,g_k)$ and since λ_0 is the direct sum of all nontrivial irreducible representations of G (all appearing with multiplicity equal to their dimension), we conclude that

$$\mathrm{P}\big(\|\pi(g_1,\ldots,g_k)\| \leq \varepsilon \;\text{ for all }\; \pi \in \mathrm{Irr}(G) \setminus \{\epsilon\} \big) \geq 1 - 4|G| \, \exp\big(-k \frac{\varepsilon^2}{16 \log 2}\big) \; .$$

Taking $G = G_n$, $k = k_n$ and $n \to \infty$, our assumption that $k_n / \log |G_n| \to \infty$ implies that for every fixed $\varepsilon > 0$,

$$|G_n| \, \exp\big(-k_n \frac{\varepsilon^2}{16 \log 2}\big) \to 0$$

and thus the lemma follows. □

On the other hand, in [GHSSV07] it is proven that random Cayley graphs of the groups $\mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ have large girth. More precisely, we say that elements $g_1,\ldots,g_k$ in a group G satisfy no relation of length $\leq \ell$ if every nontrivial reduced word of length at most ℓ with letters from $g_1^{\pm 1},\ldots,g_k^{\pm 1}$ defines a nontrivial element in G. The estimates in the proof of [GHSSV07, Lemma 10] give the following result. Again for completeness, we provide the argument.

Lemma 5.51 ([GHSSV07]). *Let p_n be a sequence of prime numbers tending to infinity and let k_n be a sequence of positive integers such that $\log k_n / \log p_n \to 0$. Put $\Gamma_{p_n} = \mathrm{PGL}_2(\mathbb{Z}/p_n\mathbb{Z})$. For every $\ell > 0$ and for a uniform and independent choice of k_n elements $g_1,\ldots,g_{k_n} \in \Gamma_{p_n}$, we have that*

$$\lim_{n\to\infty} \mathrm{P}\Big(g_1,\ldots,g_{k_n} \;\; \textit{satisfy no relation of length} \leq \ell \Big) = 1 \; .$$

Proof. Let G be a group. A *law* of length ℓ in G is a nontrivial element w in a free group $\mathbb{F}_n$ such that w has length ℓ and $w(g_1,\ldots,g_n) = e$ for all $g_1,\ldots,g_n \in G$. For example, if G is abelian, the element $w = aba^{-1}b^{-1}$ of $\mathbb{F}_2$ defines a law of length 4 in G. Since the labeling of the generators does not matter, any law of length ℓ can be defined by a nontrivial element of $\mathbb{F}_n$ with $n \leq \ell$. In particular, there are only finitely many possible laws of a certain length ℓ.

Since $\mathbb{F}_\infty \hookrightarrow \mathbb{F}_2 \hookrightarrow \mathrm{PSL}_2(\mathbb{Z})$, the group $\mathrm{PSL}_2(\mathbb{Z})$ satisfies no law. For every prime number p, write $\Gamma_p = \mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$. Using the quotient maps $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})$, we get that a given nontrivial element $w \in \mathbb{F}_n$ can be a law for at most finitely many Γ_p. So, for every $\ell > 0$, we get that Γ_p satisfies no law of length $\leq \ell$ for all large enough primes p. (Note that [GHSSV07, Proposition 11] provides a much more precise result.)

Let $w = g_{i_1}^{\varepsilon_1} \cdots g_{i_\ell}^{\varepsilon_\ell}$ with $i_j \in \{1,\ldots,k\}$ and $\varepsilon_j \in \{\pm 1\}$ be a reduced word of length ℓ in $g_1^{\pm 1},\ldots,g_k^{\pm 1}$. Let p be a prime number and assume that w is not a law of Γ_p. With the same argument as in the proof of [GHSSV07, Lemma 10], we now prove that for a uniform and independent choice of $g_1,\ldots,g_k \in \Gamma_p$, we have that

$$\mathrm{P}\big(w(g_1,\ldots,g_k) = e \text{ in } \Gamma_p \big) \leq \frac{\ell}{p} \big(1 + \frac{1}{p-1}\big)^{3k} . \tag{5.31}$$

Denote $F_p = \mathbb{Z}/p\mathbb{Z}$, not to be confused with the free group $\mathbb{F}_p$. Write $G_p = GL_2(F_p) \subset F_p^{2\times 2}$. Define the map

$$W\colon \big(F_p^{2\times 2}\big)^k \to F_p^{2\times 2} : \quad W(a_1,\ldots,a_k) = b_{i_1} \cdots b_{i_\ell}$$

where $b_{i_j} = a_{i_j}$ when $\varepsilon_j = 1$ and b_{i_j} equals the adjunct matrix of a_{i_j} when $\varepsilon_j = -1$. Note that the four components W_{st}, $s,t \in \{1,2\}$, of the map W are polynomials of degree at most ℓ in the $4k$ variables $a \in \big(F_p^{2\times 2}\big)^k$. Define the subset $\mathcal{W} \subset \big(F_p^{2\times 2}\big)^k$ given by

$$\begin{aligned}\mathcal{W} &= \big\{ a \in \big(F_p^{2\times 2}\big)^k \bigm| W(a) \text{ is a multiple of the identity matrix} \big\} \\ &= \big\{ a \in \big(F_p^{2\times 2}\big)^k \bigm| W_{11}(a) - W_{22}(a) = W_{12}(a) = W_{21}(a) = 0 \big\} .\end{aligned}$$

We also define $\mathcal{V} = \mathcal{W} \cap (G_p)^k$ and

$$\mathcal{U} = \{ g \in (\Gamma_p)^k \mid w(g_1,\ldots,g_k) = e \text{ in } \Gamma_p \} .$$

The quotient map $G_p \to \Gamma_p$ induces the $(p-1)^k$-fold covering $\pi\colon \mathcal{V} \to \mathcal{U}$.

The subset $\mathcal{W} \subset F_p^{4k}$ is the solution set of a system of three polynomial equations of degree at most ℓ. If each of these polynomials is identically zero, we get that

$\mathcal{W} = F_p^{4k}$ and thus $\mathcal{U} = (\Gamma_p)^k$. This means that w is a law of Γ_p, which we supposed not to be the case. So at least one of the polynomials is not identically zero. The number of zeros of such a polynomial is bounded above by ℓp^{4k-1} (and a better, even optimal, bound can be found in [Se89]). So, $|\mathcal{W}| \leq \ell p^{4k-1}$. Then also $|\mathcal{V}| \leq \ell p^{4k-1}$ and because π is a $(p-1)^k$-fold covering, we find that

$$|\mathcal{U}| \leq \ell\,(p-1)^{-k}\,p^{4k-1}\ .$$

Since $|\Gamma_p| = (p-1)\,p\,(p+1)$, we conclude that

$$\begin{aligned} \mathrm{P}\big(\,w(g_1,\ldots,g_k) = e \text{ in } \Gamma_p\,\big) &= \frac{|\mathcal{U}|}{|\Gamma_p|^k} \leq \frac{\ell}{p}\,(p-1)^{-2k}\,(p+1)^{-k}\,p^{3k} \\ &\leq \frac{\ell}{p}\,\big(1+\frac{1}{p-1}\big)^{3k}\ . \end{aligned}$$

So, (5.31) holds.

Now assume that p_n is a sequence of prime numbers and k_n are positive integers such that $p_n \to \infty$ and $\log k_n / \log p_n \to 0$. For all n large enough, $3k_n \leq p_n - 1$ and for all n large enough, as we explained in the beginning of the proof, Γ_{p_n} has no law of length $\leq \ell$. Since $(1+1/x)^x < 3$ for all $x > 0$ and since there are less than $(2k)^{\ell+1}$ reduced words of length $\leq l$ in $g_1^{\pm 1}, \ldots, g_k^{\pm 1}$, we find that for all n large enough and a uniform, independent choice of $g_1,\ldots,g_{k_n} \in \Gamma_{p_n}$, we have

$$\mathrm{P}\Big(\,g_1,\ldots,g_{k_n}\ \text{ satisfy a relation of length} \leq \ell \text{ in } \Gamma_{p_n}\,\Big) \leq (2k_n)^{\ell+1}\,\frac{3\ell}{p_n}\ .$$

By our assumption that $\log k_n / \log p_n \to 0$, the right hand side tends to 0 as $n \to \infty$ and the lemma is proved. □

Combining Lemmas 5.50 and 5.51, we obtain the following.

Lemma 5.52. *For all $\varepsilon > 0$ and all $k_0, p_0, \ell \in \mathbb{N}$, there exists a prime number $p \geq p_0$, an integer $k \geq k_0$ and elements $g_1,\ldots,g_k \in \Gamma_p = \mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ generating the group Γ_p such that*

(1) *$\|\pi(g_1,\ldots,g_k)\| \leq \varepsilon$ for every nontrivial irreducible representation $\pi \in \mathrm{Irr}(\Gamma_p)$,*

(2) *$g_1,\ldots,g_k$ satisfy no relation of length $\leq \ell$.*

Proof. Choose any sequence of prime numbers p_n tending to infinity. Define $k_n = \lfloor (\log p_n)^2 \rfloor$. Since $|\Gamma_{p_n}| = (p_n - 1)\,p_n\,(p_n+1)$, we get that $k_n / \log|\Gamma_{p_n}| \to$

∞. Also, $\log k_n / \log p_n \to 0$. So Lemmas 5.50 and 5.51 apply and for a large enough choice of n, properties (1) and (2) in the lemma hold for $p = p_n$, $k = k_n$ and a large portion of the k_n-tuples $(g_1, \ldots, g_{k_n}) \in \Gamma_{p_n}^{k_n}$.

The first property in the lemma is equivalent with

$$\Big\| \Big(\frac{1}{k} \sum_{j=1}^{k} \lambda(g_j) \Big)_{\ell^2(\Gamma_p) \ominus \mathbb{C}1} \Big\| \leq \varepsilon \; ,$$

where $\lambda \colon \Gamma_p \to \ell^2(\Gamma_p)$ is the regular representation. If $\varepsilon < 1$, it then follows in particular that there are no nonzero functions in $\ell^2(\Gamma_p) \ominus \mathbb{C}1$ that are invariant under all $\lambda(g_j)$, meaning that every element of Γ_p can be written as a product of elements in $\{g_1, \ldots, g_k\}$. So, we get that $g_1, \ldots, g_k$ generate Γ_p. □

Having proven Lemma 5.52, we are now ready to prove Theorem 5.49.

Proof of Theorem 5.49. As in (5.30), for every finite group G, subset $F \subset G$ and unitary representation $\pi \colon G \to \mathcal{U}(H)$, we write

$$\pi(F) := \frac{1}{|F|} \sum_{g \in F} \pi(g) \; .$$

For every prime number p, we write $\Gamma_p = \mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$. We construct by induction on n a sequence of prime numbers p_n and a generating set

$$F_n \subset K_n := \prod_{j=1}^{n} \Gamma_{p_j}$$

such that, denoting by $\theta_{n-1} \colon K_n \to K_{n-1}$ the projection onto the first $n-1$ coordinates, the following properties hold.

(1) $\theta_{n-1}(F_n) = F_{n-1}$ and the map $\theta_{n-1} \colon F_n \to F_{n-1}$ is an r_n-fold covering with $r_n \geq 2$.

(2) If $\pi \in \mathrm{Irr}(K_n)$ and π does not factor through θ_{n-1}, then $\|\pi(F_n)\| \leq 1/n$.

(3) The elements of F_n satisfy no relation of length $\leq n$.

Assume that $p_1, \ldots, p_{n-1}$ and $F_1, \ldots, F_{n-1}$ have been constructed. We have to construct p_n and F_n. Write $k_1 = |F_{n-1}|$ and put $k_0 = \max\{2n+1, k_1\}$. By Lemma 5.52, we can choose $k_2 > k_0$, a prime number p_n and a subset $F \subset \Gamma_{p_n}$ with $|F| = k_2$ such that the elements of F satisfy no relation of length $\leq 3n$ and such that $\|\pi(F)\| \leq 1/(4n)$ for every nontrivial irreducible representation π of Γ_{p_n}.

Write $F_{n-1} = \{g_1, \ldots, g_{k_1}\}$ and $F = \{h_1, \ldots, h_{k_2}\}$. Note that we have chosen $k_2 > \max\{2n+1, k_1\}$. So we can define the subset $F_n \subset K_{n-1} \times \Gamma_{p_n} = K_n$ given by

$$F_n = \{(g_i, h_i h_j h_i^{-1}) \mid 1 \leq i \leq k_1 \ , \ 1 \leq j \leq k_2 \ , \ i \neq j\} \ .$$

Note that $\theta_{n-1}(F_n) = F_{n-1}$ and that the map $\theta_{n-1} \colon F_n \to F_{n-1}$ is a (k_2-1)-fold covering.

Every irreducible representation $\pi \in \text{Irr}(K_n)$ that does not factor through θ_{n-1} is of the form $\pi = \pi_1 \otimes \pi_2$ with $\pi_1 \in \text{Irr}(K_{n-1})$ and with π_2 being a nontrivial irreducible representation of Γ_{p_n}. Note that

$$\pi(F_n) = \frac{1}{k_1} \sum_{i=1}^{k_1} \bigl(\pi_1(g_i) \otimes \pi_2(h_i)\, T_i\, \pi_2(h_i)^*\bigr) \ ,$$

where

$$T_i := \frac{1}{k_2 - 1} \sum_{1 \leq j \leq k_2\, ,\, j \neq i} \pi_2(h_j) \ .$$

For every fixed $i \in \{1, \ldots, k_1\}$, we have

$$T_i = \frac{k_2}{k_2 - 1} \pi_2(F) - \frac{1}{k_2 - 1} \pi_2(h_i) \ .$$

Therefore,

$$\|T_i\| < 2\, \|\pi_2(F)\| + \frac{1}{2n} \leq \frac{1}{n} \ . \tag{5.32}$$

It then also follows that $\|\pi(F_n)\| < 1/n$.

We next prove that F_n is a generating set of K_n. Fix $i \in \{1, \ldots, k_1\}$. For all $s, t \in \{1, \ldots, k_2\}$ with $s \neq i$ and $t \neq i$, we have

$$(g_i, h_i h_s h_i^{-1})\, (g_i, h_i h_t h_i^{-1})^{-1} = (e, h_i\, h_s h_t^{-1}\, h_i^{-1}) \ .$$

It thus suffices to prove that the set $H_i := \{h_s h_t^{-1} \mid s, t \in \{1, \ldots, k_2\} \setminus \{i\}\}$ generates Γ_{p_n} for each $i \in \{1, \ldots, k_1\}$.

Denote by λ_0 the regular representation of Γ_{p_n} restricted to $\ell^2(\Gamma_{p_n}) \ominus \mathbb{C}1$. Define

$$R_i = \frac{1}{k_2 - 1} \sum_{1 \leq j \leq k_2\, ,\, j \neq i} \lambda_0(h_j) \ .$$

By (5.32), we get that $\|R_i\| < 1$. Then also $\|R_i R_i^*\| < 1$. So, there is no nonzero function in $\ell^2(\Gamma_{p_n}) \ominus \mathbb{C}1$ that is invariant under all $\lambda(h)$, $h \in H_i$. It follows that each H_i is a generating set of Γ_{p_n}.

Denote by $\eta_n \colon K_n \to \Gamma_{p_n}$ the projection onto the last coordinate. If the elements of F_n satisfy any relation of length $\leq n$, applying η_n will give a nontrivial relation of length $\leq 3n$ between the elements of F. Since such relations do not exist, we have proved that the elements of F_n satisfy no relation of length $\leq n$.

Define $K = \prod_{n=1}^{\infty} \Gamma_{p_n}$ and still denote by $\theta_n \colon K \to K_n$ the projection onto the first n coordinates. Define

$$F = \{k \in K \mid \theta_n(k) \in F_n \text{ for all } n \geq 1\} .$$

Note that $F \subset K$ is closed and $\theta_n(F) = F_n$. Denoting by $\langle F \rangle$ the subgroup of K generated by F, we get that $\theta_n(\langle F \rangle) = K_n$ for all n. So, $\langle F \rangle$ is dense in K, meaning that F topologically generates K.

Since each map $\theta_{n-1} \colon F_n \to F_{n-1}$ is an r_n-fold covering, there is a unique probability measure ν_0 on K such that $(\theta_n)_*(\nu_0)$ is the normalized counting measure on F_n for each n. Since $r_n \geq 2$ for all n, we have that $|F_n| \to \infty$ and hence the measure ν_0 is non-atomic. Indeed, for any $x \in K$ and any $n \in \mathbb{N}$, we have that

$$\nu_0(\{x\}) \leq \nu_0(\theta_n^{-1}(\{\theta_n(x)\})) = \frac{1}{|F_n|} \to 0 .$$

Note that the support of ν_0 equals F. Define the symmetric probability measure ν on K given by $\nu(\mathcal{U}) = (\nu_0(\mathcal{U}) + \nu_0(\mathcal{U}^{-1}))/2$ for all Borel sets $\mathcal{U} \subset K$. The support of ν equals $F \cup F^{-1}$. Since $\lambda(\nu) = (\lambda(\nu_0) + \lambda(\nu_0)^*)/2$, to conclude the proof of the theorem, it suffices to prove that F is free and that ν_0 is a c_0 probability measure.

Let $g_1^{\varepsilon_1} \cdots g_m^{\varepsilon_m}$ be a reduced word of length m with $g_1, \ldots, g_m \in F$. Take $n \geq m$ large enough such that $\theta_n(g_i) \neq \theta_n(g_{i+1})$ whenever $g_i \neq g_{i+1}$. We then get that $\theta_n(g_1)^{\varepsilon_1} \cdots \theta_n(g_m)^{\varepsilon_m}$ is a reduced word of length $m \leq n$ in the elements of F_n. It follows that

$$e \neq \theta_n(g_1)^{\varepsilon_1} \cdots \theta_n(g_m)^{\varepsilon_m} = \theta_n\big(g_1^{\varepsilon_1} \cdots g_m^{\varepsilon_m}\big) .$$

So, $g_1^{\varepsilon_1} \cdots g_m^{\varepsilon_m} \neq e$ and we have proven that F is free.

We finally prove that $\|\pi(\nu_0)\| < 1/m$ for every irreducible representation π of K that does not factor through $\theta_m \colon K \to K_m$. Since there are only finitely many irreducible representations that do factor through $\theta_m \colon K \to K_m$, this will conclude the proof of the theorem. Let π be such an irreducible representation. There then exists a unique $n > m$ such that $\pi = \pi_0 \circ \theta_n$ and π_0 is an irreducible representation of K_n that does not factor through $\theta_{n-1} \colon K_n \to K_{n-1}$. Since $(\theta_n)_*(\nu_0)$ is the counting measure on F_n, get that $\pi(\nu_0) = \pi_0(F_n)$ and thus

$$\|\pi(\nu_0)\| = \|\pi_0(F_n)\| \leq \frac{1}{n} < \frac{1}{m} .$$

□

5.6 Property Gamma

Given a tracial von Neumann algebra (M, τ), a *central sequence* for M is a $\|\cdot\|$-bounded sequence $x_n \in M$ that asymptotically commutes with M in the sense that

$$\lim_{n\to\infty} \|x_n y - y x_n\|_2 = 0 \quad \text{for all } \; y \in M.$$

The central sequence $(x_n)_{n\in\mathbb{N}}$ is said to be trivial if $\lim_{n\to\infty} \|x_n - \tau(x_n)1\|_2 = 0$.

A separable II_1 factor M admits a nontrivial central sequence if and only if there exists a central sequence consisting of unitaries $u_n \in \mathcal{U}(M)$ with $\tau(u_n) = 0$ for all n (see for instance [AP16, Theorem 15.2.3]). This property is known as *property Gamma* and was first introduced by Murray and von Neumann [MvN43] in order to distinguish between the hyperfinite II_1 factor R and the free group factors $L(\mathbb{F}_n)$, $n \geq 2$. Historically, this provided the first example of a non-hyperfinite factor.

A separable II_1 factor M is called a *full factor* if it does not have property Gamma, i.e., if it has no nontrivial central sequences. Connes showed [Co75] that being a full factor is equivalent with the following stronger property: the unitary representation $(\mathrm{Ad}\, u)_{u\in\mathcal{U}(M)}$ on $L^2(M) \ominus \mathbb{C}1$ given by $(\mathrm{Ad}\, u)\xi = u\xi u^*$ has spectral gap in the following sense.

Definition 5.53. Let (π, H) be a unitary representation of a group G. We say that π has *spectral gap* if it does not weakly contain the trivial representation, i.e., if π has no almost invariant vectors.

Having nontrivial central sequences can also be expressed in terms of the ultrapower von Neumann algebra. Indeed, a separable II_1 factor M has property Gamma if and only if $M' \cap M^\omega \neq \mathbb{C}1$ for any free ultrafilter ω on $\mathbb{N}$, where M^ω denotes the von Neumann algebra ultrapower of M with respect to ω. The von Neumann algebra $M' \cap M^\omega$ is called the *central sequence algebra* of M (with respect to ω).

If $A \subset M$ is a von Neumann subalgebra, we say that a central sequence $(x_n)_{n\in\mathbb{N}}$ for M *asymptotically lies in* A if $\|x_n - E_A(x_n)\|_2 \to 0$ as $n \to \infty$. If every central sequence of M asymptotically lies in A, we have that $M' \cap M^\omega \subset A' \cap A^\omega$, for any free ultrafilter ω on $\mathbb{N}$.

Fix a separable tracial von Neumann algebra (A, τ) and a symmetric A-bimodule (H, J). Put $M = \Gamma(H, J, A, \tau)''$. Our goal in this section is to locate the central sequences of M and in particular find a criterion for when M is a full factor. Since even factoriality of M is very difficult to characterize in general (see Theorem 5.21), we will restrict ourselves to the case where H is a weakly mixing bimodule. At the end of this section, we will apply this criterion to the factors M_ν that we constructed in Section 5.5, where ν is a symmetric probability measure on a compact group K.

In the case of free Bogoljubov actions, Houdayer showed the following.

Theorem 5.54 ([Ho12b, Theorem A])**.** *Let G be any countable discrete group and $\pi\colon G \to \mathcal{O}(H_\mathbb{R})$ any faithful orthogonal representation such that $\dim H_\mathbb{R} \geq 2$.*

If $\pi(G)$ is discrete in $\mathcal{O}(H_\mathbb{R})$ with respect to the strong operator topology, then $\Gamma(H_\mathbb{R})'' \rtimes G$ is a full factor.

We can rephrase the condition $\pi(G) \subset \mathcal{O}(H_\mathbb{R})$ being discrete in the following way: There exists no sequence $g_n \in G$ with $g_n \to \infty$ such that

$$\pi(g_n)\xi \to \xi \quad \text{for all } \ \xi \in H_\mathbb{R}.$$

The natural analogue of this in the setting of symmetric A-bimodules is the following: There exists no nontrivial sequence of unitaries $a_n \in \mathcal{U}(A)$ such that

$$\|a_n \xi a_n^* - \xi\| \to 0 \quad \text{for all } \ \xi \in H.$$

We prove that this is indeed a criterion for M being a full factor. The proof is based on the proof of [Ho12b, Theorem A], as well as techniques from the proof of Theorem 5.19.

Theorem 5.55. *Let (A, τ) be a separable tracial von Neumann algebra and (H, J) a symmetric A-bimodule. Assume that ${}_AH_A$ is faithful and weakly mixing. Then any central sequence for M asymptotically lies in A and the central sequence algebra of M is exactly given by*

$$M' \cap M^\omega = \{(a_n) \in A' \cap A^\omega \mid \lim_{n\to\omega} \|a_n\xi - \xi a_n\| = 0 \text{ for all } \xi \in H\}.$$

Proof. It is enough to show that $M' \cap M^\omega \subset A^\omega$, since the rest of the statement follows trivially by looking at the commutator of $a_n \in A$ and $\ell(\xi) + \ell(J\xi)^*$ for $\xi \in \mathcal{H}$.

Let $(x_n)_{n\in\mathbb{N}}$ be a central sequence for M and put $y_n = x_n - E_A(x_n)$. We may assume that $\sup_n \|y_n\| \leq 1$. We first prove the following two claims, analogously to Claim I and Claim II in the proof of Theorem 5.19.

Claim 1. *For any $\xi \in \mathcal{H}$ and any $\varepsilon > 0$, there exists a projection $p \in A$ with $\tau(1-p) < \varepsilon$ such that*

$$\lim_{n\to\infty} \|\ell(\xi p)\ell(\xi p)^*(y_n)\|_2 < \varepsilon.$$

Proof of claim. Put $a = \sqrt{\langle \xi, \xi \rangle_A} \in A$ and let $q \in A$ be the support projection of a. Take a spectral projection $q_1 \in qAq$ of a such that aq_1 is invertible in $q_1 A q_1$ and such that $\tau(q - q_1) < \varepsilon/2$. Let $b \in q_1 A q_1$ be its inverse and put $\eta = \xi b$. Then $\ell(\eta)^* \ell(\eta) = q_1$ and $\xi q_1 = \eta a$.

Pick $N \in \mathbb{N}$ such that $2^{-N} < \frac{1}{3} \|a\|^{-4} \varepsilon^2$ and put $\kappa = 2^N$. Then choose $\delta > 0$ so small that $\delta < \frac{1}{6} \|a\|^{-2} \varepsilon \kappa^{-3/2}$. Exactly as in the proof of Theorem 5.19, we find unitaries $v_1, \ldots, v_\kappa \in \mathcal{U}(A)$ and a projection $q_2 \in q_1 A q_1$ such that $\tau(q_1 - q_2) < \varepsilon/2$ and such that the vectors $\eta_i := v_i \eta$ satisfy

$$\|q_2 \langle \eta_i, \eta_j \rangle_A q_2\| < \delta \quad \text{for all} \quad i \neq j.$$

Put $\mu_i = \eta_i q_2 = v_i \eta q_2$ and $P_i = \ell(\mu_i) \ell(\mu_i)^*$. Note that $P_i = v_i P_1 v_i^*$ and that P_i is a projection for all i. By construction, $\|P_i P_j\| < \delta$ whenever $i \neq j$. Putting $P = \sum_{i=1}^{\kappa} P_i$, it follows as in the proof of Theorem 5.19 that $\|P\| < 1 + \kappa^2 \delta$.

Since y_n asymptotically commutes with A and since P_i commutes with the right A-action, we have that

$$\lim_{n\to\infty} \|P_i(y_n)\|_2 = \lim_{n\to\infty} \|v_i P_1 v_i^* y_n\|_2 = \lim_{n\to\infty} \|v_i P_1 y_n v_i^*\|_2 = \lim_{n\to\infty} \|P_1(y_n)\|_2$$

for all $i = 1, \ldots, \kappa$. Also note that

$$\Big| \| \sum_i P_i(y_n) \|_2^2 - \sum_i \|P_i(y_n)\|_2^2 \Big| = \Big| \sum_{i \neq j} \langle P_i P_j y_n, y_n \rangle_2 \Big| < \delta \kappa (\kappa - 1).$$

It follows that

$$\begin{aligned} \lim_{n\to\infty} \|P(y_n)\|_2^2 &= \lim_{n\to\infty} \| \sum_{i=1}^{\kappa} P_i(y_n) \|_2^2 \\ &\geq \lim_{n\to\infty} \sum_{i=1}^{\kappa} \|P_i(y_n)\|_2^2 - \delta\kappa(\kappa - 1) \\ &= \kappa \lim_{n\to\infty} \|P_1(y_n)\|_2^2 - \delta\kappa(\kappa - 1). \end{aligned}$$

On the other hand, since $\|P\| < 1 + \kappa^2 \delta$, we have that $\|P(y_n)\|_2 < 1 + \kappa^2 \delta$ for all n. Thus,

$$\kappa \lim_{n\to\infty} \|P_1(y_n)\|_2^2 - \delta\kappa(\kappa - 1) < (1 + \kappa^2 \delta)^2.$$

We conclude that

$$\lim_{n\to\infty} \|P_1(y_n)\|_2 < \sqrt{\frac{(1 + \kappa^2 \delta)^2}{\kappa} + \delta(\kappa - 1)} \leq \|a\|^{-2} \varepsilon.$$

Since q_1 and a commute, the right support of $(q_1 - q_2)a$ is a projection of the form $q_1 - p_0$ where $p_0 \in q_1 A q_1$ is a projection with $\tau(q_1 - p_0) \leq \tau(q_1 - q_2) < \varepsilon/2$. By construction, $q_1 a p_0 = q_2 a p_0$. Since $p_0 \leq q_1$ and $\eta = \eta q_1$, it follows that

$$\xi p_0 = \xi q_1 p_0 = \eta a p_0 = \eta q_1 a p_0 = \eta q_2 a p_0 \ .$$

Define the projection $p \in A$ given by $p = (1-q) + p_0$. Since $\xi(1-q) = 0$, we still have $\xi p = \eta q_2 a p_0$. Because $1 - p = (q - q_1) + (q_1 - p_0)$, we get that $\tau(1-p) < \varepsilon$. Finally,

$$\begin{aligned}\lim_{n\to\infty} \|\ell(\xi p)\ell(\xi p)^*(y_n)\|_2 &= \lim_{n\to\infty} \|\ell(\eta q_2)\, a p_0 a^*\, \ell(\eta q_2)^*(y_n)\|_2 \\ &\leq \|a\|^2 \lim_{n\to\infty} \|\ell(\eta q_2)\ell(\eta q_2)^*(y_n)\|_2 \\ &= \|a\|^2 \lim_{n\to\infty} \|P_1(y_n)\|_2 < \varepsilon.\end{aligned}$$

So, we have proven the claim. □

Claim 2. *For every $\xi \in \mathcal{H}$ and $\varepsilon > 0$, there exists a projection $p \in A$ with $\tau(1-p) < \varepsilon$ such that*

$$\lim_{n\to\infty} \|\ell(\xi p)\ell(\xi p)^*(y_n)\|_2 = 0.$$

Proof of claim. For every $k \in \mathbb{N}$, Claim 1 gives us a projection $p_k \in A$ such that $\tau(1-p_k) < 2^{-k}\varepsilon$ and

$$\lim_{n\to\infty} \|\ell(\xi p_k)\ell(\xi p_k)^*(y_n)\|_2 < \frac{1}{k}.$$

Let $p = \bigwedge_{k\geq 1} p_k$. Then $\tau(1-p) < \varepsilon$ and for every $k \geq 1$, we have that

$$\begin{aligned}\lim_{n\to\infty} \|\ell(\xi p)\ell(\xi p)^*(y_n)\|_2 &= \lim_{n\to\infty} \|\ell(\xi) p \ell(\xi)^*(y_n)\|_2 \\ &\leq \lim_{n\to\infty} \|\ell(\xi) p_k \ell(\xi)^*(y_n)\|_2 \\ &= \lim_{n\to\infty} \|\ell(\xi p_k)\ell(\xi p_k)^*(y_n)\|_2 < \frac{1}{k}.\end{aligned}$$

It follows that $\lim_{n\to\infty} \|\ell(\xi p)\ell(\xi p)^*(y_n)\|_2 = 0$ as claimed. □

We are now ready to finish the proof of Theorem 5.55, which is done exactly as in [Ho12b, Proposition 6.1]. Given any right A-submodule L of H and an integer $k \geq 1$, we define

$$\chi_k(L) = \overline{\text{span}}\{\xi_1 \otimes_A \cdots \otimes_A \xi_n \mid n \geq k,\ \xi_1 \in \mathcal{L}\} \subset \mathcal{F}_A(H),$$

where $\mathcal{L}$ denotes the set of left and right A-bounded vectors in L. For simplicity, we write $\chi(L) := \chi_1(L)$. Given any closed subspace $S \subset L^2(M)$, we denote by P_S the orthogonal projection onto S.

Let $\xi \in \mathcal{H}$ be any symmetric vector. By Claim 2, we can take a projection $p \in A$ with $\tau(1-p) < \varepsilon$ such that

$$\lim_{n\to\infty} \|\ell(\xi p)\ell(\xi p)^*(y_n)\|_2 = 0.$$

Put $\xi' = p\xi p$. Note that ξ' is symmetric and since y_n asymptotically commutes with p, we have that $\lim_{n\to\infty} \|\ell(\xi')\ell(\xi')^*(y_n)\|_2 = 0$. Using the polar decomposition [AP16, Lemma 8.4.9], we may write $\xi' = \xi_0\langle \xi', \xi'\rangle_A^{1/2}$ with $\xi_0 \in H$ a right bounded vector such that $\overline{\xi' A} = \overline{\xi_0 A}$ and such that $\langle \xi_0, \xi_0\rangle_A$ is the range projection of $\langle \xi', \xi'\rangle_A^{1/2}$. Note that $P_{\chi(\overline{\xi' A})} = \ell(\xi_0)\ell(\xi_0)^*$. Take a spectral projection $q \in pAp$ of $a := \langle \xi', \xi'\rangle_A^{1/2}$ lying arbitrarily close to the support of a such that aq is invertible inside qAq. Then, $\xi_0 q = \xi' b$ for some $b \in qAq$ and thus $P_{\chi(\overline{\xi' qA})} = \ell(\xi')bb^*\ell(\xi')^*$. So, after replacing p with a slightly smaller projection but still arbitrarily close to 1, we may assume that $P_{\chi(\overline{\xi' A})} = \ell(\xi')bb^*\ell(\xi')^*$. Since $\lim_{n\to\infty} \|\ell(\xi')\ell(\xi')^*(y_n)\|_2 = 0$, we get that $\lim_{n\to\infty} \|P_{\chi(\overline{\xi' A})}(y_n)\|_2 = 0$. Since also $y_n \in M \ominus A$, it follows that

$$\lim_{n\to\infty} \|y_n - P_{\chi(H\ominus\overline{\xi' A})}(y_n)\|_2 = 0. \tag{5.33}$$

Denote by $\mathcal{J} : \mathcal{F}_A(H) \to \mathcal{F}_A(H)$ the anti-unitary involution defined in the proof of Proposition 5.7. Note that

$$W(\xi')(\chi(H \ominus \overline{\xi' A})) \subset \chi_2(\overline{\xi' A}),$$

$$\mathcal{J}W(\xi')\mathcal{J}(\chi(H \ominus \overline{\xi' A})) \subset L^2(A) \oplus \chi(H \ominus \overline{\xi' A}).$$

Indeed, for left and right bounded vectors $\eta \in H \ominus \overline{\xi' A}$ and $\zeta \in \mathcal{H}^{\otimes_A^k}$, $k \geq 0$ (note that $\zeta \in A$ if $k = 0$), we have that $\langle J\xi', \eta\rangle_A = \langle \xi', \eta\rangle_A = 0$ since ξ' is symmetric and hence

$$W(\xi')(\eta \otimes_A \zeta) = \xi' \otimes_A \eta \otimes_A \zeta + \langle J\xi', \eta\rangle_A \zeta = \xi' \otimes_A \eta \otimes_A \zeta \in \chi_2(\overline{\xi' A}),$$

$$\mathcal{J}W(\xi')\mathcal{J}(\eta \otimes_A \zeta) = \eta \otimes_A \zeta \otimes_A \xi' + \eta\langle J\zeta, \xi'\rangle_A \in \chi(H \ominus \overline{\xi' A}) \quad \text{if } k \geq 1,$$

$$\mathcal{J}W(\xi')\mathcal{J}(\eta\zeta) = \eta\zeta \otimes_A \xi' + \langle J(\eta\zeta), \xi'\rangle_A \in L^2(A) \oplus \chi(H \ominus \overline{\xi' A}) \quad \text{if } k = 0.$$

In particular,

$$W(\xi')(\chi(H \ominus \overline{\xi' A})) \perp H,$$

$$W(\xi')(\chi(H \ominus \overline{\xi' A})) \perp \mathcal{J}W(\xi')\mathcal{J}(\chi(H \ominus \overline{\xi' A})).$$

For all $n \in \mathbb{N}$, we have

$$W(\xi')x_n - x_n W(\xi') = W(\xi')(E_A(x_n) + y_n) - \mathcal{J}W(\xi')\mathcal{J}(E_A(x_n) + y_n)$$
$$= (W(\xi')E_A(x_n) - \mathcal{J}W(\xi')\mathcal{J}E_A(x_n) - \mathcal{J}W(\xi')\mathcal{J}y_n) + W(\xi')y_n.$$

Thus

$$\|W(\xi')x_n - x_n W(\xi')\|_2^2$$
$$= \|W(\xi')E_A(x_n) - \mathcal{J}W(\xi')\mathcal{J}E_A(x_n) - \mathcal{J}W(\xi')\mathcal{J}y_n\|_2^2 + \|W(\xi')y_n\|_2^2$$
$$+ 2\mathrm{Re}\Big\langle W(\xi')E_A(x_n) - \mathcal{J}W(\xi')\mathcal{J}E_A(x_n) - \mathcal{J}W(\xi')\mathcal{J}y_n, W(\xi')y_n\Big\rangle.$$

Using (5.33) along with the orthogonality properties above, we get

$$\langle W(\xi')E_A(x_n), W(\xi')y_n\rangle = \langle W(\xi')E_A(x_n), W(\xi')(y_n - P_{\chi(H\ominus\overline{\xi' A})}(y_n))\rangle \to 0,$$

as $n \to \infty$ and similarly

$$\langle \mathcal{J}W(\xi')\mathcal{J}E_A(x_n), W(\xi')y_n\rangle \to 0 \quad \text{as } n \to \infty,$$
$$\langle \mathcal{J}W(\xi')\mathcal{J}y_n, W(\xi')y_n\rangle \to 0 \quad \text{as } n \to \infty.$$

Since also $\|W(\xi')x_n - x_n W(\xi')\|_2 \to 0$, we get from the above computation that

$$\|W(\xi')E_A(x_n) - \mathcal{J}W(\xi')\mathcal{J}E_A(x_n) - \mathcal{J}W(\xi')\mathcal{J}y_n\|_2 \to 0 \quad \text{and}$$
$$\|W(\xi')y_n\|_2 \to 0.$$

Using (5.33) again, it follows that $\|W(\xi')P_{\chi(H\ominus\overline{\xi' A})}(y_n)\|_2 \to 0$. Since $\ell(\xi')^* P_{\chi(H\ominus\overline{\xi' A})}(y_n) = 0$, we have that

$$\|W(\xi')P_{\chi(H\ominus\overline{\xi' A})}(y_n)\|_2 = \|\xi' \otimes_A P_{\chi(H\ominus\overline{\xi' A})}(y_n)\|_2$$
$$= \|\langle \xi', \xi'\rangle_A^{1/2} P_{\chi(H\ominus\overline{\xi' A})}(y_n)\|_2.$$

So, we have that $\|\langle \xi', \xi'\rangle_A\, P_{\chi(H\ominus\overline{\xi' A})}(y_n)\|_2 \to 0$. Using (5.33) once more, we get $\lim_{n\to\infty} \|\langle \xi', \xi'\rangle_A\, y_n\|_2 = 0$. Recall that $\xi' = p\xi p$ and that $\tau(1-p) < \varepsilon$. Since ε can be chosen arbitrarily small, we have that $\|\langle \xi, \xi\rangle_A - \langle \xi', \xi'\rangle_A\|_2$ is arbitrarily small. It follows that $\lim_{n\to\infty} \|\langle \xi, \xi\rangle_A\, y_n\|_2 = 0$.

Since this holds for all symmetric vectors $\xi \in \mathcal{H}$ and since ${}_A H_A$ is faithful, we conclude that

$$\lim_{n\to\infty} \|x_n - E_A(x_n)\|_2 = \lim_{n\to\infty} \|y_n\|_2 = 0.$$

So, $(x_n)_{n\in\mathbb{N}}$ is asymptotically contained in A.

□

Example 5.56. Let ${}_AH_A$ be a weakly mixing symmetric A-bimodule and assume that $M = \Gamma(H, J, A, \tau)''$ is a factor. Then M is a full factor whenever H contains a nonzero mixing A-subbimodule. Indeed, assume that $K \subset H$ is a mixing A-subbimodule and that M is not a full factor. Since M is a II_1 factor, this means that there exists a central sequence consisting of unitaries $(a_n)_{n\in\mathbb{N}} \subset \mathcal{U}(M)$ with $\tau(a_n) = 0$ for all n. After passing to a subsequence if necessary, we may assume that a_n converges weakly to some $a \in M$. Then $a \in \mathcal{Z}(M) = \mathbb{C}1$. Since $\tau(a) = 0$, it follows that $a = 0$. So, $a_n \to 0$ weakly.

Since K is mixing, we now get that $\|\langle a_n\xi, \eta\rangle_A\|_2 \to 0$ for all left and right bounded vectors $\xi, \eta \in \mathcal{K}$. In particular,

$$|\langle a_n\xi, \xi a_n\rangle| = |\tau(\langle a_n\xi, \xi a_n\rangle_A)| = |\tau(\langle a_n\xi, \xi\rangle_A a_n)| \leq \|\langle a_n\xi, \xi\rangle_A\|_2\|a_n\|_2 \to 0,$$

for all $\xi \in \mathcal{K}$. Thus,

$$0 = \lim_n \|a_n\xi - \xi a_n\|^2 = \lim_n(\|a_n\xi\|^2 + \|\xi a_n\|^2 - 2\mathrm{Re}\langle a_n\xi, \xi a_n\rangle) = 2\|\xi\|^2,$$

for all $\xi \in K$, which is a contradiction.

In Section 5.5, we defined the von Neumann algebras $M_\nu = \Gamma(H_\nu, J_\nu, A, \tau)''$ associated with a symmetric probability measure ν on a compact second countable group K. Recall that $A = L^\infty(K, \mu)$, where μ denotes the Haar measure on K, and that $H_\nu = L^2(K \times K, \mu \times \nu)$ with A-bimodular actions given by (5.27) and symmetry J_ν given by (5.28). For the remainder of this section, we will use Theorem 5.55 to characterize when M_ν has property Gamma in terms of the measure ν.

In Proposition 5.47, we saw that M_ν has nontrivial center if and only if $\nu(K_0) = 1$ for some closed proper subgroup $K_0 < K$. This is equivalent with the existence of a nontrivial irreducible representation (π, L) of K and a unit vector $\xi \in L$ such that $\nu(K_{\pi,\xi}) = 1$, where

$$K_{\pi,\xi} := \{x \in K \mid \pi(x)\xi = \xi\}.$$

Indeed, this follows from the following well-known lemma.

Lemma 5.57. *Let K_0 be a closed subgroup of a compact second countable group K. Then $K_0 \neq K$ if and only if there exists a nontrivial irreducible representation π of K that has a nonzero K_0-invariant vector.*

Proof. Given two representations (π_1, H_1) and (π_2, H_2) of K, we denote by $C(\pi_1, \pi_2)$ the intertwiner space, i.e., $C(\pi_1, \pi_2)$ consists of all bounded linear operators $T\colon H_1 \to H_2$ such that $T\pi_1(x) = \pi_2(x)T$ for all $x \in K$. Note that a representation π of K has a nonzero invariant vector if and only if $C(\epsilon, \pi) \neq \{0\}$. So, we have to show that $K_0 = K$ if and only if $C(\epsilon, \pi|_{K_0}) = \{0\}$ for all $\pi \in \mathrm{Irr}(K) \setminus \{\epsilon\}$.

By the Frobenius reciprocity theorem, $C(\epsilon, \pi|_{K_0}) \cong C(\lambda_{K/K_0}, \pi)$, where λ_{K/K_0} denotes the regular representation of K on $L^2(K/K_0)$. Note that any nontrivial irreducible representation (π, H) that is contained in λ_{K/K_0} satisfies $C(\lambda_{K/K_0}, \pi) \neq \{0\}$. Indeed, the projection $P\colon L^2(K/K_0) \to H$ is a nontrivial intertwiner between λ_{K/K_0} and π. So, we have that $C(\epsilon, \pi|_{K_0}) = \{0\}$ for all $\pi \in \mathrm{Irr}(K) \setminus \{\epsilon\}$ if and only if λ_{K/K_0} is trivial, which holds if and only if $K = K_0$. □

Given a sequence of irreducible representations $\pi = (\pi_n, L_n)_{n\in\mathbb{N}}$ of K and given a sequence of unit vectors $\xi = (\xi_n)_{n\in\mathbb{N}}$ with $\xi_n \in L_n$, we define the subgroup

$$K_{\pi,\xi} := \{x \in K \mid \lim_{n\to\infty} \|\pi_n(x)\xi_n - \xi_n\| = 0\}.$$

Lemma 5.58. *The subgroup $K_{\pi,\xi}$ defined above is Borel. Moreover, if $\pi_n \to \infty$, then $K_{\pi,\xi} \neq K$.*

Proof. We have that $K_{\pi,\xi}$ is Borel since

$$K_{\pi,\xi} = \bigcap_{k=1}^{\infty} \bigcup_{n_0=1}^{\infty} \bigcap_{n=n_0}^{\infty} \{x \in K \mid \|\pi_n(x)\xi_n - \xi_n\| < \frac{1}{k}\}.$$

Next, assume that $\pi_n \to \infty$. Note that $x \in K_{\pi,\xi}$ if and only if $\langle \pi_n(x)\xi_n, \xi_n\rangle \to 1$. By the dominated convergence theorem, we have that

$$\int_{K_{\pi,\xi}} \langle \pi_n(x)\xi_n, \xi_n\rangle \,\mathrm{d}\mu(x) \to \mu(K_{\pi,\xi}) \qquad \text{as } n \to \infty.$$

But, by the Schur orthogonality relations, we have $\int_K \langle \pi_n(x)\xi_n, \xi_n\rangle \,\mathrm{d}\mu(x) = 0$ whenever $\pi_n \neq 1$. Since $\pi_n \to \infty$ as $n \to \infty$, it follows that $K_{\pi,\xi} \neq K$. □

Above, we saw that M_ν has a nontrivial central element if and only if $\nu(K_{\pi,\xi}) = 1$ for a single $\pi \in \mathrm{Irr}(K) \setminus \{\epsilon\}$ and a single unit vector ξ. In the following proposition, we will show that M_ν has a nontrivial central sequence, i.e. property Gamma, if and only if $\nu(K_{\pi,\xi}) = 1$ for sequences $\pi = (\pi_n)_{n\in\mathbb{N}}$ and $\xi = (\xi_n)_{n\in\mathbb{N}}$, under the assumption that M_ν is a factor. We will also show that this condition is equivalent with the operator $\lambda(\nu)$ not having spectral gap when restricted to $L^2(K) \ominus \mathbb{C}1$.

Given a representation π of K and a symmetric probability measure ν on K, recall that we defined the operator $\pi(\nu)$ by

$$\pi(\nu) = \int_K \pi(x)\,\mathrm{d}\nu(x).$$

Note that $\pi(\nu)$ is a self-adjoint operator with $\|\pi(\nu)\| \leq 1$. We denote by λ_0 the left regular representation of K restricted to $L^2(K) \ominus \mathbb{C}1$. In the following proposition, the equivalence between (1) and (4) states that M_ν is a full factor if and only if $\lambda_0(\nu)$ has a spectral gap, meaning that there exists $\delta > 0$ such that the spectrum of $\lambda_0(\nu)$ is contained in $[-1, 1-\delta]$. Compare this with Proposition 5.47, where we saw that H_ν is mixing if and only if $\lambda(\nu)$ is a compact operator.

Theorem 5.59. *Let ν be a symmetric non-atomic probability measure on a compact second countable group K, and let $M_\nu = \Gamma(H_\nu, J_\nu, A, \tau)''$ be as defined above. Assume that the support of ν topologically generates K, i.e., M_ν is a factor. Then the following are equivalent.*

(1) *M_ν has property Gamma.*

(2) *There exists a sequence of irreducible representations $\pi = (\pi_n)_{n\in\mathbb{N}}$ on K with $\pi_n \to \infty$ and a sequence of unit vectors $(\xi_n)_{n\in\mathbb{N}}$ such that $\nu(K_{\pi,\xi}) = 1$.*

(3) *For any $\varepsilon > 0$, any $\delta \in (0,1)$ and any finite subset $F \subset \mathrm{Irr}(K)$, there exists $(\pi, L) \in \mathrm{Irr}(K) - F$ and a unit vector $\xi_0 \in L$ such that*

$$\nu(\{x \in K \mid \mathrm{Re}\,\langle \pi(x)\xi_0, \xi_0\rangle > 1-\varepsilon\}) > 1-\delta.$$

(4) *The spectrum of $\lambda_0(\nu)$ contains 1.*

Proof. (2) $\Rightarrow$ (1): For each n, denote by L_n the finite-dimensional Hilbert space on which π_n acts. We will view each L_n as a subspace of $L := \ell^2(\mathbb{N})$. Let $f_n \colon K \to L$ be the map given by $f_n(k) = \pi_n(k)\xi_n$. Then f_n defines an element in $L^\infty(K) \otimes L \subset L^2(K) \otimes L$ for each n with $\|f_n\|_\infty = 1$. Note that

$$(\tau \otimes \mathrm{id})(f_n) = \int_K \pi_n(k)\xi_n\,\mathrm{d}\mu(k) = 0 \quad \text{if } \pi_n \neq 1,$$

by the Schur orthogonality relations. So for n so large that $\pi_n \neq 1$, we have that $f_n \in (L^2(K) \ominus \mathbb{C}1) \otimes L$.

Recall that $A = L^\infty(K)$. Fix an essentially bounded function $\eta \in L^2(K \times K, \mu \times \nu) = H_\nu$ and note that $\eta \in H_\nu$ is left and right A-bounded. Using the notation $f_n \cdot \eta := (r(\eta) \otimes 1)(f_n)$ and $\eta \cdot f_n = (\ell(\eta) \otimes 1)(f_n)$, we have that

$$\begin{aligned}\|f_n \cdot \eta - \eta \cdot f_n\|^2 &= \int_K \int_K \|\pi_n(xy)\xi_n \cdot \eta(x,y) - \eta(x,y) \cdot \pi_n(x)\xi_n\|^2 \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y) \\ &= \int_K \int_K |\eta(x,y)|^2 \|\pi_n(x)(\pi_n(y)\xi_n - \xi_n)\|^2 \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y) \\ &\leq \|\eta\|_\infty^2 \int_K \|\pi_n(y)\xi_n - \xi_n\|^2 \, \mathrm{d}\nu(y).\end{aligned}$$

Since $\nu(K_{\pi,\xi}) = 1$ and since $\|\pi_n(y)\xi_n - \xi_n\| \to 0$ for all $y \in K_{\pi,\xi}$, we get by Lebesgue's dominated convergence theorem that $\int_K \|\pi_n(y)\xi_n - \xi_n\|^2 \, \mathrm{d}\nu(y) \to 0$ as $n \to \infty$. So, we have proved that $\|f_n \cdot \eta - \eta \cdot f_n\| \to 0$ for all $\eta \in H_\nu$ with $\|\eta\|_\infty < \infty$. This means that $f_n \in (L^2(M_\nu) \ominus \mathbb{C}1) \otimes L$ asymptotically commutes with $W(\eta) \in M_\nu$. Hence, f_n asymptotically commutes with the $*$-subalgebra M_0 given by

$$M_0 = \mathrm{span}\Big(\{W(\eta_1, \ldots, \eta_n) \mid \eta_i \in H_\nu, \ \|\eta_i\|_\infty < \infty\} \cup A\Big).$$

Note that M_0 is $\|\cdot\|_2$-dense in M_ν. Also note that f_n is a tracial vector for all $n \in \mathbb{N}$, i.e., $\langle x f_n, f_n \rangle = \langle f_n x, f_n \rangle = \tau(x)$ for all $x \in M_\nu$. Indeed, for $a \in A = L^\infty(K)$, we have that

$$\begin{aligned}\langle f_n a, f_n \rangle = \langle a f_n, f_n \rangle &= \int_K \langle a(x)\pi_n(x)\xi_n, \pi_n(x)\xi_n \rangle \, \mathrm{d}\mu(x) \\ &= \int_K a(x) \|\xi_n\|^2 \, \mathrm{d}\mu(x) = \tau(a).\end{aligned}$$

Moreover, when $x \in M_\nu \ominus A$ we have that $x f_n \in L^2(M_\nu \ominus A) \otimes L$ so that $\langle x f_n, f_n \rangle = 0$. Therefore,

$$\langle x f_n, f_n \rangle = \langle E_A(x) f_n, f_n \rangle = \tau(E_A(x)) = \tau(x) \quad \text{for all } \ x \in M_\nu,$$

and similarly $\langle f_n x, f_n \rangle = \tau(x)$.

We can now conclude that f_n asymptotically commutes with all of M_ν. Indeed, given $x \in M_\nu$ and $\varepsilon > 0$, choose $x_0 \in M_0$ such that $\|x - x_0\|_2 < \frac{\varepsilon}{3}$. Then,

$$\begin{aligned}\|x f_n - f_n x\|_2 &\leq \|(x - x_0) f_n\|_2 + \|x_0 f_n - f_n x_0\|_2 + \|f_n(x_0 - x)\|_2 \\ &= 2\|x - x_0\|_2 + \|x_0 f_n - f_n x_0\|_2.\end{aligned}$$

It follows that $\|x f_n - f_n x\|_2 < \varepsilon$ for n so large that $\|x_0 f_n - f_n x_0\|_2 \leq \frac{\varepsilon}{3}$.

We have now shown that f_n is asymptotically M_ν-central. So, the unitary representation $(\operatorname{Ad} u \otimes 1)_{u \in \mathcal{U}(M_\nu)}$ on $(L^2(M_\nu) \ominus \mathbb{C}1) \otimes L$ does not have spectral gap. Since this is just a multiple of $(\operatorname{Ad} u)_{u \in \mathcal{U}(M_\nu)}$, we have that $(\operatorname{Ad} u)_{u \in \mathcal{U}(M_\nu)}$ on $L^2(M_\nu) \ominus \mathbb{C}1$ also does not have spectral gap. By [Co75], we conclude that M_ν has property Gamma.

(3) $\Rightarrow$ (2): Given (3), pick a sequence $\pi_n \to \infty$ in $\operatorname{Irr}(K)$ together with a sequence of unit vectors ξ_n such that

$$\nu(\{x \in K \mid \operatorname{Re}\langle \pi_n(x)\xi_n, \xi_n \rangle > 1 - \frac{1}{n}\}) > 1 - 2^{-n}.$$

Define

$$K_{n_0} = \bigcap_{n \geq n_0} \{x \in K \mid \operatorname{Re}\langle \pi_n(x)\xi_n, \xi_n \rangle > 1 - \frac{1}{n}\}.$$

Then $\nu(K_{n_0}) > 1 - \sum_{n=n_0}^{\infty} 2^{-n} = 1 - 2^{-n_0+1}$ and thus $\nu(\bigcup_{n_0=1}^{\infty} K_{n_0}) = 1$. If $x \in K_{n_0}$, then $\operatorname{Re}\langle \pi_n(x)\xi_n, \xi_n \rangle > 1 - \frac{1}{n}$ for $n \geq n_0$. Then $\operatorname{Re}\langle \pi_n(x)\xi_n, \xi_n \rangle \to 1$ and thus $x \in K_{\pi,\xi}$. So, $\bigcup_{n_0=1}^{\infty} K_{n_0} \subset K_{\pi,\xi}$. Since $\nu(\bigcup_{n_0=1}^{\infty} K_{n_0}) = 1$, we conclude that $\nu(K_{\pi,\xi}) = 1$.

(1) $\Rightarrow$ (3): Given $\pi \in \operatorname{Irr}(K)$, we denote by L_π the finite-dimensional Hilbert space on which π acts and we denote by d_π the dimension of L_π. Assume that (3) does not hold. Pick $\varepsilon > 0$ and $\delta \in (0,1)$ and $F \subset \operatorname{Irr}(K)$ finite such that

$$\nu(\{x \in K \mid \operatorname{Re}\langle \pi(x)\xi, \xi \rangle \leq (1-\varepsilon)\|\xi\|^2\}) \geq \delta \qquad \text{for all } \; \pi \notin F \; \text{ and } \; \xi \in L_\pi.$$

When $\pi \in \operatorname{Irr}(K)$, we denote by $\pi_{ij} \in L^\infty(K) = A$ the ij'th matrix coefficient of π, for $i,j \in \{1, \ldots, d_\pi\}$, that is, $\pi_{ij}(x) = \langle \pi(x)e_j, e_i \rangle$ where $(e_i)_i$ is an orthonormal basis for L_π.

By the Peter-Weyl theorem,

$$\{\sqrt{d_\pi} \cdot \pi_{ij} \mid \pi \in \operatorname{Irr}(K), \; 1 \leq i,j \leq d_\pi\}$$

is an orthonormal basis for $L^2(K)$. Since K is compact, we have $L^\infty(K) \subset L^2(K)$. Let $a_n \in \mathcal{U}(L^\infty(K))$ be a sequence of unitaries tending to zero weakly. Write

$$a_n = \sum_{\pi \in \operatorname{Irr}(K)} \sum_{i,j=1}^{d_\pi} (a_n)_{\pi,i,j} \pi_{ij},$$

with coefficients $(a_n)_{\pi,i,j} = d_\pi \langle a_n, \pi_{ij} \rangle \in \mathbb{C}$. We need to show that $(a_n)_{n \in \mathbb{N}}$ cannot be a central sequence for M_ν. It is enough to show that $\langle a_n \cdot (1 \otimes 1), (1 \otimes 1) \cdot a_n \rangle$ does not converge to 1 as $n \to \infty$.

For all $\pi, \pi' \in \mathrm{Irr}(K)$ and $i, j \in \{1, \dots, d_\pi\}$ and $k, l \in \{1, \dots, d_{\pi'}\}$, we have that

$$\begin{aligned}\langle \pi_{ij} \cdot (1 \otimes 1),\, (1 \otimes 1) \cdot \pi'_{kl} \rangle &= \int_K \int_K \pi_{ij}(xy)\overline{\pi'_{kl}(x)}\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \\ &= \sum_{s=1}^{d_\pi} \int_K \pi_{sj}(y) \int_K \pi_{is}(x)\overline{\pi'_{kl}(x)}\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y).\end{aligned}$$

By the Peter-Weyl theorem, we have that $x \mapsto \pi_{is}(x)$ and $x \mapsto \pi'_{kl}(x)$ are orthogonal if $\pi \neq \pi'$. Moreover, $x \mapsto \pi_{is}(x)$ and $x \mapsto \pi_{kl}(x)$ are orthogonal if $i \neq k$ or if $s \neq l$. So, if $\pi = \pi'$, we get

$$\begin{aligned}\langle \pi_{ij} \cdot (1 \otimes 1),\, (1 \otimes 1) \cdot \pi_{kl} \rangle &= \delta_{ik} \int_K \pi_{lj}(y) \int_K \pi_{il}(x)\overline{\pi_{il}(x)}\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \\ &= \delta_{ik} \|\pi_{il}\|_2^2 \int_K \pi_{lj}(y)\,\mathrm{d}\nu(y) = \delta_{ik}\frac{1}{d_\pi}\int_K \pi_{lj}(y)\,\mathrm{d}\nu(y).\end{aligned}$$

We conclude that

$$\langle \pi_{ij} \cdot (1 \otimes 1),\, (1 \otimes 1) \cdot \pi'_{kl} \rangle = \begin{cases} 0 & \text{if } \pi \neq \pi' \text{ or } i \neq k, \\ \frac{1}{d_\pi}\int_K \pi_{lj}(y)\,\mathrm{d}\nu(y) & \text{if } \pi = \pi' \text{ and } i = k. \end{cases}$$

Now,

$$\begin{aligned}\langle a_n \cdot (1 \otimes 1), (1 \otimes 1) \cdot a_n \rangle &= \sum_{\pi \in \mathrm{Irr}(K)} \frac{1}{d_\pi} \sum_{i,j,l} (a_n)_{\pi,i,j} \overline{(a_n)_{\pi,i,l}} \int_K \pi_{lj}(y)\,\mathrm{d}\nu(y) \\ &= \sum_{\pi \in \mathrm{Irr}(K)} \sum_{i=1}^{d_\pi} \int_K \langle \pi(y)\xi^n_{(\pi,i)}, \xi^n_{(\pi,i)} \rangle\,\mathrm{d}\nu(y),\end{aligned}$$

where $\xi^n_{(\pi,i)} = \sum_{j=1}^{d_\pi} \frac{1}{\sqrt{d_\pi}}(a_n)_{\pi,i,j} e_j$. Note that $\sum_\pi \sum_i \|\xi^n_{(\pi,i)}\|^2 \leq \|a_n\|_2^2 = 1$. Also, since $a_n \to 0$ weakly, we have that $\|\xi^n_{(\pi,i)}\| \to 0$ as $n \to \infty$ for every fixed π and i. Let $d = \max_{\pi \in F} d_\pi$ and choose $N \in \mathbb{N}$ so large that $\|\xi^n_{(\pi,i)}\|^2 \leq \frac{\varepsilon\delta}{2|F|d}$ for $n \geq N$, for all $\pi \in F$ and $i = 1, \dots, d_\pi$.

By the choice of F, ε and δ, we have for $\pi \notin F$ that

$$\begin{aligned}\int_K \mathrm{Re}\,\langle \pi(y)\xi^n_{(\pi,i)}, \xi^n_{(\pi,i)} \rangle\,\mathrm{d}\nu(y) &\leq \delta(1-\varepsilon)\|\xi^n_{(\pi,i)}\|^2 + (1-\delta)\|\xi^n_{(\pi,i)}\|^2 \\ &= (1 - \varepsilon\delta)\|\xi^n_{(\pi,i)}\|^2.\end{aligned}$$

Thus, for $n \geq N$,

$$\begin{aligned}\operatorname{Re}\langle a_n \cdot (1 \otimes 1), (1 \otimes 1) \cdot a_n \rangle &\leq \sum_{\pi \in F} \sum_i \|\xi^n_{(\pi,i)}\|^2 + \sum_{\pi \notin F} \sum_i (1 - \varepsilon\delta) \|\xi^n_{(\pi,i)}\|^2 \\ &\leq \frac{\varepsilon\delta}{2} + (1 - \varepsilon\delta) = 1 - \frac{\varepsilon\delta}{2}.\end{aligned}$$

It follows that

$$\limsup_{n \to \infty} \operatorname{Re}\langle a_n \cdot (1 \otimes 1), (1 \otimes 1) \cdot a_n \rangle \leq 1 - \frac{\varepsilon\delta}{2},$$

and thus $(a_n)_{n \in \mathbb{N}}$ cannot be a central sequence for M_ν.

By Theorem 5.55, all central sequences of M_ν asymptotically belong to A. So, we conclude that (1) does not hold.

(2) $\Rightarrow$ (4): In the following, we denote by $\sigma(T) \subset \mathbb{C}$ the spectrum of an operator T. Assume that $\sigma(\lambda_0(\nu)) \subset [-1, 1 - \delta]$ for some $\delta > 0$. Since λ_0 decomposes as the direct sum of all nontrivial irreducible representations of K, each occurring with multiplicity equal to its dimension, we have that $\sigma(\pi(\nu)) \subset [-1, 1 - \delta]$ for all $\pi \in \operatorname{Irr}(K) \setminus \{\epsilon\}$. So, we have that $\langle \pi(\nu)\xi, \xi \rangle \leq 1 - \delta$ for all π and all unit vectors ξ, which implies that

$$\|\pi(\nu)\xi - \xi\|^2 = 2 - 2\langle \pi(\nu)\xi, \xi \rangle \geq 2\delta.$$

In particular, there can be no sequences $(\pi_n)_{n \in \mathbb{N}}$ and $(\xi_n)_{n \in \mathbb{N}}$ with $\pi_n \to \infty$ such that $\nu(K_{\pi,\xi}) = 1$.

(4) $\Rightarrow$ (3): Assume that (3) does not hold. We claim that for any sequence $(\pi_n)_{n \in \mathbb{N}}$ in $\operatorname{Irr}(K)$ with $\pi_n \to \infty$ and any sequence of unit vectors $(\xi_n)_{n \in \mathbb{N}}$, we have that $\|\pi_n(\nu)\xi_n - \xi_n\|$ does not converge to 0. Indeed, assume that this is not the case, i.e., $\|\pi_n(\nu)\xi_n - \xi_n\| \to 0$ for some choice of $(\pi_n)_{n \in \mathbb{N}}$ and $(\xi_n)_{n \in \mathbb{N}}$. Equivalently, we have that $\langle \pi_n(\nu)\xi_n, \xi_n \rangle \to 1$, i.e.,

$$\int_K \operatorname{Re}\langle \pi_n(x)\xi_n, \xi_n \rangle \, \mathrm{d}\nu(x) \to 1.$$

Since also $\operatorname{Re}\langle \pi_n(x)\xi_n, \xi_n \rangle \leq 1$ for all $x \in K$, this implies that for any $\delta > 0$, we have

$$\nu(\{x \in K \mid \operatorname{Re}\langle \pi_n(x)\xi_n, \xi_n \rangle \geq 1 - \delta\}) \to 1.$$

This contradicts the fact that (3) does not hold.

We conclude that there exists a finite subset $F \subset \operatorname{Irr}(K)$ and $\delta > 0$ such that for any $(\pi, L) \notin F$, we have

$$\|\pi(\nu)\xi - \xi\| \geq \delta \|\xi\| \quad \text{for all } \xi \in L.$$

Fix $(\pi, L) \notin F$ and $\xi \in L$ with $\|\xi\| = 1$. Then $\langle \pi(\nu)\xi, \xi\rangle \leq 1 - \frac{\delta^2}{2}$ and hence

$$\|\pi(\nu)\xi - (1-\lambda)\xi\| \geq \langle (1-\lambda)\xi - \pi(\nu)\xi, \xi\rangle \geq \frac{\delta^2}{2} - \lambda,$$

for all $\lambda \in \mathbb{R}$. It follows that $\pi(\nu) - (1-\lambda)1$ is invertible whenever $0 < \lambda < \frac{\delta^2}{2}$. So, $\sigma(\pi(\nu)) \subset [-1, 1 - \frac{\delta^2}{2}]$ for all $\pi \notin F$.

Note that any $\pi \in \mathrm{Irr}(K) \setminus \{\epsilon\}$ satisfies $1 \notin \sigma(\pi(\nu))$. Indeed, since π is finite-dimensional, we have that $\sigma(\pi(\nu))$ consists of the eigenvalues of $\pi(\nu)$. Since π is irreducible and since the support of ν topologically generates K, we have that 1 is not an eigenvalue of $\pi(\nu)$ and hence $1 \notin \sigma(\pi(\nu))$.

For each $\pi \in F$, choose $\delta_\pi > 0$ such that $\sigma(\pi(\nu)) \subset [-1, 1-\delta_\pi]$. With $\tilde{\delta} := \min(\{\frac{\delta^2}{2}\} \cup \{\delta_\pi\}_{\pi\in F})$, we now have that $\sigma(\lambda_0) \subset [-1, 1-\tilde{\delta}]$ as wanted. $\square$

Note that in the case where K is abelian, we have that M_ν has property Gamma if and only if there exists a sequence of characters $\omega = (\omega_n)_{n\in\mathbb{N}} \subset \widehat{K}$ with $\omega_n \to \infty$ such that $\nu(K_\omega) = 1$, where

$$K_\omega = \{x \in K \mid \omega_n(x) \to 1\}.$$

Example 5.60. Even the circle $K = \mathbb{T}$ has quite a few such subgroups K_ω that are large enough to carry a non-atomic probability measure. For instance, consider the characters $\omega_n = 2^{n^2} \in \mathbb{Z} \cong \widehat{K}$. When identifying K with $\mathbb{R}/\mathbb{Z}$, the characters $\omega_n \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ are given by $\omega_n(x) = 2^{n^2}x$. We denote by $d(x, \mathbb{Z})$ the distance from a point $x \in \mathbb{R}$ to $\mathbb{Z}$. Then,

$$K_\omega = \{x \in \mathbb{R}/\mathbb{Z} \mid d(2^{n^2}x, \mathbb{Z}) \to 0\}.$$

Define $\pi\colon \{0,1\}^{\mathbb{N}} \to K$ by $\pi(\lambda) = \sum_{k=1}^\infty \lambda_k 2^{-k^2}$. Then,

$$2^{n^2}\pi(\lambda) = \sum_{k=1}^{n} \lambda_k 2^{n^2-k^2} + \sum_{k=n+1}^{\infty} \lambda_k 2^{n^2-k^2}.$$

Note that the first sum on the right hand side belongs to $\mathbb{Z}$, while the second sum can be bounded as follows:

$$\begin{aligned}\sum_{k=n+1}^{\infty} \lambda_k 2^{n^2-k^2} &= \sum_{k=n+1}^{\infty} \lambda_k 2^{-(k-n)(k+n)} \\ &\leq \sum_{k=n+1}^{\infty} \lambda_k 2^{-(k+n)} \\ &\leq 2^{-n}\cdot 2^{-n+1} \to 0.\end{aligned}$$

Thus, $\pi(\{0,1\}^{\mathbb{N}}) \subset K_\omega$. So, it suffices to find a non-atomic probability measure supported on $\pi(\{0,1\}^{\mathbb{N}})$. But then we can simply take the push-forward under π of any non-atomic probability measure on the Cantor set $\{0,1\}^{\mathbb{N}}$.

Let ν denote such a symmetric non-atomic probability measure that is supported on K_ω. Then the associated von Neumann algebra is a II_1 factor by Proposition 5.47, since K_ω is dense in K. By Theorem 5.59, M_ν has property Gamma.

Chapter 6

Conclusion

Cartan subalgebras play an important role in the classification of II_1 factors. One of the main questions of interest is that of existence and uniqueness of Cartan subalgebras and more generally, how many Cartan subalgebras a given II_1 factor has. Indeed, the number of Cartan subalgebras of a II_1 factor M tells us in how many ways M decomposes as $L(\mathcal{R})$, the II_1 factor associated to an equivalence relation $\mathcal{R}$.

The group measure space (gms) Cartan subalgebras are the ones arising as $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ for a free ergodic pmp action $\Gamma \curvearrowright X$. By Singer's theorem, the number of gms Cartan subalgebras of a II_1 factor M tells us in how many ways M decomposes as a crossed product $M = L^\infty(X) \rtimes \Gamma$, up to orbit equivalence of the actions. In particular, when M has a unique gms Cartan subalgebra, there is a unique free ergodic pmp action $\Gamma \curvearrowright X$ up to orbit equivalence such that $M = L^\infty(X) \rtimes \Gamma$. When this action is in fact unique up to *conjugacy*, then we say that $\Gamma \curvearrowright X$ is W*-superrigid. This means that the action $\Gamma \curvearrowright X$ can be completely recovered from the II_1 factor $L^\infty(X) \rtimes \Gamma$.

Uniqueness of Cartan subalgebras is thus an important step in proving W*-superrigidity and is therefore a very desirable property. The first theorem proving uniqueness of Cartan subalgebras up to unitary conjugacy was obtained by Ozawa and Popa in their breakthrough article [OP07] and since then, more and more uniqueness results have been proved. On the other hand, it remains an open problem to find II_1 factors with exactly n Cartan subalgebras, for some $n \geq 2$.

In this thesis, we solved this open problem in the case of gms Cartan subalgebras. Our main theorem describes all of the gms Cartan subalgebras for a specific class of II_1 factors. In particular, for any $n \geq 1$, we found explicit examples of II_1 factors M with exactly n gms Cartan subalgebras up to conjugacy by an automorphism of M. This means that M has exactly n crossed product decompositions up to orbit equivalence of the actions. We proved that this even holds up to conjugacy of the actions: M has exactly n crossed product decompositions up to conjugacy of the actions. We were also able to find II_1 factors with exactly n gms Cartan subalgebras up to *unitary* conjugacy but only in the case where n is a power of 2.

It would be very interesting to find examples of II_1 factors for which all Cartan subalgebras can be determined, not only the ones of group measure space type. However, one would need an entirely different approach since our proof relies on the existence of a dual coaction $\Delta\colon M \to M \mathbin{\overline{\otimes}} M$ associated to a crossed product decomposition of M.

In this thesis, we also addressed the question of when a group Γ is $\mathcal{C}$-rigid, meaning that any crossed product $L^\infty(X) \rtimes \Gamma$ by a free ergodic pmp action of Γ has a unique Cartan subalgebra up to unitary conjugacy. It is a big open problem to characterize the class of $\mathcal{C}$-rigid groups and for this reason, one is interested in finding interesting counterexamples to $\mathcal{C}$-rigidity. At the moment, all known counterexamples have an infinite amenable almost normal subgroup. We showed that all groups with an infinite abelian normal subgroup are non-$\mathcal{C}$-rigid. Moreover, we showed that all groups Γ with an infinite abelian *almost normal* subgroup satisfy a slightly weaker property: There exists a finite normal subgroup $F < \Gamma$ such that Γ/F is non-$\mathcal{C}$-rigid. Since $\mathcal{C}$-rigidity is not known to satisfy any "finite index" stability properties, we were unable to conclude that Γ itself is non-$\mathcal{C}$-rigid.

It is expected that there are also other counterexamples to $\mathcal{C}$-rigidity, that is, groups without an infinite abelian almost normal subgroup. It would be interesting to look for such examples in order to get a better understanding of $\mathcal{C}$-rigidity.

Finally, we presented new examples of II_1 factors without Cartan subalgebras. The motivation for this work was a question raised by Popa: does there exist s-thin II_1 factors without Cartan subalgebras? The s-thin approximation property was introduced by Popa in search of an intrinsic criterion for a II_1 factor to have a Cartan subalgebra. We answered Popa's question affirmatively by finding concrete examples of s-thin II_1 factors that are even strongly solid. In order to construct these examples, we studied the von Neumann algebras associated with Shlyakhtenko's A-valued semicircular systems in the case where A is a tracial von Neumann algebra. By applying Popa's deformation/rigidity theory, we proved general structural properties of such von Neumann algebras, and under certain conditions, we were able to show that the associated von Neumann algebra is a non-amenable, s-thin and strongly solid II_1 factor. Concrete examples of A-valued semicircular systems satisfying these conditions were constructed from c_0 probability measures supported on free subsets of compact groups. We were able to prove the existence of such measures in an ad hoc way and conjectured that such measures should also exist on the natural compact groups $\mathrm{SO}(n)$ for $n \geq 3$.

Appendix A

Spectral gap rigidity for co-induced actions

In [Po03, Po04], Popa discovered that all Bernoulli actions $\Gamma \curvearrowright (X, \mu)$ have a remarkable deformation property: the flip automorphism on $X \times X$ can be continuously deformed into the identity via a one-parameter group of automorphisms that commute with the Bernoulli action. This property is called *malleability.* In combination with certain rigidity properties, this allowed Popa to prove powerful W*-rigidity results for Bernoulli crossed products $L^\infty(X) \rtimes \Gamma$. In [Po05], the rigidity is given by Kazhdan's property (T) of the group Γ, whereas [Po06b] uses spectral gap rigidity of the Bernoulli action in the case where Γ is a direct product of two non-amenable groups.

In this appendix, we will show how Popa's malleability and spectral gap rigidity (more precisely, the generalization provided in [BV14, Theorems 3.1 and 3.3]) carries over to the case where $\Gamma \curvearrowright (X, \mu)$ is a *co-induced action.* In fact, the results carry over almost verbatim but we will nevertheless provide a full argument for the sake of completeness. In the end of this appendix, we will prove some useful results for controlling quasi-normalizers inside crossed products associated with co-induced actions. These results are also direct generalizations of the same results for Bernoulli actions.

Given a tracial von Neumann algebra (A_0, τ_0) and a countable set I, we denote by $(A_0, \tau_0)^I$ (or just A_0^I) the von Neumann algebra tensor product $\overline{\bigotimes}_I (A_0, \tau_0)$. For each $i \in I$, we denote by $\pi_i \colon A_0 \to A_0^I$ the embedding of A_0 as the i'th tensor factor. When $\Lambda < \Gamma$ is a subgroup with a trace-preserving action $\Lambda \curvearrowright (A_0, \tau_0)$, we get a co-induced action $\Gamma \curvearrowright (A_0, \tau_0)^{\Lambda \backslash \Gamma}$ as defined in Section 3.1. In this appendix, we will view co-induced actions as being actions “build over a set”, in the following sense.

Definition A.1. Let $\Gamma \curvearrowright I$ be an action of a countable group Γ on a countable set I. We say that a trace-preserving action $\sigma\colon \Gamma \curvearrowright (A_0, \tau_0)^I$ is *built over* $\Gamma \curvearrowright I$ if it satisfies

$$\sigma_g(\pi_i(A_0)) = \pi_{g\cdot i}(A_0) \quad \text{for all } \; g \in \Gamma,\, i \in I\;.$$

Assume that $\Gamma \curvearrowright A_0^I$ is an action built over $\Gamma \curvearrowright I$. Choose a subset $J \subset I$ that contains exactly one point in every orbit of $\Gamma \curvearrowright I$. For every $j \in J$, the group $\operatorname{Stab} j$ globally preserves $\pi_j(A_0)$. This defines an action $\operatorname{Stab} j \curvearrowright A_0$ that can be *co-induced* to an action $\Gamma \curvearrowright A_0^{\Gamma/\operatorname{Stab} j}$. The original action $\Gamma \curvearrowright A_0^I$ is conjugate with the direct product of all these co-induced actions. In particular, co-induced actions are exactly actions built over a transitive action $\Gamma \curvearrowright I = \Gamma/\Gamma_0$.

A.1 The tensor length deformation

The tensor length deformation is a variant of Popa’s malleable deformation for Bernoulli crossed products, due to [Io06]. This deformation can be defined more generally for crossed products coming from co-induced actions (or rather actions built over $\Gamma \curvearrowright I$ as defined in Definition A.1) as follows.

Let $\Gamma \curvearrowright I$ be an action on a countable set I and let $\sigma\colon \Gamma \curvearrowright (A_0, \tau)^I$ be an action built over $\Gamma \curvearrowright I$. Let $M = A_0^I \rtimes \Gamma$ be the corresponding crossed product. We can extend the action σ to an action $\tilde{\sigma}\colon \Gamma \curvearrowright (A_0 * L(\mathbb{Z}))^I$ uniquely in such a way that

$$\tilde{\sigma}_g(\pi_i(b)) = \pi_{g\cdot i}(b) \quad \text{for all } \; b \in L(\mathbb{Z}),\; g \in \Gamma,\; i \in I.$$

Indeed, for a fixed $g \in \Gamma$, the automorphism σ_g of A_0^I is the composition of the shift automorphism $\pi_i(a) \mapsto \pi_{g\cdot i}(a)$ with a certain tensor product automorphism $\bigotimes_{i\in I} \sigma_g^i$ with $\sigma_g^i \in \operatorname{Aut}(A_0)$. We can then define $\tilde{\sigma}_g$ to be the composition of $\bigotimes_{i\in I}(\sigma_g^i * \mathrm{id})$ with the same shift automorphism.

Let $\widetilde{M} = (A_0 * L(\mathbb{Z}))^I \rtimes \Gamma$ be the crossed product associated with $\tilde{\sigma}$. Let $u_1 \in L(\mathbb{Z})$ be the canonical generating unitary and let $h \in L(\mathbb{Z})$ be the self-adjoint element with spectrum $[-\pi, \pi]$ such that $u_1 = \exp(ih)$. For $t \in \mathbb{R}$, we put $u_t = \exp(ith)$. Then $(u_t)_{t\in\mathbb{R}}$ is a one-parameter group of unitaries in $L(\mathbb{Z})$ with $|\tau(u_t)| < 1$ for all $t \neq 0$.

We still denote by π_i the embedding as the i’th tensor factor $A_0 * L(\mathbb{Z}) \to (A_0 * L(\mathbb{Z}))^I$ for $i \in I$. We can then define automorphisms $\alpha_t \in \operatorname{Aut}(\widetilde{M})$, $t \in \mathbb{R}$, by $\alpha_t(u_g) = u_g$ for $g \in \Gamma$ and $\alpha_t(\pi_i(x)) = \pi_i(u_t x u_t^*)$ for $x \in A_0 * L(\mathbb{Z})$. Note that this is well defined since Γ acts trivially on $L(\mathbb{Z})$.

Now, $\widetilde{M}$ together with $(\alpha_t)_{t\in\mathbb{R}}$ is a malleable deformation called the *tensor length deformation.* To explain this terminology, consider the completely positive map $\psi_t\colon M \to M$ given by $\psi_t = E_M \circ \alpha_t$. Let $\rho_t = |\tau(u_t)|^2$ and note that $0 \leq \rho_t < 1$ for $t \neq 0$. Whenever $a \in A_0^I$ is an elementary tensor given by $a = \bigotimes_{i\in F} a_i$ with $a_i \in A_0 \ominus \mathbb{C}1$ and $F \subset I$ a finite set, we have

$$\psi_t(au_g) = \rho_t^{|F|} au_g, \quad t \in \mathbb{R},\ g \in \Gamma.$$

Note that the tensor length deformation is s-malleable in the sense of [Po06a, Section 6]. Indeed, we can define an automorphism $\beta \in \mathrm{Aut}(\widetilde{M})$ by $\beta(x) = x$ for $x \in M$ and $\beta(\pi_i(u_1)) = \pi_i(u_1^*)$ for $i \in I$. This is well defined since Γ acts trivially on $L(\mathbb{Z})$. By construction, $\beta^2 = \mathrm{id}$ and $\beta \circ \alpha_t = \alpha_{-t} \circ \beta$, meaning that the deformation $(\alpha_t)_{t\in\mathbb{R}}$ is s-malleable.

A.2 Spectral gap rigidity

Using the tensor length deformation, Popa's spectral gap rigidity [Po06b] applies to actions built over $\Gamma \curvearrowright I$. The generalization provided in [BV14, Theorems 3.1 and 3.3] carries over verbatim and this gives the following result. For completeness, we include a full proof.

Whenever $\Gamma \curvearrowright I$ and $F \subset I$, we denote by $\mathrm{Stab}(F)$ the subgroup given by $\mathrm{Stab}(F) = \{g \in \Gamma \mid g \cdot i = i \text{ for all } i \in F\}$, and we denote by $\mathrm{Norm}(F)$ the subgroup given by $\mathrm{Norm}(F) = \{g \in \Gamma \mid g \cdot F = F\}$.

Theorem A.2. *Let $\Gamma \curvearrowright I$ be an action of an icc group on a countable set. Assume that $\mathrm{Stab}\{i,j\}$ is amenable for all $i, j \in I$ with $i \neq j$. Let (A_0, τ_0) be a tracial von Neumann algebra and (N, τ) a tracial factor. Let $\Gamma \curvearrowright (A_0, \tau_0)^I$ be an action built over $\Gamma \curvearrowright I$ and put $M = A_0^I \rtimes \Gamma$.*

If $P \subset N \mathbin{\overline{\otimes}} M$ is a von Neumann subalgebra that is strongly non-amenable relative to $N \mathbin{\overline{\otimes}} A_0^I$, then the relative commutant $Q := P' \cap N \mathbin{\overline{\otimes}} M$ satisfies at least one of the following properties:

(1) *there exists an $i \in I$ such that $Q \prec N \mathbin{\overline{\otimes}} (A_0^I \rtimes \mathrm{Stab}\, i)$;*

(2) *there exists a unitary $v \in N \mathbin{\overline{\otimes}} M$ such that $v^* Q v \subset N \mathbin{\overline{\otimes}} L(\Gamma)$.*

For the rest of this section, fix M as in Theorem A.2 and let $\widetilde{M}$, $(\alpha_t)_{t\in\mathbb{R}}$ denote the tensor length deformation as defined in Section A.1. Me moreover put

$\mathcal{M} = N \mathbin{\overline{\otimes}} M$ and $\widetilde{\mathcal{M}} = N \mathbin{\overline{\otimes}} \widetilde{M}$. We will still denote by α_t the automorphism $\mathrm{id} \otimes \alpha_t$ on $\widetilde{\mathcal{M}}$.

The first step in the proof of Theorem A.2 is to show that the von Neumann subalgebra Q is "rigid" in the sense that $\alpha_t \to \mathrm{id}$ uniformly on Q. This is where spectral gap rigidity comes into play. An M-bimodule ${}_M H_M$ is said to have *spectral gap* if it does not weakly contain the trivial M-bimodule, i.e., if it has no almost M-central and almost tracial vectors. In our case, the spectral gap comes from the bimodule K defined in the following lemma. We show that K is essentially coarse relative to $N \mathbin{\overline{\otimes}} A_0^I$, so that ${}_P K_P$ has spectral gap by our non-amenability assumption on P.

Lemma A.3 ([BV14, Lemma 3.2]). *Let ${}_{\mathcal{M}} K_{\mathcal{M}} \subset {}_{\mathcal{M}} L^2(\widetilde{\mathcal{M}} \ominus \mathcal{M})_{\mathcal{M}}$ be the $\mathcal{M}$-subbimodule given by*

$$K = \overline{\mathrm{span}} \left\{ x \otimes \pi_{\mathcal{F}}(a) u_g \;\middle|\; \begin{array}{l} x \in N,\ g \in \Gamma,\ \mathcal{F} \subset I,\ 2 \leq |\mathcal{F}| < \infty, \\ a = \bigotimes_{i \in F} a_i \textit{ with } a_i \in A_0 * L\mathbb{Z} \textit{ for all } i \\ \textit{and with } a_i \in A_0 * L\mathbb{Z} \ominus A_0 \textit{ for at} \\ \textit{least } 2 \textit{ elements } i \in F \end{array} \right\}.$$

Put $\mathcal{M}_0 = N \mathbin{\overline{\otimes}} A_0^I \subset \mathcal{M}$. Then there exists an $\mathcal{M}_0$-$\mathcal{M}$-bimodule H such that ${}_{\mathcal{M}} K_{\mathcal{M}}$ is weakly contained in ${}_{\mathcal{M}}(L^2(\mathcal{M}) \otimes_{\mathcal{M}_0} H)_{\mathcal{M}}$.

Proof. Let $S \subset \mathbb{N}^{\mathbb{N}}$ denote the set of all sequences $(s_n)_{n \in \mathbb{N}}$ with $s_n \in \mathbb{N}$ for which there exists $k \in \mathbb{N}$ such that $s_n = 0$ for $n \geq k$ and $s_n \neq 0$ for $n < k$. Given such a sequence $s \in S$, we can associate an A_0-subbimodule $H_s \subset L^2(A_0 * L\mathbb{Z})$ given by

$$H_s = \overline{\mathrm{span}}\, A_0 u_1^{s_1} (A_0 \ominus \mathbb{C}1) u_1^{s_2} (A_0 \ominus \mathbb{C}1) \cdots (A_0 \ominus \mathbb{C}1) u_1^{s_k} A_0,$$

where $k \in \mathbb{N}$ is the smallest number such that $s_n = 0$ for all $n > k$. Then, $L^2(A_0 * L\mathbb{Z}) = \bigoplus_{s \in S} H_s$ as A_0-bimodules.

Given a finite subset $\mathcal{F} \subset I$ with $|\mathcal{F}| \geq 2$ and nontrivial sequences $s_i \in S$ for each $i \in \mathcal{F}$, we let $K_{(s_i)_{i \in \mathcal{F}}}$ denote the $\mathcal{M}$-subbimodule of K given by

$$K_{(s_i)_{i \in \mathcal{F}}} = \overline{\mathrm{span}}\, \mathcal{M}\big(1 \otimes \pi_{\mathcal{F}}\big(\otimes_{i \in \mathcal{F}} H_{s_i}\big)\big)\mathcal{M}.$$

Define the subgroup $\Lambda < \Gamma$ given by

$$\Lambda := \{ g \in \Gamma \mid g \cdot \mathcal{F} = \mathcal{F} \text{ and } s_{g \cdot i} = s_i \text{ for all } i \in \mathcal{F} \},$$

and let $Q = N \mathbin{\overline{\otimes}} (A_0^{I \setminus \mathcal{F}} \rtimes \Lambda)$. Note that $u_g K_{(s_i)_{i \in \mathcal{F}}} u_g^* = K_{(s_i)_{i \in \mathcal{F}}}$ for all $g \in \Lambda$. We claim that $K_{(s_i)_{i \in \mathcal{F}}}$ is contained in the $\mathcal{M}$-bimodule

$$L^2(\mathcal{M}) \otimes_Q L^2(\widetilde{\mathcal{M}}) \otimes_Q L^2(\mathcal{M}).$$

To see this, let $c, c' \in K_{(s_i)_{i\in\mathcal{F}}}$ be elements of the form $c = 1 \otimes \pi_{\mathcal{F}}(\otimes_{i\in\mathcal{F}} c_i)$ and $c' = 1 \otimes \pi_{\mathcal{F}}(\otimes_{i\in\mathcal{F}} c_i')$ with $c_i, c_i' \in u_1^{(s_i)_1}(A_0 \ominus \mathbb{C}1)u_1^{(s_i)_2} \cdots (A_0 \ominus \mathbb{C}1)u_1^{(s_i)_k} \subset H_{s_i}$ for all $i \in \mathcal{F}$. One easily checks that

$$\langle xcy, c' \rangle = \langle E_Q(x) c E_Q(y), c' \rangle = \langle x \otimes_Q c \otimes_Q y, 1 \otimes_Q c' \otimes_Q 1 \rangle$$

for all $x, y \in \mathcal{M}$. Since the union of all subsets $\mathcal{M}c\mathcal{M}$ with c as above spans a dense subspace of $K_{(s_i)_{i\in\mathcal{F}}}$, this implies that $K_{(s_i)_{i\in\mathcal{F}}}$ is contained in $L^2(\mathcal{M}) \otimes_Q L^2(\widetilde{\mathcal{M}}) \otimes_Q L^2(\mathcal{M})$, as $\mathcal{M}$-bimodules.

Given two finite subsets $\mathcal{F}, \mathcal{F}' \subset I$ with $|\mathcal{F}|, |\mathcal{F}'| \geq 2$ and sequences $(s_i)_{i\in\mathcal{F}}$, $(s_i')_{i\in\mathcal{F}'}$ as above, we have that exactly one of the following conditions holds.

1. There exists $g \in \Gamma$ such that $g \cdot \mathcal{F} = \mathcal{F}'$ and $s'_{g\cdot i} = s_i$ for all $i \in \mathcal{F}$. In this case, $K_{(s_i')_{i\in\mathcal{F}'}} = u_g K_{(s_i)_{i\in\mathcal{F}}} u_g^*$ so that $K_{(s_i)_{i\in\mathcal{F}}} = K_{(s_i')_{i\in\mathcal{F}'}}$.
2. There exists no such $g \in \Gamma$. In this case, $K_{(s_i)_{i\in\mathcal{F}}} \perp K_{(s_i')_{i\in\mathcal{F}'}}$.

Since the $K_{(s_i)_{i\in\mathcal{F}}}$'s moreover span a dense subspace of K, we can choose a sequence of $\mathcal{M}$-subbimodules $K_n \subset K$ of the form $K_n = K_{(s_i^n)_{i\in\mathcal{F}_n}}$ for some $\mathcal{F}_n$ and some $(s_i^n)_{i\in\mathcal{F}_n}$, such that $K = \bigoplus_n K_n$. Writing as above

$$\Lambda_n = \{g \in \Gamma \mid g \cdot \mathcal{F}_n = \mathcal{F}_n \text{ and } s^n_{g\cdot i} = s^n_i \text{ for all } i \in \mathcal{F}_n\},$$

$$Q_n = N \mathbin{\overline{\otimes}} (A_0^{I\setminus\mathcal{F}_n} \rtimes \Lambda_n),$$

we then have that

$$K \subset \bigoplus_n L^2(\mathcal{M}) \otimes_{Q_n} L^2(\widetilde{\mathcal{M}}) \otimes_{Q_n} L^2(\mathcal{M}).$$

Note that $\mathrm{Stab}(\mathcal{F}_n) < \Lambda_n$ is a subgroup of finite index. Since $|\mathcal{F}_n| \geq 2$, we have by assumption that $\mathrm{Stab}(\mathcal{F}_n)$ is amenable. Hence, Λ_n is also amenable. It now follows from Lemma [MP03, Proposition 6] that $N \mathbin{\overline{\otimes}} (A_0^I \rtimes \Lambda_n)$ is amenable relative to $N \mathbin{\overline{\otimes}} A_0^I = \mathcal{M}_0$. In particular, Q_n is amenable relative to $\mathcal{M}_0$. By [PV11, Proposition 2.4], this means that ${}_{\mathcal{M}}L^2(\mathcal{M})_{Q_n}$ is weakly contained in $L^2(\mathcal{M}) \otimes_{\mathcal{M}_0} L^2(\mathcal{M})$ for all $n \in \mathbb{N}$. So, if we let $H = \bigoplus_n L^2(\mathcal{M}) \otimes_{Q_n} L^2(\widetilde{\mathcal{M}}) \otimes_{Q_n} L^2(\mathcal{M})$ as an $\mathcal{M}_0$-$\mathcal{M}$-bimodule, we have that K is weakly contained in $L^2(\mathcal{M}) \otimes_{\mathcal{M}_0} H$. □

Proof of Theorem A.2. Let K be the $\mathcal{M}$-bimodule from Lemma A.3 and let P_K denote the projection of $L^2(\widetilde{\mathcal{M}})$ onto K. We start by proving the following claim.

Claim I. If $\sup_{b\in\mathcal{U}(Q)} \|P_K(\alpha_t(b))\|_2 \to 0$ as $t \to 0$, then also

$$\sup_{b\in\mathcal{U}(Q)} \|\alpha_t(b) - b\|_2 \to 0 \quad \text{as } \ t \to 0.$$

For $n \geq 0$, let $H_n \subset L^2(\mathcal{M})$ be the closed subspace defined by

$$H_n = \overline{\operatorname{span}} \left\{ x \otimes \pi_{\mathcal{F}}(a) u_g \;\middle|\; \begin{array}{l} x \in N,\ g \in \Gamma,\ \mathcal{F} \subset I,\ |\mathcal{F}| = n, \\ a = \bigotimes_{i\in F} a_i \text{ with } a_i \in A_0 \ominus \mathbb{C}1 \text{ for all } i \in F \end{array} \right\},$$

and let P_n denote the orthogonal projection of $L^2(\mathcal{M})$ onto H_n. Note that $L^2(\mathcal{M}) = \bigoplus_{n\geq 0} H_n$. By definition of the tensor length deformation α_t, we have that

$$\|\alpha_t(b) - b\|_2^2 = \sum_{n=0}^{\infty} 2(1 - \rho_t^n)\|P_n(b)\|_2^2 \quad \text{for all } \ b \in \mathcal{U}(Q),$$

where $\rho_t = |\tau(u_t)|^2$.

Exactly as in the proof of [BV14, Theorem 3.1], we get the following formula for $\|P_K(\alpha_t(y))\|_2$ when $y \in \mathcal{M}$.

$$\|P_K(\alpha_t(y))\|_2^2 = \sum_{n=0}^{\infty}(1 - c(t,n))\|P_n(y)\|_2^2 \quad \text{for all } \ y \in \mathcal{M}, \tag{A.1}$$

where

$$c(t,n) = \begin{cases} \rho_t^{2n} + n(1-\rho_t^2)\rho_t^{2(n-1)} & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Note that $c(t,n) \to 0$ as $n \to \infty$ for fixed $t > 0$, since $0 \leq \rho_t < 1$.

To prove the claim, assume that $\sup_{b\in\mathcal{U}(Q)} \|P_K(\alpha_t(b))\|_2 \to 0$ as $t \to 0$. Given $\varepsilon > 0$, choose $t > 0$ such that $\|P_K(\alpha_t(b))\|_2 < \varepsilon$ for all $b \in \mathcal{U}(Q)$. Then take $n_0 \in \mathbb{N}$ such that $c(t,n) < \frac{1}{2}$ for $n \geq n_0$. By (A.1), we get that

$$\frac{1}{2} \sum_{n=n_0}^{\infty} \|P_n(b)\|_2^2 \leq \|P_K(\alpha_t(b))\|_2^2 < \varepsilon^2 \quad \text{for all } \ b \in \mathcal{U}(Q).$$

Moreover, we choose $s_0 > 0$ such that $1 - \rho_s^n < \varepsilon^2$ for $|s| < s_0$ and $0 \leq n < n_0$. Then, for all $b \in \mathcal{U}(Q)$ and $|s| < s_0$, we get

$$\begin{aligned} \|\alpha_s(b) - b\|_2^2 &= \sum_{n=0}^{\infty} 2(1-\rho_s^n)\|P_n(b)\|_2^2 \\ &\leq \sum_{n=0}^{n_0-1} 2\varepsilon^2 \|P_n(b)\|_2^2 + 2\sum_{n=n_0}^{\infty} \|P_n(b)\|_2^2 \leq 2\varepsilon^2 + 4\varepsilon^2. \end{aligned}$$

This proves the claim.

Next, assume for contradiction that $\sup_{b\in\mathcal{U}(Q)} \|\alpha_t(b) - b\|_2$ does not converge to zero as $t \to 0$. By the claim above, we find $\varepsilon > 0$ and $t_0 > 0$ such that for every $0 < t < t_0$, there exists $b_t \in \mathcal{U}(Q)$ with $\|P_K(\alpha_t(b_t))\|_2 \geq \varepsilon$. Put $\xi_t = P_K(\alpha_t(b_t))$. Using that P_K is $\mathcal{M}$-bimodular and that $b_t \in Q = P' \cap \mathcal{M}$, we get that $\|x\xi_t - \xi_t x\|_2 \to 0$ as $t \to 0$ for each fixed $x \in P$. Indeed,

$$\begin{aligned}\|x\xi_t - \xi_t x\|_2 &\leq \|x\alpha_t(b_t) - \alpha_t(b_t)x\|_2 = \|\alpha_{-t}(x)b_t - b_t\alpha_{-t}(x)\|_2 \\ &\leq 2\|\alpha_{-t}(x) - x\|_2 + \|xb_t - b_t x\|_2 = 2\|\alpha_{-t}(x) - x\|_2 \to 0.\end{aligned}$$

We also have that $\|x\xi_t\|_2 \leq \|x\|_2$, since b_t is a unitary. From Lemma [BV14, Lemma 2.10], it now follows that there exists a nonzero projection $q \in P' \cap \mathcal{M}$ such that the $q\mathcal{M}q$-$\mathcal{M}$-bimodule qK is left Pq-amenable. By Lemma A.3, we have that qK is weakly contained in $qL^2(\mathcal{M}) \otimes_{\mathcal{M}_0} H$ so [PV11, Corollary 2.5] gives that $qL^2(\mathcal{M}) \otimes_{\mathcal{M}_0} H$ is left Pq-amenable. But then also ${}_{q\mathcal{M}q}(qL^2(\mathcal{M}))_{\mathcal{M}_0}$ is left Pq-amenable by [PV11, Proposition 2.4.4], which exactly means that Pq is amenable relative to $\mathcal{M}_0$. This contradicts our assumptions on P.

We have now shown that

$$\sup_{b\in\mathcal{U}(Q)} \|\alpha_t(b) - b\|_2 \to 0 \quad \text{as } \ t \to 0.$$

To prove Theorem A.2, assume that Q does not satisfy (1), i.e., $Q \not\prec N \mathbin{\overline{\otimes}} (A_0^I \rtimes \operatorname{Stab} i)$ for all $i \in I$. We then have to show that Q satisfies (2), i.e., we have to find a unitary $v \in \mathcal{M}$ such that $v^*Qv \subset N \mathbin{\overline{\otimes}} L(\Gamma)$. We start by proving the following claim.

Claim II. There exists a nonzero partial isometry $v \in \mathcal{M}$ such that $vv^* \in Q' \cap \mathcal{M}$ and $v^*Qv \subset N \mathbin{\overline{\otimes}} L(\Gamma)$.

Take $t > 0$ of the form $t = 2^{-n}$ such that $\|\alpha_t(b) - b\|_2 \leq \frac{1}{2}$ for all $b \in \mathcal{U}(Q)$. Then $|\tau(b\alpha_t(b)^*)| \geq \frac{7}{8}$ for all $b \in \mathcal{U}(Q)$. Let y be the unique element of minimal 2-norm in the weakly closed convex hull of $\{b\alpha_t(b)^* \mid b \in \mathcal{U}(Q)\}$. Then $|\tau(y)| \geq \frac{7}{8}$ so y is nonzero, and $by\alpha_t(b)^* = y$ for all $b \in \mathcal{U}(Q)$ by uniqueness of y. Letting $w_0 \in \widetilde{\mathcal{M}}$ be the partial isometry from the polar decomposition of y, we get that

$$xw_0 = w_0\alpha_t(x) \quad \text{for all } \ x \in Q.$$

Using the s-malleability of α_t, we can even obtain $t = 1$ in the equality above. Indeed, let $\beta \in \operatorname{Aut}(\widetilde{M})$ be the automorphism defined in Section A.1. Recall that $\beta^2 = \mathrm{id}$, $\beta(x) = x$ for $x \in M$ and that $\beta \circ \alpha_t = \alpha_{-t} \circ \beta$. Put $w_1 = \alpha_t(\beta(w_0^*)w_0) \in \widetilde{\mathcal{M}}$. Since $Q \not\prec N \mathbin{\overline{\otimes}} (A_0^I \rtimes \operatorname{Stab} i)$ for all $i \in I$, it follows from Lemma A.4(1) that $w_0w_0^* \in \mathcal{M}$. It follows that w_1 is a nonzero partial isometry and one easily checks that $xw_1 = w_1\alpha_{2t}(x)$ for all $x \in Q$. So, we have doubled t. Since $t = 2^{-n}$, we can continue inductively in this way and obtain a nonzero partial isometry $w = w_n \in \widetilde{\mathcal{M}}$ such that

$$xw = w\alpha_1(x) \quad \text{for all } \ x \in Q. \tag{A.2}$$

Next, we show that there exists a finite subset $\mathcal{F} \subset I$ (possibly empty) such that $Q \prec N \mathbin{\overline{\otimes}} (A_0^{\mathcal{F}} \rtimes \operatorname{Stab} \mathcal{F})$. Assume that this is not the case. By Theorem 2.12 (and [Va07, Remark 3.3]), this means that there exists a net of unitaries $v_n \in Q$ such that

$$\|E_{N \overline{\otimes} (A_0^{\mathcal{F}} \rtimes \operatorname{Stab} \mathcal{F})}(av_nb^*)\|_2 \to 0 \quad \text{for all } \ a, b \in \mathcal{M} \ \text{ and } \ \mathcal{F} \subset I \ \text{ finite.}$$

From this, we will deduce that

$$\|E_{\mathcal{M}}(x\alpha_1(v_n)y^*)\|_2 \to 0 \quad \text{for all } \ x, y \in \widetilde{\mathcal{M}}. \tag{A.3}$$

Let $S \subset \widetilde{\mathcal{M}}$ denote the set of all elements of the form $1_N \otimes \pi_{\mathcal{F}}(\otimes_{i \in \mathcal{F}} x_i)$, where $\mathcal{F} \subset I$ is finite and $x_i \in (A_0 * L\mathbb{Z}) \ominus A_0\alpha_1(A_0)$ for all $i \in \mathcal{F}$. Note that $\operatorname{span}\{\mathcal{M}x\alpha_1(A_0^I) \mid x \in S\}$ forms a $\|\cdot\|_2$-dense subalgebra of $\widetilde{\mathcal{M}}$. So, it is enough to show (A.3) for $x, y \in S$.

Let $x, y \in S$ and write $x = 1 \otimes \pi_{\mathcal{F}}(\otimes x_i)$ and $y = 1 \otimes \pi_{\mathcal{G}}(\otimes y_j)$ with $\mathcal{F}, \mathcal{G} \subset I$ finite subsets and $x_i, y_j \in (A_0 * L\mathbb{Z}) \ominus A_0\alpha_1(A_0)$ for $i \in \mathcal{F}$, $j \in \mathcal{G}$. We also write $v_n = \sum_{g \in \Gamma} (v_n)^g u_g$ with $(v_n)^g \in N \mathbin{\overline{\otimes}} A_0^I = \mathcal{M}_0$ for the Fourier decomposition of v_n. Then

$$\begin{aligned} E_{\mathcal{M}}(x\alpha_1(v_n)y^*) &= \sum_{g \in \Gamma} E_{\mathcal{M}}(x\alpha_1((v_n)^g)\sigma_g(y^*)u_g) \\ &= \sum_{g \in \Gamma} E_{\mathcal{M}_0}(x\alpha_1((v_n)^g)\sigma_g(y^*))u_g. \end{aligned}$$

Note that $E_{\mathcal{M}_0}(x\alpha_1((v_n)^g)\sigma_g(y^*)) = 0$ if $g \cdot \mathcal{G} \neq \mathcal{F}$. So, it is enough to sum over all $g \in \Gamma$ with $g \cdot \mathcal{G} = \mathcal{F}$. Since $\mathcal{F}$ is finite, we can take a finite set $\{g_1, \ldots, g_k\} \subset \Gamma$ such that $\{g \mid g \cdot \mathcal{G} = \mathcal{F}\}$ is the disjoint union of the sets $\operatorname{Stab}(\mathcal{F})g_i$ for $i = 1, \ldots, k$.

Now,

$$\begin{aligned} E_{\mathcal{M}}(x\alpha_1(v_n)y^*) &= \sum_{i=1}^{k} \sum_{s\in \operatorname{Stab}\mathcal{F}} E_{\mathcal{M}_0}(x\alpha_1(E_{N\overline{\otimes}A_0^{\mathcal{F}}}((v_n)^{sg_i}))\sigma_{sg_i}(y^*))u_{sg_i} \\ &= \sum_{i=1}^{k} \sum_{s\in \operatorname{Stab}\mathcal{F}} E_{\mathcal{M}}(x\alpha_1(E_{N\overline{\otimes}A_0^{\mathcal{F}}}((v_n)^{sg_i})u_s)\sigma_{g_i}(y^*))u_{g_i} \\ &= \sum_{i=1}^{k} E_{\mathcal{M}}(x\alpha_1(E_{N\overline{\otimes}(A_0^{\mathcal{F}}\rtimes \operatorname{Stab}\mathcal{F})}(v_n u_{g_i}^*))\sigma_{g_i}(y^*))u_{g_i} \\ &= E_{\mathcal{M}}(x\alpha_1(z_n)y^*), \end{aligned}$$

where $z_n = \sum_{i=1}^{k} E_{N\overline{\otimes}(A_0^{\mathcal{F}}\rtimes \operatorname{Stab}\mathcal{F})}(v_n u_{g_i}^*)u_{g_i}$. By assumption, we have that $\|z_n\|_2 \to 0$ and hence (A.3) follows.

Applying (A.3) to $x = y = w$, where w is the partial isometry from (A.2), yields

$$\|E_{\mathcal{M}}(w\alpha_1(v_n)w^*)\|_2 = \|v_n E_{\mathcal{M}}(ww^*)\|_2 = \|E_{\mathcal{M}}(ww^*)\|_2 \to 0,$$

contradicting the fact that $w \neq 0$.

We conclude that there is a finite subset $\mathcal{F} \subset I$ such that $Q \prec N\overline{\otimes}(A_0^{\mathcal{F}} \rtimes \operatorname{Stab}\mathcal{F})$. Our assumption that $Q \not\prec N \overline{\otimes} (A_0^I \rtimes \operatorname{Stab} i)$ for all $i \in I$ implies that $\mathcal{F}$ must be empty. This means that $Q \prec N \overline{\otimes} L(\Gamma)$.

By Theorem 2.12, there exists a $*$-homomorphism $\theta\colon Q \to q(M_n(\mathbb{C})\otimes N\overline{\otimes}L(\Gamma))q$ for some $n \in \mathbb{N}$ and projection $q \in M_n(\mathbb{C}) \otimes N \overline{\otimes} L(\Gamma)$, and there exists a nonzero partial isometry $v \in M_{1,n}(\mathbb{C}) \otimes \mathcal{M}$ such that $xv = v\theta(x)$ for all $x \in Q$. Since $Q \not\prec N \overline{\otimes} (A_0^I \rtimes \operatorname{Stab} i)$ for all $i \in I$, we may assume that also $\theta(Q) \not\prec N \overline{\otimes} L(\operatorname{Stab} i)$, by [Va07, Remark 3.8]. By Lemma A.4 (1), it follows that

$$\theta(Q)' \cap q(M_n(\mathbb{C}) \otimes \mathcal{M})q \subset M_n(\mathbb{C}) \otimes N \overline{\otimes} L(\Gamma).$$

In particular, $v^*v \in M_n(\mathbb{C}) \otimes N \overline{\otimes} L(\Gamma)$. We also have that $(\operatorname{Tr}\otimes\tau)(v^*v) = \tau(vv^*) \leq 1$ so since $N \overline{\otimes} L(\Gamma)$ is a II_1 factor, we may assume that $n = 1$. Then $v \in \mathcal{M}$ does the job: it is a nonzero partial isometry such that $vv^* \in Q' \cap \mathcal{M}$ and $v^*Qv \subset N \overline{\otimes} L(\Gamma)$. This finishes the proof of Claim II.

To get a unitary instead of merely a partial isometry, we use a simple maximality argument. Let $(v_j)_{j\in J} \subset \mathcal{M}$ be a maximal family of nonzero partial isometries such that $v_j v_j^* \in Q' \cap \mathcal{M}$ are mutually orthogonal and $v_j^* Q v_j \subset N \overline{\otimes} L(\Gamma)$ for all j. By Claim II (applied to $Q(1 - \sum_j v_j v_j^*)$) and by maximality of $(v_j)_{j\in J}$, we have that $\sum_j v_j v_j^* = 1$. Since $v_j^* v_j \in N \overline{\otimes} L(\Gamma)$ and since $N \overline{\otimes} L(\Gamma)$ is a II_1

factor, we can take partial isometries $w_j \in N \mathbin{\overline{\otimes}} L(\Gamma)$ such that $w_j w_j^* = v_j^* v_j$ and such that the projections $w_j^* w_j$ are mutually orthogonal. It is now easy to check that $u := \sum_j v_j w_j$ is a unitary in $\mathcal{M}$ such that $u^* Q u \subset N \mathbin{\overline{\otimes}} L(\Gamma)$. □

A.3 Controlling quasi-normalizers

Recall that for a von Neumann subalgebra $P \subset M$, we define $\mathcal{QN}_M(P) \subset M$ as the set of elements $x \in M$ for which there exist $x_1, \dots, x_n, y_1, \dots, y_m \in M$ satisfying

$$xP \subset \sum_{i=1}^{n} P x_i \quad \text{and} \quad Px \subset \sum_{j=1}^{m} y_j P \; .$$

Then $\mathcal{QN}_M(P)$ is a $*$-subalgebra of M containing P. Its weak closure is called the *quasi-normalizer* of P inside M.

The following lemma is proved in exactly the same way as [IPV10, Lemma 4.1] and goes back to [Po03, Theorem 3.1]. We will provide a proof, which is essentially the same as the proof of [Va07, Lemma 4.2].

Lemma A.4. *Let $A_0 \subset B_0$ be tracial von Neumann algebras and let $\Gamma \curvearrowright B_0^I$ be an action built over $\Gamma \curvearrowright I$ that leaves A_0^I globally invariant. Let (N, τ) be an arbitrary tracial von Neumann algebra and put $\mathcal{M} = N \mathbin{\overline{\otimes}} (A_0^I \rtimes \Gamma)$ and $\widetilde{\mathcal{M}} = N \mathbin{\overline{\otimes}} (B_0^I \rtimes \Gamma)$.*

(1) *If $P \subset p\mathcal{M}p$ is a von Neumann subalgebra such that*

$$P \not\prec_{\mathcal{M}} N \mathbin{\overline{\otimes}} (A_0^I \rtimes \operatorname{Stab} i) \quad \textit{for all} \;\; i \in I,$$

then $\mathcal{QN}_{p\widetilde{\mathcal{M}}p}(P)$ is contained in $p\mathcal{M}p$.

(2) *Fix $i_0 \in I$ and assume that $Q \subset q(N \mathbin{\overline{\otimes}} (A_0^I \rtimes \operatorname{Stab} i_0))q$ is a von Neumann subalgebra such that*

$$Q \not\prec_{N \overline{\otimes} (A_0^I \rtimes \operatorname{Stab} i_0)} N \mathbin{\overline{\otimes}} (A_0^I \rtimes \operatorname{Stab}\{i_0, j\}) \quad \textit{for all} \;\; j \neq i_0.$$

Then $\mathcal{QN}_{q\mathcal{M}q}(Q)$ is contained in $q(N \mathbin{\overline{\otimes}} (A_0^I \rtimes \operatorname{Stab} i_0))q$.

The proof of Lemma A.4 relies on the following general lemma based on Popa's intertwining technique. For a proof, we refer to [Va07].

Lemma A.5 ([Va07, Lemma 4.1])**.** *Let (M, τ) be a tracial von Neumann algebra and let $P \subset N \subset M$ be von Neumann subalgebras. Assume that there is a net of unitaries $v_i \in P$ such that*

$$\|E_P(x v_i y)\|_2 \to 0 \quad \textit{for all} \;\; x \in M, \; y \in M \ominus N.$$

Then $\mathcal{QN}_M(P)'' \subset N$.

Proof of Lemma A.4. To prove (1), assume that $P \nprec N \,\overline{\otimes}\, (A_0^I \rtimes \operatorname{Stab} i)$ for all $i \in I$. By Theorem 2.12 (and [Va07, Remark 3.3]), there exists a net of unitaries $v_n \in P$ such that

$$\|E_{N\overline{\otimes}(A_0^I \rtimes \operatorname{Stab} i)}(xv_ny)\|_2 \to 0 \quad \text{for all } \ x, y \in p\mathcal{M}p \ \text{ and all } \ i \in I.$$

By Lemma A.5, it is enough to show that

$$\|E_{\mathcal{M}}(av_nb)\|_2 \to 0 \quad \text{for all } \ a \in p\widetilde{\mathcal{M}}p, \ \ b \in p\widetilde{\mathcal{M}}p \ominus p\mathcal{M}p.$$

We may assume that a and b are of the form $a = x \otimes \pi_{\mathcal{F}}(\otimes_{i\in\mathcal{F}} a_i)$ and $b = y \otimes \pi_{\mathcal{G}}(\otimes_{j\in\mathcal{G}} b_j)$ with $x, y \in N$, $\mathcal{F}, \mathcal{G} \subset I$ finite subsets and $a_i, b_j \in B_0$. Moreover, we have that $b_{j_0} \in B_0 \ominus A_0$ for at least one $j_0 \in \mathcal{G}$.

Write $v_n = \sum_{g\in\Gamma} (v_n)^g u_g$ with $(v_n)^g \in N \,\overline{\otimes}\, A_0^I$. Then

$$\|E_{\mathcal{M}}(av_nb)\|_2^2 = \sum_{g\in\Gamma} \|E_{N\overline{\otimes}A_0^I}((x \otimes \pi_{\mathcal{F}}(\otimes a_i))(v_n)^g(y \otimes \sigma_g(\pi_{\mathcal{G}}(\otimes b_j))))\|_2^2.$$

Note that $\tau(\pi_{\mathcal{F}}(\otimes a_i)(v_n)^g\sigma_g(\pi_{\mathcal{G}}(\otimes b_j))) = 0$ if $g \cdot j_0 \notin \mathcal{F}$ so it is enough to sum over the set $\{g \in \Gamma \mid g \cdot j_0 \in \mathcal{F}\}$. Since $\mathcal{F}$ is finite, we can take a finite set of elements $g_1, \dots, g_k \in \Gamma$ such that $\{g \mid g \cdot j_0 \in F\}$ is the disjoint union of the sets $g_s \operatorname{Stab} j_0$ for $s = 1, \dots, k$. Then

$$\begin{aligned}
\|E_{\mathcal{M}}(av_nb)\|_2^2 &\leq \sum_{s=1}^{k} \sum_{h\in\operatorname{Stab} j_0} \|(x \otimes \pi_{\mathcal{F}}(\otimes a_i))(v_n)^{g_sh}(y \otimes \sigma_{g_sh}(\pi_{\mathcal{G}}(\otimes b_j)))\|_2^2 \\
&\leq \sum_{s=1}^{k} \sum_{h\in\operatorname{Stab} j_0} \|\pi_{\mathcal{F}}(\otimes a_i)\|^2 \|\pi_{\mathcal{G}}(\otimes b_j)\|^2 \|x(v_n)^{g_sh}y\|_2^2 \\
&= \|\pi_{\mathcal{F}}(\otimes a_i)\|^2 \|\pi_{\mathcal{G}}(\otimes b_j)\|^2 \sum_{s=1}^{k} \|E_{N\overline{\otimes}(A_0^I \rtimes \operatorname{Stab} j_0)}(xv_nu_{g_s}^* y)\|_2^2 \\
&\to 0.
\end{aligned}$$

This concludes the proof of (1).

To prove (2), take instead a net of unitaries $v_n \in \mathcal{U}(Q)$ such that

$$\|E_{N\overline{\otimes}(A_0^I \rtimes \operatorname{Stab}\{i_0,j\})}(xv_ny)\|_2 \to 0 \quad \text{for all } \ x, y \in N \,\overline{\otimes}\, (A_0^I \rtimes \operatorname{Stab} i_0), \ j \neq i_0.$$

Again by Lemma A.5, it suffices to show that

$$\|E_{N\overline{\otimes}(A_0^I \rtimes \operatorname{Stab} i_0)}(av_n b)\|_2 \to 0 \quad \text{for all } \ a \in \mathcal{M},\ b \in \mathcal{M} \ominus (A_0^I \rtimes \operatorname{Stab} i_0).$$

We may assume that a and b are of the form $a = a_0 u_g$ and $b = b_0 u_h$ where $g, h \in \Gamma$ and $a_0 = x \otimes \pi_{\mathcal{F}}(\otimes_{i\in\mathcal{F}} a_i)$, $b_0 = y \otimes \pi_{\mathcal{G}}(\otimes_{j\in\mathcal{G}} b_j)$ with $x, y \in N$, $\mathcal{F}, \mathcal{G} \subset I$ finite subsets and $a_i, b_j \in A_0$ for all $i \in \mathcal{F}$, $j \in \mathcal{G}$. Since b is orthogonal to $A_0^I \rtimes \operatorname{Stab} i_0$, we have that $h \notin \operatorname{Stab} i_0$. Let $j_0 = h \cdot i_0 \neq i_0$.

Writing $v_n = \sum_{k\in\operatorname{Stab} i_0} (v_n)^k u_k$ with $(v_n)^k \in N \overline{\otimes} A_0^I$, we get

$$\|E_{N\overline{\otimes}(A_0^I \rtimes \operatorname{Stab} i_0)}(av_n b)\|_2^2 = \sum_{k\in(\operatorname{Stab} i_0)\cap g^{-1}(\operatorname{Stab} i_0)h^{-1}} \|a_0 \sigma_g((v_n)^k)\sigma_{gk}(b_0)\|_2^2.$$

If $(\operatorname{Stab} i_0) \cap g^{-1}(\operatorname{Stab} i_0)h^{-1} = \emptyset$ we are done, so assume that this set is nonempty. For fixed $k_0 \in (\operatorname{Stab} i_0) \cap g^{-1}(\operatorname{Stab} i_0)h^{-1}$, we then have that

$$(\operatorname{Stab} i_0) \cap g^{-1}(\operatorname{Stab} i_0)h^{-1} = k_0(\operatorname{Stab} i_0 \cap h(\operatorname{Stab} i_0)h^{-1}) \subset k_0 \operatorname{Stab}\{i_0, j_0\}.$$

Hence,

$$\begin{aligned}
\|E_{N\overline{\otimes}(A_0^I \rtimes \operatorname{Stab} i_0)}(av_n b)\|_2^2 &\leq \sum_{k\in\operatorname{Stab}\{i_0,j_0\}} \|a_0 \sigma_g((v_n)^{k_0 k})\sigma_{gk_0 k}(b_0)\|_2^2 \\
&\leq \|a_0\|^2 \|b_0\|^2 \sum_{k\in\operatorname{Stab}\{i_0,j_0\}} \|(v_n)^{k_0 k}\|_2^2 \\
&= \|a_0\|^2 \|b_0\|^2 \|E_{N\overline{\otimes}(A_0^I \rtimes \operatorname{Stab}\{i_0,j_0\})}(v_n u_{k_0}^*)\|_2^2 \to 0.
\end{aligned}$$

□

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- A class of II_1 factors with exactly two group measure space decompositions. With Stefaan Vaes. *Journal de Mathématiques Pures et Appliquées* **108** (2017), 88-110.
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Index

FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
ANALYSIS SECTION
Celestijnenlaan 200A box 2402
B-3001 Leuven