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Numerical analysis for elliptic optimal control problems. (English) Zbl 08074085

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The purpose of this book is to provide the theoretical and numerical algorithmic tools to solve control problems of systems described by elliptic PDEs.

The first part provides the necessary elements from functional analysis (mainly the relevant function spaces) and discusses the solution of the (semilinear) Poisson equation on $\Omega \subset \mathbb{R}^n$, homogeneous or not, and with several types of boundary conditions (Dirichlet, Neumann) and how finite element methods can produce a discretized solution that approximates the exact solution.

Part II is introducing the control component of the problem. In the simplest case of the homogeneous (semilinear) Poisson equation, a controler q is chosen from a control space in such a way that the solution u of the controled PDE approximates a desired solution u_d, by minimizing a cost function like $J(u,q) = (\|u-u_d\|^2 + \alpha\|q\|^2)/2$, using appropriate norms and where $\alpha > 0$ is a relaxation parameter. Optimality conditions (first and second order) need to be derived, taking into account the different discretization methods that produce an approximation u_h to u so that corresponding error estimates also need to be involved. Several types of control are discussed like controls that have to stay within boundaries everywhere in Ω, or bang-bang control where it is either on or off.

Parts III and IV discusses cases where the control does not apply to the whole of Ω or it is limited in some sense. Part III focuses on boundary control problems (Neumann and Dirichlet) where the controler only applies on the boundary $\partial\Omega$ and in part IV other kinds of control apply on a finite number of points $X = \{x_k\}_{k=1}^M \subset \Omega$, like pointwise (when the control only applies in X), or pointwise tracking (when the goal is to approach a desired value in X), or pointwise state constraints (when the solution has to take predefined values in X).

The last part V discusses other general contraints that can be imposed on the problem. General state constraints can impose that the solution has to stay between a lower and an upper bound everywhere in Ω. For sparse controls, the controler will be taken from a measure space $\mathcal{M}$. Two subcases are considered: either $\mathcal{M}$ is defined on the whole domain Ω and the observation is also known everywhere in Ω or control and observation are restricted to disjoint subsets of Ω.

The treatment of all these different cases is always following the same steps which investigate: (1) existence and properties of the solution; (2) formulation of the cost function; (3) existence of optimal controler; (4) formulate optimality conditions; (5) regularity of the optimal controler; (6) the discretization and (7) its effect on the cost function; (8) existence and (9) conditions for a discrete optimal controler and (10) bounding the discretization error. All this requires several theorems that need to be proved, but in the end the focus is on the practical solution of the problems. So, each of the cases results in a full numerical example that illustrates the performance of the method with some discussion of the results.

Although many problems are considered, the book cannot be exhaustive. The authors chose to cover a reasonable representation of practical situations, keeping a balance with insight and readability. The intended readership are students, researchers and even practitioners that need to solve these kinds of control problems. The contents is accessible for science or engineering students and as much as possible self-contained so that it can be used for self study, while it is also possible to use it as a basis for a graduate course, although no exercises are provided.

Reviewer: Adhemar Bultheel (Leuven)

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MSC:

65-02	Research exposition (monographs, survey articles) pertaining to numerical analysis	Cited in **1** Document
49-02	Research exposition (monographs, survey articles) pertaining to calculus of variations and optimal control	
49M25	Discrete approximations in optimal control	
49J20	Existence theories for optimal control problems involving partial differential equations	
65M15	Error bounds for initial value and initial-boundary value problems involving PDEs	
49M29	Numerical methods involving duality	

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