WWW: Wavelets What and Why
A tool for signal and image prcessing

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Abstract

We shall give some basics about wavelet theory and stress its advantages in several applications. We illustrate the usefulness in image and signal processing.

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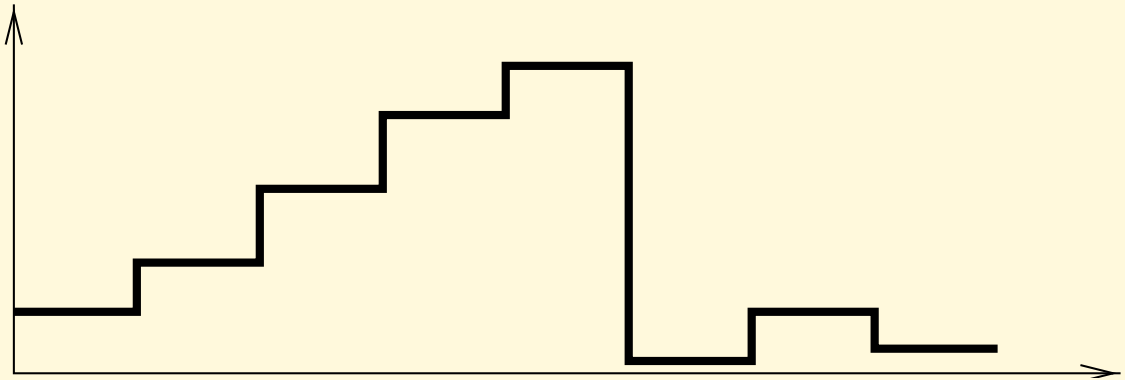
Fourier transforms
Digital filters
Sampling theorem
Subsampling
Filter banks
Reconstruction . . .

1. Introduction

Suppose we are given a *digital* signal:



This is a representation of *discrete* data.

Typically: strong correlation between adjacent points.
e.g.: pixel grey values
→ redundancy
→ sparse representation possible?

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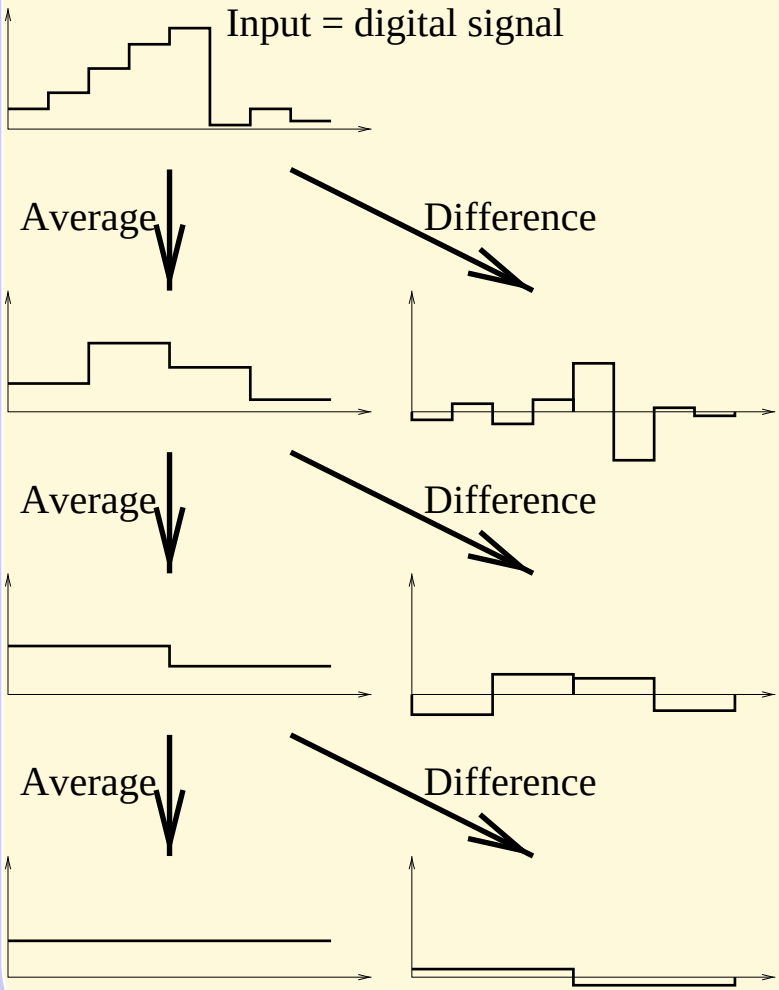

This transform contains 2 operations:

1. average
   → low resolution version of input
   → lowpass filter
2. difference
   → captures local details
   → highpass filter

The transform is invertible (perfect reconstruction):
if $a = \frac{f_1+f_2}{2}$ and $d = \frac{f_1-f_2}{2}$
then $f_1 = a + d$ and $f_2 = a - d$.

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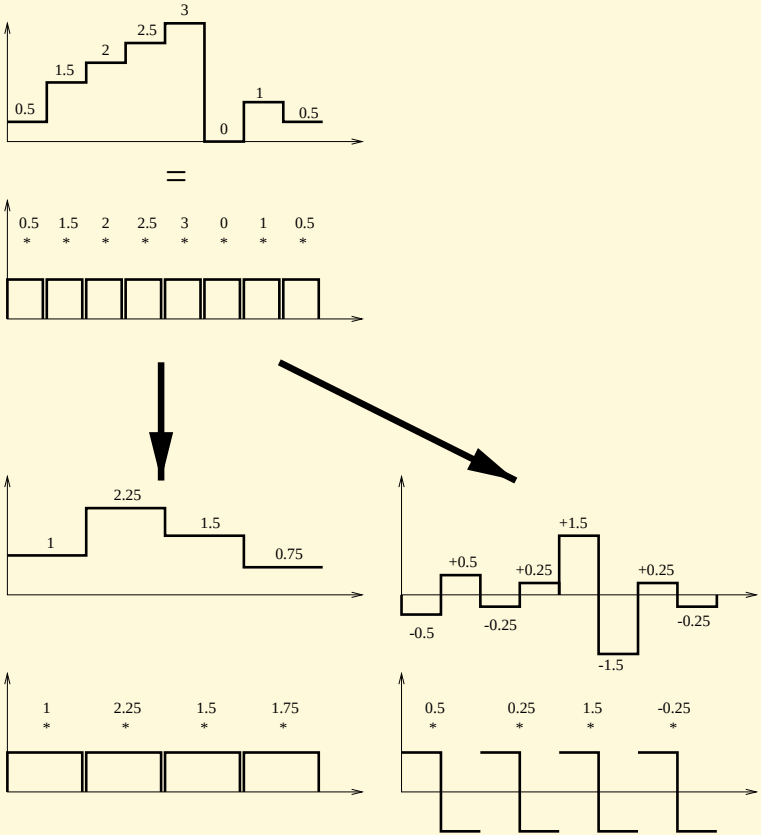
1.1. Basis functions



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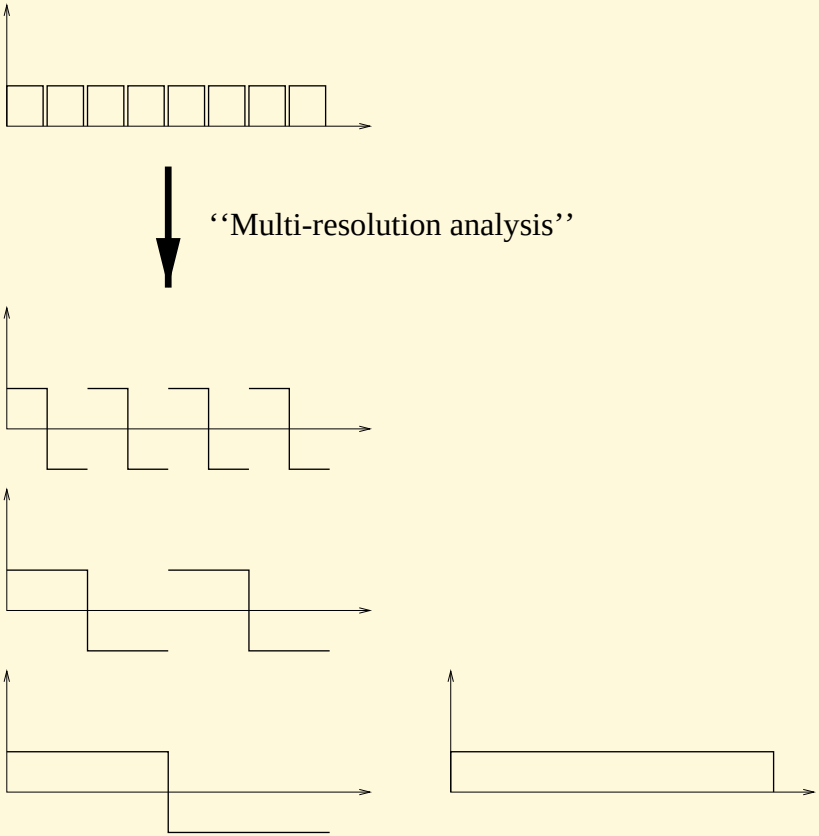

decomposition into a basis of translations and dilations of one “mother”function

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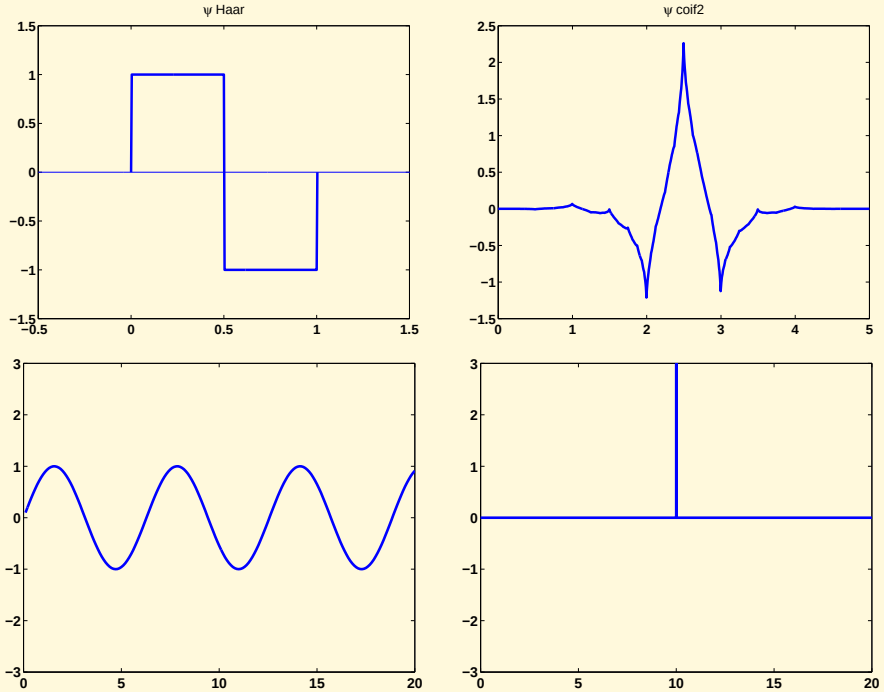
→ basis functions are “short” waves (wave-let):



sine is local in frequency

Dirac is local in time

wavelets are local in time/place and frequency.

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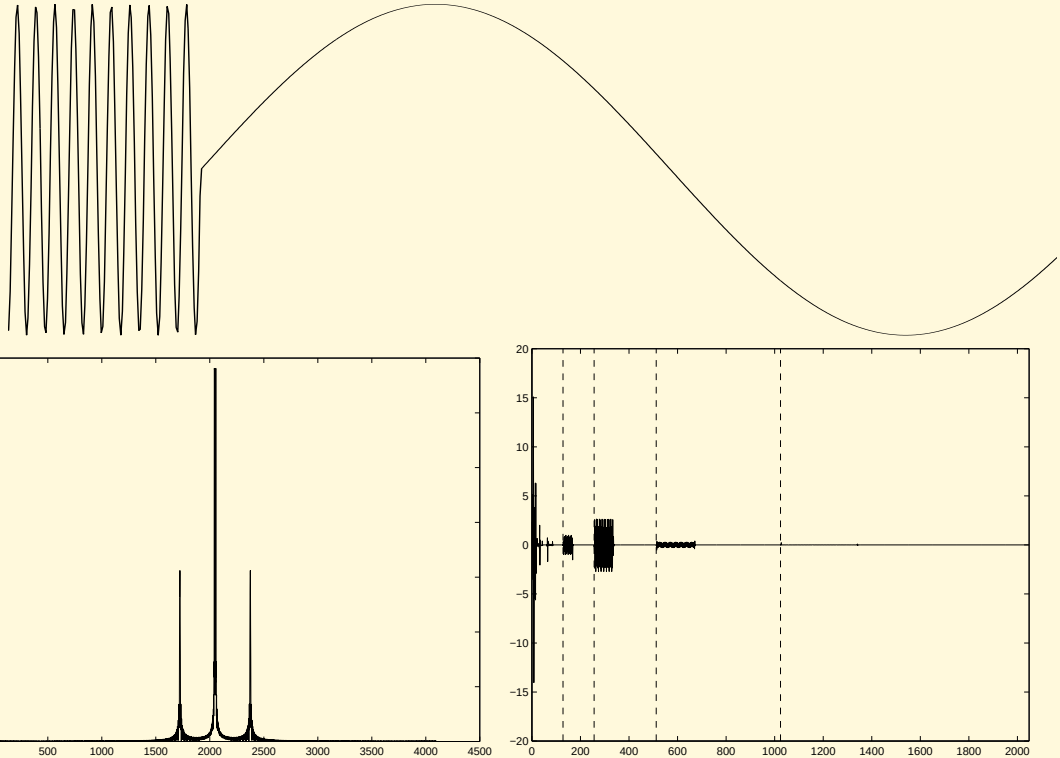

- A wavelet coefficient carries *local* information.
- manipulation of wavelet coefficients has local effect.

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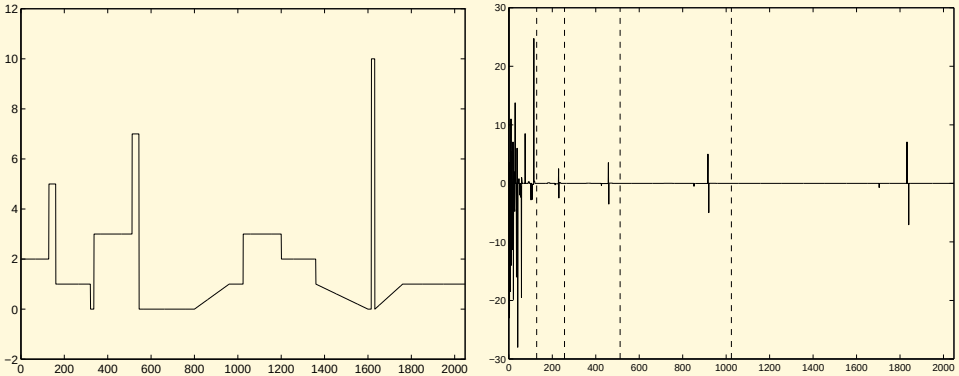
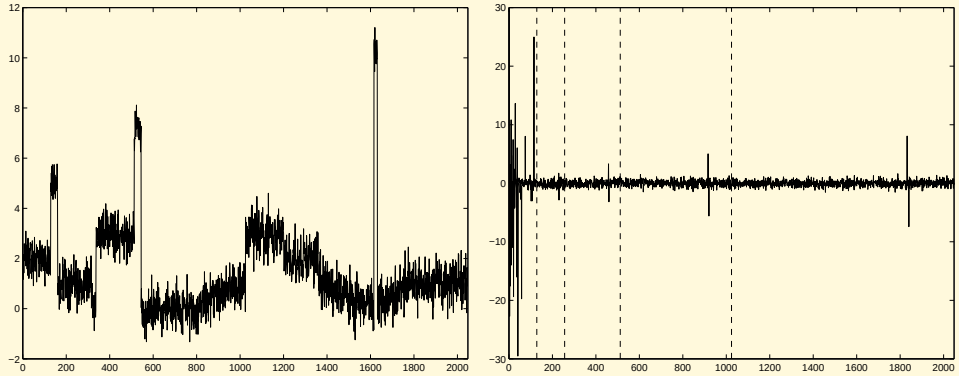
1.2. Decorrelation



Decorrelation not complete



uncorrelated noise remains uncorrelated

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1.3. Wavelet properties

1. multiresolution, pyramidal algorithm
2. decorrelation → sparse representation
3. local in time/space *and* frequency
   → manipulation has local effect
4. fast algorithm (linear complexity)
5. orthogonal or bi-orthogonal transform;
   numerically well conditioned
6. many types of basisfunctions
7. based on a nice mathematical background



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2. Signals

either digital (discrete) or analog (continuous)
either nonperiodic (infinite) or periodic (finite)

	digital	analog
infin	$(f_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$	$f(t) \in L^2(\mathbb{R})$
$\langle f, g\rangle$	$\sum_{n=-\infty}^{\infty} \bar{f}_n g_n$	$\int_{-\infty}^{\infty} \overline{f(t)} g(t) dt$
finite	$(f_n)_{n=0}^{N-1} \in \mathbb{C}^N$	$f(t) \in L^2_T$
$\langle f, g\rangle$	$\sum_{n=0}^{N-1} \bar{f}_n g_n$	$\int_0^T \overline{f(t)} g(t) dt$

examples:

1. The impulse
   $\boldsymbol{\delta} = (\ldots 00000100000 \ldots)$
   (a 1 at the 0-th position)

2. The Dirac impulse $\delta(t)$:
   $$\int f(\tau)\delta(t-\tau)d\tau = f(t)$$

3. $f(t) = e^{i\omega t}$: period $T = 2\pi$

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3. Fourier transforms

- Duality digital $\leftrightarrow$ analog signal
- Duality periodic $\leftrightarrow$ not periodic
- So there are 4 cases

time-domain	digital	analog
infinite	$\ell^2(\mathbb{Z})$	$L^2(\mathbb{R})$
finite/ periodic	$\mathbb{R}^n$	$L^2_T = L^2(-\frac{T}{2}, \frac{T}{2})$

freq-domain	digital	analog
infinite	$L^2(\mathbb{T})$	$L^2_{2\pi} = L^2(-\pi, \pi)$
finite/ periodic	$\mathbb{C}^n$	$\ell^2(\mathbb{Z})$

- A note on notation DTFT:

$$\mathsf{F}(\omega) = F(e^{i\omega}) = \sum_n f_n e^{-in\omega}$$

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digital	**analog**
infinite	infinite
$(f_n) \in \ell^2(\mathbb{Z}) \leftrightarrow F(e^{i\omega}) \in L^2(\mathbb{T})$	$f(t) \in L^2(\mathbb{R}) \leftrightarrow F(\omega) \in L^2_{2\pi}$
$F(e^{i\omega}) = \sum_{n\in\mathbb{Z}} f_n e^{-in\omega}$	$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\omega t} dt$
$f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\omega}) e^{in\omega} d\omega$	$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F(\omega) e^{i\omega t} d\omega$
finite/periodic	finite/periodic
$(f_n)_{n=0}^{N-1} \in \mathbb{C}^N \leftrightarrow (F_k)_{k=0}^{N-1} \in \mathbb{C}^N$	$f(t) \in L^2_T \leftrightarrow (F_n) \in \ell^2(\mathbb{Z})$
$F_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n e^{-ink\frac{2\pi}{N}}$	$F_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\frac{2\pi}{T}} dt$
$f_n = \sum_{k=0}^{N-1} F_k e^{ink\frac{2\pi}{N}}$	$f(t) = \sum_{n\in\mathbb{Z}} F_n e^{in\frac{2\pi}{T}}$

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3.1. Z-transform

$$(f_n) \mapsto F(z) = \sum_n f_n z^{-k}$$

Note that $F(e^{i\omega})$ is the DTFT

3.2. Convolution

$$\begin{array}{lllll} \text{C.T.} & y = h * x \equiv & y(t) & = & \int_{\mathbb{R}} h(t-\tau)x(\tau)d\tau \quad \Leftrightarrow \quad Y = HX \\ \text{D.T.} & \boldsymbol{y} = \boldsymbol{h} * \boldsymbol{x} \equiv & y_n & = & \sum_{k\in\mathbb{Z}} h_{n-k}x_k \quad \Leftrightarrow \quad Y = HX \end{array}$$

$$\boxed{\mathcal{F}(h * x) = \mathcal{F}(h)\mathcal{F}(x)}$$

note: $h * x = x * h$

4. Digital filters

A *filter* maps an input signal into an output signal. A *digital* filter maps a discrete input into a discrete output.

$$\boldsymbol{y} = \mathcal{H}(\boldsymbol{x})$$

4.1. Important classes of filters

1. *Linear* filters:

$$\mathcal{H}(\alpha \boldsymbol{f} + \beta \boldsymbol{g}) = \alpha \mathcal{H}(\boldsymbol{f}) + \beta \mathcal{H}(\boldsymbol{g})$$

The operation $\mathcal{H}$ can be represented by a matrix $\mathbf{H}$ (with inifinite size)

$$\boldsymbol{y} = \mathcal{H}\boldsymbol{x} \Leftrightarrow \mathbf{y} = \mathbf{H}\mathbf{x}$$

2. *Time-invariant* filters

Call $\mathcal{D}$ the delay (backward shift) operator:

$$\boldsymbol{y} = \mathcal{D}\boldsymbol{x} \Leftrightarrow y_k = x_{k-1}, \forall k \in \mathbb{Z}$$

$\mathcal{H}$ is shift-invariant or time-invariant if

$$\mathcal{H}\mathcal{D} = \mathcal{D}\mathcal{H}$$

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This means: if $u_k = x_{k+m}, \forall k \in \mathbb{Z}$
and $\boldsymbol{y} = \mathcal{H}(\boldsymbol{x})$ and $\boldsymbol{v} = \mathcal{H}(\boldsymbol{u})$,
then: $v_k = y_{k+m}, \forall k \in \mathbb{Z}$.

A time-invariant filter is characterized by its *impulse response* $\boldsymbol{h}$:

$$\boldsymbol{h} = \mathcal{H}(\boldsymbol{\delta})$$

3. *Causal* filters

A filter is called *causal* if the following is true:
if $x_k = 0, \forall k < l, l \in \mathbb{Z}$ and $\boldsymbol{y} = \mathcal{H}(\boldsymbol{x})$,
then $y_k = 0, \forall k < l$.
The output does not start *before* the input.

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4.2. Linear, time-invariant filters

All filters in this course are linear and time-invariant.

$$\boldsymbol{x} = \{\ldots x_k \ldots\} = \sum_{k=-\infty}^{\infty} (\mathcal{D}^k \boldsymbol{\delta}) x_k$$

($\mathcal{D}^k \boldsymbol{\delta}$ is a signal with 1 on position k and zero otherwise)

$$\begin{aligned} \boldsymbol{y} &= \mathcal{H}(\boldsymbol{x}) = \mathcal{H}(\sum_k (\mathcal{D}^k \boldsymbol{\delta}) x_k) \\ &= \sum_k (\mathcal{H}\mathcal{D}^k(\boldsymbol{\delta})) x_k = \sum_k (\mathcal{D}^k \mathcal{H}(\boldsymbol{\delta})) x_k \\ &= \sum_k (\mathcal{D}^k \boldsymbol{h}) x_k \end{aligned}$$

So:

$$y_n = \sum_{k=-\infty}^{\infty} h_{n-k} x_k$$

This is Convolution: $\boldsymbol{y} = \boldsymbol{h} * \boldsymbol{x}$

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In matrix notation

$$\begin{bmatrix} \vdots \\ y_{n-1} \\ y_n \\ y_{n+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & h_{n-k} & h_{n-k-1} & h_{n-k-2} & \cdots \\ \cdots & h_{n-k+1} & h_{n-k} & h_{n-k-1} & \cdots \\ \cdots & h_{n-k+2} & h_{n-k+1} & h_{n-k} & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ x_{k-1} \\ x_k \\ x_{k+1} \\ \vdots \end{bmatrix}$$

$$\mathbf{y} = \mathbf{H}\mathbf{x}$$

→ Filter matrix $\mathbf{H}$ is Toeplitz.

$$\mathbf{H} = [\cdots|\mathbf{Z}^{-1}\mathbf{h}|\mathbf{h}|\mathbf{Z}\mathbf{h}|\cdots]$$

$\mathbf{Z}$ = down-shift matrix,
$\mathbf{x}$, $\mathbf{y}$, $\mathbf{h}$ = stacking vectors

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4.3. Different descriptions

1. *time domain*: convolution

$$\boldsymbol{y} = \boldsymbol{h} * \boldsymbol{x}$$

$\mathcal{H}(\boldsymbol{\delta}) = \boldsymbol{h} = (h_k)$: impulse response

2. *linear algebra*: matrix-vector prod.

$$\mathbf{y} = \mathbf{H}\mathbf{x}$$

Toeplitz matrix $\mathbf{H} = \mathbf{T}_h$

3. *Fourier domain*: product

$$Y(e^{i\omega}) = H(e^{i\omega})X(e^{i\omega})$$

or $\mathsf{Y}(\omega) = \mathsf{H}(\omega)\mathsf{X}(\omega)$,

with

$$\mathsf{H}(\omega) = H(e^{i\omega}) = \sum_k h_k e^{-ik\omega}$$

$=$ *frequency response*
$H(z) =$ *transfer function* of the filter

4.4. Examples of filters: moving average and moving difference

1. *Moving average filter*:
   $h_0 = 1/2 = h_1$ (all other zero)

$$\mathsf{H}(\omega) = \frac{1}{2} + \frac{1}{2}\mathrm{e}^{-i\omega} = \cos(\frac{\omega}{2})\mathrm{e}^{-i\omega/2}$$

   modulus: $|\mathsf{H}(\omega)| = \cos\frac{\omega}{2}$; phase: $-\frac{\omega}{2}$ (*linear phase*)

2. *Moving difference filter*:
   $h_0 = 1/2, h_1 = -1/2$ (all other zero)

$$\mathsf{H}(\omega) = \frac{1}{2} - \frac{1}{2}\mathrm{e}^{-i\omega} = \sin(\frac{\omega}{2})\mathrm{e}^{i(\pi/2-\omega/2)}$$

   modulus: $|\mathsf{H}(\omega)| = |\sin\frac{\omega}{2}|$;
   phase: $\frac{\pi}{2} - \frac{\omega}{2}$ (*generalized linear phase*)

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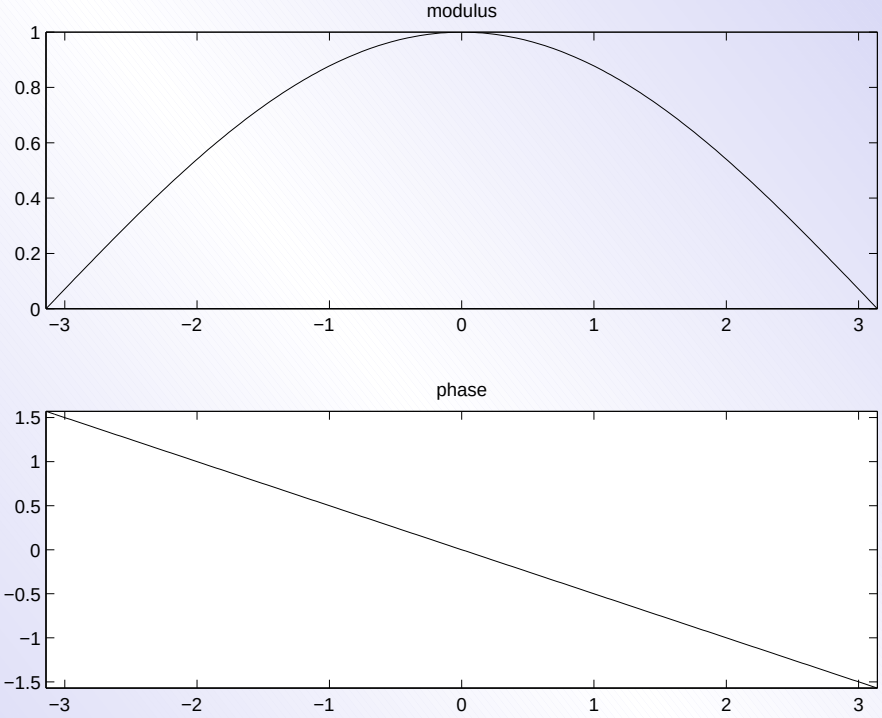
moving average: modulus and phase



approximate *low pass*, linear phase filter

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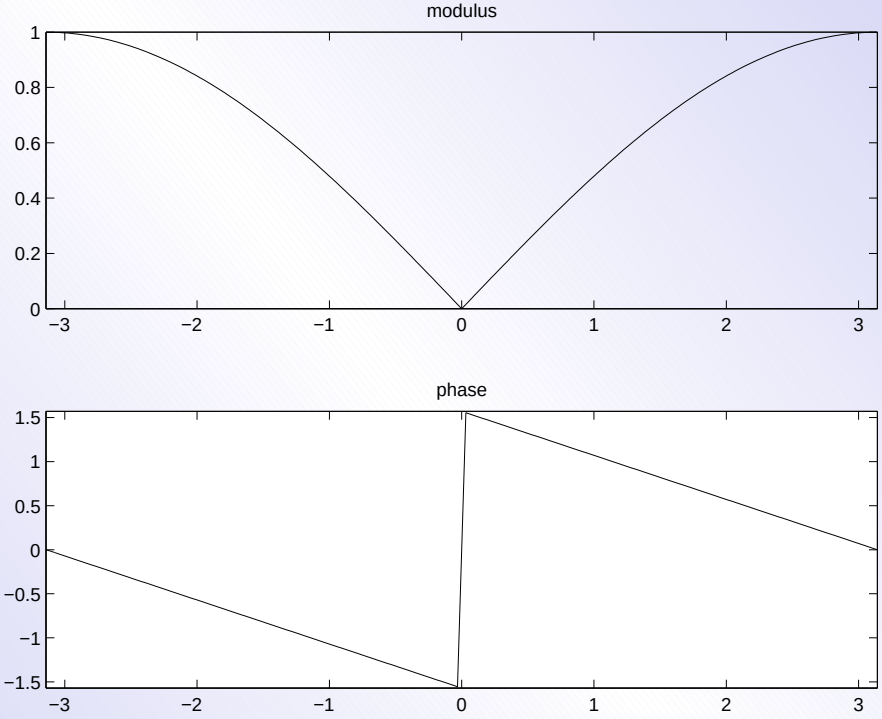
moving difference: modulus and phase



approx. *high pass*, generalized linear phase filter

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5. Sampling theorem

Continuous or discrete?

real world is continuous (macroscopicly)	$\leftrightarrow$	digital (discrete) processing/manipulation

solution:
sample C.T. signal/image $\rightarrow$ D.T. samples/pixels

$$f_k = f(t_k)$$

note: *analog* signal $f(t) \rightarrow$ *digital* signal f_k

1. discretized: $(t \in \mathbb{R}) \rightarrow (k \in \mathbb{Z})$
2. quantized: $(f(k) \in \mathbb{R}) \rightarrow (f_k \in \mathbb{Z})$

here only step 1

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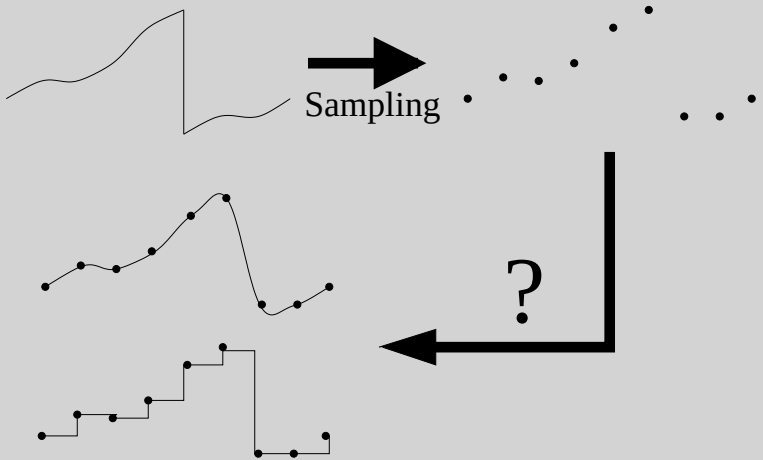
5.1. sampling: (step 1)



→ We need both the D.T. and the C.T. model. sampling period $T \rightarrow f(nT)$

Can we reconstruct $f(t)$ from $f(nT)$?

Yes if

1. limited frequency content in $f(t)$
2. $f(t)$ is sampled sufficiently often

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5.2. sampling mathematically

pulse train

$$\delta^T = \sum_{n\in\mathbb{Z}} \delta(t - nT)$$

is 1 at $\ldots, -2T, -T, 0, T, 2T, \ldots$, otherwise 0
sampled signal:

$$f_s(t) = f(t)\delta^T(t)$$

Can we reconstruct $f(t)$ from $f_s(t)$?
Answer = the *Nyquist theorem.*

5.3. Nyquist theorem

Yes if f is bandlimitted ($F(\omega) = 0$ for \omega $> \omega_m$)
and if $\omega_s = \frac{2\pi}{T} \geq 2\omega_m$
proof By Fourier transform

$$\delta^T(t) = \frac{1}{T}\sum_n e^{in\omega_s t} \Rightarrow f_s(t) = \frac{1}{T}\sum_n f(t)e^{in\omega_s t}$$

$$\Rightarrow F_s(\omega) = \frac{1}{T}\sum_n F(\omega - n\omega_s)$$

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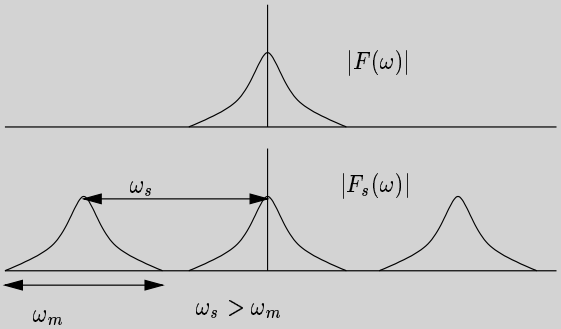
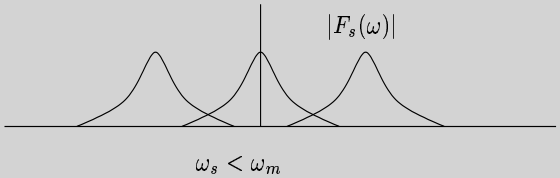
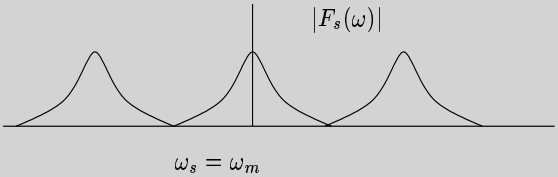
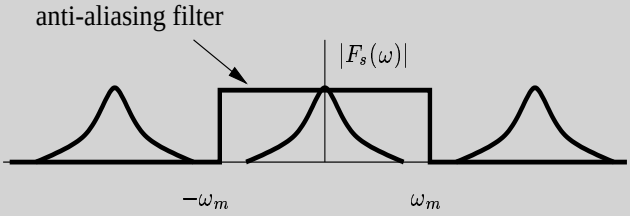
5.4. sampling graphically

$$F_s(\omega) = \tfrac{1}{T} \sum_n F(\omega - n\omega_s)$$









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6. Subsampling

6.1. A coarse view

Suppose we look at a signal on a coarser grid.
We select every m-th sample and neglect the others.
This is *subsampling*. Notation: $(\downarrow m)$

$$\boldsymbol{y} = (\downarrow m)\boldsymbol{x} \Leftrightarrow y_k = x_{mk}$$

6.2. Mathematical description

For a mathematical description, it is easy to consider it as a subsequence of two steps:

1. replace all in-between samples by zero $(\boldsymbol{x} \to \boldsymbol{z})$
   !!loss of information possible (aliasing)
2. remove all in-between samples $(\boldsymbol{z} \to \boldsymbol{y})$
   no loss of information (rescaling ω-axis)

consider step 2 first

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2. removing in-between zeroes

not "dangerous" (no loss of information)

is merely a rescaling of the ω-axis

$\cdots$	$z_0\ 0\cdots0$	$z_m\ 0\cdots0$	$z_{2m}\ 0\cdots0$	$\cdots$
$\cdots$	y_0	y_1	y_2	$\cdots$

In frequency domain, this corresponds to:

$$\begin{aligned} \mathsf{Z}(\omega) &= \sum_{l=-\infty}^{\infty} z_l e^{-i\omega l} = \sum_{k=-\infty}^{\infty} z_{mk} e^{-i\omega mk} \\ \mathsf{Y}(\omega) &= \sum_{k=-\infty}^{\infty} y_k e^{-i\omega k} = \sum_{k=-\infty}^{\infty} z_{mk} e^{-i\omega k} \\ &= \sum_{k=-\infty}^{\infty} z_{mk} e^{-i(\omega/m)mk} = \mathsf{Z}(\omega/m) \end{aligned}$$

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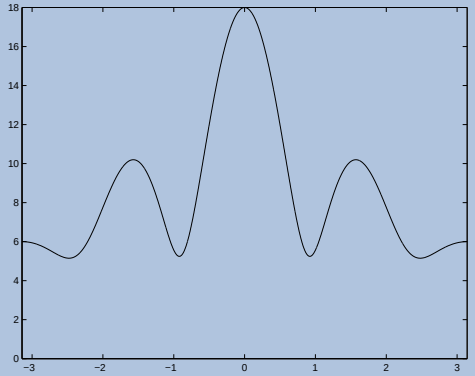
graphical example:



← plot of $|\mathsf{Y}(\omega)|$

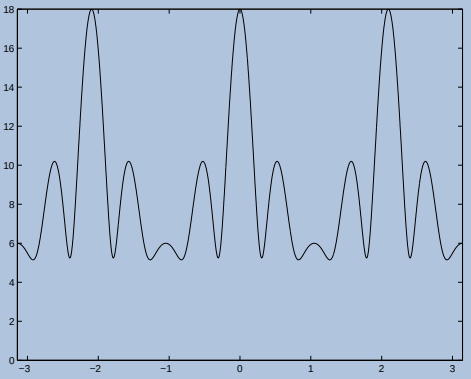
$\boldsymbol{y} = \{\ldots, 0, 1, 3, 2, 1, 3, 8, 0, \ldots\}$

$\boldsymbol{y}^{[3]} =$

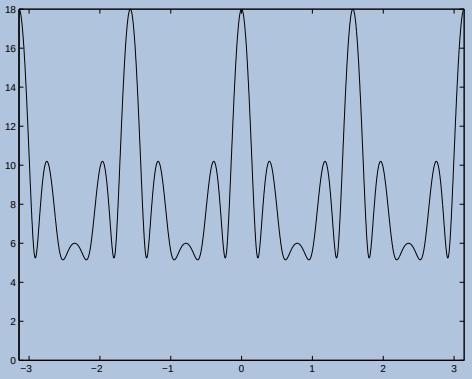
$\{\ldots, 0, 1, 0, 0, 3, 0, 0, 2, 0, 0, 1, 0, 0, 3, 0, 0, 8, 0, \ldots\}$

$\boldsymbol{y}^{[4]} =$

$\{\ldots, 0, 1, 0, 0, 0, 3, 0, 0, 0, 2, 0, 0, 0, 1, 0, 0, 0, 3, 0, 0, 0, 8, 0, \ldots\}$



plot of $|\mathsf{Y}^{[3]}(\omega)|$



plot of $|\mathsf{Y}^{[4]}(\omega)|$

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1. replace in-between samples by zero

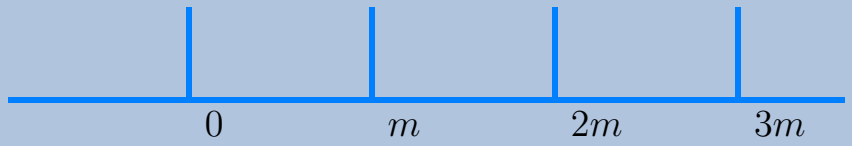
given m a fixed integer and replace all x_l by 0, except for $l = km$

$$\begin{aligned} z_{km} &= x_{km}, \quad k \in \mathbb{Z} \\ z_l &= 0, \quad \text{for all other } l \in \mathbb{Z} \end{aligned}$$

hence $z_l = x_l \delta_l^m$ with

$$\delta_l^m = \begin{cases} 1, & l = mk \in m\mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

this is a *pulse train*



pulse train

pulse train $\boldsymbol{\delta}^m$ = signal with samples

$$\delta_k^m = \begin{cases} 1, & k = ml \in m\mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

periodic signal, period $m \rightarrow$ consider one period $(\delta_k^m)_{k=0}^{m-1} = (1, 0, \ldots, 0)$.

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its DTFT is

$$\Delta_n^m = \frac{1}{m}\sum_{k=0}^{m-1} \delta_k^m \mathrm{e}^{-ink\frac{2\pi}{m}} = \frac{1}{m}, \quad n = 0, \ldots, m-1$$

its inverse DTFT is

$$\delta_k^m = \sum_{n=0}^{m-1} \Delta_n^m \mathrm{e}^{ink\frac{2\pi}{m}} = \frac{1}{m}\sum_{n=0}^{m-1} \mathrm{e}^{ink\frac{2\pi}{m}}, \quad k = 0, \ldots, m-1$$

and repeated periodically.
So if $z_{mk} = x_{mk}$ and all other z_l are zero, then:

$$\begin{aligned} \mathsf{Z}(\omega) &= \sum_{l=-\infty}^{\infty} z_l \mathrm{e}^{-i\omega l} = \sum_{l=-\infty}^{\infty} x_l \delta_l^m \mathrm{e}^{-i\omega l} = \sum_{l=-\infty}^{\infty} x_l \left(\frac{1}{m}\sum_{n=0}^{m-1} e^{iln\frac{2\pi}{m}}\right) \mathrm{e}^{-i\omega l} \\ &= \frac{1}{m}\sum_{n=0}^{m-1}\sum_{l=-\infty}^{\infty} x_l e^{-il(\omega - \frac{2\pi n}{m})} = \frac{1}{m}\sum_{n=0}^{m-1} \mathsf{X}(\omega - \frac{2\pi n}{m}) \end{aligned}$$

$\mathsf{Z}(\omega)$ is the sum of m copies of $\mathsf{X}(\omega)$, each shifted over $2\pi/m$ and scaled by $1/m$.

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For example $m = 2$

$$\mathsf{Z}(\omega) = \frac{1}{2}\mathsf{X}(\omega) + \frac{1}{2}\mathsf{X}(\omega - \pi)$$

!! overlap (aliasing) if $|\text{support}\mathsf{X}(\omega)| > 2\pi/m$!!

6.3. Two steps together

If $\boldsymbol{y} = (\downarrow m)\boldsymbol{x}$, then

$$\mathsf{Y}(\omega) = \frac{1}{m}\sum_{d=1}^{m} \mathsf{X}(\frac{\omega + 2\pi d}{m})$$

This can be seen in two ways:

1. directly: $y_k = x_{mk}$, so

$$\begin{aligned}\mathsf{Y}(\omega) &= \sum_{k\in\mathbb{Z}} x_{mk}\mathrm{e}^{-i\omega k} = \sum_{k\in\mathbb{Z}} x_{mk}\mathrm{e}^{-i\frac{\omega}{m}km} = \sum_{l\in\mathbb{Z}} x_l \mathrm{e}^{-i\frac{\omega}{m}l}\left(\frac{1}{m}\sum_{n=0}^{m-1} e^{\frac{2\pi l}{m}in}\right)\\ &= \frac{1}{m}\sum_{n=0}^{m-1}\sum_{l\in\mathbb{Z}} x_l \mathrm{e}^{-i\frac{\omega-2\pi n}{m}l} = \frac{1}{m}\sum_{n=0}^{m-1}\mathsf{X}(\frac{\omega - 2\pi n}{m})\end{aligned}$$

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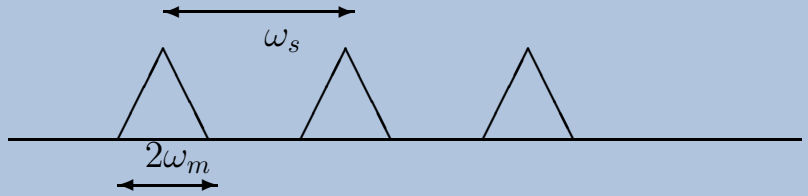
2. in two steps:

$$
\begin{array}{rcl}
\boldsymbol{x} & \Rightarrow & \mathsf{X}(\omega) \\
\downarrow & & \\
\boldsymbol{z} & \Rightarrow & \mathsf{Z}(\omega) = \dfrac{1}{m}\displaystyle\sum_{n=1}^{m-1} \mathsf{X}(\omega - \frac{2\pi}{m}n) \\
\downarrow & & \mathsf{Y}(\omega) = \mathsf{Z}(\omega/m) \\
\boldsymbol{y} & \Rightarrow & \mathsf{Y}(\omega) = \dfrac{1}{m}\displaystyle\sum_{n=1}^{m-1} \mathsf{X}(\frac{\omega - 2\pi n}{m})
\end{array}
$$

Nyquist theorem

To avoid aliasing, the shift $(2\pi/m)$ of the spectrum should be larger than the width of the spectrum.



This is expressed in the

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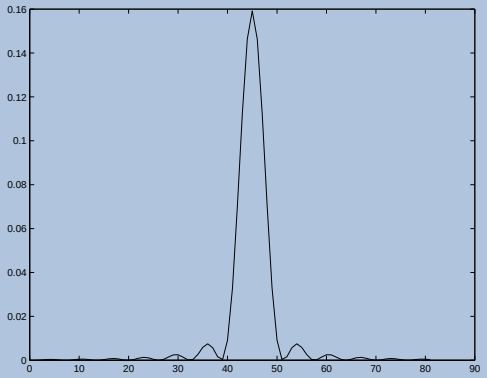
Nyquist theorem:
To avoid aliasing, the subsampling frequency must be at least two times the highest signal frequency.

An example for $m = 2$

Consider $x_k = b\,\frac{1-\cos(ak)}{\pi a k^2}$ (and $x_0 = \frac{ba}{2\pi}$)

A plot of $\boldsymbol{x}$ for $a = 1 = b$:



One period of the DTFT of this signal looks like this (prove this!!):

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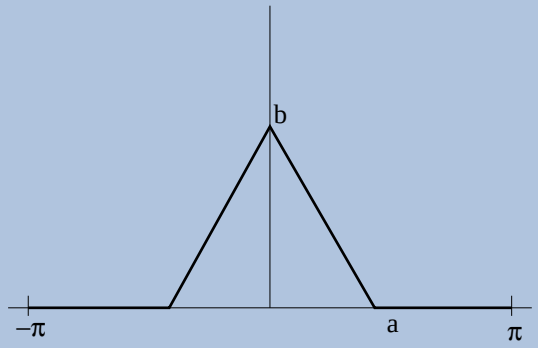
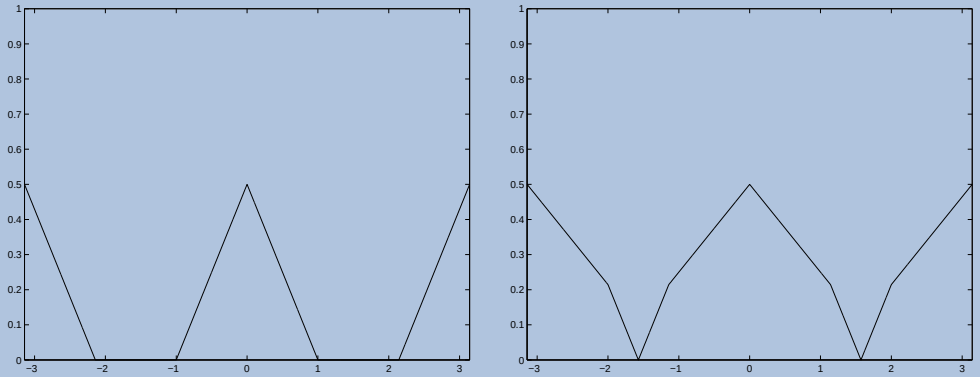


Replacing the odd samples by zero gives:

for $a = 1$: for $a = 2$:



In the first case, we can reconstruct the original signal $\boldsymbol{x}$ from the subsampled version $\boldsymbol{y}$:

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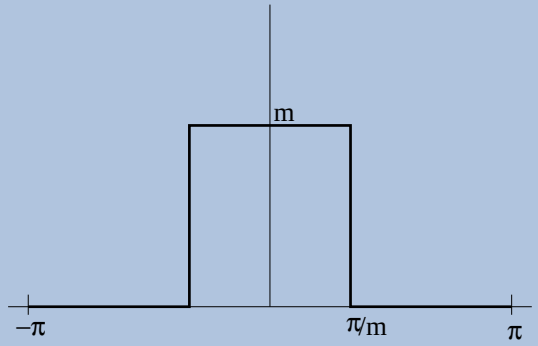
multiply $\mathsf{Y}(\omega)$ by this function:



This corresponds in time-domain to a convolution with

$$\left\{ \frac{\sin(\pi k/m)}{(\pi k/m)} \right\}_{k \in \mathbb{Z}}$$

Thus:

$$x_k = \sum_{l=-\infty}^{\infty} \frac{\sin(\pi k/m)}{(\pi k/m)} \; y_{k-l} \quad \text{or} \quad x_k = \sum_{l=-\infty}^{\infty} \frac{\sin\left[\pi(k-l)/m\right]}{\left[\pi(k-l)/m\right]} \; y_l$$

This reconstruction is not possible in case 2.

The highest pulsation ω_h in the signal is higher than π/m

$\rightarrow$ overlap = aliasing

Aliasing introduces spurious frequencies after subsampling.

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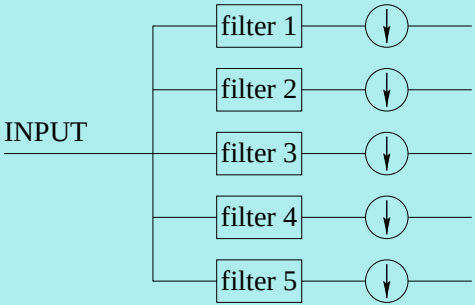
7. Filter banks

Image preprocessing tasks like noise-reduction, contrast enhancement, ...
Image features frequency-related: → operations based on frequency-analysis
e.g.: filter design

other possibility: make operation frequency-dependent.
i.e. split signal/image into frequency subbands and apply different operation on each subband



After filtering often downsampling: one subband of a signal contains less information than the whole signal.

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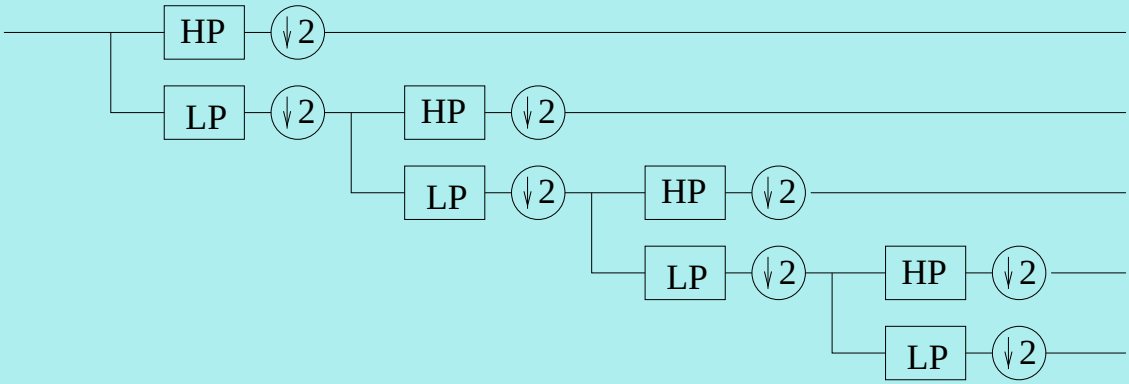
- adaptivity
- efficiency (parallellism)

7.1. Recursive filterbanks

Instead of one big M-channel filter bank, we analyse a filter with a recursive 2-channel filterbank.



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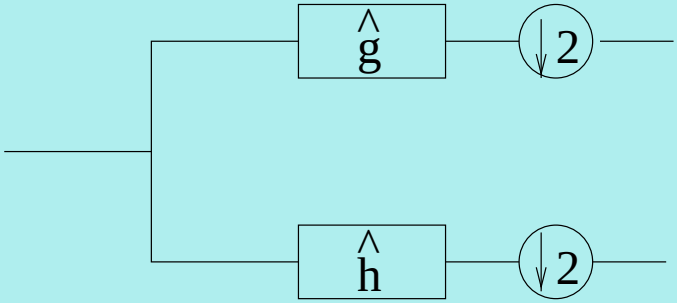
7.2. The basic element: a two channel filterbank



A lowpass filter $\hat{\boldsymbol{h}}$ and a highpass filter $\hat{\boldsymbol{g}}$, both followed by downsampling with factor two.

This is *critical downsampling*: if input has N points, the two filtered signals both have N points. The downsampled signals have $\frac{N}{2}$ points. (We suppose N even !!) Together, this makes $\frac{N}{2} + \frac{N}{2} = N$ output data. From N output data, *maybe* we can reconstruct the input. With less than N output points, this is certainly not generally possible.

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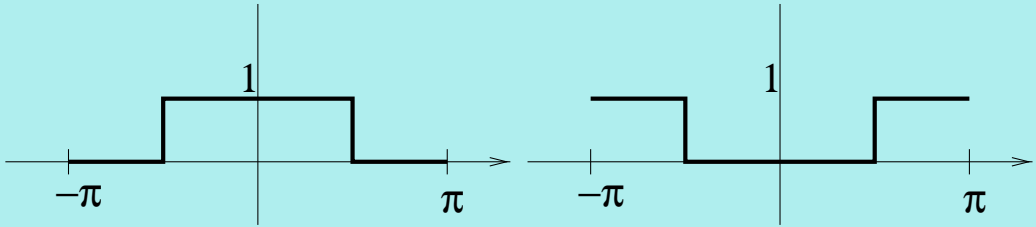
7.3. The ideal lowpass and highpass

The ideal lowpass and highpass filters have this frequency response:



In time domain, this corresponds to:

$$\begin{aligned} \hat{h}_0 &= 1/2 &&= \hat{g}_0 \\ \hat{h}_{2n} &= 0 &&= \hat{g}_{2n} \\ \hat{h}_{2n+1} &= (-1)^n/(2n+1)\pi \quad , \quad \hat{g}_{2n+1} &&= (-1)^{n+1}/(2n+1)\pi \end{aligned}$$

These filters have infinite length (FIR) and are non-causal. (So, they are not possible in reality)

What happens with the input on the lowpass branch? In frequency domain:

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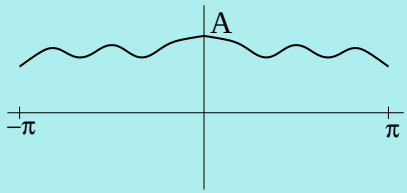
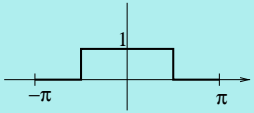
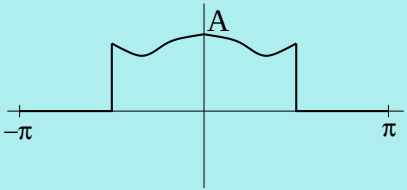
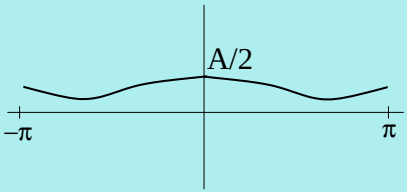


Filtering = multiplication with:





Subsampling



This result can be filtered again → recursive filter bank.

A two channel filterbank has two "complimentary" filters: If $\hat{\mathsf{H}}(\omega)$ and $\hat{\mathsf{G}}(\omega)$ are mirror images with respect to the middle frequencies $\frac{\pi}{2}$ and $-\frac{\pi}{2}$, then $\hat{\boldsymbol{h}}$ and $\hat{\boldsymbol{g}}$ are called

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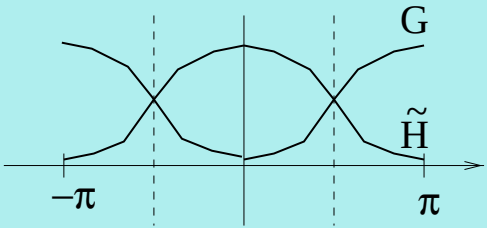
quadrature mirror filters



examples:

- the ideal lowpass and highpass filter
- the moving average and moving difference filter

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7.4. Power Complementary Filters (PCF)

$\hat{\boldsymbol{h}}$ and $\hat{\boldsymbol{g}}$ are called
power complementary filters (PCF) if:

$$|\hat{\mathsf{H}}(\omega)|^2 + |\hat{\mathsf{G}}(\omega)|^2 = 1$$

examples:

- the ideal lowpass and highpass filter
- the moving average and moving difference filter



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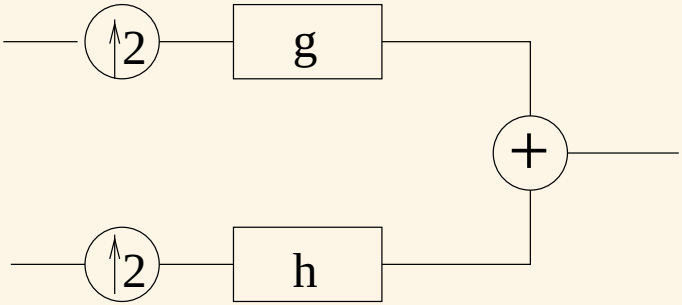
8. Reconstruction (Synthesis)

For reconstruction, we use *upsampling* and two reconstruction filters $\boldsymbol{h}$ and $\boldsymbol{g}$.



8.1. Upsampling

The upsampling operator inserts zeros between subsequent samples:

$$\begin{array}{ccc} \boldsymbol{y} & = & (\uparrow m)\boldsymbol{x} \\ & \Updownarrow & \\ y_k = 0 \text{ for } k \neq mn & \text{and} & y_{mn} = x_n \end{array}$$

Upsampling is *not the inverse* of downsampling:

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recall two steps in downsampling:

1. replace all in-between samples by zero
2. remove all in-between samples

The second step is undone by upsampling, but not the first. (This first step is irreversible)

So

$$(\uparrow m)(\downarrow m)\boldsymbol{x} \neq \boldsymbol{x} \quad \text{but} \quad (\downarrow m)(\uparrow m)\boldsymbol{x} = \boldsymbol{x}$$

Downsampling and upsampling are *conjugate* operations. *Conjugate* operations A and A^* in a unitary space:
$\forall u, v : \langle Au, v\rangle = \langle u, A^* v\rangle$
(For linear, matrix operations, the conjugate is the (complex) transpose)
Now, consider two signals $\boldsymbol{x}$ and $\boldsymbol{y}$, then

$$\langle (\downarrow m)\boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{n\in\mathbb{Z}} x_{nm}\overline{y_n} = \sum_{k\in\mathbb{Z}} x_k \overline{z_k}$$

where $z_k = y_n$ if $k = nm$ and $z_k = 0$ otherwise. So

$$\langle (\downarrow m)\boldsymbol{x}, \boldsymbol{y}\rangle = \langle \boldsymbol{x}, (\uparrow m)\boldsymbol{y}\rangle$$

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The matrix for downsampling is:

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & 1 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 1 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The matrix for upsampling is the transpose:

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \dots & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & \dots \\ \dots & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & \dots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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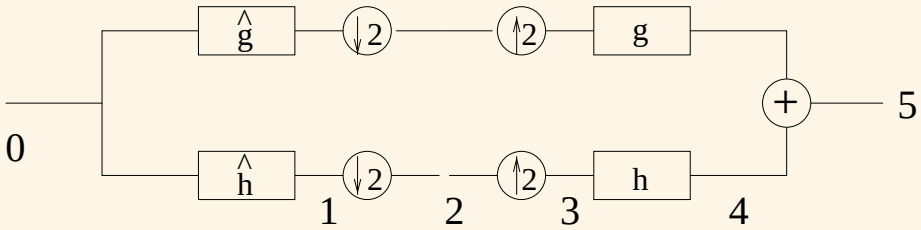
8.2. Perfect reconstruction

After upsampling we filter to get smoother signals. These are added.



We follow the signal in frequency domain throughout the lowpass-branch of the filterbank:
input: $X(\omega)$

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1. filtering $\quad X(\omega)\hat{H}(\omega)$
2. subsampling $\quad \frac{1}{2}\left(X(\frac{\omega}{2})\hat{H}(\frac{\omega}{2}) + X(\frac{\omega}{2}+\pi)\hat{H}(\frac{\omega}{2}+\pi)\right)$
3. upsampling $\quad \frac{1}{2}\left(X(\omega)\hat{H}(\omega) + X(\omega+\pi)\hat{H}(\omega+\pi)\right)$
4. filtering $\quad \frac{1}{2}\left(X(\omega)\hat{H}(\omega)H(\omega) + X(\omega+\pi)\hat{H}(\omega+\pi)H(\omega)\right)$
5. Adding $\quad X(\omega)\cdot\frac{1}{2}\left(\hat{H}(\omega)H(\omega) + \hat{G}(\omega)G(\omega)\right) +$
$X(\omega+\pi)\cdot\frac{1}{2}\left(\hat{H}(\omega+\pi)H(\omega) + \hat{G}(\omega+\pi)G(\omega)\right)$

For perfect reconstruction:
the output must be equal to $X(\omega)$, so

$$\begin{aligned}\hat{H}(\omega)H(\omega) + \hat{G}(\omega)G(\omega) &= 2\\ \hat{H}(\omega+\pi)H(\omega) + \hat{G}(\omega+\pi)G(\omega) &= 0\end{aligned}$$

How are $\hat{G}$, $\hat{H}$, G, H related to guarantee PR?

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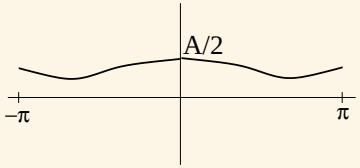
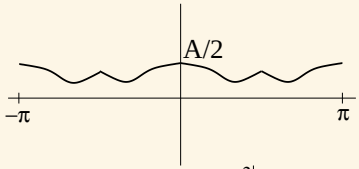
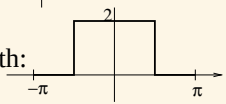
8.3. Reconstruction from ideal filterbank

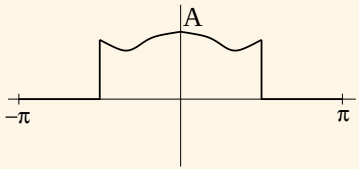
The lowpass channel:



Upsampling



Filtering with:  to remove redundancy



Adding high frequency gives reconstruction

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