Computing orthogonal rational functions on a halfline

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Abstract

We present some transformation formulas which relate orthogonal rational functions (ORF) on $[0, \infty)$ to ORF on $[-1, 1]$. From a computational point of view, we can then limit our attention to the interval case, which has been well studied. We also use these formulas to obtain quadrature rules and asymptotic behaviour for ORF on $[0, \infty)$. Some examples serve as illustration.

1 Introduction

Orthogonal rational functions (ORF) are widely used in applications such as signal processing and system identification. To approximate the (unknown) rational transfer function in system theory, researchers first used rational functions with one pole [17] or a pair of complex conjugate poles [18], then with a finite number of poles that are cyclically repeated [3, 8, 10] and finally with an arbitrary pole sequence [5, 6, 7]. These rational functions are orthogonal with respect to the Lebesgue measure on the complex unit circle and are special cases of the so-called Malmquist basis. Later on this was generalized to arbitrary orthogonality measures on the unit circle and (subsets of) the real line [1].

The accurate computation of orthogonal rational functions for arbitrary measures is a far from trivial problem. For the case of orthogonality on a finite real interval, several results were obtained in [13, 14]. It is the purpose of the present paper to show that the case of orthogonality on a halfline can be completely reduced to the case of the interval, using a suitable transformation. We will start by reviewing the basic theory of orthogonal rational

functions on subsets of the real line, including the three term recurrence relation. Next we derive a relation between functions orthogonal on a halfline and on an interval. Some examples are given to illustrate the theory. As an application, we study the asymptotic behaviour of ORF on a halfline, using known results from the interval case, and we look at quadrature formulas.

2 Preliminaries

The real line is denoted by $\mathbb{R}$ and the extended real line by $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

By a measure μ we will mean a positive bounded Borel measure whose support $\mathrm{supp}(\mu) \subset \overline{\mathbb{R}}$ is an infinite set and normalized such that $\mu(\overline{\mathbb{R}}) = 1$. The inner product in the metric space $L_2(\mu)$ is then defined as

$$\langle f\,, g\rangle = \int f\overline{g}d\mu. \tag{1}$$

Next we will introduce the spaces of rational functions with real poles. Let a sequence $A = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{R}} \setminus \{0\}$ be given such that $A \cap \mathrm{supp}(\mu) = \emptyset$. As a consequence we cannot have $\mathrm{supp}(\mu) = \overline{\mathbb{R}}$. Define factors

$$Z_n(z) = \frac{z}{1 - z/\alpha_n}$$

and basis functions

$$b_0(z) \equiv 1, \quad b_n(z) = b_{n-1}(z)Z_n(z), \quad n = 1, 2, \ldots$$

Then the space of rational functions with poles in A is defined as

$$\mathcal{L}_n = \mathrm{span}\{b_0, \ldots, b_n\}.$$

Note that we recover the polynomial case if we take all poles equal to infinity.

Let $\mathcal{P}_n$ denote the space of polynomials of degree at most n and define

$$\pi_n(z) = \prod_{k=1}^{n}(1 - z/\alpha_k),$$

then we may write equivalently

$$\mathcal{L}_n = \{p_n/\pi_n, p_n \in \mathcal{P}_n\}.$$

Orthonormalizing the basis $\{b_0, \ldots, b_n\}$ with respect to μ we obtain orthogonal rational functions $\{\varphi_0, \ldots, \varphi_n\}$ where we choose the leading coefficient κ_n in the expansion $\varphi_n(z) = \kappa_n b_n(z) + \ldots$ to be real. The φ_n will be uniquely determined once the sign of κ_n is fixed. We will get back to this later on. The following lemma from [1] will be useful.

Lemma 2.1. *The orthonormal functions φ_n have real coefficients with respect to the basis $\{b_k\}$.*

It follows in particular that $\varphi_n(z)$ is real for real z.

The orthogonal rational function φ_n is called *regular* if its numerator polynomial satisfies $p_n(\alpha_{n-1}) \neq 0$. The system $\{\varphi_n\}$ is regular if φ_n is regular for every n. We now mention the most important theorem for the computation of orthogonal rational functions on the real line, which states that they satisfy a three term recurrence relation, analogous to the one for the polynomial case. For the proof of the theorem we refer to [1].

Theorem 2.2. *Put by convention $\alpha_{-1} = \alpha_0 = \infty$. Then for $n = 1, 2, \ldots$ the orthonormal rational functions φ_n satisfy the following three term recurrence relation if and only if φ_n and φ_{n-1} are regular,*

$$\varphi_n(z) = \left(E_n Z_n(z) + B_n \frac{Z_n(z)}{Z_{n-1}(z)} \right) \varphi_{n-1}(z) - \frac{E_n}{E_{n-1}} \frac{Z_n(z)}{Z_{n-2}(z)} \varphi_{n-2}(z). \quad (2)$$

The initial conditions are $\varphi_{-1}(z) \equiv 0$, $\varphi_0(z) \equiv 1$ and the coefficients E_n are nonzero.

Note that the coefficient E_0 is never used and can be arbitrarily chosen. We take it equal to $E_0 = 1$. If we take the coefficient E_n to be positive, then the functions φ_n will be uniquely determined. This amounts to fixing the sign of κ_n.

If we take all poles outside the convex hull of $\mathrm{supp}(\mu)$, then the system $\{\varphi_n\}$ will be regular and thus the recurrence relation will hold for every n. This follows from the fact that in this case the zeros of φ_n are inside the convex hull of $\mathrm{supp}(\mu)$. Therefore, if $\mathrm{supp}(\mu)$ is connected then $\{\varphi_n\}$ will be regular (because of the assumptions we made on the location of the poles).

3 Mapping the halfline to the interval

In what follows, we only consider two different cases: orthogonality on the interval $[-1, 1]$ and orthogonality on the halfline $[0, \infty)$. To avoid the intro-

duction of more notation, we will denote the case of the halfline merely by adding a tilde to all quantities involved. So suppose we are given a measure $\tilde{\mu}$ on the halfline $[0,\infty]$ and a set of poles $\tilde{A} = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots\} \subset (-\infty, 0)$. With this measure and poles we associate a set of orthonormal rational functions $\{\tilde{\varphi}_n\}$. Since $\mathrm{supp}(\tilde{\mu})$ is connected, the recurrence relation holds for every n. According to the conventions from section 2, we cannot have poles at infinity, because this is in the support of the measure.

We map the halfline to the interval $[-1,1]$ using the transformation

$$\tau(x) = \frac{1-x}{1+x}, \quad x \in [0,\infty]. \tag{3}$$

Note that $y = \tau(x)$ implies $x = \tau(y)$. Then associate to $\tilde{\mu}$ and $\tilde{A}$ a measure μ on $[-1,1]$ and a set of poles $A = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{R}} \setminus [-1,1]$ in the following way. For every Borel measurable set $E \subset [-1,1]$ set

$$\mu(E) = \tilde{\mu}(\{\tau(y), y \in E\}) \tag{4}$$

and for the poles A set

$$\alpha_n = \tau(\tilde{\alpha}_n), \quad n = 1, 2, \ldots \tag{5}$$

From (4) we have $\mu' = |\tau'|(\tilde{\mu}' \circ \tau)$, where the prime means derivative (in case of a measure this is of course the Radon-Nikodym derivative with respect to the Lebesgue measure). More explicitly this yields

$$\mu'(y) = \frac{2}{(1+y)^2} \tilde{\mu}' \left(\frac{1-y}{1+y} \right) .$$

We will need several simple lemmas before we can prove the main theorem. Let $\tilde{\mathcal{L}}_n$ denote the space of rational functions of degree n with poles in $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\}$ and $\tilde{b}_n(z) = \prod_{k=1}^n \tilde{Z}_k(z)$ the corresponding basis functions. As before, the symbols without a tilde refer to the interval. The proof of the following lemma is a matter of straightforward computation and we omit it.

Lemma 3.1. *Put $y = \tau(x)$ then we have*

$$\tilde{Z}_n(x) = -\frac{1}{2} \left(1 - \frac{1}{\alpha_n} \right) \left[\left(1 - \frac{1}{\alpha_n} \right) Z_n(y) - 1 \right], \quad n \geq 1$$

$$\frac{\tilde{Z}_n(x)}{\tilde{Z}_m(x)} = \frac{1 - 1/\alpha_n}{1 - 1/\alpha_m} \frac{Z_n(y)}{Z_m(y)}, \quad n, m \geq 1$$

and for every pair of complex numbers (a, b) *we have*

$$aZ_n(y) + b = \left(a - \frac{b}{Z_{n-1}(\alpha_n)}\right) Z_n(y) + b\frac{Z_n(y)}{Z_{n-1}(y)}, \quad n \geq 1$$

Next we derive a relation between the rational functions in $\mathcal{L}_n$ orthonormal with respect to μ and those in $\tilde{\mathcal{L}}_n$ orthonormal with respect to $\tilde{\mu}$.

Lemma 3.2. *Let* $\{\tilde{\varphi}_n\}_{n=1}^{\infty}$ *denote the set of orthonormal rational functions associated with* $(\tilde{A}, \tilde{\mu})$ *and* $\{\varphi_n\}_{n=1}^{\infty}$ *those associated with* (A, μ)*. Then we have*

$$\tilde{\varphi}_n \circ \tau = \pm\varphi_n, \quad n \geq 1$$

where the sign is determined by the normalization $\tilde{E}_n > 0$ *for* $\tilde{\varphi}_n$ *and* $E_n > 0$ *for* φ_n*.*

Proof. It is clear that $\tilde{\varphi}_n \circ \tau \in \mathcal{L}_n$. Because $\tilde{\alpha}_k \neq 0$ for all k we have $\alpha_k \neq 1$ and then it follows from lemma 3.1 that the basis functions b_k and $\tilde{b}_k$ satisfy the following relation,

$$\tilde{b}_k = \sum_{j=0}^{k} c_j^{(k)} (b_j \circ \tau), \quad c_k^{(k)} \neq 0$$

for some constants $\{c_j^{(k)}\}$. Next expand $\tilde{\varphi}_n$ in the basis $\{\tilde{b}_0, \ldots \tilde{b}_n\}$,

$$\tilde{\varphi}_n = \sum_{k=0}^{n} d_k^{(n)} \tilde{b}_k$$

where $d_n^{(n)} \neq 0$ because $\tilde{\varphi}_n \in \tilde{\mathcal{L}}_n$ and $\tilde{\varphi}_n \perp \tilde{\mathcal{L}}_{n-1}$. Then with the previous relation and the fact that $\tilde{b}_k \circ \tau \circ \tau = \tilde{b}_k$ it follows that

$$\begin{aligned}
\tilde{\varphi}_n \circ \tau &= \sum_{k=0}^{n} d_k^{(n)} (\tilde{b}_k \circ \tau) \\
&= \sum_{k=0}^{n} d_k^{(n)} \sum_{j=0}^{k} c_j^{(k)} b_j \\
&= \sum_{k=0}^{n} \tilde{d}_k^{(n)} b_k
\end{aligned}$$

and $\tilde{d}_n^{(n)} = d_n^{(n)} c_n^{(n)} \neq 0$. This shows that $\tilde{\varphi}_n \circ \tau \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ and thus $\{\tilde{\varphi}_0 \circ \tau, \ldots, \tilde{\varphi}_n \circ \tau\}$ forms a basis for $\mathcal{L}_n$.

Because of orthogonality we have

$$\int \tilde{\varphi}_n \tilde{\varphi}_m d\tilde{\mu} = \delta_{nm}, \quad n, m \geq 0$$

where δ_{nm} is the Kronecker symbol. Using the definition of τ this becomes

$$\int (\tilde{\varphi}_n \circ \tau)(\tilde{\varphi}_m \circ \tau) d\mu = \delta_{nm}, \quad n, m \geq 0.$$

This means that $\tilde{\varphi}_n \circ \tau \perp \mathcal{L}_{n-1}$ and $\|\tilde{\varphi}_n \circ \tau\| = 1$. But we also have $\varphi_n \perp \mathcal{L}_{n-1}$ and $\|\varphi_n\| = 1$ so it follows that $\tilde{\varphi}_n \circ \tau = \gamma_n \varphi_n$ with $|\gamma_n| = 1$. Using lemma 2.1 we then get

$$\tilde{\varphi}_n \circ \tau = \pm \varphi_n$$

which proves the lemma. □

Using the previous two lemmas we can express the recurrence coefficients $\{\tilde{E}_n, \tilde{F}_n\}$ for $\{\tilde{\varphi}_n\}$ in terms of the recurrence coefficients $\{E_n, F_n\}$ for $\{\varphi_n\}$. We will need one more lemma before we can prove our main theorem.

Lemma 3.3. *If $n = 1$ then the recurrence coefficients for φ_1 satisfy*

$$E_1 > F_1.$$

Proof. From the recurrence relation, taking the inner product with $\varphi_0 = 1$ on both sides, we obtain,

$$-F_1 = E_1 \frac{\int_{-1}^{1} Z_1(x) d\mu(x)}{\int_{-1}^{1} \frac{Z_1(x)}{x} d\mu(x)}$$

or writing Z_1 explicitly

$$E_1 - F_1 = E_1 \frac{\int_{-1}^{1} \frac{x+1}{1-x/\alpha_1} d\mu(x)}{\int_{-1}^{1} \frac{1}{1-x/\alpha_1} d\mu(x)} .$$

Both integrands are positive on $[-1, 1]$ and so is E_1. Therefore also $E_1 - F_1 > 0$, proving the lemma. □

We now state and prove our main theorem.

Theorem 3.4. *With the definitions of this section we obtain the following relations between* $\{E_n, F_n\}$ *and* $\{\tilde{E}_n, \tilde{F}_n\}$ *for* $n \geq 1$,

$$\tilde{E}_n = [2E_n - \delta_{n1}(E_1 + F_1)] \left(1 - \frac{1}{\alpha_{n-1}}\right)^{-1} \left(1 - \frac{1}{\alpha_n}\right)^{-1},$$
$$\tilde{F}_n = -\left(1 - \frac{1}{\alpha_n}\right)^{-1} \left[F_n \left(1 - \frac{1}{\alpha_{n-1}}\right) + E_n - \delta_{n2} \frac{E_2}{E_1} \frac{1 - 1/\alpha_1}{F_1 - E_1}\right].$$

Note that the formulas simplify for $n > 2$ *because of the Kronecker symbols.*

Proof. First we will prove the theorem for the cases $n = 1$ and $n = 2$ and then for general $n > 2$. The first two cases are special because by convention we have $\alpha_{-1} = \alpha_0 = \infty$ but also $\tilde{\alpha}_{-1} = \tilde{\alpha}_0 = \infty$ which shows that $\alpha_k \neq \tau(\tilde{\alpha}_k)$ for $k = -1, 0$.

For $n = 1$ use lemma 3.2 to write $(\tilde{\varphi}_1 \circ \tau)(x) = c\varphi_1(x)$ where $c = \pm 1$. Then write down the recurrence relations for $\tilde{\varphi}_1$ and φ_1, use the definition of τ and equate the coefficients of like powers of x to obtain (recall that $\tilde{\varphi}_0 = \varphi_0 = 1$)

$$\left(1 - \frac{1}{\alpha_1}\right) \tilde{E}_1 = c(F_1 - E_1),$$
$$\left(1 - \frac{1}{\alpha_1}\right) \tilde{F}_1 = c(F_1 + E_1),$$

Now it follows from $|\alpha_1| > 1$, $\tilde{E}_1 > 0$ and lemma 3.3 that $c = -1$, proving the theorem for $n = 1$.

For $n = 2$ we proceed in the same way. Write down the recurrence relation for $\tilde{\varphi}_2 \circ \tau$, using the fact that we already know that $\tilde{\varphi}_1 \circ \tau = -\varphi_1$ and of course $\tilde{\varphi}_0 \circ \tau = \varphi_0$. Then use the relation

$$E_2 = \left.\frac{\varphi_2(x)}{\varphi_1(x) Z_2(x)}\right|_{x=\alpha_1}$$

and $E_2 > 0$ to find

$$E_2 = \frac{\tilde{E}_2}{2} \left(1 - \frac{1}{\alpha_2}\right) \left(1 - \frac{1}{\alpha_1}\right).$$

To find the relation for $\tilde{F}_2$ use $\varphi_2(0) = F_2\varphi_1(0) - E_2/E_1$ and $\varphi_1(0) = F_1$, again comparing the recurrence relations for $\tilde{\varphi}_2 \circ \tau$ and φ_2 and using all the previous results.

The general case $n > 2$ is the easiest to prove. Write down the recurrence relation for φ_n and for $\tilde{\varphi}_n \circ \tau$ using $\tilde{\varphi}_k \circ \tau = c_k\varphi_k$ for $k = n, n-1, n-2$ and $c_k = \pm 1$ and the formulas from lemma 3.1. Comparing the factors in front of φ_{n-1} and φ_{n-2} and using $E_n > 0$ immediately yields the result, thus proving the theorem. □

Remark. If we take all poles $\tilde{\alpha}_k = -1$ then transforming to the interval we obtain the orthogonal polynomials on $[-1, 1]$ with respect to μ. For the case of Legendre polynomials we would have to take $\tilde{\mu}$ absolutely continuous with weight $\tilde{\mu}'(x) = (1 + x)^{-2}$. See [2] for an application of this special case.

We could also transform orthogonal polynomials on $[0, \infty)$ to get orthogonal rational functions on $[-1, 1]$. In this case the poles $\tilde{\alpha}_n$ are in the support of the measure (polynomials have poles at infinity) and thus far we have always excluded this case. However, as long as all the integrals

$$\mu_{kl} = \int \tilde{b}_k(x)\tilde{b}_l(x)d\tilde{\mu}(x)$$

exist and are finite, the recurrence relation remains valid whenever the $\tilde{\varphi}_n$ are regular, and so do the transformation formulas from theorem 3.4. Next we look at an example of this case.

Example 3.5. The Laguerre polynomials $L_n(x)$ are orthonormal on $[0, \infty)$ with respect to the weight function e^{-x} and usually normalized such that the coefficient of x^n has sign $(-1)^n$, see e.g. [9, p. 100]. In that case they satisfy the recurrence formula

$$nL_n(x) = (-x + 2n - 1)L_{n-1}(x) - (n - 1)L_{n-2}(x).$$

with initial values $L_{-1}(x) = 0$ and $L_0(x) = 1$. If we regard them as orthogonal rational functions on the halfline and normalize them according to our convention $\tilde{E}_n > 0$, then it follows from the above recurrence relation that

$$\tilde{E}_n = \frac{1}{n}, \quad \tilde{F}_n = \frac{1}{n} - 2, \quad n = 1, 2, \ldots$$

Because of the remark following theorem 3.4, we may still use the transformation formulas from that theorem to obtain ORF on $[-1, 1]$. Of course

n	E_n	F_n
1	$\frac{10}{3}$	$-\frac{2}{3}$
2	1	$\frac{37}{40}$
$n > 2$	$\frac{2}{n}$	$2\left(1 - \frac{1}{n}\right)$

Table 1: Recurrence coefficients for φ_n

now we have to use them in the opposite direction to obtain $\{E_n, F_n\}$ from $\{\tilde{E}_n, \tilde{F}_n\}$. We then get rational functions $\varphi_n(x)$ on $[-1, 1]$ with all poles in -1 and orthogonal with respect to the absolutely continuous measure

$$d\mu(x) = \frac{2}{(1+x)^2} \exp\left\{\frac{x-1}{x+1}\right\}.$$

Some computations then yield the recurrence coefficients as given in table 1.

Example 3.6. Among the very few examples of explicitly known orthogonal rational functions (and in fact the only known example on $[-1, 1]$ with arbitrary poles), are ORF with respect to the Chebyshev weight functions $(1-x)^\gamma(1+x)^\delta$ on $[-1, 1]$, with γ and δ belonging to $\{\pm 1/2\}$ and real poles outside $[-1, 1]$. These ORF are studied in detail in [16]. For $\gamma = \delta = -1/2$ they are related to the Malmquist basis on the complex unit circle, which we mentioned in the introduction.

For $n > 2$ the recurrence coefficients are independent of the specific choice of γ and δ. Just as in the polynomial case, they are the same for the different weight functions. The general expressions are quite complicated, but if we limit our attention to the case of one multiple pole $A = \{\alpha, \alpha, \alpha, \ldots\}$ then we have for $n > 2$

$$E_n = 2\left(1 - \frac{1}{\alpha^2}\right),$$
$$F_n = -\frac{2}{\alpha}.$$

Note that we obtain the well-known recurrence coefficients for Chebyshev polynomials if we take $\alpha = \infty$.

Using theorem 3.4, some computations yield the recurrence coefficients for ORF on the halfline $[0, \infty)$ with respect to the weight functions

$$\tilde{w}(x) = \frac{2}{(1+x)^2} \left(\frac{2x}{1+x} \right)^{\gamma} \left(\frac{2}{1+x} \right)^{\delta}$$

and with a multiple pole in $\tilde{\alpha}$. They are given by

$$\tilde{E}_n = -\frac{4}{\tilde{\alpha}},$$
$$\tilde{F}_n = -2,$$

for $n > 2$.

4 Quadrature formulas

We can use orthogonal rational functions to construct gaussian quadrature formulas, analogous to the polynomial case. If we consider orthogonality on the interval $[-1, 1]$, we have the following theorem from [1, 11] which we give without proof.

Theorem 4.1. *1. For each $n = 1, 2, \ldots$, the function $\varphi_n(z)$ has exactly n distinct zeros on $(-1, 1)$.*

2. Let $\{x_{nk}\}_{k=1}^{n}$ be the zeros of the n-th orthogonal rational function φ_n and let $\{\lambda_{nk}\}_{k=1}^{n}$ be defined by

$$\lambda_{nk} = \left[\sum_{k=0}^{n-1} (\varphi_k(x_{nk}))^2 \right]^{-1}.$$

Then the quadrature formula for

$$I_\mu(f) = \int_{-1}^{1} f(x) d\mu(x)$$

given by

$$I_n(f) = \sum_{k=1}^{n} \lambda_{nk} f(x_{nk})$$

is exact in $\mathcal{L}_n \cdot \mathcal{L}_{n-1} = \{gh : g \in \mathcal{L}_n, h \in \mathcal{L}_{n-1}\}$, i.e. $I_n(f) = I_\mu(f)$ for any f in $\mathcal{L}_n \cdot \mathcal{L}_{n-1}$.

Of course, using lemma 3.2 and the formulas from section 3, we can compute quadrature formulas on the halfline, given the nodes and weights for quadrature on the interval, as given in the following corollary.

Corollary 4.2. *Suppose we are given nodes* $\{x_{nk}\}_{k=1}^{n}$ *and weights* $\{\lambda_{nk}\}_{k=1}^{n}$ *such that the quadrature formula*

$$\int_{-1}^{1} f(x)d\mu(x) \approx \sum_{k=1}^{n} \lambda_{nk} f(x_{nk})$$

is exact in $\mathcal{L}_n \cdot \mathcal{L}_{n-1}$. *Then putting*

$$\begin{aligned} \Lambda_{nk} &= \lambda_{nk}, \\ \xi_{nk} &= \tau(x_{nk}), \end{aligned}$$

the formula

$$\int_{0}^{\infty} f(x)d\tilde{\mu}(x) \approx \sum_{k=1}^{n} \Lambda_{nk} f(\xi_{nk})$$

is exact in $\tilde{\mathcal{L}}_n \cdot \tilde{\mathcal{L}}_{n-1}$.

5 Asymptotic behaviour

In this section we look at several types of asymptotic behaviour for orthogonal rational functions on a halfline. This is another very straightforward application of lemma 3.2, since we can simply translate results about the convergence of ORF on an interval to the halfline case. So far, we have studied ratio asymptotics, convergence of the recurrence coefficients [12] and some weak-star convergence results [15] for orthogonal rational functions on $[-1, 1]$. From these two references we can get the following theorem, which we give without proof.

Theorem 5.1. *Let* $x = J(z)$ *denote the Joukowski transform* $x = \frac{1}{2}(z+z^{-1})$. *The inverse mapping is denoted by* $z = J^{-1}(x)$ *and is chosen so that* $|z| < 1$ *if* $x \notin [-1, 1]$. *Assume the sequence* $A = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{R}} \setminus [-1, 1]$ *is bounded away from* $[-1, 1]$ *and let* μ *satisfy the condition* $\mu' > 0$ *a.e. Then uniformly on compact subsets of* $\mathbb{C} \setminus [-1, 1]$ *we have*

$$\lim_{n\to\infty} \frac{z - \beta_{n+1}}{1 - \beta_n z} \sqrt{\frac{1 - \beta_n^2}{1 - \beta_{n+1}^2}} \frac{\varphi_{n+1}(x)}{\varphi_n(x)} = 1$$

where $z = J^{-1}(x)$ *and* $\beta_k = J^{-1}(\alpha_k)$ *for* $k = n, n+1$. *Under the same assumptions, the following relations hold in the sense that the ratio of the left hand side and the right hand side tends to 1 as* n *tends to infinity,*

$$\begin{aligned} E_n &\sim 2\frac{\sqrt{(1-\beta_{n-1}^2)(1-\beta_n^2)}(1-\beta_{n-1}\beta_n)}{(1+\beta_{n-1}^2)(1+\beta_n^2)}, \\ F_n &\sim -\sqrt{\frac{1-\beta_n^2}{1-\beta_{n-1}^2}}\frac{(1-\beta_{n-1}^2)(\beta_n+\beta_{n-2})+2\beta_{n-1}(1-\beta_n\beta_{n-2})}{(1+\beta_n^2)(1-\beta_{n-1}\beta_{n-2})}, \end{aligned}$$

Furthermore, still under the same assumptions, we have

$$\frac{1-x/\alpha_n}{\sqrt{1-1/\alpha_n^2}}\,\varphi_n^2(x)d\mu(x) \xrightarrow{*} \frac{1}{\pi}\frac{dx}{\sqrt{1-x^2}}.$$

where $\xrightarrow{*}$ *means convergence in the weak-star topology for measures.*

Using lemma 3.2 and theorem 3.4 it is very easy to obtain convergence results for orthogonal rational functions on the halfline $[0,\infty)$ when the poles stay away from infinity (this excludes the polynomial case). In fact, because of lemma 3.2 and the remark following (3) we know that $\tilde{\varphi}_n = \pm\varphi_n \circ \tau$, so we can rephrase the previous theorem, putting $\tau(x)$ in place of x, writing $\tau(\tilde{\alpha}_n)$ for α_n, etc. Instead of repeating all the formulas in their general form, we feel it is more informative to look at the special case where all poles are equal to each other, because then we can rewrite the expressions to obtain explicit representations for the limit functions. The computations are very straightforward and we only give the result in the following corollary.

Corollary 5.2. *Let* $\tilde{\mu}$ *be a measure on* $[0,\infty)$ *satisfying the condition* $\tilde{\mu}' > 0$ *a.e. and assume that all poles are equal to each other,* $\tilde{A} = \{\tilde{\alpha}, \tilde{\alpha}, \ldots\}$ *where* $\tilde{\alpha} \in (-\infty, 0]$ *(so the pole is finite). Then locally uniformly in* $\mathbb{C} \setminus [0,\infty)$ *we have*

$$\lim_{n\to\infty} \frac{\tilde{\varphi}_n(x)}{\tilde{\varphi}_{n-1}(x)} = \frac{(\sqrt{-x}+\sqrt{-\tilde{\alpha}})^2}{x-\tilde{\alpha}}$$

where the branch of the square root is such that the limit is real for $x < 0$. *For the recurrence coefficients we have the limits*

$$\begin{aligned} \lim_{n\to\infty} \tilde{E}_n &= -\frac{4}{\tilde{\alpha}}, \\ \lim_{n\to\infty} \tilde{F}_n &= -2. \end{aligned}$$

We also have the weak-star convergence result

$$(x-\tilde{\alpha})\tilde{\varphi}_n^2(x)d\tilde{\mu}(x) \xrightarrow{*} \frac{\sqrt{-\tilde{\alpha}}}{\pi}\frac{dx}{(1+x)\sqrt{x}}.$$

Remark. Note that the asymptotic values for the recurrence coefficients are exactly the values from example 3.6. This is not surprising; also in the polynomial case it is well known that the recurrence coefficients behave asymptotically like the coefficients of the Chebyshev polynomials if the weight is strictly positive almost everywhere.

6 Conclusion

In this paper we presented some formulas to transform orthogonal rational functions on $[0,\infty)$ to $[-1,1]$ and applied these formulas to obtain quadrature formulas and asymptotics for ORF on a halfline. The main reason for studying these formulas is that they allow us to limit our attention to the interval case. Especially from a computational point of view, there no longer exists a fundamental difference between orthogonal rational functions on a halfline and on an interval. However, the asymptotic results obtained through this transformation suffer from the limitation that the poles cannot tend to infinity, which excludes the polynomial case. As is well known, the study of convergence for polynomials orthogonal on a halfline (or on the whole real line), is much more complicated than the case of the interval, see e.g. [4] and the references therein.

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