KU LEUVEN

The “best” axiom system for teaching geometry to secondary school students: A source of controversy in the early 1960s

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Introduction to the workshop

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Start of the controversy

- The Royaumont Seminar 1959: Jean Dieudonné's "Euclid must go!" opens the debates
- Geometry education needs to be modernized
- No agreement on how to modernize geometry education on subsequent international meetings and conferences (Aarhus, Denmark (1960), Zagreb-Dubrovnik, Yugoslavia (1960), Bologna, Italy (1961), Stockholm 1962)
- Two protagonists in the debates: Gustave Choquet and Jean Dieudonné
- Both "*Bourbakists"* were convinced linear algebra is the royal road to geometry
- They disagreed on how this should be implemented in geometry education

Peak in the controversy

- Search for the “best” axiom system for secondary school geometry
- In 1964, both Choquet (“L’Enseignement de la Géométrie”) and Dieudonné (“Algèbre Linéaire et Géométrie Elémentaire”) publish a seminal book with an exposition of their respective viewpoint
- Analysis of these publications reveal key differences and similarities in their opinions (content of the workshop)

Settlement of the controversy

- At a meeting in Milano Marittima, Italy in 1965, and titled “The place of geometry in modern mathematical teaching”, André Revuz tried to reconcile Choquet’s and Dieudonné’s points of view
- Revuz presented a statement about the role of geometry in the education of 12- to 18-year-olds, based on Georges Papy’s two-stages approach, which was agreed upon by the attendees
- In the same year, on a colloquium in Echternach, Luxembourg, this proposal was solemnly approved by Choquet and Dieudonné

Settlement of the controversy, continued

- Papy's approach to the teaching of geometry consisted of two stages
  - In the lower grades (12–15-year-olds), a system of (synthetic) axioms for affine plane geometry was gradually introduced, leading to the structure of the vector plane, an approach showing strong parallels with Choquet's; while in the third year of secondary school (14–15-year-olds), this structure was equipped with an inner product, leading to the so-called Euclidean vector plane
  - At the beginning of the upper grades (15–18-year-olds) the achieved goal of the first stage—the Euclidean vector space structure of the plane—was taken as a new and "unique" axiom for the second stage, i.e., the further development of geometry (of the plane and of higher dimensions) from a purely algebraic perspective

Material for the workshop

(All translations are made by the presenting author)

Additional material for the workshop

- Apart from our paper the following additional material is available:
  - The following slides compare the content of the books of Choquet and Dieudonné
    - First the content of the introductions of the two books is compared
    - Then a brief comparison of the mathematical content follows
  - A copy of the introduction, bibliography and table of contents of the two books is available (one pdf file, intended for use in the context of the workshop only, not suited for publication), in French

General information on the authors and books

Choquet (1915-2006)

- Professor at “la Faculté des Sciences de Paris”
- “L’Enseignement de la Géométrie” first edition 1964, Hermann
- Acknowledgement to several mathematicians and teachers, French and foreigners
- Explicit acknowledgement to André Revuz

Dieudonné (1906-1992)

- Professor at “la Faculté des sciences de Nice” (assistant professor in 1964)
- “Algèbre Linéaire et Géométrie Elémentaire” second edition (corrected) 1964, Hermann
- I did not manage to locate a first edition, if it exists and somebody can lead me to it, let me know

Titles of the books

Choquet (1964)

- “L’Enseignement de la Géométrie”
- Education
- Geometry

Dieudonné (1964)

- “Algèbre Linéaire et Géométrie Elémentaire”
- Linear algebra
- Elementary geometry

Target audience of the books

Choquet (1964)

- Mathematics teachers higher cycle secondary education (Choquet, 1964, p. 7)
- Students in teacher training program (Choquet, 1964, p. 7)
- All who love mathematics (Choquet, 1964, p. 7)
- Possibly also 15–18-year-olds, under supervision of their teacher (Choquet, 1964, p. 7)

Dieudonné (1964)

- Mathematics teachers higher cycle secondary education (Dieudonné, 1964, p. 7, p. 12-13, p. 15)

Out of target audience

Choquet (1964)

- Young children: geometry should not be deductive, it should be based on observation (Choquet, 1964, p. 9)
- Mathematicians and 17-year-olds (after they have studied the scalar product): the plane or space is a 2- or 3-dimensional vector space over the real numbers, equipped with a scalar product (Choquet, 1964, p. 9)

Dieudonné (1964)

- First years of secondary education: use a mixture of experiments and reasoning on the results of these, no idea how to do this exact, see the good examples of our “Belgian neighbours” (reference to Georges Papy) (Dieudonné, 1964, p. 16-17)

What do secondary school students need?

Choquet (1964)

- Geometry education for secondary school students should start with notions as Euclid did, from the sensitive world, but which allow them to swiftly use the flexible tools from linear algebra (Choquet, 1964, p. 7)
- Start deductive reasoning as soon as children start to understand what a proof is and they want to “play” mathematics according to strict rules (Choquet, 1964, p. 9-10)

Dieudonné (1964)

- Familiarise secondary school students as soon as possible with thinking linearly (Dieudonné, 1964, p. 12)
- Learn children how to think and start as soon as possible with deductive reasoning (Dieudonné, 1964, p. 10, p. 16)
- Deductive reasoning is the true motor of mathematical thinking (and not the idea of axioms) (Dieudonné, 1964, p. 16-17)

Reference to Euclid

Choquet (1964)

- Euclid based plane geometry on the equality of triangles (Choquet, 1964, p. 7)
- This does not correspond to our logic requirements any more (Choquet, 1964, p. 10)
- The plane is a 2-dimensional affine space equipped with a scalar product (Choquet, 1964, p. 7)

Dieudonné (1964)

- It was much easier to obtain an agreement on the introduction of analysis in elementary education than it was to agree on the reform of geometry education because calculus does not profit from the sacred character that geometry bestows on Euclid (Dieudonné, 1964, p. 18)

Reference to Hilbert's axiom system

Choquet (1964)

- Hilbert's axiom system completes and expands Euclid's axiom system (Choquet, 1964, p. 10)
- It is not suited for secondary education (that was not Hilbert's concern) (Choquet, 1964, p. 10)
- It hides the vector structure of the space (Choquet, 1964, p. 10)

Dieudonné (1964)

- In the Euclidean tradition (Dieudonné, 1964, p. 13, p. 17)
- Of such complexity and of such subtlety that it is too difficult to teach, even at university before the third cycle at least (Dieudonné, 1964, p. 13, p. 17)

A royal road to geometry

Choquet (1964)

- We know a royal road based on vector spaces and scalar product (Choquet, 1964, p. 11)
- But these notions cannot be introduced at an age on which one has not yet grasped well enough the notion of algebraic operation (Choquet, 1964, p. 11)

Dieudonné (1964)

- We dispose of, as Choquet said so well, a royal road, starting from very simple axioms (contrary to those of Euclid-Hilbert) (Dieudonné, 1964, p. 11)

The “best” axiom system

Choquet (1964)

- A simple axiom system is needed with (Choquet, 1964, p. 10)
  - Strong axioms leading to non-trivial theorems
  - Intuitive axioms translating properties of our space that are easily verified
- It is not a big problem if the axioms are not independent, as long as it are not too many axioms (Choquet, 1964, p. 10)

Dieudonné (1964)

- We know, since Felix Klein’s Erlangen Program, that linear algebra is hiding behind geometry (Dieudonné, 1964, p. 12)
- One of the advantages of linear algebra is that is allows to present all developments of elementary geometry in a perfectly rigorous way and without effort (Dieudonné, 1964, p. 13)

The “best” axiom system (continued)

Choquet (1964)

- In a form that leads naturally and fast to the algebraic structure of the plane and space (Choquet, 1964, p. 7)
- Axiom system based on (Choquet, 1964, p. 11)
  - Additive structure straight line
  - Parallelism (addition of vectors)
  - Symmetry (scalar product)
  - Order on a straight line (positive scalar product)

Dieudonné (1964)

- Start with a clean slate, away from all traditions and guided by the programs of university or technical schools [grandes Ecoles techniques], while restricting to 2- and 3-dimensional space (Dieudonné, 1964, p. 13-14)
- Avoid pronouncing the word axiom (Dieudonné, 1964, p. 16)

Dieudonné's comment on Choquet's proposal

- Based on the excuse that the axioms of linear algebra are too abstract, one suggests to start from another, more accessible, axiom system, and then deduce the axioms of linear algebra (Dieudonné, 1964, p. 17)
- That is in short what Hilbert did in his famous works on the foundations of geometry when he started from Euclid's axioms and completed these (Dieudonné, 1964, p. 17)
- One tries a compromise between the scaffolding of Euclid-Hilbert and the "naked" axiom system of linear algebra (Dieudonné, 1964, p. 17)
- "The last known is the system of axioms recently proposed by Choquet, of a remarkable ingenuity which testifies to the great talent of its author, but which I consider to be perfectly useless and even harmful" (Dieudonné, 1964, p. 17)

Distinction affine and metric notions

Choquet (1964)

- Separate affine and metric notions (Choquet, 1964, p. 11)
- The first axioms in the presented system are sufficient to study the vector structure of the plane and the space (Choquet, 1964, p. 12)
- As well do they result in several properties one usually links to affine and metric notions (Choquet, 1964, p. 12)

Dieudonné (1964)

- Distinction between geometric properties of affine nature and properties of metric nature (Dieudonné, 1964, p. 13)
- In linear algebra, this distinction follows easily since the two types of properties depend respectively on two groups of axioms that are separated from the beginning (Dieudonné, 1964, p. 13)

Triangle and constructions

Choquet (1964)

- The fact that a triangle is half of a parallelogram has not prevented one to put emphasis for more than twenty centuries on a detailed study […] of triangles […] (Choquet, 1964, p. 10)
- One saw the triangle, but not the parallelogram which could have led to vectors (Choquet, 1964, p. 10)
- The triangle still occupies its interesting position, but avoid to study special points of it or several elegant but useless metric relations (Choquet, 1964, p. 10)

Dieudonné (1964)

- Previously one had to erect a complex and artificial scaffolding of constructions of auxiliary triangles in order to come back somehow to the sacred saints of equal or similar triangles (Dieudonné, 1964, p. 11)
- "It is certainly advisable to teach children the art of geometric constructions, but avoid as the plague what is undoubtedly the most gigantic "joke" of classical education, the limitation of the drawing instruments to ruler and compass…" (Dieudonné, 1964, p. 16)

Mathematical content

- Both authors first treat the plane and then continue with the space
- Both authors first treat affine concepts and then continue with metric concepts

Plane and straight lines

Choquet (1964)

- The *plane* is a set, equipped with a structure, induced by a set of subsets, called *straight lines* (Choquet, 1964, p. 15)
- A straight line is equipped with a structure (Choquet, 1964, p. 15)
- Axiom Ia: For every two different points, there exists a unique straight line through the two points (Choquet, 1964, p. 16)
- Corollary: A line passing through the origin is a linear subspace (Choquet, 1964, p. 33)

Dieudonné (1964)

- The *plane* is a set on which there exist three operations (addition, scalar multiplication and scalar product) (Dieudonné, 1964, p. 29)
- A *line through the origin passing through a* is the smallest linear subspace of the vector space which contains the non-null vector *a* (Dieudonné, 1964, p. 46)
- A *straight line* is an affine subspace with as *direction* a line through the origin (Dieudonné, 1964, p. 46)
- Proposition: For every two different points, there exists a unique straight line through the two points (Dieudonné, 1964, p. 46)

Parallel straight lines and the parallel postulate

Choquet (1964)

- Two straight lines are *parallel* either if they are equal or if they have no intersection (Choquet, 1964, p. 15)
- Axiom Ib (parallel postulate): For every straight line *L* and every point *p*, there exists exactly one straight line parallel to *L* which passes through *p* (Choquet, 1964, p. 16)
- Do not attempt to prove the existence in the parallel postulate since it is as mysterious to 16-year-olds as is the uniqueness (Choquet, 1964, p. 12)

Dieudonné (1964)

- Two straight lines are *parallel* if they have the same direction (Dieudonné, 1964, p. 46)
- Corollary: Two different straight lines which are not parallel intersect in a point (Dieudonné, 1964, p. 50)
- Corollary (parallel postulate): For every straight line *L* and every point *p*, there exists exactly one straight line parallel to *L* which passes through *p* (Dieudonné, 1964, p. 47)

Direction of a straight line and addition of vectors

Choquet (1964)

- The *direction* of a straight line is the equivalence class (associated to the equivalence relation “being parallel”) the straight line belongs to (Choquet, 1964, p. 16)
- The addition of vectors is defined using parallelograms (Choquet, 1964, p. 33)

Dieudonné (1964)

- A direction is a line through the origin (see definition of a straight line) (Dieudonné, 1964, p. 46)
- The addition of vectors is part of the definition of a vector space (Dieudonné, 1964, p. 31)

Affine transformations

- Up to order and attributed importance the two authors define and examine the same affine transformations, namely, translations, homotheties and dilatations
- Loyal to his choice of the axiom system, Choquet first defines a translation as an operation on a straight line, before extending this definition to the plane (Choquet, 1964, p. 30)
- Dieudonné first defines these transformations as mappings on the vector space, even before defining a straight line (Dieudonné, 1964, p. 34)

Definition vector space

Choquet (1964)

- A vector space is a commutative group equipped with a scalar multiplication that satisfies certain conditions (Choquet, 1964, p. 40)

Dieudonné (1964)

- First dedicates some pages to the necessary basic properties of real numbers and to the axioms of Euclidean geometry (Dieudonné, 1964, p. 21-30)
- A vector space is a commutative group on which there exist three operations (addition, scalar multiplication and scalar product) satisfying certain properties (Dieudonné, 1964, p. 31-32)

Scalar product, norm and distance

Choquet (1964)

- Axiom IIIa: *Distance* is a symmetric mapping on the plane (Choquet, 1964, p. 28)
- Axiom IVa: *Being perpendicular* is a symmetric relation on the set of straight lines in the plane (Choquet, 1964, p. 56)
- *Orthogonal projection* on a straight line is defined using perpendicular straight lines (Choquet, 1964, p. 57)

Dieudonné (1964)

- The *scalar product* of two vectors is defined as a symmetric, bilinear, positive and non-degenerate form (Dieudonné, 1964, p. 85)
- The *norm* of a vector is defined using the scalar product (Dieudonné, 1964, p. 85)
- The *distance* between two points is defined using the norm of a vector (Dieudonné, 1964, p. 85)

Scalar product, norm and distance, continued

Choquet (1964)

- The *norm* of a vector is defined using the distance between two points (Choquet, 1964, p. 58)
- The *scalar product* of two vectors is defined using orthogonal projection and the norm (Choquet, 1964, p. 59)
- Corollary: The scalar product is symmetric, bilinear and positive (Choquet, 1964, p. 59)

Dieudonné (1964)

- Corollary: The distance between two points is a symmetric mapping which satisfies some properties (Dieudonné, 1964, p. 85)
- Two vectors are *orthogonal* if their scalar product is zero (Dieudonné, 1964, p. 87)

Scalar product, norm and distance, continued

- Choquet mentions that the axioms he presented allow to show that the concept of distance that he introduced is equivalent to the concept of distance that is associated to the scalar product (which is the way Dieudonné defines distance) (Choquet, 1964, p. 148)
- "Theorem 96.3. For any point $o \in E$, one can define on E, in a unique way, a structure of a vector space with origin o, equipped with a scalar product, whose affine subspaces of dimension 1 and 2 are the lines and planes of E, and whose distance associated with the scalar product is identical to the distance given on E" (Choquet, 1964, p. 148)

Choquet's second axiom system

- The axiom system described in this book is presented at the 1959 Royaumont Seminar (Choquet, 1964, p. 12)
- In the appendix a second axiom system is presented, focusing on metric properties (Choquet, 1964, p. 12)
- "The axiom system that I developed in the previous chapters is vector based; … Certain teachers think that the metric notions are more intuitive and more easily understood by young students than are vector based notions. In fact, no regular education based on vector notions is given up to now to young children; …" (Choquet, 1964, p. 153)

Euclidean transformations and angles

- Both authors treat Euclidean transformations (i.e., translations, rotations and reflections) very similar
- Both authors first define a rotation and use the group of rotations to define an angle
- Measuring angles, trigonometry and trigonometric formulas follow later
- Choquet notes that a substantial part of geometry can be constructed without the use of angles (Choquet, 1964, p. 97)
- Dieudonné opposes strongly to overestimating the importance of trigonometric formulas:
  - “The myriad of trigonometric formulas …” (Dieudonné, 1964, p. 9)
  - “It is also quite true that the trigonometric formulas are absolutely indispensable to three eminently respectable professions: 1° astronomers; 2° surveyors; 3° authors of handbooks on trigonometry.” (Dieudonné, 1964, p. 10)

Appendices to the books

- Both Choquet (1964) and Dieudonné (1964) end their exposition on the “best” axiom system by looking beyond secondary school geometry by examining in the appendices some extra topics, such as angles from a different viewpoint, non-Euclidean geometry, and quaternions and rotations

Contemporary critique on the books

- Freudenthal wrote a fierce review of Dieudonné's book (Freudenthal, 1967, p. 746) : "I believe that Dieudonné's basic ideas on teaching mathematics are sound and healthier than the classical ones (if there are any) and than most of the current attempts at modernizing. It is a great pity that they are often uttered in a bizarre, preposterous and misleading way."
  - Freudenthal, H. (1967). [Review of Algèbre linéaire et géométrie élémentaire by J. Dieudonné]. The American Mathematical Monthly, 74(6), 744–748.
- The books of Choquet (1964) and Dieudonné (1964) were presumably not often used in secondary education

A true settlement of the controversy?

- In the third edition of Dieudonné's (1964) book, which was printed in 1968, the sharp attack on Choquet (slide "Dieudonné's comment on Choquet's proposal") is still present, without any alteration or addendum.

Possible further research

- What is the influence of Emil Artin's (1957) book "Geometric Algebra" on Choquet and Dieudonné? It is the only common reference both protagonists took up in their own work
- How exactly did Revuz come to the settlement between Choquet and Dieudonné? How is the position of Papy used in this and how was Papy influenced by Artin?