ON THE STRONG RELATIVE DEGREE OF TIME-DELAY SYSTEMS WITH NON-COMMENSURATE DELAYS

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Abstract. The fundamental notion of relative degree is studied for a class of linear time-delay systems of retarded type, where the standard assumption of commensurate delays is dropped. Algebraic conditions are provided that fully exploit the delay dependence structure. Even though the analyzed systems do not have a non-trivial feedthrough from input to output, it is shown that the relative degree may be sensitive to infinitesimal delay perturbations, which is at the basis of a novel notion of relative degree, called strong relative degree. This notion is characterized algebraically and computationally in both a SISO and a MIMO setting. By means of the obtained characterizations and a benchmark problem, which illustrates that invariant zeros may be characterized as zeros of quasi-polynomials of retarded, neutral or advanced type, new light is shed on existence conditions of a normal form. The novel concepts and theoretical results play a key role in the design and analysis of extended PD controllers, for which an algorithm and numerical experiments are presented.

Key words. Relative degree, time-delay systems, normal form, PID control

AMS subject classifications. 93B10, 93B35, 93B52, 93C23, 93D15

1. Introduction. The relative degree is a fundamental notion in systems theory which has for instance a major influence on the mechanisms, possibilities and limitations of feedback control. It plays a role in input-output feedback linearization of nonlinear systems [4, 20], possibly combined with a decoupling [10, 12, 11, 26, 27], inversion and tracking [2, 6, 18]. In this paper, we provide several definitions and mathematical characterizations, considering the linear time-delay system described by

$$\begin{cases} \dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$ and $y(t) \in \mathbb{R}^{n_y}$ are the state, input and output at time t, the system matrices are assumed to be real valued and the time-delays τ_i are ordered such that

$$0 < \tau_1 < \tau_2 < \cdots < \tau_m. \tag{1.2}$$

Intuitively, the relative degree of a system is equal to the number of times the output must be differentiated to reveal an explicit relation with the input. For delay systems, this creates some ambiguity as delayed inputs may appear by differentiation. In article [12], addressing a class of nonlinear systems with single delay, a classification is made into Type-I relative degree (any explicit relation with the input), Type-II relative degree (this relation reveals the input at the present time) and Type-III relative degree (this relation reveals the input at the present time only). If $m = 1$ in (1.1), the adopted definition in the paper corresponds to the Type-I relative degree.

In most works addressing the relative degree of time-delay systems or control problems for which this notion is crucial, such as [4, 2, 3, 6, 7, 11, 18] and the references therein, it is assumed that there is a single delay in the model or that the

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delays in (1.1) are commensurate, i.e., they are all multiples of a common basis delay. For linear time-delays systems, this assumption gives rise to algebraic characterizations in terms of univariate matrix polynomials whose variable corresponds to the shift or delay operator. As a first contribution in the paper, we present mathematical characterizations of the relative degree in the single input-single output (SISO) setting ($n_u = n_y = 1$), for the case where the assumption of commensurate delays is dropped. We namely allow an arbitrary dependency structure on the delays (1.2), and build the subsequent analysis on Theorem 2.1, which states that any given set of delays can be expressed as linear combinations of a possibly smaller number of rationally independent basis delays, such that the coefficients in these linear combinations are nonnegative integers. This fundamental result gives rise to novel algebraic characterizations in terms of multivariate matrix polynomials, and in terms of multi-powers of the coefficient matrices $(A_1, \ldots, A_m)$, as they are defined in [14] in the context of obtaining finiteness conditions for the $\mathcal{H}_2$ norm for systems governed by delay-differential algebraic equations (DDAEs). Also in articles [26, 27] on input-output decoupling of nonlinear time-delay systems, the state delays are not assumed commensurate, yet their interdependence is not exploited, and delay sensitivity is not considered, which we address next.

The relative degree is a structural property of the system that is known to be sensitive to small perturbations on the coefficient matrices in case the perturbations do not respect the underlying structure, inter-dependencies or sparsity pattern of the coefficient matrices induced by inherent properties of the modeled system (e.g., if they induce interconnections between inputs, outputs and different parts of the system that are physically not present). What is far less intuitive, but we demonstrate with an example, is that the relative degree of an output may be sensitive to infinitesimal perturbations of the delay values in (1.1) and that, as a consequence, important structural information may also be encoded in the inter-dependence between the delay values. This example brings us to the second contribution, in the form of definition and algebraic characterizations of a robustified notion of relative degree called *strong* relative degree, whose terminology is inspired by the notion of strong stability of delay equations of neutral type and DDAEs [15, 21, 23]. We note that also delay-independent stability along a particular direction in the delay-parameter space may be sensitive to infinitesimal perturbations to the direction [22].

Third, we analyze the multi input-multi output (MIMO) setting, where we rely on the introduced notion of strong relative degree and on recent results derived in [8, 9] for delay-free MIMO systems, thereby allowing the number of outputs to be larger than the number of inputs (called hyperoutput systems in [8]). We define the notion of leading vector of strong relative degrees of (1.1), which can be seen as a natural generalization of the notion of leading incomplete relative degree (LIRD) vector, introduced in [8]. The term "incomplete" refers to the fact that, even for square systems (having the same number of inputs and outputs), additional conditions are needed for invertibility of the input-output map, to which we will come back in Section 6. Note that the notion of delay vector relative degree in [10] requires such an invertibility condition. Similarly to a LIRD vector for delay-free MIMO systems, a vector of strong relative degrees does not always exists. However, its existence can always be induced by an output transformation (in $\mathbb{R}^{n_p}$), while the resulting vector of strong relative degrees is independent from the transformation that induces its existence. We will present an algorithm to construct such an output transformation along with the vector of strong relative degrees. This algorithm generalizes the algorithm in [8] for

LIRD vectors and the construction in [28], in the context of input-output inversion, towards time-delays systems and towards strong relative degrees. It relies on operations on block matrices (corresponding to collections of outputs) and on orthogonal transformations, rather than on operations on scalar equations.

As an illustration of the applicability of the presented results, we enlarge the scope of a control design method for linear systems with delays in states, inputs, and outputs, where all above concepts and transformations play a key-role. Recently, an increasing interest in P(I)D controllers for time-delay systems can be observed [5, 13, 19, 24, 25, 31, 32]. These low-order controllers offer as advantages the simplicity of an implementation (compared to full order, infinite-dimensional controllers) and the need to tune only a small amount of parameters, while for particular classes of systems, even analytic design procedures are available. In case the relative degree of an output is larger than one, the range of a PD controller can be extended by also using higher order output derivatives for feedback (leading to *extended* PD controllers, following the terminology of [30]). However, there is a pitfall since the resulting closed-loop system may be ill-posed or fragile if an infinitesimal parametric perturbation would induce a decrease of the relative degree. Therefore, the order of employed output derivatives should be determined from the strong relative degree rather than the standard relative degree. The output transformation inducing the vector of strong relative degrees, on its turn, assures that the relative degrees of the outputs are maximized (a precise statement will be made in Theorem 4.5), allowing to maximally use the information contained in the outputs and their derivatives. We will present an algorithm for tuning such extended PD controllers and demonstrate the increased performance compared to standard PD controllers. Relying on minimizing the spectral abscissa of the closed-loop system as a function of the controller parameters, the algorithm bears conceptual similarities to the algorithms for computing fixed-order controllers for DDAEs and non-fragile PID controllers for retarded systems, presented in [21] and [1], respectively.

Finally, for delay-free systems the relative degree is almost invariably coupled with a state transformation to a normal form, revealing the zero dynamics subsystem. For square and hyperoutput MIMO systems, the transformation to a normal form is addressed in [8, 9] and the references therein. For classes of delay systems of retarded type with commensurate delays, conditions for the existence of a similar transformation, along with algebraic construction procedures, are given in [3, 17]. A major limitation is that in the adopted structure preserving setting, aiming at a transformed system also in the form of delay equations of retarded type, the matrix polynomial describing the feedthrough between inputs and output derivatives needs to be unimodular, as well as the matrix polynomial describing the transformation, to enable a realization of the transformed system. The translations of these requirements to delay systems with incommensurate delays are even more restrictive, as we shall see. In order to shed light on the origin of these restrictions and, at the same time, unify the different parts of the paper, we will finally couple back to the motivating example used to illustrate the potential sensitivity of the relative degree with respect to infinitesimal delay changes. For certain values of parameters, we will show that the zeros of the transfer function are described by quasipolynomials of neutral and even advanced type, in the latter case even prohibiting a realization in the time-domain by a causal dynamical system model.

The structure of the paper is as follows. In Section 2 we present some preliminary results. In Section 3 we define and analyze two notions of relative degree for SISO time-delay systems of the form (1.1), and in Section 4 we make the leap towards MIMO

$\mathbb{R}$	set of real numbers
$\mathbb{R}_{\geq 0},\ \mathbb{R}_{>0}$	sets of nonnegative and positive real numbers
$\mathbb{Z}$	set of integers
$\mathbb{Z}_{\geq 0},\ \mathbb{Z}_{>0}$	sets of non-negative integers and natural numbers
$\bar{\mathbb{Z}}_{\geq 0}$	$\mathbb{Z}_{\geq 0} \cup \infty$
$\mathbb{C}$	set of complex numbers
$\mathcal{D}_p,\ p \in \mathbb{Z}_{>0}$	$\{(z_1, ..., z_p) \in \mathbb{C}^p :\ \|z_i\| = 1, i = 1, .., p\}$
$\imath$	imaginary unit, $\imath = \sqrt{-1}$
$\Re(s), \Im(s), \|s\|,\quad s \in \mathbb{C}$	real part, imaginary part and modulus of s
$\vec{\tau},\ \vec{\tau}_e$	short notation for $(\tau_1, \ldots, \tau_m)$ and for $(0, \tau_1, \ldots, \tau_m)$
$\vec{k}, \vec{\ell}, \ldots \in \mathbb{Z}_{\geq 0}^{m+1}$	short notation for $(k_0, k_1, \ldots, k_m), (\ell_0, \ell_1, \ldots, \ell_m), \ldots$
$\vec{k} \cdot \vec{\ell},\quad \vec{k}, \vec{\ell} \in \mathbb{Z}_{\geq 0}^{m+1}$	inner product, $\sum_{i=0}^{m} k_i \ell_i$
$f(s;\ p)$	function of variable s, parametrized by p
$g(s) \equiv 0,\ g : \Omega \subseteq \mathbb{C} \to \mathbb{C}^k$	$g(s) = 0$ for all $s \in \Omega$
$y^{(j)}(t),\ j \in \mathbb{Z}_{\geq 0}$	j-th derivative of univariate function y at t, $y^{(0)}(t) = y(t)$
$\mathrm{card}(S)$	cardinality of set S
$r_\sigma(A)$	spectral radius of matrix A

TABLE 2.1
Notations.

systems. In Section 5 we present the design of extended PD controllers. Finally in Section 6 we comment on the existence and properties of the normal form, while in Section 7 we present some concluding remarks. The technical proofs are collected in Appendix B.

2. Preliminaries. The notations adopted in the paper are listed in Table 2.1.

Of major importance in the remainder of the paper are the *interdependencies* between the delays. The delays $(\tau_1, \ldots, \tau_m)$ are said to be *rationally independent* if

$$\sum_{i=1}^{m} z_i \tau_i = 0,\quad z_i \in \mathbb{Z},\ i = 1, \ldots, m$$

implies $z_i = 0,\ i = 1, \ldots, m$. For example, two delays are rationally independent if their ratio is no rational number.

In general, any given set of delay values can be expressed in terms of a possibly smaller number of independent delays, as clarified in the following theorem.

THEOREM 2.1. *For any set of delays $(\tau_1, \ldots, \tau_m) \in \mathbb{R}_{>0}^m$, there exist $p \leq m$ rationally independent numbers $(r_1, \ldots, r_p) \in \mathbb{R}_{>0}^p$ and matrix $\Gamma \in \mathbb{Z}_{\geq 0}^{m \times p}$ of full column rank such that*

$$\begin{bmatrix} \tau_1 \\ \vdots \\ \tau_m \end{bmatrix} = \underbrace{\begin{bmatrix} n_{1,1} & \cdots & n_{1,p} \\ \vdots & & \vdots \\ n_{m,1} & \cdots & n_{m,p} \end{bmatrix}}_{\Gamma} \begin{bmatrix} r_1 \\ \vdots \\ r_p \end{bmatrix} \tag{2.1}$$

holds.

If (2.1) holds with $p < m$, the delays $(\tau_1, \ldots, \tau_m)$ are called rationally dependent. If (2.1) holds with $p = 1$, they are called commensurate (multiples of the same basis delay).

EXAMPLE 2.2. *For $m = 3$, delays $(\tau_1, \tau_2, \tau_3) = (1, e, \pi)$ are rationally independent. Delays $(1, \pi, 1 + \pi)$ are rationally dependent and can be expressed in the form (2.1) with rank-two matrix*

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

and $(r_1, r_2) = (1, \pi)$. Delays $(\tau_1, \tau_2, \tau_3) = (1, 2, 3)$ are commensurate, because they are all multiples of basis delay $r_1 = 1$ and (2.1) is satisfied with $r_1 = 1$ and $\Gamma = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$.

Throughout the paper we strongly rely on properties spelled out in the following two propositions, where the latter corresponds to an extension of Kronecker's theorem.

PROPOSITION 2.3. *For any $p \in \mathbb{Z}_{>0}$, the set*

$$\left\{(r_1, \ldots, r_p) \in \mathbb{R}^p_{\geq 0} : \ (r_1, \ldots, r_p) \text{ are rationally independent}\right\}$$

is dense in $\mathbb{R}^p_{\geq 0}$.

PROPOSITION 2.4. *If the delays $(r_1, \ldots, r_p)$ are rationally independent, then for all $(\theta_1, \ldots, \theta_p) \in \ [0, \ 2\pi]^p$, $\epsilon > 0$ and $d_0 \geq 0$, there exists a number $d > d_0$ such that $\left|e^{-\imath d r_i} - e^{-\imath \theta_i}\right| < \epsilon, \ i = 1, \ldots, p$.*

The transfer function of system (1.1) is given by

$$H(s, \ \vec{\tau}) = C \left(sI - A_0 - \sum_{i=1}^{m} A_i e^{-s\tau_i} \right)^{-1} B. \tag{2.2}$$

When the delays satisfy relation (2.1), it can also be expressed as

$$H(s, \ \vec{\tau}) = G\left(s, e^{-sr_1}, \ldots, e^{-sr_p}\right) \tag{2.3}$$

where G is the multivariate rational function

$$G(s, z_1, \ldots, z_p) =: C\left(sI - \mathcal{A}(z_1, \ldots, z_p)\right)^{-1} B$$

with

$$\mathcal{A}(z_1, \ldots, z_p) := A_0 + \sum_{i=1}^{m} A_i \left(z_1^{n_{i,1}} \cdots z_p^{n_{i,p}}\right). \tag{2.4}$$

In these representations, the inter-dependence between the delays is explicitly taken into account.

3. Strong relative degree of a SISO system. Throughout the section, we consider system (1.1) with $n_u = n_y = 1$. Intuitively, the relative degree is the number of times the output must be differentiated in order to get direct access to the input. Before we present formal definitions in Section 3.2, we provide some examples, which illustrate non-intuitive properties induced by the delayed terms and inspire a robustified notion of relative degree.

3.1. Motivating examples. We consider the case where $m = 3$ and the matrices are spelled out as

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\kappa \\ 0 & -2 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \tag{3.1}$$

with $\kappa \in \mathbb{R}$ a parameter. It can be readily verified that $\dot{y}(t)$ and $\ddot{y}(t)$ do not explicitly depend on the input. In terms of the decomposition $x(t) = \begin{bmatrix} x_1(t) & \cdots & x_4(t) \end{bmatrix}^T$, we get

$$\begin{aligned} \dddot{y}(t) &= x_1(t) + 2x_3(t-\tau_2) + x_4(t-\tau_2) + x_3(t-\tau_3) + x_4(t-\tau_3) \\ &\quad - 2\kappa x_4(t-(\tau_1+\tau_2)) + x_2(t-2\tau_2) - x_2(t-(\tau_1+\tau_3)) - \kappa x_4(t-(\tau_1+\tau_3)) \\ &\quad + 4\kappa\, x_2(t-(2\tau_1+\tau_2)) + 2\kappa\, x_2(t-(\tau_1+2\tau_2)) + 2\kappa\, x_2(t-(2\tau_1+\tau_3)) \\ &\quad + \kappa\, x_2(t-(\tau_1+\tau_2+\tau_3)) + u(t-2\tau_2) - u(t-(\tau_1+\tau_3)), \end{aligned}$$

from which we conclude that the relative degree, in the sense of Type-I relative degree in [12], is equal to three whenever $2\tau_2 \neq \tau_1 + \tau_3$, and larger than three otherwise.

If the equality $2\tau_2 = \tau_1 + \tau_3$ holds, then we differentiate the output once more, which brings us to

$$\begin{aligned} y^{(4)}(t) &= F(x_t) + 4\kappa u(t-(2\tau_1+\tau_2)) + 2\kappa u(t-(\tau_1+2\tau_2)) \\ &\quad + 2\kappa u(t-(2\tau_1+\tau_3)) + \kappa u(t-(\tau_1+\tau_2+\tau_3)) \\ &= F(x_t) + 4\kappa u(t-(2\tau_1+\tau_2)) + 4\kappa u(t-(\tau_1+2\tau_2)) + \kappa u(t-3\tau_2), \end{aligned}$$

with $x_t := x(t+\theta),\ \theta \in [-4\tau_3,\ 0]$ and F some bounded linear functional. We can further distinguish between two cases. If $\kappa \neq 0$, then the relative degree is equal to four. If $\kappa = 0$, then it can be observed that $\ddot{y}(t) = \dot{y}(t)$, inducing

$$y^{(k)}(t) = \dot{y}(t), \quad k \geq 1, \tag{3.2}$$

hence differentiating the output does not provide access to the input (in such a situation we say the relative degree is equal to infinity). Thus, the relative degree, RD, satisfies

$$RD(\vec{\tau}) = \begin{cases} 3, & 2\tau_2 \neq \tau_1 + \tau_3, \\ 4, & 2\tau_2 = \tau_1 + \tau_3 \text{ and } \kappa \neq 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.3}$$

These computations are in agreement with the transfer function

$$H(s,\ \vec{\tau}) = \frac{e^{-2s\tau_2} - e^{-s(\tau_1+\tau_3)}}{s^2(s-1)} + \kappa \frac{4e^{-s(2\tau_1+\tau_2)} + 2e^{-s(\tau_1+2\tau_2)} + 2e^{-s(2\tau_1+\tau_3)} + e^{-s(\tau_1+\tau_2+\tau_3)}}{s^3(s-1)}. \tag{3.4}$$

For delay-free systems, an infinite relative degree and zero transfer function are associated with linear dependencies between output derivatives, which is also the case for (3.2). The situation may be different, since for system matrices

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}^T, \quad (3.5)$$

we can express

$$\begin{array}{rcl} \dot{y}(t) & = & \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} x(t-\tau_1) + \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t-\tau_2), \\ \ddot{y}(t) & = & \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} x(t-(\tau_1+\tau_2)) = y(t-(\tau_1+\tau_2)), \end{array}$$

which induces

$$y^{(2k)}(t) = y(t-k(\tau_1+\tau_2)), \quad k \geq 1.$$

In conclusion, the relative degree may be sensitive to *infinitesimal* delay perturbations, as is the case at nominal delay values satisfying $2\tau_2 = \tau_1 + \tau_3$ in the first example, while different mechanisms may give rise to an infinite relative degree.

REMARK 3.1. *The relative degree is sensitive with respect to infinitesimal perturbations to matrices A_i, B, C as well. However, matrix perturbations inducing a change in relative degree are, as a rule, perturbations that do not respect the underlying structure of the system. For instance, replacing a zero element by a nonzero element may correspond to introducing an interconnection that is physically not present.*

Surprisingly, the examples illustrate that important structural information is encoded in the inter-dependence between the different delay values, and motivate us to take small delay perturbations into account, in particular in applications where the delays in model (1.1) correspond to different physical parameters. Note in this context that if system (1.1) with matrices (3.1) is combined with controller $u(t) = K_0 y(t-r) + \sum_{i=1}^{4} K_i y^{(i)}(t-r)$ and $K_4 \neq 0$, the closed-loop system is well-posed if $2\tau_2 = \tau_1 + \tau_3$ (the model can be recast in terms of DDAEs of index one), but ill-posed if $2\tau_2 \neq \tau_1 + \tau_3$ (delay equations of advanced type).

3.2. Definition and properties. The adopted definition of relative degree is rooted in explicit expressions for output derivatives. Instrumental to these expressions are matrix polynomials $P_{k_0,\ldots,k_m}$, with $k_j \in \mathbb{Z}_{\geq 0}, j = 0,\ldots,m$, which are recursively defined through

$$P_{0,\ldots,0} := I, \quad (3.6)$$

$$P_{k_0,\ldots,k_m} := A_0 P_{k_0-1,k_1,\ldots,k_m} + A_1 P_{k_0,k_1-1,\ldots,k_m} + \ldots + A_m P_{k_0,k_1,\ldots,k_m-1} \quad (3.7)$$

and

$$P_{k_0,\ldots,k_m} := 0 \text{ if any } k_j \in \mathbb{Z}_{<0}, j = 0,\ldots,m. \quad (3.8)$$

For instance, for $m = 1$ and $k_0 + k_1 \leq 3$, these matrix polynomials are

$$\begin{array}{l} P_{0,0} = I, \\ P_{1,0} = A_0, \quad P_{0,1} = A_1, \\ P_{2,0} = A_0^2, \quad P_{1,1} = A_0A_1 + A_1A_0, \quad P_{0,2} = A_1^2, \\ P_{3,0} = A_0^3, \quad P_{2,1} = A_0^2A_1 + A_0A_1A_0 + A_1A_0^2, \\ P_{1,2} = A_0A_1^2 + A_1A_0A_1 + A_1^2A_0, \quad P_{0,3} = A_1^3. \end{array}$$

Note that $P_{k_0,\ldots,k_m}$ can be interpreted as the sum of all monomials in $(A_0,\ldots,A_m)$ that have power k_i in matrix A_i for $i=0,\ldots,m$. If we further define

$$\Omega_j := \left\{(k_0,\ldots,k_m)\in\mathbb{Z}_{\geq 0}^{m+1}:\ k_0+\cdots+k_m=j\right\}$$

for $j\in\mathbb{Z}_{\geq 0}$, we can express

$$\begin{array}{rcl}
\dot{y}(t) &=& CA_0x(t)+\sum_{i=1}^m CA_ix(t-\tau_i)+CBu(t)\\
&=& \sum_{\vec{k}\in\Omega_1} CP_{k_0,\ldots,k_m}x(t-k_1\tau_1\cdots-k_m\tau_m)+CP_{0,\ldots,0}Bu(t)\\
\ddot{y}(t) &=& CA_0^2x(t)+\sum_{i=1}^m C(A_0A_i+A_iA_0)\,x(t-\tau_i)+CA_0Bu(t)\\
&& +\sum_{i_1=1}^m\sum_{i_2=1}^m CA_{i_1}A_{i_2}x((t-\tau_{i_1}-\tau_{i_2})+CB\dot{u}(t)+\sum_{i=1}^m CA_iBu(t-\tau_i)\\
&=& \sum_{\vec{k}\in\Omega_2} CP_{k_0,\ldots,k_m}x(t-k_1\tau_1-k_2\tau_2\cdots-k_m\tau_m)\\
&& +\sum_{i=0}^1\left(\sum_{\vec{k}\in\Omega_i} CP_{k_0,\ldots,k_m}Bu^{(1-i)}(t-k_1\tau_1-k_2\tau_2\cdots-k_m\tau_m)\right),
\end{array}$$

which can be generalized to

$$\begin{aligned}
y^{(j)}(t) = &\sum_{\vec{k}\in\Omega_j} CP_{k_0,\ldots,k_m}x(t-\vec{k}\cdot\vec{\tau}_e)\\
&+\sum_{i=0}^{j-1}\left(\sum_{\vec{k}\in\Omega_i} CP_{k_0,\ldots,k_m}B\ u^{(j-1-i)}(t-\vec{k}\cdot\vec{\tau}_e)\right),\quad\forall j\geq 1. \qquad (3.9)
\end{aligned}$$

If the relative degree of the output is larger than one, it is determined by the highest value of j for which

$$\begin{array}{l}
\sum_{\vec{k}\in\Omega_0} CP_{k_0,\ldots,k_m}B\ u^{(j-1)}(t-\vec{k}\cdot\vec{\tau}_e)=0,\\
\qquad\vdots\\
\sum_{\vec{k}\in\Omega_{j-2}} CP_{k_0,\ldots,k_m}B\ u^{(1)}(t-\vec{k}\cdot\vec{\tau}_e)=0,
\end{array}\qquad (3.10)$$

independently of the input signal - recall that we look for an explicit relation between output derivatives and the (possibly delayed) input, as is the case for the Type-I relative degree in [12]. The formulations in terms of matrix polynomials (3.6)-(3.8) allowed us to combine contributions corresponding to permutations of the same delays into one term (see the expression for $\ddot{y}$). However, it is still possible that terms in (3.10) share the same input delay, as illustrated with system (3.1) for $2\tau_2=\tau_1+\tau_3$. Therefore, we first introduce the sets

$$\Upsilon_i := \bigcup_{\vec{k}\in\Omega_i}\left\{\vec{k}\cdot\vec{\tau}_e\right\},\quad i\geq 0,$$

whose cardinalities are equal to the effective number of input delays in (3.10).

DEFINITION 3.2. *The output has relative degree (RD) equal to $r\geq 1$ if*

$$\sum_{\vec{k}\in\Omega_i,\ \vec{k}\cdot\vec{\tau}_e=\alpha} CP_{k_0,\ldots,k_m}B=0,\quad\forall\alpha\in\Upsilon_i \qquad (3.11)$$

holds for all $i\in\{0,\ldots,r-2\}$, and there exists an $\alpha\in\Upsilon_{r-1}$ such that

$$\sum_{\vec{k}\in\Omega_{r-1},\ \vec{k}\cdot\vec{\tau}_e=\alpha} CP_{k_0,\ldots,k_m}B\neq 0.$$

The output has relative degree $RD = +\infty$ if (3.11) holds for all $i \geq 0$.

Note that for any $i \geq 0$, the set

$$\left\{ \sum_{\vec{k} \in \Omega_i,\ \vec{k} \cdot \vec{\tau}_e = \alpha} P_{k_0,\ldots,k_m} : \quad \alpha \in \Upsilon_i \right\}$$

can be identified as the set of coefficients of the multivariate polynomial $\mathcal{A}(z_1, \ldots, z_p)^i$, with $\mathcal{A}$ given by (2.4). This yields a simple characterization, which directly extends the definition of relative degree for delay-free systems, for which $\mathcal{A}$ reduces to a constant (system) matrix.

PROPOSITION 3.3. *The output has relative degree (RD) equal to $r \geq 1$ if and only if*

$$\begin{array}{l} C\mathcal{A}(z_1, \ldots, z_p)^i B \equiv 0, \quad \forall i \in \{0, \ldots, r-2\}, \\ C\mathcal{A}(z_1, \ldots, z_p)^{r-1} B \not\equiv 0. \end{array}$$

The output has relative degree $RD = +\infty$ if $C\mathcal{A}(z_1, \ldots, z_p)^i B \equiv 0$ for all $i \geq 0$.

Note that (2.4) connects with the frequency domain description of equation (1.1) via the relation $z_i = e^{-sr_i},\ i = 1, \ldots, p$. By Proposition A.1, $C\mathcal{A}(z_1, \ldots, z_p)^{r-1} B \not\equiv 0$ implies $C\mathcal{A}(z_1, \ldots, z_p)^{r-1} B \neq 0$ for some $z \in \mathcal{D}_p$. As delays $(r_1, \ldots, r_p)$ are rationally independent, Proposition 2.4 implies on its turn that $C\mathcal{A}\left(e^{-\imath\omega r_1}, \ldots, e^{-\imath\omega r_p}\right)^{r-1} B \neq 0$, which is consistent with a direct feedthrough from $y^{(r)}$ to u.

The following proposition provides a characterization in terms of transfer function (2.2).

PROPOSITION 3.4. *The output has relative degree $r \geq 0$ if and only if*

$$\lim_{\omega \to +\infty} \omega^i H(\imath\omega,\ \vec{\tau}) = 0, \quad i \in \{0, \ldots, r-1\}$$

and

$$\limsup_{\omega \to +\infty} \omega^r \left| H(\imath\omega,\ \vec{\tau}) \right| > 0.$$

It has $RD = +\infty$ if $H(s,\ \vec{\tau}) \equiv 0$.

In terms of representation (2.3), the frequency domain description is as follows.

PROPOSITION 3.5. *The output has relative degree $r \geq 0$ if and only if*

$$\lim_{\omega \to +\infty} \max_{\vec{z} \in \mathcal{D}_p} \omega^i G(\imath\omega, z_1, \ldots, z_p) = 0, \quad i \in \{0, \ldots, r-1\}$$

and

$$\lim_{\omega \to +\infty} \max_{\vec{z} \in \mathcal{D}_p} \omega^r \left| G(\imath\omega, z_1, \ldots, z_p) \right| > 0.$$

EXAMPLE 3.6. *We consider system (1.1) and (3.1) with $\kappa = 0.1$ and commensurate delays*

$$\vec{\tau} = (1, 2.1, 3) = (10, 21, 30)\ r_1, \quad r_1 = 0.1,$$

implying

$$G(s, z_1) = C\left(sI - A_0 - A_1 z_1^{10} - A_2 z_1^{21} - A_3 z_1^{30}\right)^{-1} B.$$

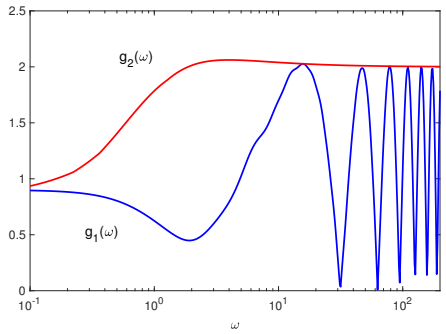


FIG. 3.1. *Functions (3.12) for delays* $\vec{\tau} = (1, 2.1, 3)$.

The relative degree is equal to three. In Figure 3.1, we display the functions

$$g_1 : \ \omega \mapsto \omega^3 \left| H(\imath\omega, \ \vec{\tau}) \right|, \quad g_2 : \ \omega \mapsto \omega^3 \max_{z \in \mathcal{D}_1} \left| G(\imath\omega, z) \right|. \tag{3.12}$$

Both $\limsup_{\omega \to +\infty} g_1(\omega)$ *and* $\lim_{\omega \to \infty} g_2(\omega)$ *exist and are strictly positive, in accordance with Propositions 3.4-3.5.*

Note that the delays can be seen as perturbations of delays $(1, 2, 3)$*, for which the relative degree is equal to four. This explains why function* g_1 *decreases towards zero in the low frequency range and* $\lim_{\omega \to +\infty} g_1(\omega)$ *does not exist. However, this issue of sensitivity is removed by using the representation in terms of* G*, in which the interdependence of the nominal delays is precisely encoded.*

The sensitivity of the relative degree with respect to infinitesimal delay perturbations, apparent from (3.3), motivates the following robustified notion.

DEFINITION 3.7. *The strong relative degree of SISO system (1.1) is given by*

$$sRD(\vec{\tau}) := \lim_{\epsilon \to 0+} \min \left\{ RD(\vec{\nu}) : \ \vec{\nu} \in \mathcal{B}(\vec{\tau}, \epsilon) \right\}$$

with

$$\mathcal{B}(\vec{\tau}, \epsilon) := \left\{ \vec{\nu} \in \mathbb{R}^m_{\geq 0} : \ \|\vec{\nu} - \vec{\tau}\|_2 < \epsilon \right\}.$$

Expression (3.9) for output derivatives is key to a simple algebraic characterization. Definition 3.7 explicitly takes small delay perturbations into account, and by Proposition 2.3, there are rationally independent delays arbitrarily close to the nominal delays. For rationally independent delays $\vec{\tau}$, it is not possible that different terms in (3.10) share the same input delay. Indeed, assuming that $\vec{k} \cdot \vec{\tau}_e = \vec{l} \cdot \vec{\tau}_e$ with $\vec{k} \neq \vec{l}$, we would have $(\vec{k} - \vec{l}) \cdot \vec{\tau}_e = 0$, contradicting that the delays $\vec{\tau}$ are rationally independent. Hence, (3.10) can only be zero for any sufficiently smooth input signal if all coefficients are zero. This brings us to the following result.

THEOREM 3.8. *The output has strong relative degree (sRD) equal to* r *if and only if*

$$CP_{k_0,\ldots,k_m} B = 0, \quad \forall (k_0, \ldots, k_m) \in \mathbb{Z}^{m+1}_{\geq 0} : \ k_0 + \cdots + k_m < r - 1 \tag{3.13}$$

and there exist some $(k_0, \ldots, k_m)$ *satisfying* $k_0 + \cdots + k_m = r - 1$ *such that* $CP_{k_0,\ldots,k_m} B \neq 0$.

The output has strong relative degree $sRD = \infty$ if (3.13) holds for all $r \geq 1$.

EXAMPLE 3.9. *System (1.1) with matrices (3.1) has strong relative degree equal to three, which can directly be seen from the transfer function (3.4). We have*

$$\begin{array}{l} CP_{0,0,0,0}B = CB = 0, \\ CP_{1,0,0,0}B = CA_0B = 0, \quad CA_1B = 0, \quad CA_2B = 0, \quad CA_3B = 0, \\ CP_{2,0,0,0}B = CA_0^2B = 0, \quad CA_1^2B = 0, \quad CA_2^2B = 1, \; CA_3^2B = 0, \\ CP_{1,1,0,0}B = C(A_0A_1 + A_1A_0)B = 0, \quad C(A_0A_2 + A_2A_0)B = 0, \quad C(A_0A_3 + A_3A_0)B = 0, \\ C(A_1A_2 + A_2A_1)B = 0, \quad C(A_1A_3 + A_3A_1)B = -1, \quad C(A_2A_3 + A_3A_2)B = 0, \end{array}$$

hence, (3.13) is satisfied for $r \leq 3$ but not for $r = 4$. Note that the drop in the (standard) relative degree whenever $2\tau_2 = \tau_1 + \tau_3$ is due to the relation

$$CP_{0,0,2,0}B + CP_{0,1,0,1}B = 0.$$

According to Theorem 3.8, determining whether the strong relative degree is equal to infinity, corresponds to verifying an infinite number of equalities, which can, however, be reduced to checking a finite number of equalities using a generalization of the Cayley-Hamilton theorem to multi-powers (3.6)-(3.8), as spelled out in [14, Theorem 5]. A combination with Theorem 3.8 leads us to the following proposition.

PROPOSITION 3.10. *The output has strong relative degree $sRD = +\infty$ if and only if $CP_{k_0,\ldots,k_m}B = 0, \quad \forall (k_0, \cdots, k_m) : \; k_0 + \ldots + k_m < (m+1)n$.*

EXAMPLE 3.11. *For system (1.1) with matrices (3.5), it can be readily verified that*

$$CP_{k_0k_1,k_2}B = 0, \; \forall (k_0, k_1, k_2) \in \mathbb{Z}^3_{\geq 0} : \; k_0 + k_1 + k_2 < 12,$$

hence $sRD = +\infty$.

We conclude the section by listing some properties of the strong relative degree.

THEOREM 3.12. *SISO system (1.1) has the following properties:*

1. $\mathrm{sRD} \leq \mathrm{RD}(\vec{\tau})$*;*
2. sRD *is independent of the delays;*
3. $\mathrm{sRD} = \mathrm{RD}(\vec{\tau})$ *for rationally independent $\vec{\tau}$;*
4. *the output has $\mathrm{sRD} = r$ if and only if $C\left(A_0 + \sum_{i=1}^m A_i z_i\right)^i B \equiv 0$ for $i \in \{0, \ldots, r-2\}$ and $C\left(A_0 + \sum_{i=1}^m A_i z_i\right)^{r-1} B \not\equiv 0$;*
5. *the output has $\mathrm{sRD} = r$ if and only if*

$$\begin{array}{l} \lim_{\omega \to +\infty} \max_{\vec{z} \in \mathcal{D}_m} \omega^i \hat{G}(\imath\omega, z_1, \ldots, z_m) = 0, \qquad i \in \{0, \ldots, r-1\} \\ \lim_{\omega \to +\infty} \max_{\vec{z} \in \mathcal{D}_m} \omega^r \left| \hat{G}(\imath\omega, z_1, \ldots, z_m) \right| > 0. \end{array}$$

with

$$\hat{G}(s, z_1, \ldots, z_m) := C \left(sI - A_0 - \sum_{i=1}^m A_i z_i \right)^{-1} B.$$

4. Vector of strong relative degrees for MIMO systems. We consider system (1.1), without assumptions on the number of inputs and outputs, and we decompose output equation $y(t) = Cx(t)$ as

$$y_i(t) = C_i x_i(t), \; i = 1, \ldots, n_y.$$

In Section 4.1 we introduced the notion of leading vector of strong relative degrees, inspired by the work in [8, 18, 9] on generalizations of the relative degree to MIMO

systems. As we shall see, such a vector does not always exist. However, the set of the relative degrees of individual outputs is not invariant under an output transformation and there always exist an output transformation such that a leading vector of relative degree uniquely exists. This transformation is addressed in Section 4.2.

4.1. Definition. We first define the strong relative degree of one individual output.

DEFINITION 4.1. *The strong relative degree of output $y_i,\ i \in \{1,\dots,n_y\}$, is the minimum of its strong relative degrees with respect to all inputs.*

We further denote by $H_k,\ k \geq 0$, a matrix such that

$$\operatorname{range}(H_k) = \operatorname{span}\left\{ P_{\ell_0,\dots,\ell_m} B :\ \vec{\ell} \in \Omega_k \right\}. \tag{4.1}$$

Similar to the concept of LIRD vector from [8] for delay-free systems, we introduce the following definition.

DEFINITION 4.2. *Vector $(\rho_1,\dots,\rho_{n_y}) \in \bar{\mathbb{Z}}_{>0}^{n_y}$ is called a leading vector of strong relative degrees for system (1.1) if the following properties holds.*

- *The strong relative degree of output y_i is equal to ρ_i for $i = 1,\dots,n_y$, such that*

$$\rho_1 \leq \rho_2 \leq \cdots \leq \rho_{n_y}.$$

- *For any $i,j \in \{1,\dots,n_y\}$ such that*

$$\rho_i = \rho_{i+1} = \cdots = \rho_j < \infty,$$

matrix

$$\begin{bmatrix} C_i \\ \vdots \\ C_j \end{bmatrix} H_{\rho_i - 1} \tag{4.2}$$

is of full row rank.

Expression (3.9) and Theorem 3.8 reveal the meaning of the rank condition on matrix (4.2). This condition expresses that the ρ_i-th derivative of all outputs with strong relative degree ρ_i explicitly depend on vector

$$\operatorname{col}\left(u(t - \vec{k}\cdot\vec{\tau}_e) \right)_{\vec{k}\in\Omega_{\rho_i-1}}$$

in linearly independent ways. On the one hand, it should be noted that a leading vector of strong relative degrees does not always exist, as illustrated with the following example.

EXAMPLE 4.3. *We reconsider system (1.1) and (3.1) with $\kappa \neq 0$, but replace the output matrix by*

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The three corresponding outputs have (strong) relative degree one. However, $(1,1,1)$ is not a leading vector of strong relative degrees since matrix $CB = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$ has rank one, violating the rank condition on (4.2) for $i = 1$ and $j = 3$.

On the other hand, there always exists a transformation of the outputs, inducing a vector of strong relative degrees, as we show by construction in the remainder of the section.

4.2. Output transformation. We start with a simple example.

EXAMPLE 4.4. *We continue with Example 4.3. The left null space of CB and its orthogonal complement, spanned by the columns of the following matrices,*

$$U_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}, \quad U_{\perp,1} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix},$$

allow us to split-off one combination of outputs with strong relative degree one, $Y_1(t) := U_{\perp,1}^T y(t)$, and two other ones,

$$\tilde{y}(t) := U_1^T y(t) = U_1^T C x(t),$$

whose strong relative degree is at least two, inducing that, for these new outputs, the rank condition on (4.2) already holds for the strong relative degree one outputs. Taking the second derivative of $\tilde{y}(t)$ provides access to the input via matrices $U_1^T C A_i B$, $i = 0, \ldots, 3$. A basis of the column space of $\begin{bmatrix} A_0B & A_1B & A_2B & A_3B \end{bmatrix}$ is given by the column space of

$$H_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ -2 & -1 \end{bmatrix}$$

and we have

$$U_1^T C H_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}. \tag{4.3}$$

This matrix has rank one, implying that with $(Y_1, \tilde{y})$ as outputs and $(\rho_1, \rho_2, \rho_3) = (1, 2, 2)$, the rank condition on (4.2) is violated for $i = 2$ and $j = 3$. We can then proceed as before. Letting

$$U_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \quad U_{\perp,2} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

correspond to the left null space of (4.3) and its orthogonal complement, allows us to split-off a number of outputs equal to the rank of (4.3), i.e., scalar output

$$Y_2(t) := U_{\perp,2}^T \tilde{y}(t) = U_{\perp,2}^T U_1^T y(t) = U_{\perp,2}^T U_1^T C x(t)$$

with strong relative degree two, while output

$$\hat{y}(t) := U_2^T \tilde{y}(t) = U_2^T U_1^T C x(t)$$

has relative degree at least three. Differentiating $\hat{y}(t)$ three times provides us with access to the input via matrices $U_2^T U_1^T C P_{k_0,k_1,k_2,k_3} B$, with $\vec{k} \in \Omega_2$. We can choose

$$H_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},$$

such that

$$U_2^T U_1^T C H_2 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

This row matrix has full rank. Its left null space is empty and the complement is spanned by $U_{\perp,3} = 1$. *Hence, output* $Y_3 := U_{\perp,3}\hat{y}$ *has strong relative degree three.*

We conclude that the transformed outputs

$$Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \\ Y_3(t) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} y(t) := \begin{bmatrix} U_{\perp,1}^T \\ U_{\perp,2}^T U_1^T \\ U_{\perp,3}^T U_2^T U_1^T \end{bmatrix} y(t) = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \sqrt{3} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{bmatrix} x(t)$$

are such that $(\rho_1, \rho_2, \rho_3) = (1, 2, 3)$ *is a leading vector of strong relative degrees. Note that, up to a factor,* Y_3 *corresponds to the original output matrix in (3.1).*

The procedure followed in Example 4.4 can be generalized to any system of the form (1.1). It is spelled out in Algorithm 1, whose application leads us to transformed outputs

$$\begin{aligned} \begin{bmatrix} Y_1(t) \\ \vdots \\ Y_\ell(t) \end{bmatrix} &= \begin{bmatrix} M_1 y(t) \\ \vdots \\ M_\ell y(t) \end{bmatrix} \\ &= \left[\begin{array}{ccc|c} I & & & \\ & \ddots & & \\ & & I & \\ \hline & & & \begin{matrix} U_{\perp,k}^T \\ U_k^T \end{matrix} \end{array}\right] \cdots \left[\begin{array}{c|c} I & \\ \hline & \begin{matrix} U_{\perp,2}^T \\ U_2^T \end{matrix} \end{array}\right] \begin{bmatrix} U_{\perp,1}^T \\ U_1^T \end{bmatrix} y(t), \end{aligned}$$

with $M_i \in \mathbb{R}^{d_i \times n},\ i = 1, \ldots, \ell$, such that vector

$$(\rho_1, \ldots, \rho_{n_y}) = (\underbrace{R_1, \ldots, R_1}_{d_1 \text{ times}}, \underbrace{R_2, \ldots, R_2}_{d_2 \text{ times}}, \cdots, \underbrace{R_\ell, \ldots, R_\ell}_{d_\ell \text{ times}})$$

is a leading vector of strong relative degrees.

Two important considerations on the algorithm need to be made. First, the rationale behind the algorithm is to assure a minimal number of outputs with strong relative degree one, subsequently to assure a minimal number of outputs with strong relative degree two etc. Although the strong relative degree of an output may be equal to infinity, the algorithm always terminates after a finite number of iterations (line 13), induced by Proposition 3.10. If Example 4.4 is reconsidered with parameter $k = 0$, the algorithm reveals as leading vector of strong relative degrees $(\rho_1, \rho_2, \rho_3) = (1, 2, +\infty)$. Second, the role of matrices U_i is merely to provide bases for null-spaces, and the role of $U_{\perp,i}$ to extend these bases to bases of $\mathbb{R}^{n_y}$ (in our implementation they are obtained from a singular value decomposition). The dimensions of M_i and the values of R_i do not depend on the choice of the basis vectors. We summarize the main results in a theorem.

THEOREM 4.5. *The following assertions hold.*

1. *There always exists a static output transformation of (1.1) such that a leading vector of strong relative degrees,* $(\rho_1, \ldots, \rho_{n_y})$, *exists. Furthermore, different transformations cannot lead to different leading vectors of strong relative degrees.*
2. *If the strong relative degree of output* y_i *is given by* $s_i,\ i = 1, \ldots, n_y$, *such that* $s_1 \leq \cdots \leq s_{n_y}$, *then it holds that* $\rho_i \geq s_i,\ i = 1, \ldots, n_y$.

Algorithm 1 Transformation of output according to the strong relative degrees

Input: system matrices $(C, A_0, A_1, \ldots, A_m, B)$
Output: numbers ℓ and k, new output matrices and strong relative degrees (M_i, R_i), $i = 1, \ldots, \ell$, for which the leading vector of relative degrees exist.
$U_0 = I_p$, $\ell = 1$
for $k = 1$ to $(m+1)n$ **do**
    Let $[U_{\perp,k}\ U_k]$ be an orthogonal matrix, such that the columns of U_k span the left null space of matrix
$$U_{k-1}^T \cdots U_0^T C H_{k-1},$$
    with H_k defined by (4.1).
    if $U_k^\perp$ is no empty matrix **then**
        $M_\ell = U_{\perp,k}^T U_{k-1}^T \cdots U_0^T$
        $R_\ell = k$
        $\ell = \ell + 1$
    end if
    if U_k is an empty matrix **then** break
    end if
    if $k = (m+1)n$ **then**
        $M_\ell = U_k^T U_{k-1}^T \cdots U_0^T$
        $R_\ell = \infty$
    end if
end for

5. Extended PD control. In order to sketch the idea behind the extension, consider a system with three outputs (y_1, y_2, y_3), with strong relative degrees $(1, 2, 3)$. Then static output feedback can be directly generalized to

$$u(t) = K_1 \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} + K_2 \begin{bmatrix} \dot{y}_2(t) \\ \dot{y}_3(t) \end{bmatrix} + K_3 \ddot{y}_3(t), \tag{5.1}$$

since the output derivatives in (5.1) can be explicitly expressed as a function of (delayed) states only. It is also possible to use, in addition, output derivatives $(\dot{y}_1, \ddot{y}_2, \dddot{y}_3)$ for feedback, which can be expressed in terms of (delayed) states and the input, but more care is needed design due to the resulting implicit expression for the input, as we shall see. In both scenarios, the existence of a leading vector of strong relative degrees allows us to maximally use information from output derivatives, by the second assertion in Theorem 4.5.

Letting the highest derivatives of the output, used for feedback, depend on the strong rather than the standard relative degree assures that the closed-loop system remains well-posed in the presence of small deviations of the delays. From an application of view it is also important that model mismatch affecting the system matrices does not alter the strong relative degrees (see Remark 3.1).

In Section 5.1 we present the general setting for the extended PD control along with stability conditions and a procedure to compute the gain values. In Section 5.2 we illustrate the methodology on a numerical example.

5.1. Computing strongly stable feedback gains. We start from model (1.1), with outputs possibly reordered,

$$\begin{cases} \dot{x}(t) = A_0 x(t) + \sum_{i=1}^m A_i x(t - \tau_i) + B u(t), \\ Y_i(t) = \mathcal{C}_i x(t),\ i = 1, \ldots, \ell, \end{cases} \tag{5.2}$$

such that Y_i groups d_i scalar outputs with the same strong relative degree R_i for $i = 1, \ldots, \ell$, such that $R_1 < R_2 < \cdots < R_m$. We assume $R_m < \infty$ and consider an extended PD controller of the form

$$u(t) = \sum_{i=1}^{\ell} \sum_{j=1}^{R_i} K_{i,j}\, Y^{(j)}(t - \tau_f), \tag{5.3}$$

where $\tau_f \geq 0$ models the delay in the feedback control loop.

If $K_{i,R_i} = 0$ for $i = 1, \ldots, \ell$, the closed-loop system is of retarded type. Otherwise, it may correspond to a DDAE of neutral type [21] with associated delay-difference equation

$$u(t) = \sum_{i=1}^{\ell} K_{i,R_i} \left(\sum_{\vec{k} \in \Omega_{R_i - 1}} \mathcal{C}_i P_{k_0,\ldots,k_m} B u(t - \vec{k} \cdot \vec{\tau}_e - \tau_f) \right), \tag{5.4}$$

where we used (3.9). For neutral type systems, it is well known that exponential stability may be lost when applying infinitesimal delay perturbations, which led in [15] to the notion of strong stability. Inspired by the analysis in [23], we define strong stability in the context of the paper as follows.

Definition 5.1. *Closed-loop system (5.2) and (5.3) is strongly exponentially stable if the null solution is exponentially stable and the exponential stability is robust against small perturbations of delays* $(\tau_1, \ldots, \tau_m, \tau_f)$.

In order to arrive at a mathematical characterization, let us define matrices $\mathcal{M}_i,\ i = 1, \ldots, m+1$, depending on the gains values, such that

$$\det \left(I - \sum_{i=1}^{\ell} K_{i,R_i} \mathcal{C}_i \left(A_0 + A_1 z_1 + \cdots + A_m z_m \right)^{R_i - 1} z_0 B \right) = \det \left(I - \sum_{i=0}^{m} \mathcal{M}_i z_i \right)$$

holds for all $(z_0, \ldots, z_m) \in \mathbb{C}^{m+1}$. Such matrices can always be constructed using the lifting procedure described in [23, Section 4.2].

Theorem 5.2. *Closed-loop system (5.2)-(5.3) is strongly exponentially stable if and only if its null solution is exponentially stable and*

$$\left(\max_{\vec{\theta} \in [0,\ 2\pi]^{m+1}} r_\sigma \left(\sum_{i=0}^{m} \mathcal{M}_i e^{\imath \theta_i} \right) \right) < 1. \tag{5.5}$$

Proof. The characteristic function of delay-difference equation (5.4) can be expressed as

$$\det \left(I - \sum_{i=1}^{\ell} K_{i,R_i} \mathcal{C}_i \left(A_0 + A_1 e^{-s\tau_1} + \cdots + A_m e^{-s\tau_m} \right)^{R_i - 1} e^{-s\tau_f} B \right) = \det \left(I - \sum_{i=1}^{m} \mathcal{M}_i e^{-s\tau_i} - \mathcal{M}_0 e^{-s\tau_f} \right).$$

The assertion now follows from the theory developed in [15, 21]. ∎

Expressing the employed output derivatives explicitly as a function of delayed state variables and possibly inputs, enables the design approach of [21] to compute

feedback	$K_{1,1}$	$K_{1,2}$	$K_{1,3}$	(robust) spectral abscissa
no				0.8636
Y	-0.6347			0.2303
$Y,\ \dot{Y}$	-0.6141	-0.3865		-0.3882
$Y,\ \dot{Y}, \ddot{Y}$	-0.6688	-0.4185	-0.061306	-0.5599

TABLE 5.1

The (robust) spectral abscissa of the open loop system and after a minimization as a function of the extended PD controller gains.

the gain values. This spectral approach is based on minimizing the so-called robust spectral abscissa [21, Section 3.2] of the closed-loop system as a function of the controller parameters, since a negative value of the robust spectral abscissa is a necessary and sufficient condition for strong exponential stability. In this way, the additional constraint (5.5) for robustness of stability against small delay perturbations is taken into account.

5.2. Numerical example. We take the system described by

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \\ 1 & 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} x\left(t-\tfrac{1}{2}\right) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \\ Y(t) &= \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} x(t),\end{aligned}$$

whose strong relative degree is equal to two. For zero input, the system is unstable with a strictly positive spectral abscissa. We can express

$$\begin{aligned}\dot{Y}(t) &= \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -4 & 4 \end{bmatrix} x\left(t-\tfrac{1}{2}\right), \\ \ddot{Y}(t) &= \begin{bmatrix} 0 & 1 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 8 & -2 & -6 \end{bmatrix} x\left(t-\tfrac{1}{2}\right) + \begin{bmatrix} 1 & -2 & 8 \end{bmatrix} x(t-1) \\ &\quad +2u(t-1) + 4u\left(t-\tfrac{3}{2}\right)\end{aligned}$$

and consider a feedback control law of the form

$$u(t) = K_{1,1}Y(t-1) + K_{1,2}\dot{Y}(t-1) + K_{1,3}\ddot{Y}(t-1).$$

The results of minimizing the robust spectral abscissa are shown in Table 5.1, where we gradually increase the number of gains considered as free parameters (the others are set to zero), while the open and closed-loop spectra are visualized in Figures 5.1-5.2. In the last considered case, the parameters related to gain $K_{1,3}$ were rescaled by a factor 100 before the optimization was done, to counter a significantly higher sensitivity of the robust spectral abscissa with respect to this gain.

Using output feedback only, achieving exponential stability is not possible, in contrast to the other scenarios. When using the output and its first derivative, the closed-loop system is of retarded type, such that the robust spectral abscissa equals the spectral abscissa. When using the second-order derivative of the output, the closed-loop system is of neutral type, which is consistent with the characteristic shape of the spectrum shown in Figure 5.2.

6. Comments on the existence of a normal form. For delay-free square systems, the notion of the relative degree is tightly connected with a transformation to the normal form, which reveals and provides an explicit description of the zero

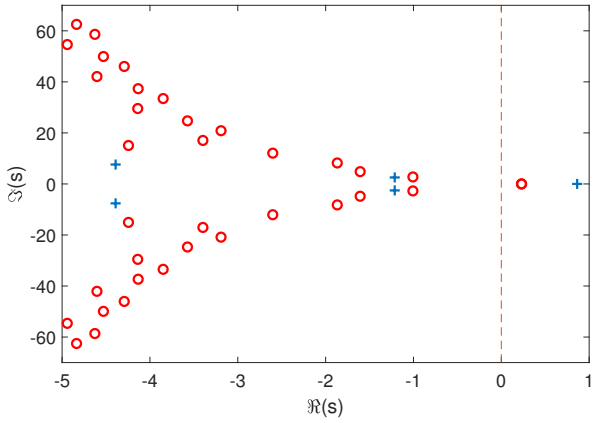


FIG. 5.1. *Rightmost characteristic roots of the uncontrolled system ('+') and after minimizing the spectral abscissa using static output feedback ('o').*

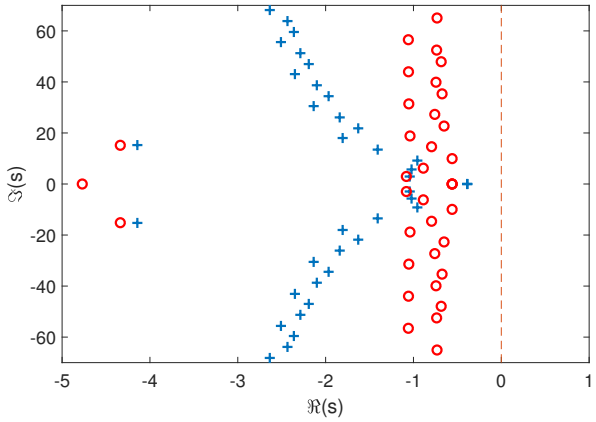


FIG. 5.2. *Rightmost characteristic after minimizing the (robust) spectral abscissa using static feedback of Y, $\dot{Y}$ ('+'), and using Y, $\dot{Y}$ and $\ddot{Y}$ ('o').*

dynamics. For delay system (1.1) with $n_u = n_y$ and with (standard) relative degrees $(\lambda_1, \ldots, \lambda_{n_y})$, a natural, structure preserving generalization of this normal form would take the form

$$\left\{\begin{array}{rcl} \dot{z}(t) & = & f_1(z_t) + g_1(y_t), \\ \dot{y}_{i,1}(t) & = & y_{i,2}(t) \\ & \vdots & \\ \dot{y}_{i,\lambda_i - 1}(t) & = & y_{i,\lambda_i}(t), \quad i = 1, \ldots, n_y, \\ \begin{bmatrix} \dot{y}_{1,\lambda_1} \\ \vdots \\ \dot{y}_{n_y,\lambda_{n_y}} \end{bmatrix} & = & f_2(z_t, \bar{y}_t) + g_2(u_t), \\ y_i(t) & = & y_{i,1}(t), \quad i = 1, \ldots, n_y, \end{array}\right. \tag{6.1}$$

where $f_1,\ f_2$ and $g_1,\ g_2$ are bounded linear functionals,

$$\bar{y}(t) = \left[\ y_{1,1}(t) \quad \cdots \quad y_{1,\lambda_1}(t) \quad \cdots \quad y_{n_y,1}(t) \quad \cdots \quad y_{n_y,\lambda_{n_y}}(t)\ \right]^T,$$

the notation x_t stand for $x_t := x(t+\theta),\ \theta \in [-\tau_{\max},\ 0]$ for some maximum delay $\tau_{\max}$ ($u_t,\ y_t$ and $\bar{y}(t)$ are similarly defined), and g_2 is assumed to be invertible. In what follows, we illustrate why a transformation to a canonical form having the structure (6.1) does not exist in general, and we shed light on existence conditions and obstructions.

Having the interdependence of the delays in (1.1) described by (2.1), a first key element in the construction of a normal form consists of the invertibility of the input-output map, which by Proposition 3.3 boils down to having matrix polynomial

$$D(z_1,\dots,z_p) := \begin{bmatrix} C_1\mathcal{A}(z_1,\dots,z_p)^{\lambda_1-1}B \\ \vdots \\ C_{n_y}\mathcal{A}(z_1,\dots,z_p)^{\lambda_{n_y}-1}B \end{bmatrix}, \tag{6.2}$$

invertible, with $\mathcal{A}$ given by (2.4). The second key element consists of a transformation where outputs, up to their relative degree minus one, correspond to new state variables. The advantage of using representations in terms of matrix polynomials, which are connected with the characteristic matrix of (1.1) via $z_i = e^{-sr_i},\ i = 1,\dots,p$, is that the transformation can be applied in the frequency domain and only involves algebraic manipulations. However, to realize the transformed transfer matrix in the time-domain, it is necessary that the matrix polynomials connecting inputs and outputs and defining the state transformation, are unimodular, such that their inverses are also matrix polynomials, enabling to realize the transformed transfer matrix by delay-differential equations. In fact, for the case of commensurate delays ($p = 1$ in (2.1)), it is proven in [17] that D being unimodular implies the existence of a transformation to a normal form in terms of a delay equation with commensurate delays in state, inputs and outputs, which fits within representation (6.1).

REMARK 6.1. *For the purpose of input-output linearization, less restrictive invertibility conditions on $\mathcal{D}$ are needed. If the feedback transformation is allowed to contain memory terms, it is sufficient that $\mathcal{D}(0,\dots,0)$ is invertible (i.e., the matrix relating output derivatives with $u(t)$), see [26, 12]. Particular for a normal form is that the z-subsystem in (6.1) does not depend on the input.*

However, the conditions for unimodular D are restrictive. In the single-input case, it implies that D is constant. In general, the following necessary conditions hold.

PROPOSITION 6.2. *If matrix D in (6.2) is unimodular, then*

1. *there does not exist $(z_1,\dots,z_p) \in \mathbb{C}^p$ and $i \in \{1,\dots,n_y\}$ such that*
$$C_i\mathcal{A}(z_1,\dots,z_p)^{\lambda_i-1}B = 0;$$
2. *the relative degrees are invariant with respect to delays parameterized by (2.1) if the entries of $(r_1,\dots,r_p) \in \mathbb{R}^p_{\geq 0}$ are considered as free parameters.*

Proof. The first assertion is trivial. The proof of the second assertion is by contradiction. Consider the i-th output, with $i \in \{1,\dots,n_y\}$. By Proposition 3.3, its relative degree equals λ_i for all rationally independent $(r_1,\dots,r_p)$. Assume that the relative degree is larger for some $(r_1,\dots,r_p) = (\hat{r}_1,\dots,\hat{r}_p)$. As these delays must be rationally dependent, there exists an integer $q < p$ rationally independent numbers $(s_1,\dots,s_q) \in \mathbb{R}^q_{>0}$ and nonnegative integers $\nu_{i,j}$ such that

$$r_i = \sum_{j=1}^{q} \nu_{i,j}s_j, \quad i = 1,\dots,p$$

and we must have, again by Proposition 3.3, that

$$C_i \mathcal{A} \left(w_1^{\nu_{1,1}} \dots w_q^{\nu_{1,q}}, \dots, w_1^{\nu_{p,1}} \dots w_q^{\nu_{p,q}} \right)^{\lambda_i - 1} B \equiv 0,$$

which is in contradiction with the first assertion. □

The second assertion implies that if the delays $(\tau_1, \dots, \tau_m)$ are rationally independent, a necessary condition for unimodular D is that the relative degrees are independent of the delay values. As we have seen in Section 3, the latter is not always the case.

One of the underlying causes of the restrictions of the procedure sketched above is that invariant zeros of (1.1) can in general *not* be expressed as characteristic roots of a delay-differential equation of retarded type, which would be implied by the intended normal form with structure (6.1). We illustrate this, once more, with system (1.1) and (3.1).

The invariant zeros are the zeros of function

$$\det \left(\begin{bmatrix} sI - A_0 - \sum_{i=1}^m A_i e^{-s\tau_i} & B \\ C & 0 \end{bmatrix} \right) = s \left(e^{-2s\tau_2} - e^{-s(\tau_1+\tau_3)} \right) + \kappa \left(4e^{-s(2\tau_1+\tau_2)} + 2e^{-s(\tau_1+2\tau_2)} + 2e^{-s(2\tau_1+\tau_3)} + e^{-s(\tau_1+\tau_2+\tau_3)} \right),$$

which correspond to the numerator of transfer function (3.4). We now specify the delays as $(\tau_1, \tau_2, \tau_3) = (1, 3, 6)$, such that the invariant roots are described by equation

$$s \left(1 - e^{-s} \right) + \kappa \left(4e^{s} + 2e^{-s} + 2e^{-2s} + e^{-4s} \right) = 0, \tag{6.3}$$

where we divided by the common factor e^{-6s}. We make the following observations.

- For $\kappa = 0$, the zeros of (6.3) can be interpreted as characteristic roots of neutral equation $\dot{z}(t) = \dot{z}(t-1)$, or equivalently, as characteristic roots of the DDAE $\dot{z}_1(t) = 0,\ z_2(t) = z_1(t) + z_2(t-1)$.
- For $\kappa \neq 0$, the zeros would correspond to a delay-differential equation of *advanced type*, due to the positive power of s.

In Figure 6.1, we display the invariant zeros for $\kappa = -0.03$, highlighting the typical spectrum distribution for advanced equations, and for $\kappa = 0$. We conclude that even though (1.1) is a delay-differential equation of retarded type, the isolation of zero dynamics, which is inherent to a description in normal form (the z-subsystem in (6.1)), may require DDAE models or even equations of advanced type, which are however a-causal [29].

7. Conclusions. The decomposition (2.1) gave rise to algebraic characterizations of the relative degree of system (1.1) if the common assumption of commensurate delays in the literature is dropped. It also played a key role in understanding the potential sensitivity of the relative degree with respect to infinitesimal delay perturbation, illustrated with example (1.1) and (3.1). This sensitivity problem led to the introduction of the notion of strong relative degree, which was analyzed in both the SISO and the MIMO setting. This notion and the algorithm revealing the leading vector of strong relative degree were instrumental in the feedback design approach of Section 5, relying on extended PD control.

A direction of future research consists of extending the definitions and characterizations of strong relative degree to systems having also delays in inputs and outputs, starting from the characterizations in Propositions 3.3-3.5 and Theorem 3.12 that rely

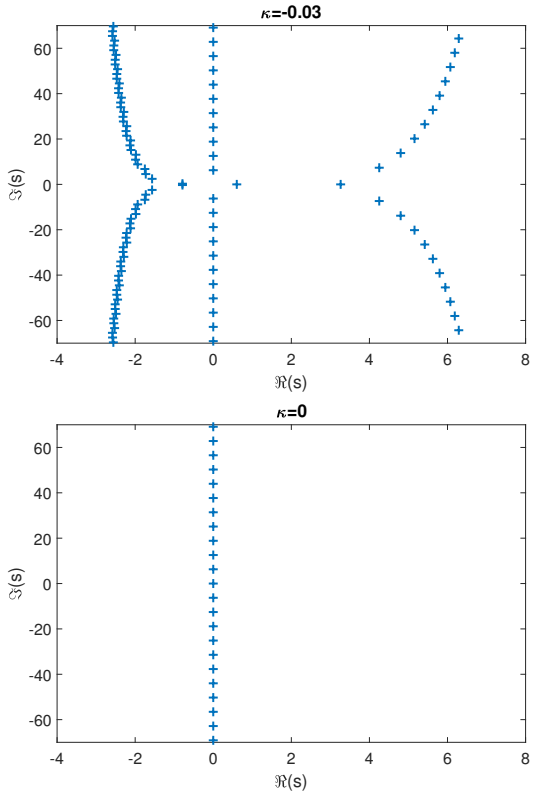


FIG. 6.1. *Invariant zeros of system (1.1) with matrices (3.1) and delays* $(\tau_1, \tau_2, \tau_3) = (1, 3, 6)$, *for* $\kappa = -0.03$ *(top pane)* $\kappa = 0$ *(bottom pane).*

on multivariate matrix polynomial $\mathcal{A}$. The extension of the control design approach toward extended PID regulation is trivial, by combining the results with [1]. Another, important direction for future research consists of determining, at a finer resolution, under which conditions and in what form (in terms of equation type) generalizations of the normal form can be obtained, elaborating on the preliminary results presented in Section 6.

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Appendix A. A technical lemma.

LEMMA A.1. *Let* P *be a multivariate polynomial on* $\mathbb{C}^p$, *with* $p \geq 1$. *Then we have*

$$P \equiv 0 \quad \Leftrightarrow \quad P(z_1, \ldots, z_p) = 0, \ \forall (z_1, \ldots, z_p) \in \mathcal{D}_p.$$

Proof. For $p = 1$, the statement is trivial. For $p > 1$, we use induction on p.

Assume the assertion holds with p replaced by $p-1$. If $P(z_1,...,z_p)=0$ for all $(z_1,...,z_p)\in\mathcal{D}_p$, we know that for any $w_p\in\mathcal{D}_1$, $P(z_1,...,z_{p-1},w_p)$ is the zero polynomial on $\mathbb{C}^{p-1}$. We can express $P(z_1,...,z_{p-1},w_p)=\sum_{i\geq 0}c_i(w_p)m_i(z_1,\ldots,z_{p-1})$, with c_i univariate polynomials and $m_1,m_2,\ldots$ (different) monomials on $\mathbb{C}^{p-1}$. We conclude that $c_i(w_p)=0$ for all i and $w_p\in\mathcal{D}_1$. As the assertion holds for $p=1$, we must have $c_i\equiv 0$ for all $i\geq 0$, and hence $p\equiv 0$. □

Appendix B. Proofs of theorems and propositions.

Proof of Theorem 2.1. If delays $(\tau_1,\ldots,\tau_m)\in\mathbb{R}^m_{>0}$ are rationally independent, then (2.1) is satisfied with $p=m$, $\Gamma=I$ and $r_i=\tau_i,\ i=1,\ldots,m$. So in the remainder of the proof we restrict ourselves to the case where the delays are rationally dependent.

Due to the rational dependence of $(\tau_1,\ldots\tau_m)$, there exists a re-ordering of these delays, integers $r\geq s\geq 1$ and positive integers $n_i,\ i=1,\ldots,r+s$, such that after reordering,

$$\sum_{i=1}^{r} n_i\tau_i-\sum_{i=r+1}^{r+s} n_i\tau_i=0. \tag{B.1}$$

Assume for the moment that $s>1$. Choose rational numbers $a_i,\ i=1,\ldots,r$, such that

$$0<\tau_i-a_i\tau_{r+1}<\frac{1}{rn_i}\sum_{j=r+2}^{r+s} n_j\tau_j,\quad i=1,\ldots,r. \tag{B.2}$$

Relation (B.1) can now be rewritten as

$$\sum_{i=1}^{r} n_i(\tau_i-a_i\tau_{r+1})+\left(-n_{r+1}+\sum_{i=1}^{r} n_ia_i\right)\tau_{r+1}-\sum_{i=r+2}^{r+s} n_i\tau_i=0.$$

We have by (B.2),

$$-n_{r+1}\tau_{r+1}+\sum_{i=1}^{r} n_ia_i\tau_{r+1}>-n_{r+1}\tau_{r+1}+\sum_{i=1}^{r} n_i\tau_i-\sum_{j=r+2}^{r+s} n_j\tau_j=0,$$

inducing $-n_{r+1}+\sum_{i=1}^{r} n_ia_i>0$. In view of the transformation

$$\hat{\tau}_i=\tau_i-a_i\tau_{r+1},\ i\in\{1,\ldots,r\},\hat{\tau}_i=\tau_i,\ i\in\{r+1,\ldots,r+s\},$$

we can substitute $\tau_i=\hat{\tau}_i+a_i\hat{\tau}_{i+1},\ i=1,\ldots,r$, and the relation between the transformed delays becomes

$$\sum_{i=1}^{r} n_i\hat{\tau}_i+\left(-n_{r+1}+\sum_{i=1}^{r} n_ia_i\right)\hat{\tau}_{r+1}-\sum_{i=r+2}^{r+s} n_i\hat{\tau}_i=0.$$

The main difference with (B.1) is that the number of terms with negative coefficients has been reduced by one, as by a pre-multiplication, rational coefficients can be turned into integers. We can repeat these steps until there is only *one* negative term left.

If (eventually) a relation of the form (B.1) holds with $s=1$, we can use this equality to express delay τ_{r+1} as a combination of $(\tau_1,\ldots,\tau_r)$ and eliminate this variable.

If numbers $(\tau_1, \ldots, \tau_r, \tau_{r+2}, \ldots, \tau_m)$ are rationally independent, we end up with a relation of the form (2.1) possibly after rescaling the transition matrix and the independent delays to ensure integer coefficients in Γ. If the delays $(\tau_1, \ldots, \tau_r, \tau_{r+2}, \ldots, \tau_m)$ are still rationally dependent, we can repeat the overall procedure, which reduces the number of independent variables one by one until rationally independent basis delays are obtained.

Proof of Proposition 2.3. We use induction on p. For $p = 1$, the assertion holds. Suppose it holds for $p-1$, with $p > 1$, and consider an arbitrary p-tuple $(s_1, \ldots, s_p) \in \mathbb{R}^p_{\geq 0}$, which we wish to approximate by rationally independent numbers. Let $\epsilon \in \mathbb{R}_{>0}$. By the hypothesis, there exist rationally independent numbers $(r_1, \ldots, r_{p-1})$ such that $|s_i - r_i| < \epsilon,\ i = 1, \ldots, p-1$. Consider now the set $\mathcal{S}$ of linear combinations of $(r_1, \ldots, r_{p-1})$ with coefficients in $\mathbb{Q}$. As set $\mathcal{S}$ is countable, there exists a number $r_p \in (s_p - \epsilon,\ s_p + \epsilon) \setminus \mathcal{S}$. By construction, numbers $(r_1, \ldots, r_p)$ are rationally independent, while precision ϵ can be chosen arbitrarily small.

Proof of Proposition 2.4. By fixing $d_0 = 0$, the proposition reduces to the standard Kronecker's theorem [16, Theorem 444]. The extension follows from this theorem, when applied to $(\hat{\theta}_1, \ldots, \hat{\theta}_p)$ such that

$$\hat{\theta}_i = (\theta_i - d_0 r_i) \bmod 2\pi,\ i = 1, \ldots, p.$$

The assertion directly follows.

Proof of Proposition 3.4. Taking into account the characterization (2.1) and (2.3), we can expand

$$H(s,\ \vec{\tau}) = \frac{\sum_{j=1}^{n} q_j(s) s^{n-j}}{s^n + \sum_{j=1}^{n} p_j(s) s^{n-j}},$$

where $p_j(s) = P_j\left(e^{-sr_1}, \ldots, e^{-sr_p}\right),\ q_j(s) = Q_j\left(e^{-sr_1}, \ldots, e^{-sr_p}\right)$ and $P_j,\ Q_j$ are multivariate complex polynomials, for $j = 1, \ldots, p$. From the definition of the relative degree, grounded in successive output differentiation, it directly follows that the relative degree is $+\infty$ if $q_j \equiv 0,\ j = 1, \ldots, n$, while it is equal to r if $q_1 \equiv 0, \ldots, q_{r-1} \equiv 0$ but $q_r \not\equiv 0$. These conditions are equivalent to the assertions of the proposition, provided that for any $i \in \{1, \ldots, n\}$, the implication

$$q_i(s) \not\equiv 0 \Rightarrow \lim \sup_{\omega \to \infty} |q_i(\imath\omega)| > 0$$

holds, whose demonstration is subject of the remainder of the proof.

From $q_i(s) \not\equiv 0$ it follows that $Q_i\left(z_1, \ldots, z_p\right) \not\equiv 0$. Then, by Lemma A.1, there exist $(\theta_1, \ldots, \theta_p) \in [0,\ 2\pi]^p$ such that $\left|Q_i\left(e^{-\imath\theta_1}, \ldots, e^{-\imath\theta_p}\right)\right| > 0$. At the same time, from Proposition 2.4 we conclude that for any $\epsilon > 0$ and $d_0 > 0$, there exists an $\omega > d_0$ such that $\left|e^{-\imath\theta_i} - e^{-\imath\omega r_i}\right| < \epsilon,\ i = 1, \ldots, p$. A combination of these properties leads us to the existence of a sequence $\{\omega_n\}_{\geq 1}$ with $\lim_{n\to\infty} \omega_n = +\infty$ and of a constant $\gamma > 0$ such that $|Q_i\left(e^{-\imath\omega_n r_1}, \ldots, e^{-\imath\omega_n r_p}\right)| > \gamma,\ \forall n \geq 1$. Hence, the non-increasing function $\omega_0 \mapsto \sup_{\omega \geq \omega_0} q_i(\imath\omega)$ is bounded from below by γ. As a consequence, $\lim \sup_{\omega \to +\infty} q_i(\imath\omega)$ exists and is strictly positive.

Proof of Proposition 3.4. Similar arguments as in the proof of Proposition 3.4 can be used. The key implication is given by

$$q_i(s) \not\equiv 0 \Rightarrow \lim_{\omega \to +\infty} \left(\max_{\vec{z} \in \mathcal{D}_p} |Q_i(z_1, \ldots, z_m)| \right) > 0$$

for any $i \in \{1, \ldots, n\}$, which directly follows from Proposition A.1.

Proof of Theorem 3.12. Assertion 1 follows from the definition and Assertion 2 from Theorem 3.8. For rationally independent delays, we have $\mathrm{card}(\Upsilon_i) = \mathrm{card}(\Omega_i)$ and the conditions in Definition 3.7 and Theorem 3.8 coincide, inducing Assertion 3. Assertion 4 follows from Assertion 3 and Proposition 3.3. Finally, Assertion 5 follows from Assertion 3 and Proposition 3.5.

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