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# Recurrence and Asymptotics for Orthonormal Rational Functions on an Interval

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Let $\mu$ be a positive bounded Borel measure on a subset $I$ of the real line, and $\mathscr{A} = \{\alpha_1,\ldots,\alpha_n\}$ a sequence of arbitrary *complex* poles outside $I$. Suppose $\{\varphi_1,\ldots,\varphi_n\}$ is the sequence of rational functions with poles in $\mathscr{A}$ orthonormal on $I$ with respect to $\mu$. First, we are concerned with reducing the number of different coefficients in the three term recurrence relation satisfied by these orthornormal rational functions. Next, we consider the case in which $I = [-1,1]$ and $\mu$ satisfies the Erdős-Turán condition $\mu' > 0$ a.e. on $I$ (where $\mu'$ is the Radon-Nikodym derivative of the measure $\mu$ with respect to the Lebesgue measure), to discuss the convergence of $\varphi_{n+1}(x)/\varphi_n(x)$ as $n$ tends to infinity and to derive asymptotic formulas for the recurrence coefficients in the three term recurrence relation. Finally, we give a strong convergence result for $\varphi_n(x)$ under the more restrictive condition that $\mu$ satisfies the Szegő condition $(1-x^2)^{-1/2}\log\mu'(x) \in L^1([-1,1])$.

*Keywords*: Orthogonal rational functions, complex poles, three term recurrence relation, asymptotics, ratio convergence, strong convergence.

## 1. Introduction

By using the Joukowski Transformation $x = (z+z^{-1})/2$, which maps the unit circle onto the interval $[-1,1]$, orthogonal polynomials (OPs) on the interval $[-1,1]$ can be related to OPs on the unit circle. In this way Szegő [10] obtained convergence results for weights satisfying Szegő's condition. Later on, Rakhmanov [7, 8] derived asymptotic results for the weaker Erdős-Turán condition, while López [5, 6] derived results for polynomials orthogonal with respect to varying measures.

Orthogonal rational functions (ORFs) are a generalisation of OPs in such a way that the OPs are recovered if all the poles are at infinity. Asymptotics for ORFs on the unit circle (or, using a Cayley Transformation, on the extended real line) are studied in [1]. Using a relation between ORFs on the unit circle and ORFs on the interval with all *real* poles, as described in [16], convergence results are derived for ORFs on the interval as well, in [13].

Just as in the polynomial case, ORFs satisfy a three term recurrence relation. If all poles are *real*, the number of different recurrence coefficients can be reduced from three to two (see [1]), and asymptotics for these remaining recurrence coefficients have been derived in [13] from the results for the ratio asymptotics of ORFs on the interval.

The aim of this paper is to generalise these results for ORFs whose poles are all *real* to ORFs with arbitrary *complex* poles, based on the extended relation between ORFs on the unit circle and ORFs on the interval, as described in [2]. After giving the necessary theoretical preliminaries in Section 2, Section 3 deals with reducing the number of different coefficients in the three term recurrence relation for ORFs on a subset of the real line with arbitrary *complex* poles. Section 4 then contains an extended result for

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ratio convergence and strong convergence in the case of ORFs on the interval. Next, in Section 5 we derive asymptotic formulas for the recurrence coefficients. Finally, in Section 6 we give some numerical examples.

## 2. Preliminaries

The field of complex numbers will be denoted by $\mathbb{C}$ and the Riemann sphere by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For the real line we use the symbol $\mathbb{R}$ and for the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. The unit circle and the open unit disc are denoted respectively by

$$\mathbb{T} = \{z : |z| = 1\} \quad \text{and} \quad \mathbb{D} = \{z : |z| < 1\}.$$

Let $\mu$ be a positive bounded Borel measure, with $\operatorname{supp}(\mu) \subset \overline{\mathbb{R}}$ an infinite set, and assume a sequence of poles $\mathscr{A} = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{C}} \setminus \{0\}$ is given so that $\mathscr{A} \cap \operatorname{supp}(\mu) = \emptyset$. The support of a measure $\mu$ is defined here as the smallest closed set whose complement with respect to $\overline{\mathbb{C}}$ has $\mu$-measure zero. Define the factors

$$Z_k(x) = Z_{\alpha_k}(x) = \frac{x}{1 - x/\alpha_k}, \quad k = 1, 2, \ldots \tag{2.1}$$

and the basis functions

$$b_0 = 1, \quad b_k(x) = b_{k-1}(x) Z_k(x), \quad k = 1, 2, \ldots \quad . \tag{2.2}$$

Then the space of rational functions with poles in $\mathscr{A}$ is defined as

$$\mathscr{L}_n = \operatorname{span}\{b_0, \ldots, b_n\}.$$

In the special case of all $\alpha_k = \infty$, the factor (2.1) becomes $Z_k(x) = x$ and the basis functions (2.2) become $b_k(x) = x^k$.

Orthonormalising the basis $\{b_0, b_1, \ldots, b_n\}$ with respect to the measure $\mu$ and inner product

$$\langle f, g \rangle = \int f(x) \overline{g(x)} d\mu(x)$$

on a subset of the real line, we obtain the orthonormal rational functions (ORFs) $\{\varphi_0, \varphi_1, \ldots, \varphi_n\}$. In the case of orthogonality on a subset of the real line, we define the involution operation or substar conjugate of a function $f \in \mathscr{L}_n$ as

$$f_*(x) = \overline{f(\overline{x})}.$$

Suppose $\varphi_n(x) = \frac{p_n(x)}{\pi_n(x)}$, then $\varphi_n(x)$ is degenerate (respectively exceptional) iff $p_n(\overline{\alpha}_{n-1}) = 0$ (respectively $p_n(\alpha_{n-1}) = 0$). In [12, Thm. 2.1.1], and [1, Chapter 11.1] for the special case of all *real* poles, the following recurrence relation has been proven.

THEOREM 2.1 Take by convention $\alpha_{-1} = \alpha_0 = \infty$. Then $\varphi_{n-1}(x)$ is not degenerate and $\varphi_n(x)$ is not exceptional for $n \geqslant 1$ iff there exists a three term recurrence relation of the form

$$\varphi_n(x) = \left( E_n Z_n(x) + F_n \frac{Z_n(x)}{Z_{n-1}(x)} \right) \varphi_{n-1}(x) + C_n \frac{Z_n(x)}{Z_{n-2*}(x)} \varphi_{n-2}(x), \tag{2.3}$$

with $E_n \neq 0$ and $C_n \neq 0$. The initial conditions are $\varphi_{-1}(x) \equiv 0$ and $\varphi_0(x) \equiv \frac{1}{\sqrt{\mu_0}}$ with $\mu_0 = \mu(\overline{\mathbb{R}})$. In the special case of all real poles, it holds that

$$E_n = -C_n E_{n-1}. \tag{2.4}$$

When all the poles are chosen outside the convex hull of $\mathrm{supp}(\mu)$, the zeros of $\varphi_n$ are inside the convex hull of $\mathrm{supp}(\mu)$. Therefore, if $\mathrm{supp}(\mu)$ is connected (a closed interval), the system $\{\varphi_n\}$ will be not degenerate and not exceptional and thus the recurrence relation will hold for every $n$. Note that for every $a,b\in\mathbb{R}$, with $-\infty<a<b<\infty$, the interval $[a,b]$ can be mapped onto the interval $[-1,1]$ using the transformation

$$x=\frac{2t-b-a}{b-a},\quad t\in[a,b].$$

Even the case of orthogonality on a halfline can be completely reduced to the case of the interval, using a suitable transformation (see [14]). Thus, when studying the asymptotic behaviour of ORFs on an interval, we can restrict ourselves to the interval $[-1,1]$.

In the case of ORFs on the interval $I=[-1,1]$ with arbitrary *complex* poles outside $I$, a relation exists with ORFs on the unit circle. Given a sequence of complex numbers $\mathscr{B}=\{\beta_1,\beta_2,\ldots\}\subset\mathbb{D}$, define the Blaschke factors

$$\zeta_k(z)=\frac{z-\beta_k}{1-\overline{\beta}_k z},\quad k=1,2,\ldots$$

and the Blaschke products

$$B_0=1,\quad B_k(z)=B_{k-1}(z)\zeta_k(z),\quad k=1,2,\ldots\quad .$$

Then the space of rational functions associated with $\mathscr{B}$ is defined as

$$\dot{\mathscr{L}}_n=\mathrm{span}\{B_0,\ldots,B_n\}.$$

Orthonormalising this basis with respect to $\dot{\mu}$ and inner product

$$\langle f,g\rangle_{\mathbb{T}}=\frac{1}{2\pi}\int_{\mathbb{T}}f(z)\overline{g(z)}d\dot{\mu}(z),$$

we obtain the ORFs $\{\phi_0,\phi_1,\ldots,\phi_n\}$. When considering the sequence $\mathscr{B}^c=\{\overline{\beta}_1,\ldots,\overline{\beta}_n\}\subset\mathbb{D}$, instead of $\mathscr{B}$, we obtain the ORFs $\{\phi_0^c,\phi_1^c,\ldots,\phi_n^c\}$ in $\dot{\mathscr{L}}_n^c$, where $\phi_n^c(z)=\overline{\phi_n(\overline{z})}$. And if we consider the sequence $\tilde{\mathscr{B}}=\{\tilde{\beta}_1,\ldots,\tilde{\beta}_{2n}\}\subset\mathbb{D}$, with

$$\tilde{\beta}_{2k}=\beta_k \text{ and } \tilde{\beta}_{2k-1}=\overline{\beta}_k,\quad k=1,\ldots,n, \tag{2.5}$$

we obtain the ORFs $\{\tilde{\phi}_0,\tilde{\phi}_1,\ldots,\tilde{\phi}_{2n}\}$ in $\tilde{\mathscr{L}}_{2n}=\dot{\mathscr{L}}_n\cdot\dot{\mathscr{L}}_n^c$. In the case of orthogonality on the unit circle, we define the involution operation or substar conjugate of a function $f\in\dot{\mathscr{L}}_n$ as

$$f_*(z)=\overline{f(1/\overline{z})}$$

and the superstar transformation as

$$f^*(z)=B_n(z)f_*(z).$$

Note that the factor $B_n(z)$ merely replaces the polynomial with zeros $\{\beta_k\}_{k=1}^n$ in the denominator of $f_*(z)$ by a polynomial with zeros $\left\{1/\overline{\beta}_k\right\}_{k=1}^n$ so that $\dot{\mathscr{L}}_n^*=\dot{\mathscr{L}}_n$.

The complement of the interval $I$ with respect to a set $X$ will be given by $X^I$, e.g.

$$\overline{\mathbb{C}}^I=\overline{\mathbb{C}}\setminus I.$$

Although $x$ and $z$ are both complex variables, we reserve the notation $x$ for ORFs on the interval, and $z$ for ORFs on the unit circle. We denote the Joukowski Transformation $x = \frac{1}{2}(z + z^{-1})$ by $x = J(z)$, mapping the open unit disc $\mathbb{D}$ onto the cut Riemann sphere $\overline{\mathbb{C}}^I$ and the unit circle $\mathbb{T}$ onto the interval $I$. When $z = e^{\mathbf{i}\theta}$, then $x = J(z) = \cos\theta$. The inverse mapping is denoted by $z = J^{inv}(x)$ and is chosen so that $z \in \mathbb{D}$ if $x \in \overline{\mathbb{C}}^I$. With the sequence $\mathscr{A} = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{C}}^I$ we associate the sequence $\mathscr{B} = \{\beta_1, \beta_2, \ldots\} \subset \mathbb{D}$ so that $\beta_k = J^{inv}(\alpha_k)$.

Next, let the measure $\dot{\mu}$ on $\mathbb{T}$ be given by

$$\dot{\mu}(E) = \mu\left(\{\cos\theta, \theta \in E \cap [0,\pi)\}\right) + \mu\left(\{\cos\theta, \theta \in E \cap [\pi, 2\pi)\}\right), \tag{2.6}$$

which can also be written as $\dot{\mu}(E) = \int_E |d\mu(\cos\theta)|$. Using the Lebesgue decomposition of $\mu$ and the change-of-variables theorem (see e.g. [9, p. 153]) it is not difficult to see that $\dot{\mu}'(\theta) = \mu'(\cos\theta)\,|\sin\theta|$. Then the following theorem gives a relation between the ORFs on $I$ and the ORFs on $\mathbb{T}$, which has been proven in [2, Thm. 4.2] [1].

THEOREM 2.2 Let $\{\varphi_n\}$ be a set of orthonormal rational functions on $I$ with respect to the measure $\mu$, and $\{\phi_n\}$ the corresponding set of orthonormal rational functions on $\mathbb{T}$ with respect to the measure $\dot{\mu}$ as defined above. Then they are related by

$$\varphi_n(x) = \frac{\rho_n}{\sqrt{2\pi}} \left[1 + \Re\left\{\frac{\tilde{\phi}^c_{2n}(\beta_n)}{\tilde{\phi}^*_{2n}(\beta_n)}\right\}\right]^{-\frac{1}{2}} \frac{\tilde{\phi}^*_{2n}(z)}{B_n(z)} \left(1 + \frac{\tilde{\phi}^c_{2n}(z)}{\tilde{\phi}^*_{2n}(z)}\right),$$

where $x = J(z)$, $\rho_n$ is a unimodular constant that can be chosen arbitrarely, and the tilde refers to the sequence of complex numbers given by (2.5).

The following two convergence results for ORFs on the unit circle can be found in [1, Chapter 9]. With $\dot{\mu}'$ (respectively $\mu'$) we denote the Radon-Nikodym derivative of the measure $\dot{\mu}$ (respectively $\mu$) with respect to the Lebesgue measure, and hence the 'almost everywhere' is also with respect to the Lebesgue measure.

THEOREM 2.3 Let $\dot{\mu}$ satisfy the Erdős-Turán condition $\dot{\mu}' > 0$ a.e. on $\mathbb{T}$ and assume that the sequence $\mathscr{B}$ is compactly included in $\mathbb{D}$. Then we have

$$\lim_{n\to\infty} \frac{\phi_n(z)}{\phi^*_n(z)} = 0,$$

locally uniform in $\mathbb{D}$.

THEOREM 2.4 Let $\dot{\mu}$ satisfy the Erdős-Turán condition $\dot{\mu}' > 0$ a.e. on $\mathbb{T}$ and assume that the sequence $\mathscr{B}$ is compactly included in $\mathbb{D}$. Then we have

$$\lim_{n\to\infty} \frac{\varepsilon_{n+1}\phi^*_{n+1}(z)(1-\overline{\beta}_{n+1}z)\sqrt{1-|\beta_n|^2}}{\varepsilon_n\phi^*_n(z)(1-\overline{\beta}_n z)\sqrt{1-|\beta_{n+1}|^2}} = 1,$$

where $\varepsilon_n$ is a unimodular normalisation constant such that $\varepsilon_n\phi^*_n(0) > 0$, i.e. $\varepsilon_n = |\phi^*_n(0)|/\phi^*_n(0)$. Again convergence is locally uniform in $\mathbb{D}$.

[1] In [2] the measure $\mu$ was assumed to be absolutely continuous, but this can easily be extended to arbitrary positive Borel measures whose support is an infinite set. See also [4, p. 190] for the polynomial case.

Note that, if $\dot{\mu}$ and $\mu$ are related through (2.6), the condition $\dot{\mu}' > 0$ a.e. on $\mathbb{T}$ is equivalent with the condition $\mu' > 0$ a.e. on $I$.

Finally, the following strong convergence result for ORFs on the unit circle can also be found in [1, Chapter 9].

THEOREM 2.5 Let $\dot{\mu}$ satisfy the Szegő condition

$$\int_0^{2\pi} \log \dot{\mu}'(\theta) d\theta > -\infty$$

and assume that the sequence $\mathscr{B}$ is compactly included in $\mathbb{D}$. Then locally uniform in $\mathbb{D}$

$$\lim_{n\to\infty} \varepsilon_n \frac{\phi_n^*(z)(1-\overline{\beta}_n z)}{\sqrt{1-|\beta_n|^2}} = \frac{1}{\sigma(z)},$$

where $\varepsilon_n$ is the same as in Theorem 2.4 and $\sigma(z)$ is the Szegő function given by

$$\sigma(z) = \exp\left\{ \frac{1}{4\pi} \int_0^{2\pi} \frac{e^{\mathbf{i}\theta}+z}{e^{\mathbf{i}\theta}-z} \log \dot{\mu}'(\theta) d\theta \right\}, \quad z \in \mathbb{D}.$$

Note again that for the interval, with $x = \cos\theta$ and $\dot{\mu}$ given by (2.6), the Szegő condition is equivalent with the condition

$$\int_{-1}^{1} \frac{\log \mu'(x)}{\sqrt{1-x^2}} dx > -\infty.$$

## 3. Three term recurrence relation

In Theorem 2.1 a simple relation has been given between the third coefficient $C_n$ and the first coefficient $E_n$ for all *real* poles. Due to this relation, the number of coefficients can be reduced from three to two. Equation (2.4), however, does not hold in general for arbitrary *complex* poles. In this section we will prove a slightly different, but still simple, relation between $C_n$ and the other two coefficients $E_n$ and $F_n$ that holds in general. Next, with $F_n = E_n \hat{F}_n$, we will illustrate how $E_n$, apart from a unimodular normalisation constant $\eta_n$, can be defined recursively in function of $\hat{F}_n$ as well, when the last pole in the sequence $\alpha_n$ is not real. The special case in which $\alpha_n \in \overline{\mathbb{R}} \setminus (\{0\} \cup \mathrm{supp}(\mu))$ will appear as a limiting case $\Im\{\alpha_n\} \to 0$, where $\Im\{.\}$ refers to the imaginary part. We will conclude this section with a Favard type theorem. First we will need the following partial results. The first lemma is easily verified, and hence we will omit the proof.

LEMMA 3.1 Let $A(\alpha,\beta)$ be given by

$$A(\alpha,\beta) = \frac{1}{Z_\alpha(x)} - \frac{1}{Z_\beta(x)}.$$

Then the following statements hold:

1. $A(\alpha,\beta) = \frac{1}{Z_\alpha(\beta)}$ and hence is independent of $x$,
2. $A(\alpha,\beta) = -A(\beta,\alpha)$,
3. $\overline{A(\alpha,\beta)} = A(\overline{\alpha},\overline{\beta})$,

4. $A(\alpha,\beta) - A(\gamma,\beta) = A(\alpha,\gamma)$,

5. $A(\alpha,\beta) + A(\alpha,\gamma) = 2A\left(\alpha, \frac{2\beta\gamma}{\beta+\gamma}\right)$,

6. $\frac{Z_\beta(x)}{Z_\alpha(x)} = A(\alpha,\beta)Z_\beta(x) + 1$,

7. $\frac{b_k(x)}{Z_\alpha(x)} = A(\alpha,\alpha_k)b_k(x) + b_{k-1}(x) \quad \in \begin{cases} \mathscr{L}_k \setminus \mathscr{L}_{k-1}, & \alpha \neq \alpha_k \\ \mathscr{L}_{k-1}, & \alpha = \alpha_k \end{cases}$ .

THEOREM 3.1 Suppose $\varphi_n(x) = \kappa_n b_n(x) + \kappa_n' b_{n-1}(x) + f_{n-2}(x)$, where $\kappa_n, \kappa_n' \in \mathbb{C}$, $\kappa_n \neq 0$ and $f_{n-2} \in \mathscr{L}_{n-2}$. Then the following statements hold:

1. $\langle b_n, \varphi_n \rangle = \frac{1}{\kappa_n} = \overline{\langle \varphi_n, b_n \rangle}$,

2. $\left[\frac{\varphi_n(x)}{b_n(x)}\right]_{x=\alpha_n} = \kappa_n$,

3. $\left[\frac{\varphi_n(x)}{b_n(x)}\right]_{x=\alpha_{n-1}} = \kappa_n + \kappa_n' A(\alpha_n, \alpha_{n-1})$,

4. $E_n = \frac{\kappa_n + \kappa_n' A(\alpha_n,\alpha_{n-1})}{\kappa_{n-1}}$.

*Proof.* First, note that

$$1 = \langle \varphi_n, \varphi_n \rangle = \langle \kappa_n b_n, \varphi_n \rangle + \left\langle \left(\kappa_n' b_{n-1} + f_{n-2}\right), \varphi_n \right\rangle = \kappa_n \langle b_n, \varphi_n \rangle,$$

which proves the first statement. Next, we have that

$$\left[\frac{\varphi_n(x)}{b_n(x)}\right]_{x=\alpha_n} = \kappa_n + \left[\frac{1}{Z_n(x)}\left(\kappa_n' + \frac{f_{n-2}(x)}{b_{n-1}(x)}\right)\right]_{x=\alpha_n} = \kappa_n,$$

and

$$\begin{aligned} \left[\frac{\varphi_n(x)}{b_n(x)}\right]_{x=\alpha_{n-1}} &= \kappa_n + \frac{\kappa_n'}{Z_n(\alpha_{n-1})} + \left[\frac{1}{Z_n(x)Z_{n-1}(x)}\left(\frac{f_{n-2}(x)}{b_{n-2}(x)}\right)\right]_{x=\alpha_{n-1}} \\ &= \kappa_n + \kappa_n' A(\alpha_n, \alpha_{n-1}), \end{aligned}$$

proving the second and third statement. Finally, it holds that

$$\begin{aligned} \left[\frac{\varphi_n(x)}{b_n(x)}\right]_{x=\alpha_{n-1}} &= \left[\left(E_n + \frac{F_n}{Z_{n-1}(x)}\right)\frac{\varphi_{n-1}(x)}{b_{n-1}(x)}\right]_{x=\alpha_{n-1}} \\ &\qquad + \left[\left(\frac{C_n}{Z_{n-2*}(x)}\right)\frac{\varphi_{n-2}(x)}{b_{n-1}(x)}\right]_{x=\alpha_{n-1}} \\ &= E_n \left[\frac{\varphi_{n-1}(x)}{b_{n-1}(x)}\right]_{x=\alpha_{n-1}}. \end{aligned}$$

Using the second and third statement then proves the last statement. □

In order to reduce the number of coefficients in Theorem 2.1, we are now able to prove our first main result.

THEOREM 3.2 The coefficient $C_n$ in (2.3) is given by

$$C_n = -\frac{E_n + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}}. \tag{3.1}$$

*Proof.* From the last statement in Lemma 3.1 it follows that $\frac{b_{n-1}(x)}{Z_{n*}(x)} \in \mathscr{L}_{n-1}$, and hence

$$\begin{aligned}
0 &= \langle \varphi_n, b_{n-1}/Z_{n*} \rangle = \langle \varphi_n/Z_n, b_{n-1} \rangle \\
&= E_n \langle \varphi_{n-1}, b_{n-1} \rangle + F_n \langle \varphi_{n-1}/Z_{n-1}, b_{n-1} \rangle + C_n \langle \varphi_{n-2}/Z_{n-2*}, b_{n-1} \rangle \\
&= \frac{E_n}{\overline{\kappa}_{n-1}} + F_n \langle \varphi_{n-1}, b_{n-1}/Z_{n-1*} \rangle + C_n \langle \varphi_{n-2}, b_{n-1}/Z_{n-2} \rangle \\
&= \frac{E_n}{\overline{\kappa}_{n-1}} + \frac{F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{\kappa}_{n-1}} + C_n \left( A(\overline{\alpha}_{n-2}, \overline{\alpha}_{n-1}) \langle \varphi_{n-2}, b_{n-1} \rangle + \frac{1}{\overline{\kappa}_{n-2}} \right).
\end{aligned}$$

Furthermore, with $b_{n-1}(x) = \frac{1}{\kappa_{n-1}} \left( \varphi_{n-1}(x) - \kappa'_{n-1} b_{n-2}(x) - f_{n-3}(x) \right)$, we get that

$$\begin{aligned}
0 &= \frac{1}{\overline{\kappa}_{n-1}} \left[ E_n + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1}) + C_n \left( \frac{\overline{\kappa}'_{n-1} A(\overline{\alpha}_{n-1}, \overline{\alpha}_{n-2})}{\overline{\kappa}_{n-2}} + \frac{\overline{\kappa}_{n-1}}{\overline{\kappa}_{n-2}} \right) \right] \\
&= \frac{1}{\overline{\kappa}_{n-1}} \left[ E_n + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1}) + C_n \overline{\left( \frac{\kappa_{n-1} + \kappa'_{n-1} A(\alpha_{n-1}, \alpha_{n-2})}{\kappa_{n-2}} \right)} \right] \\
&= \frac{1}{\overline{\kappa}_{n-1}} \left[ E_n + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1}) + C_n \overline{E}_{n-1} \right].
\end{aligned}$$

□

Consequently, using the new parameter $\hat{F}_n = F_n/E_n$ instead of $F_n$, we can now reformulate Theorem 2.1 as follows.

THEOREM 3.3 Take by convention $\alpha_{-1} = \alpha_0 = \infty$. Then $\varphi_{n-1}(x)$ is not degenerate and $\varphi_n(x)$ is not exceptional for $n \geqslant 1$ iff there exists a three term recurrence relation of the form

$$\begin{aligned}
\varphi_n(x) &= E_n Z_n(x) \left( \left[ 1 + \frac{\hat{F}_n}{Z_{n-1}(x)} \right] \varphi_{n-1}(x) - \frac{1 + \hat{F}_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1} Z_{n-2*}(x)} \varphi_{n-2}(x) \right) \\
&= E_n \hat{\varphi}_n(x),
\end{aligned} \tag{3.2}$$

with $E_n \neq 0$ and $1 + \hat{F}_n A(\alpha_{n-1}, \overline{\alpha}_{n-1}) \neq 0$. The initial conditions are $\varphi_{-1}(x) \equiv 0$ and $\varphi_0(x) \equiv \frac{1}{\sqrt{\mu_0}}$ with $\mu_0 = \mu(\overline{\mathbb{R}})$.

Explicit expressions can easily be found for the recurrence coefficients, but first we will need the following lemma.

LEMMA 3.2 Let $a_j(x), b_j(x), c_j(x), d_j(x), A_j, B_j$ and $C_j$, with $j = 1, \ldots, 4$, be given by Table 1. Then for every function $f(x)$ and $g(x)$ it holds that

$$\left\langle \frac{a_j}{b_j} f, \frac{c_j}{d_j} g \right\rangle = A_j \langle a_j f, g \rangle + B_j \langle f, c_j g \rangle + C_j \langle f, g \rangle. \tag{3.3}$$

If $\alpha = \overline{\gamma}$ in Table 1, then the equality holds in the sense that the limit of the right hand side for $(\alpha, \gamma) \to (a, \overline{a})$ tends to the left hand side with $\alpha = \overline{\gamma} = a$.

TABLE 1 *Definition of* $a_j(x), b_j(x), c_j(x), d_j(x), A_j, B_j$ *and* $C_j$ *for* $j = 1, \ldots, 4$*, with* $\{\alpha, \beta, \gamma, \delta\} \subset \overline{\mathbb{C}} \setminus \{0\}$ *and* $\{\alpha, \gamma\} \cap supp(\mu) = \emptyset$*.*

| $j$ | $a_j(x)$ | $b_j(x)$ | $c_j(x)$ | $d_j(x)$ | $A_j$ | $B_j$ | $C_j$ |
|---|---|---|---|---|---|---|---|
| 1 | $Z_\alpha(x)$ | $Z_\beta(x)$ | 1 | 1 | $A(\beta,\alpha)$ | 0 | 1 |
| 2 | $Z_\alpha(x)$ | 1 | $Z_\gamma(x)$ | 1 | $\frac{1}{A(\overline{\gamma},\alpha)}$ | $\frac{1}{A(\alpha,\overline{\gamma})}$ | 0 |
| 3 | $Z_\alpha(x)$ | $Z_\beta(x)$ | $Z_\gamma(x)$ | 1 | $\frac{A(\beta,\alpha)}{A(\overline{\gamma},\alpha)}$ | $\frac{A(\beta,\overline{\gamma})}{A(\alpha,\overline{\gamma})}$ | 0 |
| 4 | $Z_\alpha(x)$ | $Z_\beta(x)$ | $Z_\gamma(x)$ | $Z_\delta(x)$ | $\frac{A(\beta,\alpha)A(\overline{\delta},\alpha)}{A(\overline{\gamma},\alpha)}$ | $\frac{A(\overline{\delta},\overline{\gamma})A(\beta,\overline{\gamma})}{A(\alpha,\overline{\gamma})}$ | 1 |

*Proof.* First, note that for $j = 1$, the equality directly follows from the sixth statement in Lemma 3.1. Secondly, for $j = 2$ we have that

$$\begin{aligned} \langle Z_\alpha f, g\rangle &= \left\langle \frac{Z_\alpha}{Z_{\gamma*}} f, Z_\gamma g\right\rangle = \left\langle \{A(\overline{\gamma},\alpha) Z_\alpha + 1\} f, Z_\gamma g\right\rangle \\ &= A(\overline{\gamma},\alpha)\left\langle Z_\alpha f, Z_\gamma g\right\rangle + \left\langle f, Z_\gamma g\right\rangle, \end{aligned}$$

so that

$$\left\langle Z_\alpha f, Z_\gamma g\right\rangle = \frac{\langle Z_\alpha f, g\rangle - \left\langle f, Z_\gamma g\right\rangle}{A(\overline{\gamma},\alpha)} = \frac{\langle Z_\alpha f, g\rangle}{A(\overline{\gamma},\alpha)} + \frac{\left\langle f, Z_\gamma g\right\rangle}{A(\alpha,\overline{\gamma})}.$$

Thirdly, for $j = 3$ it holds that

$$\begin{aligned} \left\langle \frac{Z_\alpha}{Z_\beta} f, Z_\gamma g\right\rangle &= A(\beta,\alpha)\left\langle Z_\alpha f, Z_\gamma g\right\rangle + \left\langle f, Z_\gamma g\right\rangle \\ &= A(\beta,\alpha)\left(\frac{\langle Z_\alpha f, g\rangle}{A(\overline{\gamma},\alpha)} + \frac{\left\langle f, Z_\gamma g\right\rangle}{A(\alpha,\overline{\gamma})}\right) + \left\langle f, Z_\gamma g\right\rangle \\ &= \frac{A(\beta,\alpha)}{A(\overline{\gamma},\alpha)}\langle Z_\alpha f, g\rangle + \frac{A(\beta,\alpha) - A(\overline{\gamma},\alpha)}{A(\alpha,\overline{\gamma})}\left\langle f, Z_\gamma g\right\rangle \\ &= \frac{A(\beta,\alpha)}{A(\overline{\gamma},\alpha)}\langle Z_\alpha f, g\rangle + \frac{A(\beta,\overline{\gamma})}{A(\alpha,\overline{\gamma})}\left\langle f, Z_\gamma g\right\rangle. \end{aligned}$$

Next, note that for $j = 4$ we get that

$$\left\langle \frac{Z_\alpha}{Z_\beta} f, \frac{Z_\gamma}{Z_\delta} g\right\rangle = \left\langle \{A(\beta,\alpha) Z_\alpha + 1\} f, \{A(\gamma,\delta) Z_\gamma + 1\} g\right\rangle.$$

Further computations, similarly as for $j = 3$, now prove the equality for $j = 4$.
Finally, because the functions $Z_\alpha(x)$ and $Z_\gamma(x)$ are bounded for $x \in \text{supp}(\mu)$, and $\alpha$ and $\gamma$ are in a compact subset of $\overline{\mathbb{C}} \setminus (\{0\} \cup \text{supp}(\mu))$, the dominated convergence theorem implies that the left hand side of (3.3) is continuous for any $\alpha$ and $\gamma$ in $\overline{\mathbb{C}} \setminus (\{0\} \cup \text{supp}(\mu))$. Hence, the limit of the right hand side must coincide with the limit of the left hand side at the points $(\alpha,\gamma) = (a,\overline{a})$, with $a \in \overline{\mathbb{C}} \setminus (\{0\} \cup \text{supp}(\mu))$, because these are the only points where the right hand side can not be evaluated due to the denominators $A(\alpha,\overline{\gamma})$ and $A(\overline{\gamma},\alpha)$. $\square$

COROLLARY 3.1 In the special case of $f = g$, $\alpha = \gamma$ and $\beta = \delta$, it holds for $j = 2$, respectively $j = 4$, in Lemma 3.2 that

$$\|Z_\alpha f\|^2 = \langle Z_\alpha f, Z_\alpha f\rangle = 2\Re\left\{\frac{\langle Z_\alpha f, f\rangle}{A(\overline{\alpha},\alpha)}\right\}, \tag{3.4}$$

respectively

$$\left\|\frac{Z_\alpha}{Z_\beta} f\right\|^2 = \left\langle \frac{Z_\alpha}{Z_\beta} f, \frac{Z_\alpha}{Z_\beta} f\right\rangle = 2\Re\left\{\frac{A(\beta,\alpha)A(\overline{\beta},\alpha)}{A(\overline{\alpha},\alpha)}\langle Z_\alpha f, f\rangle\right\} + \|f\|^2, \tag{3.5}$$

where $\Re\{.\}$ refers to the real part. Equation (3.4) and (3.5) also hold for $\alpha \in \overline{\mathbb{R}} \setminus (\{0\} \cup \mathrm{supp}(\mu))$ in the sense that the limit of the right hand side for $\Im\{\alpha\} \to 0$ tends to the left hand side with $\Im\{\alpha\} = 0$.

Explicit representations for the recurrence coefficients in terms of inner products are now given by the following theorem.

THEOREM 3.4 The coefficients $E_n$ and $\hat{F}_n$ in the recurrence relation (3.2) have the following explicit representation in terms of inner products:

$$E_n = \frac{\eta_n}{\|\hat{\varphi}_n\|}, \quad \eta_n \in \mathbb{T} \tag{3.6}$$

and

$$\hat{F}_n = \frac{K_{n,k} - L_{n,k}}{A(\alpha_{n-1},\alpha_n)L_{n,k} - A(\alpha_{n-1},\overline{\alpha}_{n-1})K_{n,k} + \delta_{n-1,k}\overline{E}_{n-1}}, \quad k < n \tag{3.7}$$

where

$$\begin{aligned} K_{n,k} &= A(\overline{\alpha}_{n-2},\alpha_n)\langle Z_n\varphi_{n-2},\varphi_k\rangle + \delta_{n-2,k} \\ L_{n,k} &= \overline{E}_{n-1}\langle Z_n\varphi_{n-1},\varphi_k\rangle, \end{aligned}$$

and

$$\delta_{n,k} = \langle\varphi_n,\varphi_k\rangle.$$

*Proof.* Using the fact that $\langle\varphi_n,\varphi_n\rangle = 1$ yields the first equation. Next, when taking the inner product on both sides of (3.2) with $\varphi_k$ for $k < n$ and solving for $\hat{F}_n$, we get that

$$\hat{F}_n = \frac{\left\langle \frac{Z_n}{Z_{n-2*}}\varphi_{n-2},\varphi_k\right\rangle - \overline{E}_{n-1}\langle Z_n\varphi_{n-1},\varphi_k\rangle}{\overline{E}_{n-1}\left\langle\frac{Z_n}{Z_{n-1}}\varphi_{n-1},\varphi_k\right\rangle - A(\alpha_{n-1},\overline{\alpha}_{n-1})\left\langle\frac{Z_n}{Z_{n-2*}}\varphi_{n-2},\varphi_k\right\rangle}.$$

Using the results from Lemma 3.2 then completes the proof. □

As a consequence of Theorem 3.4, we have the following corollary.

COROLLARY 3.2 Let $M_n$ be given by

$$M_n = \frac{\overline{E}_{n-1}\left[1+\hat{F}_nA(\alpha_{n-1},\alpha_n)\right]}{\left[1+\hat{F}_nA(\alpha_{n-1},\overline{\alpha}_{n-1})\right]}.$$

Then it holds that

$$A(\overline{\alpha}_{n-2},\alpha_n)\langle Z_n\varphi_{n-2},\varphi_{n-2}\rangle = M_n\langle Z_n\varphi_{n-1},\varphi_{n-2}\rangle - \delta_{n-2,n-2}$$

and

$$\langle Z_n\varphi_{n-1},\varphi_{n-1}\rangle = \frac{A(\overline{\alpha}_{n-2},\alpha_n)\,\langle Z_n\varphi_{n-2},\varphi_{n-1}\rangle}{M_n} - \frac{\hat{F}_n}{1+\hat{F}_nA(\alpha_{n-1},\alpha_n)}.$$

The following theorem now illustrates how $E_n$, apart from a unimodular normalisation constant, can be defined recursively in function of $\hat{F}_n$ when $\alpha_n$ is not real.

THEOREM 3.5 The coefficient $E_n = \eta_n|E_n|$ in the recurrence relation (3.2) is given by

$$|E_n|^2 = \frac{|E_{n-1}|^2}{2\Re\{a_n\}}, \tag{3.8}$$

where

$$a_n = \frac{|E_{n-1}|^2\,\hat{F}_n\left[1+\overline{\hat{F}}_nA(\overline{\alpha}_{n-1},\omega_n)\right] + \left|1+\hat{F}_nA(\alpha_{n-1},\overline{\alpha}_{n-1})\right|^2 A(\alpha_{n-2},\omega_n)}{A(\alpha_n,\overline{\alpha}_n)}$$

and $\omega_n = \frac{|\alpha_n|^2}{\Re\{\alpha_n\}}$. If $\alpha_n$ is real, the equality holds in the sense that the limit of the right hand side for $\Im\{\alpha_n\}\to 0$ tends to the left hand side.

*Proof.* From Equation (3.6) and (3.2) it follows that

$$\begin{aligned}|E_n|^{-2} = {}& \left\|Z_n\left(1+\frac{\hat{F}_n}{Z_{n-1}}\right)\varphi_{n-1}\right\|^2 + \left|\frac{1+\hat{F}_nA(\alpha_{n-1},\overline{\alpha}_{n-1})}{\overline{E}_{n-1}}\right|^2 \left\|\frac{Z_n}{Z_{n-2*}}\varphi_{n-2}\right\|^2 \\ & -2\Re\left\{\left(\frac{1+\hat{F}_nA(\alpha_{n-1},\overline{\alpha}_{n-1})}{\overline{E}_{n-1}}\right)\left\langle \frac{Z_n}{Z_{n-2*}}\varphi_{n-2}, Z_n\left(1+\frac{\hat{F}_n}{Z_{n-1}}\right)\varphi_{n-1}\right\rangle\right\}. \end{aligned} \tag{3.9}$$

Based on the results in Lemma 3.1 and 3.2, together with Corollary 3.1 and 3.2, we get that

1.

$$\begin{aligned}\left\|Z_n\left(1+\frac{\hat{F}_n}{Z_{n-1}}\right)\varphi_{n-1}\right\|^2 &= \left\|\left\{Z_n\left(1+\hat{F}_nA(\alpha_{n-1},\alpha_n)\right)+\hat{F}_n\right\}\varphi_{n-1}\right\|^2 \\ &= 2\left|1+\hat{F}_nA(\alpha_{n-1},\alpha_n)\right|^2\Re\left\{\frac{\langle Z_n\varphi_{n-1},\varphi_{n-1}\rangle}{A(\overline{\alpha}_n,\alpha_n)}\right\} + \left|\hat{F}_n\right|^2 \\ &\quad + 2\Re\left\{\left(1+\hat{F}_nA(\alpha_{n-1},\alpha_n)\right)\overline{\hat{F}}_n\langle Z_n\varphi_{n-1},\varphi_{n-1}\rangle\right\} \\ &= 2\Re\left\{\frac{\left[1+\hat{F}_nA(\alpha_{n-1},\alpha_n)\right]\left[1+\overline{\hat{F}}_nA(\overline{\alpha}_{n-1},\alpha_n)\right]}{A(\overline{\alpha}_n,\alpha_n)}\langle Z_n\varphi_{n-1},\varphi_{n-1}\rangle\right\} + \left|\hat{F}_n\right|^2 \\ &= 2\Re\left\{b_{n1}\langle Z_n\varphi_{n-2},\varphi_{n-1}\rangle - \frac{\hat{F}_n}{A(\overline{\alpha}_n,\alpha_n)}\left[1+\overline{\hat{F}}_nA(\overline{\alpha}_{n-1},\omega_n)\right]\right\}\end{aligned}$$

where

$$b_{n1} = \left(\frac{1+\hat{F}_nA(\alpha_{n-1},\overline{\alpha}_{n-1})}{\overline{E}_{n-1}}\right)\frac{A(\overline{\alpha}_{n-2},\alpha_n)\left[1+\overline{\hat{F}}_nA(\overline{\alpha}_{n-1},\alpha_n)\right]}{A(\overline{\alpha}_n,\alpha_n)}.$$

2.
$$\left\| \frac{Z_n}{Z_{n-2*}} \varphi_{n-2} \right\|^2 = 2\Re \left\{ \frac{A(\overline{\alpha}_{n-2}, \alpha_n) A(\alpha_{n-2}, \alpha_n)}{A(\overline{\alpha}_n, \alpha_n)} \langle Z_n \varphi_{n-2}, \varphi_{n-2} \rangle \right\} + \delta_{n-2,n-2},$$

so that

$$\left| \frac{1+\hat{F}_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \right|^2 \left\| \frac{Z_n}{Z_{n-2*}} \varphi_{n-2} \right\|^2 = 2\Re \left\{ b_{n2} \langle Z_n \varphi_{n-1}, \varphi_{n-2} \rangle - \delta_{n-2,n-2} \left| \frac{1+\hat{F}_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \right|^2 \frac{A(\alpha_{n-2}, \omega_n)}{A(\overline{\alpha}_n, \alpha_n)} \right\}$$

where

$$b_{n2} = \left( \frac{1+\overline{\hat{F}}_n A(\overline{\alpha}_{n-1}, \alpha_{n-1})}{E_{n-1}} \right) \frac{A(\alpha_{n-2}, \alpha_n) \left[ 1+\hat{F}_n A(\alpha_{n-1}, \alpha_n) \right]}{A(\overline{\alpha}_n, \alpha_n)}.$$

3.
$$\left\langle \frac{Z_n}{Z_{n-2*}} \varphi_{n-2}, Z_n \left( 1+\frac{\hat{F}_n}{Z_{n-1}} \right) \varphi_{n-1} \right\rangle = \frac{A(\overline{\alpha}_{n-2}, \alpha_n) \left[ 1+\overline{\hat{F}}_n A(\overline{\alpha}_{n-1}, \alpha_n) \right]}{A(\overline{\alpha}_n, \alpha_n)} \langle Z_n \varphi_{n-2}, \varphi_{n-1} \rangle + \frac{A(\overline{\alpha}_{n-2}, \overline{\alpha}_n) \left[ 1+\overline{\hat{F}}_n A(\overline{\alpha}_{n-1}, \overline{\alpha}_n) \right]}{A(\alpha_n, \overline{\alpha}_n)} \langle \varphi_{n-2}, Z_n \varphi_{n-1} \rangle,$$

so that

$$-2\Re \left\{ \left( \frac{1+\hat{F}_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \right) \left\langle \frac{Z_n}{Z_{n-2*}} \varphi_{n-2}, Z_n \left( 1+\frac{\hat{F}_n}{Z_{n-1}} \right) \varphi_{n-1} \right\rangle \right\} = -2\Re \left\{ b_{n1} \langle Z_n \varphi_{n-2}, \varphi_{n-1} \rangle \right\} - 2\Re \left\{ \overline{b_{n2} \langle Z_n \varphi_{n-1}, \varphi_{n-2} \rangle} \right\}.$$

Substituting this back into (3.9), taking into account that $\Re \left\{ \frac{A(\alpha_{-1}, \omega_1)}{A(\overline{\alpha}_1, \alpha_1)} \right\} = 0$ so that $\delta_{n-2,n-2}$ can be replaced with 1 even for $n = 1$, completes the proof. □

Clearly, the relation between $E_n$ and $\hat{F}_n$ is not as simple anymore as it is for the relation between $C_n$, and $E_n$ and $F_n = E_n \hat{F}_n$. It can, however, be simplified a little bit further by noticing that $A(\alpha, \overline{\alpha}) = \mathbf{i} \frac{2\Im\{\alpha\}}{|\alpha|^2}$, and that $\Im \{A(\alpha, \beta)\} = \frac{\Im\{\alpha\}}{|\alpha|^2}$ when $\beta$ is real. This way we get that

$$2\Re\{a_n\} = \frac{\left[ \Im\{\hat{F}_n\} - \left|\hat{F}_n\right|^2 \frac{\Im\{\alpha_{n-1}\}}{|\alpha_{n-1}|^2} \right] \left[ |E_{n-1}|^2 - 4 \frac{\Im\{\alpha_{n-1}\}}{|\alpha_{n-1}|^2} \frac{\Im\{\alpha_{n-2}\}}{|\alpha_{n-2}|^2} \right] + \frac{\Im\{\alpha_{n-2}\}}{|\alpha_{n-2}|^2}}{\frac{\Im\{\alpha_n\}}{|\alpha_n|^2}}. \tag{3.10}$$

Finally, we have the following Favard type theorem. For the complete proof, we refer to [1, p. 307–319].

THEOREM 3.1 Let $\{\varphi_n\}$ be a sequence of rational functions, and assume that the following conditions are satisfied:

1. $\alpha_k \neq 0$, $k = 1,2,\ldots$;
2. $\varphi_n$ is generated by the recurrence (3.2);
3. $\varphi_n \in \mathscr{L}_n \setminus \mathscr{L}_{n-1}$, $n = 1,2,\ldots$, and $\phi_0 \neq 0$;
4. $E_n \neq 0$, $n = 1,2,\ldots$;
5. $1+\hat{F}_n A(\alpha_{n-1},\overline{\alpha}_{n-1}) \neq 0$, $n = 2,3,\ldots$.

Then there exists a functional $M$ on $\mathscr{L}_\infty \cdot \mathscr{L}_{\infty *}$ so that

$$\langle f,g\rangle = M\{fg_*\}$$

defines a real positive inner product on $\mathscr{L}_\infty$ for which the $\varphi_n$ form an orthonormal system.

*Proof.* The outline of the proof is exactly the same as in the case of all real poles (see [1, p. 307–319]), with the following adaptations:

1. the inner products $M\{fg\}$ have to be replaced with the inner products $M\{fg_*\}$;
2. the factors $\frac{Z_n(x)}{Z_{n-2}(x)}$ in [1, Eq. (11.39) and (11.40)], respectively $\frac{Z_n(x)}{Z_j(x)}$ in [1, Thm. 11.9.2] and $\frac{Z_n(x)}{Z_{j-1}(x)}$ in [1, Thm. 11.9.3], have to be replaced with the factors $\frac{Z_n(x)}{Z_{n-2*}(x)}$, respectively $\frac{Z_n(x)}{Z_{j*}(x)}$ and $\frac{Z_n(x)}{Z_{j-1*}(x)}$;
3. the equality given by [1, Eq. (11.42)] becomes
$$M\{|\varphi_{n-1}|^2\} = -\frac{C_n\overline{E}_{n-1}}{E_n(1+\hat{F}_nA(\alpha_{n-1},\overline{\alpha}_{n-1}))}M\{|\varphi_{n-2}|^2\};$$
4. in the proof by induction (see [1, p. 313–318]), the assumption that $\varphi_n\varphi_{j*} \in \mathscr{R}_{n,j-1*}$ implies that $\alpha_m = \overline{\alpha}_n$ (instead of $\alpha_m = \alpha_n$) when $m \geqslant j+2$.

□

## 4. Asymptotic behaviour

Ratio asymptotics and a strong convergence result for ORFs on the interval $I$ have been derived in [13, Section 6] in the case of all *real* poles outside the interval. These derivations were based on the relation between ORFs on the interval and ORFs on the unit circle, a relation that was at that time only proven for all *real* poles by Van gucht et al. in [16, Thm. 4.1]. With the generalisation of this relation to arbitrary *complex* poles in Theorem 2.2 we are able to extend these results to the case of arbitrary *complex* poles outside the interval. But we first need the following lemma.

LEMMA 4.1 Let $\dot{\mu}$ satisfy the Erdős-Turán condition $\dot{\mu}' > 0$ a.e. on $\mathbb{T}$ and assume that the sequence $\mathscr{B}$ is compactly included in $\mathbb{D}$. Then we have

$$\lim_{n\to\infty}\frac{\tilde{\phi}^c_{2n}(z)}{\tilde{\phi}^*_{2n}(z)} = 0,$$

locally uniform in $\mathbb{D}$, where the tilde refers to the sequence of complex numbers given by (2.5).

*Proof.* Note that $\tilde{\phi}^c_{2n} \in \tilde{\mathscr{L}}_{2n}$ and $\tilde{\phi}^c_{2n} \perp \tilde{\mathscr{L}}_{2n-2}$ so that

$$\tilde{\phi}^c_{2n}(z) = a_{2n}\tilde{\phi}_{2n}(z) + b_{2n}\tilde{\phi}_{2n-1}(z),$$

where

$$a_{2n} = \left\langle \tilde{\phi}^c_{2n}, \tilde{\phi}_{2n} \right\rangle_{\mathbb{T}}, \quad b_{2n} = \left\langle \tilde{\phi}^c_{2n}, \tilde{\phi}_{2n-1} \right\rangle_{\mathbb{T}} \text{ and } 1 = |a_{2n}|^2 + |b_{2n}|^2.$$

And hence

$$\frac{\tilde{\phi}^c_{2n}(z)}{\tilde{\phi}^*_{2n}(z)} = a_{2n}\frac{\tilde{\phi}_{2n}(z)}{\tilde{\phi}^*_{2n}(z)} + b_{2n}\frac{\tilde{\phi}_{2n-1}(z)}{\tilde{\phi}^*_{2n}(z)}.$$

Furthermore, we have that there exist functions $A_{2n}(z)$ and $B_{2n}(z)$ with $B_{2n}(z) \neq 0$ for $z \in \mathbb{D}$, so that $\tilde{\phi}^*_{2n}(z) = A_{2n}(z)\tilde{\phi}_{2n-1}(z) + B_{2n}(z)\tilde{\phi}^*_{2n-1}(z)$ (see [1, p. 77]). Thus, it holds that

$$\begin{aligned}
\frac{\tilde{\phi}^c_{2n}(z)}{\tilde{\phi}^*_{2n}(z)} &= a_{2n}\frac{\tilde{\phi}_{2n}(z)}{\tilde{\phi}^*_{2n}(z)} + b_{2n}\frac{\tilde{\phi}_{2n-1}(z)}{A_{2n}(z)\tilde{\phi}_{2n-1}(z) + B_{2n}(z)\tilde{\phi}^*_{2n-1}(z)} \\
&= a_{2n}\frac{\tilde{\phi}_{2n}(z)}{\tilde{\phi}^*_{2n}(z)} + b_{2n}\frac{\frac{\tilde{\phi}_{2n-1}(z)}{\tilde{\phi}^*_{2n-1}(z)}}{A_{2n}(z)\frac{\tilde{\phi}_{2n-1}(z)}{\tilde{\phi}^*_{2n-1}(z)} + B_{2n}(z)}.
\end{aligned}$$

From Theorem 2.3 it now follows that

$$\lim_{n\to\infty}\frac{\tilde{\phi}^c_{2n}(z)}{\tilde{\phi}^*_{2n}(z)} = \lim_{n\to\infty}\left[a_{2n}\frac{\tilde{\phi}_{2n}(z)}{\tilde{\phi}^*_{2n}(z)} + b_{2n}\frac{\frac{\tilde{\phi}_{2n-1}(z)}{\tilde{\phi}^*_{2n-1}(z)}}{A_{2n}(z)\frac{\tilde{\phi}_{2n-1}(z)}{\tilde{\phi}^*_{2n-1}(z)} + B_{2n}(z)}\right] = 0,$$

locally uniform in $\mathbb{D}$. □

With this, we get the following results about the ratio convergence and strong convergence of ORFs on $I$.

THEOREM 4.1 Assume the sequence $\mathscr{A} = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{C}}^I$ is bounded away from $I$ and let $\mu$ be a positive bounded Borel measure with $\operatorname{supp}(\mu) = I$, which satisfies the Erdős-Turán condition $\mu' > 0$ a.e on $I$. If $\{\varphi_n\}$ are the ORFs on $I$ associated with $\mathscr{A}$ and $\mu$, then locally uniform in $\overline{\mathbb{C}}^I$ we have

$$\lim_{n\to\infty}\lambda_{n+1}\frac{z-\beta_{n+1}}{1-\overline{\beta}_n z}\sqrt{\frac{1-|\beta_n|^2}{1-|\beta_{n+1}|^2}}\frac{\varphi_{n+1}(x)}{\varphi_n(x)} = 1,$$

where $z = J^{inv}(x)$, $\beta_k = J^{inv}(\alpha_k)$ for $k = n, n+1$, and

$$\lambda_{n+1} = \frac{\tilde{\varepsilon}_{2n+2}}{\tilde{\varepsilon}_{2n}}\frac{\rho_n}{\rho_{n+1}} \quad \in \mathbb{T},$$

with $\varepsilon_n$ and $\rho_n$ the same as in Theorem 2.4, respectively Theorem 2.2, and the tilde referring to the sequence of complex numbers given by (2.5).

*Proof.* Define $\mathring{\mu}$ on $\mathbb{T}$ by (2.6) and use Theorem 2.2 to write

$$\frac{\varphi_{n+1}(x)}{\varphi_n(x)} = \frac{\rho_{n+1}}{\rho_n}\frac{1}{\zeta_{n+1}(z)}\frac{\tilde{\phi}^*_{2n+2}(z)}{\tilde{\phi}^*_{2n}(z)}\sqrt{\frac{1+\Re\left\{\frac{\tilde{\phi}^c_{2n}(\beta_n)}{\tilde{\phi}^*_{2n}(\beta_n)}\right\}}{1+\Re\left\{\frac{\tilde{\phi}^c_{2n+2}(\beta_{n+1})}{\tilde{\phi}^*_{2n+2}(\beta_{n+1})}\right\}}\frac{1+\frac{\tilde{\phi}^c_{2n+2}(z)}{\tilde{\phi}^*_{2n+2}(z)}}{1+\frac{\tilde{\phi}^c_{2n}(z)}{\tilde{\phi}^*_{2n}(z)}}}.$$

Using Lemma 4.1 and Theorem 2.4 we then obtain

$$\lim_{n\to\infty}\frac{\tilde{\varepsilon}_{2n+2}}{\tilde{\varepsilon}_{2n}}\frac{\rho_n}{\rho_{n+1}}\frac{z-\beta_{n+1}}{1-\overline{\beta}_n z}\sqrt{\frac{1-|\beta_n|^2}{1-|\beta_{n+1}|^2}}\frac{\varphi_{n+1}(x)}{\varphi_n(x)}=1,$$

locally uniform in $\overline{\mathbb{C}}^I$. □

THEOREM 4.2 Assume the sequence $\mathscr{A}=\{\alpha_1,\alpha_2,\ldots\}\subset\overline{\mathbb{C}}^I$ is bounded away from $I$ and let $\mu$ be a positive bounded Borel measure with $\operatorname{supp}(\mu)=I$, which satisfies the Szegő condition

$$\int_{-1}^{1}\frac{\log\mu'(x)}{\sqrt{1-x^2}}dx>-\infty.$$

Let $\dot{\mu}$ be given by (2.6) and suppose $\sigma(z)$ is the associated Szegő function as defined in Section 2. If $\{\varphi_n\}$ are the ORFs on $I$ associated with $\mathscr{A}$ and $\mu$, then locally uniform in $\overline{\mathbb{C}}^I$ we have

$$\lim_{n\to\infty}\lambda_n B_n(z)\frac{1-\overline{\beta}_n z}{\sqrt{1-|\beta_n|^2}}\varphi_n(x)=\frac{1}{\sqrt{2\pi}\sigma(z)},$$

where $z=J^{inv}(x)$, $\beta_k=J^{inv}(\alpha_k)$ and $\lambda_n=\frac{\tilde{\varepsilon}_{2n}}{\rho_n}\in\mathbb{T}$, with $\varepsilon_n$ and $\rho_n$ the same as in Theorem 2.4, respectively Theorem 2.2, and the tilde referring to the sequence of complex numbers given by (2.5). In particular we have

$$\lim_{n\to\infty}\varphi_n(x)=\infty$$

pointwise for $x\in\overline{\mathbb{C}}^I$.

*Proof.* From Theorem 2.2, 2.5 and Lemma 4.1 it follows that

$$\begin{aligned}\lim_{n\to\infty}\lambda_n B_n(z)\frac{1-\overline{\beta}_n z}{\sqrt{1-|\beta_n|^2}}\varphi_n(x) &= \lim_{n\to\infty}\frac{\tilde{\varepsilon}_{2n}}{\sqrt{2\pi}}\frac{\tilde{\phi}^*_{2n}(z)(1-\overline{\beta}_n z)}{\sqrt{1-|\beta_n|^2}}\\ &= \frac{1}{\sqrt{2\pi}\sigma(z)},\end{aligned}$$

locally uniform in $\mathbb{D}$. The last statement in the theorem follows from the fact that the Blaschke product $B_n(z)$ diverges to zero for $z\in\mathbb{D}$. □

## 5. Asymptotics for $E_n$ and $F_n$

In this section we wish to derive asymptotic formulas for the recurrence coefficients $E_n$ and $F_n=E_n\hat{F}_n$. Explicit formulas for the coefficients in terms of the ORFs $\varphi_n$ are given in the next theorem.

THEOREM 5.1 The explicit formulas for the recurrence coefficients $E_n$ and $F_n=E_n\hat{F}_n$ in terms of the orthonormal rational functions $\varphi_n$ are given by

$$\begin{aligned}
E_n &= \lim_{x\to\alpha_{n-1}} \frac{\varphi_n(x)}{Z_n(x)\varphi_{n-1}(x)} \\
F_n &= \lim_{x\to\overline{\alpha}_{n-2}} \left(\frac{Z_{n-1}(x)\varphi_n(x)}{Z_n(x)\varphi_{n-1}(x)} - E_n Z_{n-1}(x)\right) \\
&= \lim_{x\to\alpha} \frac{\left[\frac{\varphi_n(x)}{Z_n(x)\varphi_{n-1}(x)} - E_n\left(1-\frac{\varphi_{n-2}(x)}{\overline{E}_{n-1}Z_{n-2*}(x)\varphi_{n-1}(x)}\right)\right]}{\left[\frac{1}{Z_{n-1*}(x)} - \frac{1}{Z_{n-1*}(\alpha_{n-1})}\left(1-\frac{\varphi_{n-2}(x)}{\overline{E}_{n-1}Z_{n-2*}(x)\varphi_{n-1}(x)}\right)\right]}, \quad \forall\alpha\in\overline{\mathbb{C}}.
\end{aligned}$$

*Proof.* Using Theorem 2.1 we obtain that

$$\begin{aligned}
\lim_{x\to\alpha_{n-1}} \frac{\varphi_n(x)}{Z_n(x)\varphi_{n-1}(x)} &= E_n + \lim_{x\to\alpha_{n-1}} \left(\frac{F_n}{Z_{n-1}(x)} + C_n \frac{\varphi_{n-2}(x)}{Z_{n-2*}(x)\varphi_{n-1}(x)}\right) \\
&= E_n + 0,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{x\to\overline{\alpha}_{n-2}} \left(\frac{Z_{n-1}(x)\varphi_n(x)}{Z_n(x)\varphi_{n-1}(x)} - E_n Z_{n-1}(x)\right) &= F_n + C_n \lim_{x\to\overline{\alpha}_{n-2}} \frac{Z_{n-1}(x)\varphi_{n-2}(x)}{Z_{n-2*}(x)\varphi_{n-1}(x)} \\
&= F_n + 0.
\end{aligned}$$

Finally, the last equality for $F_n$ directly follows from Theorem 3.3, with $F_n = E_n\hat{F}_n$, and from the fact that

$$\begin{aligned}
\frac{1}{Z_{n-1}(x)} - A(\alpha_{n-1},\overline{\alpha}_{n-1}) &= \frac{1}{Z_{n-1*}(x)} \\
A(\alpha_{n-1},\overline{\alpha}_{n-1}) &= -\frac{1}{Z_{n-1*}(\alpha_{n-1})}.
\end{aligned}$$

□

Now we can use Theorem 4.1 to find the asymptotic formulas for $E_n$ and $F_n$.

THEOREM 5.2 Let $\beta_k = J^{inv}(\alpha_k)$ for $k = n, n-1, n-2$. Under the assumptions of Theorem 4.1, the following relation holds for $E_n$ in the sense that the ratio of the left hand side and the right hand side tends to 1 as $n$ tends to infinity,

$$E_n \sim 2\overline{\lambda}_n \frac{\sqrt{(1-|\beta_{n-1}|^2)(1-|\beta_n|^2)}\,(1-\beta_{n-1}\beta_n)}{\left(1+\beta_{n-1}^2\right)(1+\beta_n^2)}. \tag{5.1}$$

Further, the following relation holds for $F_n$,

$$\lim_{n\to\infty}\left\{F_n + \overline{\lambda}_n\sqrt{\frac{(1-|\beta_n|^2)}{(1-|\beta_{n-1}|^2)}}\,\frac{(1-|\beta_{n-1}|^2)\left(\beta_n+\overline{\beta}_{n-2}\right)+2\Re\{\beta_{n-1}\}\left(1-\beta_n\overline{\beta}_{n-2}\right)}{(1+\beta_n^2)\left(1-\beta_{n-1}\overline{\beta}_{n-2}\right)}\right\} = 0. \tag{5.2}$$

In the special case in which

$$\exists N\in\mathbb{N}: \forall n > N : \left|\left(1-|\beta_{n-1}|^2\right)\left(\beta_n+\overline{\beta}_{n-2}\right) + 2\Re\{\beta_{n-1}\}\left(1-\beta_n\overline{\beta}_{n-2}\right)\right| \geqslant \delta > 0, \tag{5.3}$$

the relation given by (5.2) is equivalent with

$$F_n \sim -\overline{\lambda}_n \sqrt{\frac{(1-|\beta_n|^2)}{(1-|\beta_{n-1}|^2)}} \frac{(1-|\beta_{n-1}|^2)\left(\beta_n+\overline{\beta}_{n-2}\right)+2\Re\{\beta_{n-1}\}\left(1-\beta_n\overline{\beta}_{n-2}\right)}{(1+\beta_n^2)\left(1-\beta_{n-1}\overline{\beta}_{n-2}\right)}. \tag{5.4}$$

*Proof.* It holds that

$$Z_\alpha(x) = \frac{\alpha x}{\alpha - x} = \frac{1}{2}\frac{(1+\beta^2)(1+z^2)}{z(\beta^2+1)-\beta(z^2+1))} = \frac{(1+\beta^2)(1+z^2)}{2(z-\beta)(1-\beta z)},$$

Further, note that the uniform convergence ensured by Theorem 4.1 permits us to interchange the limits $x \to \alpha_{n-1}$ and $n \to \infty$. Consequently, we can substitute $\varphi_n(x)/\varphi_{n-1}(x)$ in the expression of $E_n$, given by Theorem 5.1, by its asymptotic equivalent expression, given by Theorem 4.1, to find that

$$\lim_{x\to\alpha_{n-1}} \frac{\varphi_n(x)}{Z_n(x)\varphi_{n-1}(x)} \sim 2\overline{\lambda}_n \sqrt{\frac{1-|\beta_n|^2}{1-|\beta_{n-1}|^2}} \lim_{z\to\beta_{n-1}} \left[\frac{(1-\overline{\beta}_{n-1}z)(1-\beta_n z)}{(1+\beta_n^2)(1+z^2)}\right].$$

For $F_n$, it follows from Theorem 5.1 that

$$F_n = \lim_{z\to\beta} \frac{A_n(z)-E_nB_n(z)}{\frac{1}{Z_{n-1*}(x)} - \frac{1}{Z_{n-1*}(\alpha_{n-1})}B_n(z)}, \quad \beta = J^{inv}(\alpha),$$

where

$$A_n(z) = \frac{\varphi_n(x)}{Z_n(x)\varphi_{n-1}(x)}, \qquad B_n(z) = 1 - \frac{\varphi_{n-2}(x)}{\overline{E}_{n-1}Z_{n-2*}(x)\varphi_{n-1}(x)},$$

and $z = J^{inv}(x)$. Next, let $A_n'(z)$ and $B_n'(z)$ be given by

$$\begin{aligned} A_n'(z) &= 2\overline{\lambda}_n \sqrt{\frac{1-|\beta_n|^2}{1-|\beta_{n-1}|^2}} \frac{(1-\overline{\beta}_{n-1}z)(1-\beta_n z)}{(1+\beta_n^2)(1+z^2)} \\ B_n'(z) &= \frac{(1-\overline{\beta}_{n-1}z)\left[2\Re\{\beta_{n-1}\}(z-\overline{\beta}_{n-2})+(1-|\beta_{n-1}|^2)(1+\overline{\beta}_{n-2}z)\right]}{(1-|\beta_{n-1}|^2)(1-\overline{\beta}_{n-2}\overline{\beta}_{n-1})(1+z^2)}. \end{aligned}$$

Supposing that $\beta_k$, with $k = n-2, n-1, n$, and $z$ are compactly included in $\mathbb{D}$, it holds that $0 \leqslant |A_n'(z)| < \infty$ and $0 \leqslant |B_n'(z)| < \infty$. From Theorem 4.1 it now follows that $A_n(z) - A_n'(z) \to 0$ and $B_n(z) - B_n'(z) \to 0$. Further, with

$$E_n' = 2\overline{\lambda}_n \frac{\sqrt{(1-|\beta_{n-1}|^2)(1-|\beta_n|^2)}(1-\beta_{n-1}\beta_n)}{\left(1+\beta_{n-1}^2\right)(1+\beta_n^2)}$$

and

$$V_n = (1-|\beta_{n-1}|^2)(\beta_n+\overline{\beta}_{n-2}) + 2\Re\{\beta_{n-1}\}(1-\beta_n\overline{\beta}_{n-2}),$$

it holds that

$$A_n'(z) - E_n'B_n'(z) = -\frac{2\overline{\lambda}_n\sqrt{\frac{1-|\beta_n|^2}{1-|\beta_{n-1}|^2}}(z-\beta_{n-1})(1-\overline{\beta}_{n-1}z)V_n}{(1+\beta_{n-1}^2)(1-\overline{\beta}_{n-2}\overline{\beta}_{n-1})(1+\beta_n^2)(1+z^2)}$$

and

$$1/Z_{n-1*}(x) - B_n'(z)/Z_{n-1*}(\alpha_{n-1}) = \frac{2(z-\beta_{n-1})(1-\overline{\beta}_{n-1}z)(1-\beta_{n-1}\overline{\beta}_{n-2})}{(1+z^2)(1+\beta_{n-1}^2)(1-\overline{\beta}_{n-2}\overline{\beta}_{n-1})}.$$

We now have that $0 \leqslant |A_n'(z) - E_n'B_n'(z)| < \infty$ and $0 \leqslant |1/Z_{n-1*}(x) - B_n'(z)/Z_{n-1*}(\alpha_{n-1})| < \infty$, so that

$$\begin{aligned} [A_n(z) - E_nB_n(z)] - [A_n'(z) - E_n'B_n'(z)] &\rightarrow 0 \\ \left[\frac{1}{Z_{n-1*}(x)} - \frac{1}{Z_{n-1*}(\alpha_{n-1})}B_n(z)\right] - \left[\frac{1}{Z_{n-1*}(x)} - \frac{1}{Z_{n-1*}(\alpha_{n-1})}B_n(z)\right] &\rightarrow 0. \end{aligned} \tag{5.5}$$

Furthermore, if $z$ is bounded away from $\beta_{n-1}$, the relation given by (5.5) is equivalent with

$$\frac{1}{Z_{n-1*}(x)} - \frac{1}{Z_{n-1*}(\alpha_{n-1})}B_n(z) \sim \frac{1}{Z_{n-1*}(x)} - \frac{1}{Z_{n-1*}(\alpha_{n-1})}B_n'(z).$$

Consequently, supposing that $z$ is bounded away from $\beta_{n-1}$, we find that

$$\begin{aligned} & F_n - \lim_{z\to\beta} \frac{A_n'(z) - E_n'B_n'(z)}{\frac{1}{Z_{n-1*}(x)} - \frac{1}{Z_{n-1*}(\alpha_{n-1})}B_n'(z)} \to 0 \\ \Rightarrow \quad & F_n + \lim_{z\to\beta} \overline{\lambda}_n \sqrt{\frac{1-|\beta_n|^2}{1-|\beta_{n-1}|^2}} \frac{V_n}{(1+\beta_n^2)(1-\beta_{n-1}\overline{\beta}_{n-2})} \to 0 \\ \Rightarrow \quad & F_n + \overline{\lambda}_n \sqrt{\frac{1-|\beta_n|^2}{1-|\beta_{n-1}|^2}} \frac{V_n}{(1+\beta_n^2)(1-\beta_{n-1}\overline{\beta}_{n-2})} \to 0. \end{aligned}$$

Finally, if $V_n$ is bounded away from zero, we get that

$$F_n \sim -\overline{\lambda}_n \sqrt{\frac{1-|\beta_n|^2}{1-|\beta_{n-1}|^2}} \frac{V_n}{(1+\beta_n^2)(1-\beta_{n-1}\overline{\beta}_{n-2})}.$$

□

Note that for $n$ large enough the coefficients $E_n$ and $F_n$ will only depend on respectively the last two or three poles. If the last two poles are *real*, $E_n$ is bounded by $0 < E_n \leqslant 2$, but this will not be the case for $|E_n|$ if these two poles are *complex*. Take for example $\beta_{n-1} = \beta_n = \pm(1-\varepsilon)\mathbf{i}$, where $\varepsilon$ is a small positive number. Then for large $n$ we have that $|E_n| \approx \frac{2}{\varepsilon}$. Nevertheless, assuming that $\beta_{n-1}$ and $\beta_n$ are compactly included in $\mathbb{D}$, it follows that there exists a $\beta \in [0,1)$ so that $|\beta_{n-1}| < \beta$ and $|\beta_n| < \beta$. Consequently, for large $n$, it follows from (5.1) that

$$2\left(\frac{1-\beta^2}{1+\beta^2}\right)^2 < |E_n| < \frac{2(1+\beta^2)}{(1-\beta^2)^2}. \tag{5.6}$$

Finally, asymptotic formulas for $\hat{F}_n = \frac{F_n}{E_n}$ and $C_n$, given by (3.1), can be found as well. Depending on whether condition (5.3) is satisfied, we get for $\hat{F}_n$ that

$$\hat{F}_n \sim -\frac{(1+\beta_{n-1}^2)\left[(1-|\beta_{n-1}|^2)\left(\beta_n + \overline{\beta}_{n-2}\right) + 2\Re\{\beta_{n-1}\}\left(1-\beta_n\overline{\beta}_{n-2}\right)\right]}{2(1-\beta_{n-1}\beta_n)(1-|\beta_{n-1}|^2)\left(1-\beta_{n-1}\overline{\beta}_{n-2}\right)} \tag{5.7}$$

or

$$\lim_{n\to\infty}\left\{\hat{F}_n+\frac{(1+\beta_{n-1}^2)\left[\left(1-|\beta_{n-1}|^2\right)\left(\beta_n+\overline{\beta}_{n-2}\right)+2\Re\left\{\beta_{n-1}\right\}\left(1-\beta_n\overline{\beta}_{n-2}\right)\right]}{2\left(1-\beta_{n-1}\beta_n\right)\left(1-|\beta_{n-1}|^2\right)\left(1-\beta_{n-1}\overline{\beta}_{n-2}\right)}\right\}=0, \tag{5.8}$$

which can be readily obtained from the previous theorem. While for $C_n$ a series of computations eventually leads to

$$C_n\sim-\frac{\overline{\lambda}_n}{\lambda_{n-1}}\sqrt{\frac{1-|\beta_n|^2}{1-|\beta_{n-2}|^2}}\frac{(1+\overline{\beta}_{n-2}^2)(1-\beta_n\overline{\beta}_{n-1})}{(1+\beta_n^2)(1-\overline{\beta}_{n-2}\overline{\beta}_{n-1})}. \tag{5.9}$$

Note that the asymptotic formula for $C_n$ holds as well if condition (5.3) is not satisfied, due to the fact that the right hand side of (5.9) is bounded from above and bounded away from zero for every $\beta_k$, $k=n-2,n-1,n$, compactly included in $\mathbb{D}$.

## 6. Numerical examples

Explicit expressions are known for the so-called Chebyshev ORFs on $I$ with respect to the weight function $\mu'(x)=(1-x)^a(1+x)^b$, where $a,b\in\left\{\pm\frac{1}{2}\right\}$, and are given in [3, Thm. 3.2]. It has been proven (first in [15, Thm. 3.5] for all *real* poles and afterwards, only for $\mu'(x)=1/\sqrt{1-x^2}$, in [11, Section 4] for *complex* conjugate poles [2]) that for every $n>1$ the recurrence coefficients $E_n$, $F_n$ and $C_n$ are given by respectively the right hand side of (5.1), (5.4) and (5.9). This allows us to compute $|E_n|$ with Equation (5.1) and Equation (3.8), and to compare the results. From now on, we will assume that $E_n$ is positive real for $n=1,2,\ldots$. Furthermore, with $E_{n(i)}$ we denote the result for $E_n$ when $E_n$ is computed with Equation ($i$), where $i=3.6$, 3.8 or 5.1. The computations in the examples that follow are performed in Maple 8 [3] with 10 digits.

EXAMPLE 6.1 Assume that $\mu'(x)=(1-x)^a(1+x)^b$ so that '$\sim$' can be replaced with '$=$' in the asymptotic formulas for the recurrence coefficients when $n>1$, and let $\beta_1=-\beta_2=0.3+0.2\mathbf{i}$ and $\beta_3=C_r+C_{im}\mathbf{i}$, with $|\beta_3|\leqslant 0.99$. Figure 1 then shows the graph of $E_{3(5.1)}$, while Figure 2 shows the graph of $E_{3(3.8)}$. For the latter, $\hat{F}_3$ and $E_2$ are computed using Equation (5.7) and (5.1). These graphs clearly illustrate that the result for $E_3$ is the same for both formulas as long as $\beta_3$ (and hence, $\alpha_3=J(\beta_3)$ as well) is not real.

To get a better idea of what happens if $\alpha_3$ is real, we take a closer look at the case in which $\Re\{\beta_3\}$ is constant (Figure 3 and 4) or $\Im\{\beta_3\}$ is constant (Figure 5 and 6). Note that

$$\frac{\Im\{\alpha\}}{|\alpha|^2}=-\frac{2\Im\{\beta\}\left(1-|\beta|^2\right)}{\left(1+|\beta|^2\right)^2-4\left[\Im\{\beta\}\right]^2},$$

so that $\Im\{\alpha\}/|\alpha|^2$ is (close to) zero iff $\beta$ is (close to) real or $|\beta|$ is (close to) one. The figures on the right show the relative error of $E_{3(3.8)}$ compared to $E_{3(5.1)}$, given by

$$r_E=\left|1-\frac{E_{3(3.8)}}{E_{3(5.1)}}\right|. \tag{6.1}$$

[2] Neither the restriction to the weight function $\mu'(x)=1/\sqrt{1-x^2}$, nor the restriction to complex conjugate poles is in fact necessary, and hence we may assume that $\mu'(x)=(1-x)^a(1+x)^b$, with $a,b\in\left\{\pm\frac{1}{2}\right\}$, and that the poles are arbitrary complex as well.

[3] Maple and Maple V are registered trademarks of Waterloo Maple Inc.

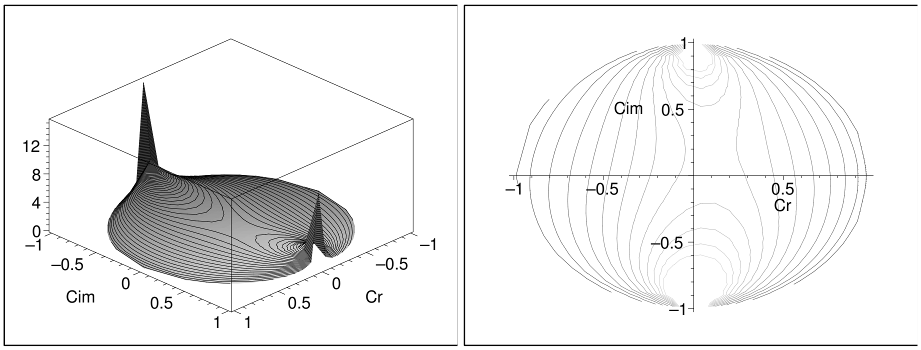


FIG. 1. Graph of $E_{3(5.1)}$ in function of $\beta_3 = C_r + C_{im}\mathbf{i}$. The figure on the left gives a 3D representation of the graph, while the figure on the right shows the contours $E_{3(5.1)} = 0.1(1+2k)$ for $k = 0,\ldots,12$.

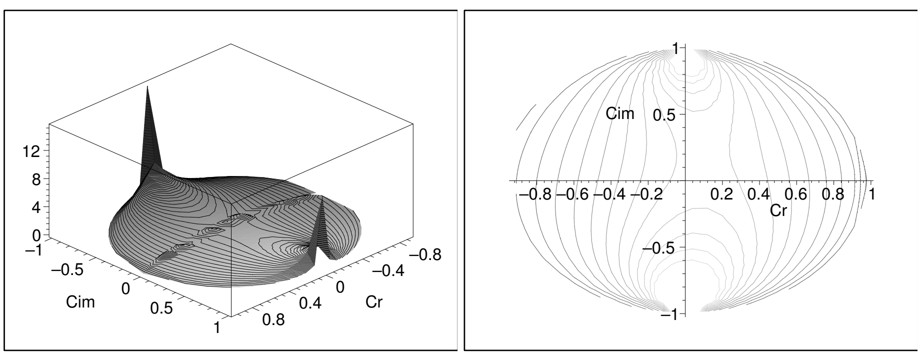


FIG. 2. Graph of $E_{3(3.8)}$ in function of $\beta_3 = C_r + C_{im}\mathbf{i}$. The figure on the left gives a 3D representation of the graph, while the figure on the right shows the contours $E_{3(3.8)} = 0.1(1+2k)$ for $k = 0,\ldots,12$.

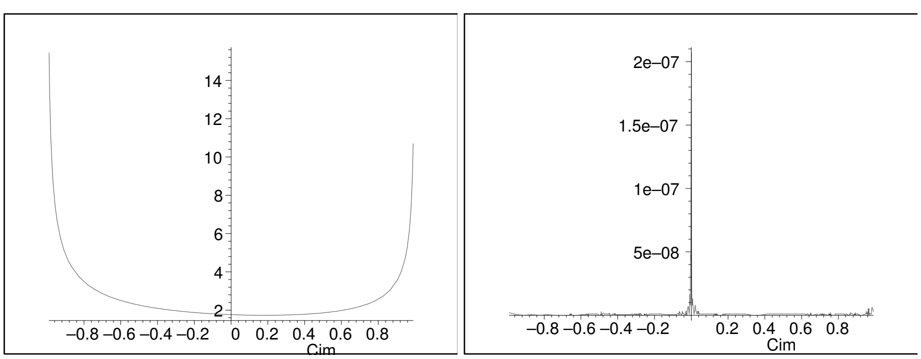


FIG. 3. Figure on the left: Graph of $E_{3(3.8)}$ in function of $\Im\{\beta_3\}$ with $\Re\{\beta_3\} = 0$. Figure on the right: The relative error given by (6.1).

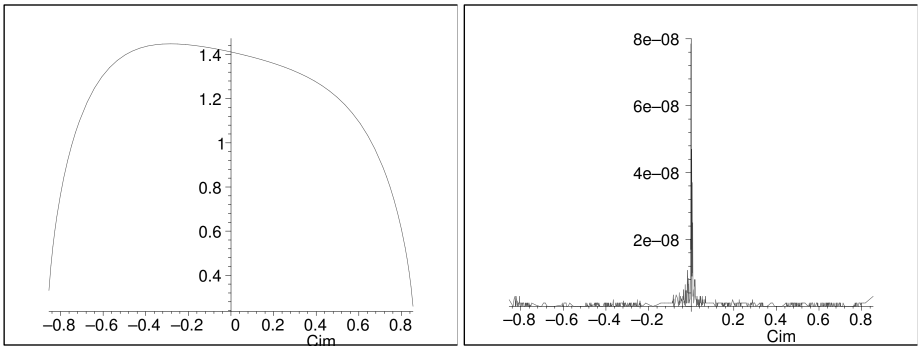


FIG. 4. Figure on the left: Graph of $E_{3(3.8)}$ in function of $\Im\{\beta_3\}$ with $\Re\{\beta_3\} = 0.5$. Figure on the right: The relative error given by (6.1).

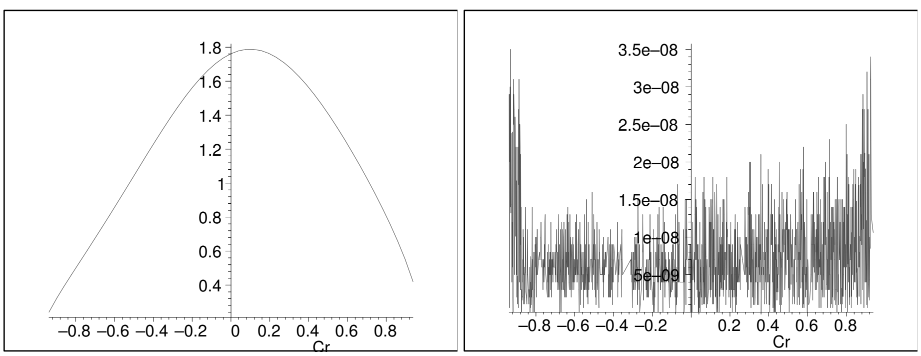


FIG. 5. Figure on the left: Graph of $E_{3(3.8)}$ in function of $\Re\{\beta_3\}$ with $\Im\{\beta_3\} = 10^{-2}$. Figure on the right: The relative error given by (6.1).

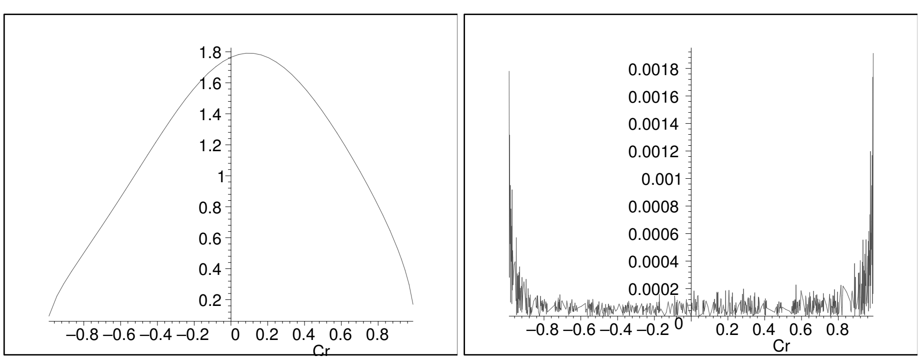


FIG. 6. Figure on the left: Graph of $E_{3(3.8)}$ in function of $\Re\{\beta_3\}$ with $\Im\{\beta_3\} = 10^{-6}$. Figure on the right: The relative error given by (6.1).

TABLE 2 *Results for $E_{n(3.8)}$ and $\hat{F}_n$, with $n = 1,\ldots,9$, when $\mu'(x) = [\arccos(x)]^2$ and $\alpha_n = (-1)^{n+1}\mathbf{i}$.*

| $n$ | $\hat{F}_n$ | $E_n$ | $\lvert\langle\varphi_n,\varphi_0\rangle\rvert$ |
|---|---|---|---|
| 1 | $0.4110241305 + 0.1725060290\mathbf{i}$ | 2.407674987 | $2.8\times10^{-10}$ |
| 2 | $0.07063851560 - 0.1925226262\mathbf{i}$ | 2.064699855 | $1.7\times10^{-10}$ |
| 3 | $0.01806270322 + 0.003063182959\mathbf{i}$ | 2.036303603 | $3.7\times10^{-10}$ |
| 4 | $0.01137541442 - 0.001494840571\mathbf{i}$ | 2.022945557 | $3.4\times10^{-10}$ |
| 5 | $0.007764863080 + 0.0008250747269\mathbf{i}$ | 2.015731886 | $2.6\times10^{-10}$ |
| 6 | $0.005620120537 - 0.0004991519144\mathbf{i}$ | 2.011430574 | $6.8\times10^{-10}$ |
| 7 | $0.004250190603 + 0.0003236351299\mathbf{i}$ | 2.008670444 | $1.0\times10^{-9}$ |
| 8 | $0.003324576249 - 0.0002213314747\mathbf{i}$ | 2.006797426 | $4.9\times10^{-10}$ |
| 9 | $0.002670807580 + 0.0001578585701\mathbf{i}$ | 2.005469615 | $6.1\times10^{-10}$ |

TABLE 3 *Results for $E_{n(3.6)}$ and $\hat{F}_n$, with $n = 1,\ldots,9$, when $\mu'(x) = [\arccos(x)]^2$ and $\alpha_n = (-1)^{n+1}\mathbf{i}$.*

| $n$ | $\hat{F}_n$ | $E_n$ | $\lvert\langle\varphi_n,\varphi_0\rangle\rvert$ |
|---|---|---|---|
| 1 | $0.4110241305 + 0.1725060290\mathbf{i}$ | 2.407674987 | $2.8\times10^{-10}$ |
| 2 | $0.07063851560 - 0.1925226262\mathbf{i}$ | 2.064699855 | $1.7\times10^{-10}$ |
| 3 | $0.01806270322 + 0.003063182959\mathbf{i}$ | 2.036303604 | $3.7\times10^{-10}$ |
| 4 | $0.01137541444 - 0.001494840915\mathbf{i}$ | 2.022945556 | $3.4\times10^{-10}$ |
| 5 | $0.007764863061 + 0.0008250744303\mathbf{i}$ | 2.015731888 | $2.5\times10^{-10}$ |
| 6 | $0.005620120577 - 0.0004991523606\mathbf{i}$ | 2.011430573 | $2.9\times10^{-10}$ |
| 7 | $0.004250190558 + 0.0003236348818\mathbf{i}$ | 2.008670444 | $6.8\times10^{-10}$ |
| 8 | $0.003324576292 - 0.0002213314742\mathbf{i}$ | 2.006797427 | $5.0\times10^{-10}$ |
| 9 | $0.002670807541 + 0.0001578587700\mathbf{i}$ | 2.005469613 | $2.7\times10^{-11}$ |

Repeating the computations in Example 6.1 with other values for $\beta_1$ and $\beta_2$ gives similar results as long as $\beta_1$ and/or $\beta_2$ are not too close to $\pm\mathbf{i}$. And hence we may assume that for more general weight functions, satisfying the assumptions in Theorem 4.1, Equation (3.8) is a fast but reliable way to get accurate results for $E_n$, with $n$ large enough so that the ratios in Theorem 5.1 are close to one, as long as $\Im\{\beta_n\}$ is not too close to zero, and $\beta_{n-2}$ and $\beta_{n-1}$ are not too close to $\pm\mathbf{i}$. In other words, $\frac{\Im\{\alpha_n\}}{|\alpha_n|^2}$ may not be too small, while $\frac{\Im\{\alpha_{n-2}\}}{|\alpha_{n-2}|^2}$ and $\frac{\Im\{\alpha_{n-1}\}}{|\alpha_{n-1}|^2}$ may not be too large.

EXAMPLE 6.2 Consider the weight function $\mu'(x) = [\arccos(x)]^2$ and let $\alpha_n = (-1)^{n+1}\mathbf{i}$ (or equivalently $\beta_n = (-1)^{n+1}(1-\sqrt{2})\mathbf{i}$) for $n = 1,2,\ldots$. From (5.1) and (5.8) we can deduce that $E_n$ tends to 2 and that $\hat{F}_n$ tends to 0 as $n$ tends to infinity. Table 2, respectively Table 3, shows the results for $E_{n(3.8)}$, respectively $E_{n(3.6)}$, and $\hat{F}_n$ (using Equation (3.7) with $k = n-1$), for $n = 1,\ldots,9$. To verify the correctness of the results, $|\langle\varphi_n,\varphi_0\rangle|$ (which has to equal zero) is computed as well. These tables confirm that $E_n$ tends to 2 and that $\hat{F}_n$ tends to 0 with increasing $n$.

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