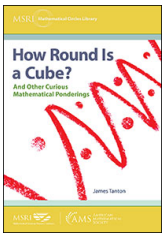
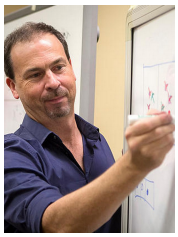
How round is a cube? And other curious mathematical ponderings, MSRI-AMS, 2019 (xiv+262 p.)
by *James Tanton*



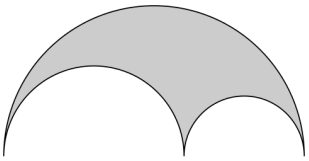


James Tanton

James Tanton is a mathematician whose main activity is related to raising mathematical awareness for a general public and helping teachers and students in secondary schools by providing joyful mathematical problems. He writes books, has a website (*jamestanton.com*), he tweets, posts videos, gives lectures, etc.

This book has 34 chapters that collect mathematical problems and puzzles and their solution and each chapter has one or more "research topics" that coud be used for further exploration, either individually or in class. So, it is a collection of mainly curious fun problems, the ones that are usually not on the school curriculum, but it is more than just fun because some mathematical rigour is maintained and the research challenges are very instructive. The usual aversion that is triggered by an overload of abstraction, technicalities, and formulas, is avoided a much as possible. There are formulas, but the many colourful figures make most of them unnecessary. Though the graphics, the proofs look like almost obvious. This influences the selection of problems because they often have some "geometric flavour" even though the original problem does not seem to have any geometric interpretation.

Examples of geometric problems are related to the classical arbelos. The figure was studied by Archimedes from Syracuse and one of the problems is to prove that the area of the arbelos equals the area of the circle whose diameter is the vertical common tangent to the small semi-circles inside the big one. This vertical divides the arbelos in two parts. The Archimedean circles are the largest ones that fit inside each part and Archmedes proved that these are equal, which is a less simple problem. Many other problems can be invented just based on three circles, For example the curve described by the centers of the circles that are tangent to two of the three semi-cicles generate conics.



arbelos

Another geometric problem is Feynman's triangle: Connect every vertex with a point on the opposite side cutting it in a ratio $1/3$. What is the area of the inner triangle defined by these lines with respect to the original one? What if the ratio is $1/r$ instead or $1/3$? How about squares instead of circles?

The title of the book is also the title of a chapter on 3D geometric objects. Consider an arbitrary polyhedron. How spherical is it? For this one has to study circles whose center is on a vertex of solids with flat faces. A flat circle spans an arc of 360°. At the vertex of a cube such a circle consists of three, and not four quarter circles. For a tetrahedron, this is three arcs of 60°, etc. In general we get for vertex v $(1 - x_v)360°$ and x is called the "pointiness" of that vertex. The "pointiness" of a polyhedron can then be defined as $\sum_v x_v$, a sum over all vertices. Step by step, the reader is led to the Euler characteristic $V - E + F$ where $(V, E, F) = \#$(vertices, edges, faces).

But there are also many less geometric examples involving Ramsey theory, L^p metrics, binary numbers, combinatorics, number theory, graph theory,.... However for all of these, there is some graphical representation, even if it is not really a geometrical problem. What I mean by this is best illustrated with the number theoretical problems. The Pythagorean triples form an excellent example. Similarly for the chapter on trapizoidal numbers (the number of nodes in a regular trapizoidal grid) or the chapter about triangular numbers with some side conditions. It is about numbers, but there is a geometry of triangles and squares behind.

Constructions on grids are used a number of times. There is a chapter on grid polygons, which are polygons whose vertices are on the nodes of a regular square grid. This is introduced as Gauss' shoelace problem. It was indeed Gauss who discovered a formula for the area of such a polygon given the (integer) coordinates of its vertices. Another one in this vein is to construct equilateral grid polygons, and to compute the area of grid triangles, which is given by Pick's theorem (by counting the number of gridpoints inside or on the boundary). There is also a chapter on the Prouhet-Thue-Morse sequence, with original proofs of its recursive properties, one of which is based on folding paper with red and blue sides in a particular way, rater than construction a binary sequence based on the binary representation of integers. Some problems are more demanding than other, but there are a lot of fun ones to be found.

Adhemar Bultheel