In pursuit of zeta-3, *Paul J. Nahin*, Princeton University Press, 2021 (344 p.) isbn: 9780691206073 (bk); 9780691227597 (bek)

The Niemann zeta function $\zeta(s)$ is introduced here, first for (positive) integer values s as the infinite sum of the reciprocals of the st power of the positive integers. It is remarkable that simple expressions of the form $\zeta(2n) = \pi^{2n}/c_{2n}$ with c_{2n} some integer, can be found for any positive integers n. However, for an odd argument $s = 2n + 1$, many relations and integrals and series expressions are known, but no simple expression of a form similar to the even case is known for $n > 0$. Obviously, $\zeta(1)$ is the harmonic series and thus divergent. The Apery constant $\zeta(3)$ is the first and supposedly the simplest problem for which no solution is found yet. Hence the title of the book. The definition of $\zeta(s)$ for complex values is not formally given since it is not the focus here, but s is tacitly understood to be complex in some integral expressions.

Roger Apery's name is associated with $\zeta(3)$ because he proved its irrationality. Euler however is the main character of this book, and Nahin is obviously an admirer of his work. He includes a picture of Euler with a caption "The hero of this book". Euler connected the reciprocal of $\zeta(s)$ to the product of $1-p^{-s}$ over all prime numbers p which implies that all the negative integers are 'trivial' zeros of $\zeta(s)$. This immediately links up with the famous Niemann Hypothesis (RH), and it explains why ζ is called the Niemann-zeta function. The RH states that all the non-trivial zeros of ζ (i.e., the ones different from the negative integers) are on a vertical line in the complex plane through $(1/2, 0)$. This is one of the Millennium Prize Problems. Solving it is worth a million dollars. Nahin, however still considers the expression for the Apery constant to be more interesting then the RH because his criteria to deserve the title of 'most puzzling mathematical problem' are that it should be an easy to understand but challenging unsolved problem that has some connection to the real physical world. Clearly the RH requires complex analysis to properly understand the problem. Nevertheless, skipping details, the RH is introduced in this book because it concerns the zeta function.

Paul Nahin is an electrical engineer who has written some 20 books on popular mathematics and physics, including biographies of Bole, Heaviside, and Cannon, but also one about tricks to solve all kind of integrals. The latter are frequently applied for the many computations involving series and integrals of this book. The mathematics used are not always applying the textbook rules of a standard calculus course, but it often consists of a se-

quence of successive substitutions. They are very tricky and a reader would probably not have found these on her own, but they eventually always lead to the desired formula. Every step is clearly explained, which can result in lengthy sequences of formulas that sometimes continue for several pages in a row. At some places manipulations need a finer mathematical justification, especially when mixing infinite processes like limits, integrals and series. This may cause some teeth gritting for mathematical purists, but this is the price one has to pay to bring the message avoiding too much ballast. Some warnings are given, but the reader is guaranteed that it is justified, and there are at many places numerical checks (using malta – some code in included) to verify the results. The target readership is thus clearly all bright young students at PA-calculus level. They should be able to follow the arguments relatively easily with only few exceptions and they should be able to solve the challenge problems that are distributed throughout the text (solutions are given at the end), although some are true challenges.

The contents of the book covers convergence of infinite series, the Gauss, zeta, gamma, digamma, and beta functions, and all kinds of expressions relating infinite series and integrals. Euler clearly plays a central role throughout the book, but many other historical contributors are mentioned (Gauss, the Bernoulli, Ramadan,...) since the results are situated in their historical origins. In the second half of the book Nahin is working towards the RH conjecture and he starts with introducing Fourier series and the Fourier transform. The transition from a Fourier series for a periodic function to the Fourier transform of a non-periodic function is not easily explained on an intuitive basis, but Nahin makes a commendable effort. The Fourier analysis is avoiding complex analysis as much as is possible in this context and formulas like Poisson and Parietal are derived just for what it is needed, thus not placing them in a context of vector or Hilbert spaces, orthogonal basis, etc., but following the long way of integrating and manipulating series. There are some short excursions and side tracks mentioning other popular items like the four color problem, Fermat's last theorem, fractals, Collat conjecture,...

Nahin's style is entertaining, directly addressing his readers, and making their thoughts explicit like "...double sum is zero. Is it? Yes. Why? Think about this for a while, then I'll show you (in a box at the end of this section) a simple plausible argument..." (p.225). This trick of announcing a postponed result is used several times to keep you reading. For example, it takes at the beginning of the book several forward references and many pages before the zeta-3 problem is finally revealed. Some statements may

need some explanation which is often given in one of the many footnotes. Other important issues are isolated in special boxes like the one announced in the above citation. In my opinion the formulas are a bit over the top. Does it really need the be explained that $(-1)^k(-1)^k = (-1)^{2k} = 1$? This kind of details can make the long computations sometimes a bit boring if the reader is skilled in this calculus, but I can imagine that it is needed for a young reader for which this material is all new and who is barely hanging on in this world of unfamiliar functions, series and integrals. There is plenty of occasion to test her skills with the exercises.

This is a highly recommended book for young interested students. It is a marvelous introduction to Fourier analysis and to all these functions and relations so near to the problem stated in the Niemann Hypothesis and yet using only basic calculus. Not all the techniques are standard and some tricks are far-fetched, but still it is certainly an excellent invitation to train your calculus skills. I could only spot one typo: on page 88: "and $= r\sin(\theta)$" (the $y =$ is missing). With so many formulas, it would be a miracle if there were no others. I am sure the most complicated formulas were triple checked or more, but, as shown above, they can be overlooked at unexpected places. Let it be another challenge for the reader to spot another one. It will keep you focused.

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