Powell–Sabin Spline Prewavelets on the Hexagonal Lattice

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Outline

1. Motivation
   - Preconditioning
   - Stability
2. Powell–Sabin spline prewavelets
   - Powell–Sabin splines
   - Prewavelets

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Preconditioning

- Simple example:

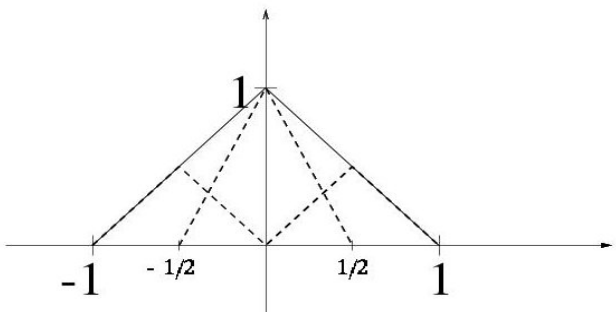
$$-u'' + qu = f \quad \text{on } [0,1], \quad u(0) = u(1) = 0$$

- Weak formulation:

$$\langle u', v' \rangle + q\langle u, v \rangle = \langle f, v \rangle, \quad v \in H_0^1([0,1])$$

- Galerkin scheme:

$$\begin{aligned} \phi(x) &:= (1 - |x|)_+ \\ \phi_{j,k}(x) &= 2^{j/2}\phi(2^j \cdot -k) \\ S_j &= \operatorname{span}\{\phi_{j,k} \mid k = 1, \ldots, 2^j - 1\} \end{aligned}$$



$$\langle u_j', v' \rangle + q\langle u_j, v \rangle = \langle f, v \rangle, \quad v \in S_j$$

Preconditioning: some results

- Multilevel basis:

$$W_j = \operatorname{span}\{\psi_{j,k} := \phi_{j+1,2k+1} \mid k = 1, \ldots, 2^j - 1\}$$

$$S_{j+1} = S_j \oplus W_j$$

- System to solve:

$$\langle \sum_{j,k} c_{j,k} \psi'_{j,k}, \psi'_{l,m} \rangle + q \langle \sum_{j,k} c_{j,k} \psi_{j,k}, \psi_{l,m} \rangle = \langle f, \psi_{l,m} \rangle$$

- Take $q = 1$, then $\kappa(A) \approx 1.4$ and we need 4 iterations to converge to the approximate solution.
- Take $q = 10^8$, then $\kappa(A) \approx 3386$ and we need 125 iterations.

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The key is stability

- General form:

$$a(u,v) = \langle f, v \rangle \qquad \text{(bilinear form)}$$

- Ellipticity:

$$a(v,v) \sim \|v\|_{\mathcal{H}}^2 \quad \text{with } \mathcal{H} \text{ a Hilbert space}$$

- e.g.: $a(u,v) = \langle u', v' \rangle + \langle u, v \rangle, \qquad a(v,v) \sim \|v\|_{H^1}^2$

Stability determines condition

Suppose that

$$\gamma \sum_{j,k} |c_{j,k}|^2 \leq \left\| \sum_{j,k} c_{j,k} \psi_{j,k} \right\|_{\mathcal{H}}^2 \leq \Gamma \sum_{j,k} |c_{j,k}|^2$$

then

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A large range of stability is preferable

$$a(u,v) = \langle u', v' \rangle + q\langle u, v \rangle$$

- For q small we have $a(v,v) \sim \|v\|_{H^1}^2$
- For q large we practically have $a(v,v) \sim \|v\|_{L_2}^2$

Topic of this talk

We construct a multiscale basis for $\mathbb{R}^2$ that is stable in the Sobolev spaces H^s for all $-\frac{5}{2} < s < \frac{5}{2}$.

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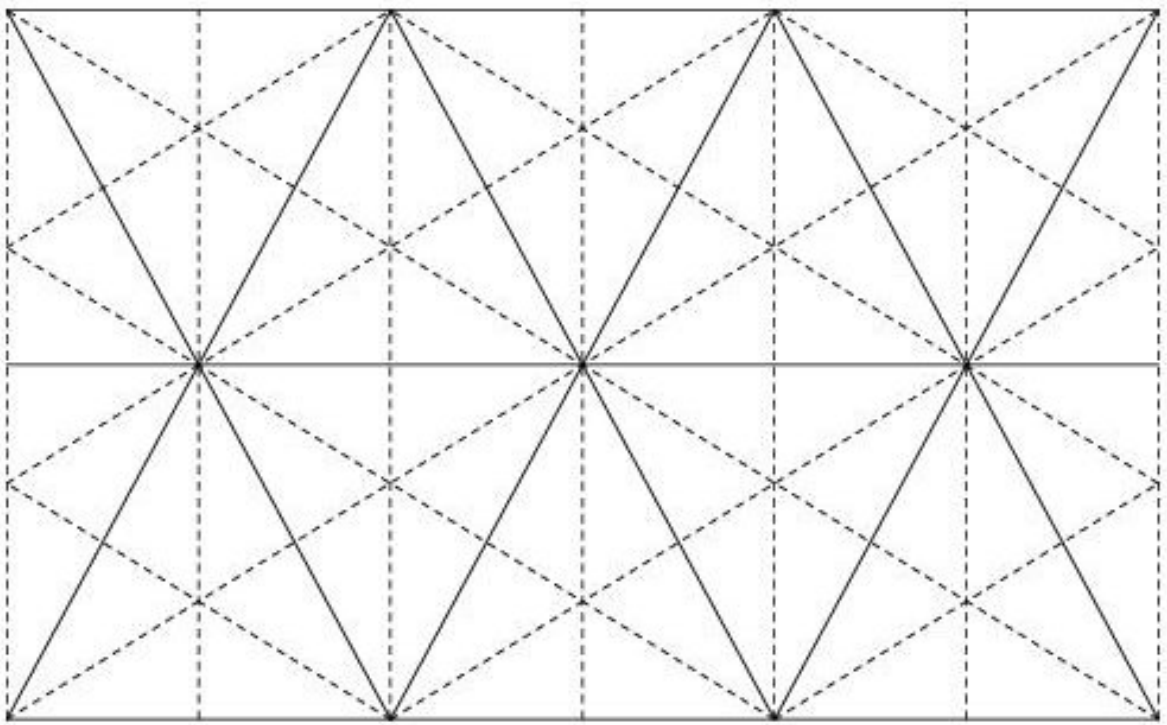
Hexagonal lattice Δ and PS 6-split Δ^*

The space of Powell–Sabin splines $S_2^1(\Delta^*)$ is the space of real-valued functions in $C^1(\mathbb{R}^2)$ whose restriction on each triangle of the triangulation Δ^* is a bivariate quadratic polynomial.



A basis for $S_2^1(\Delta^*)$

- A spline in $S_2^1(\Delta^*)$ is uniquely determined by its function values and gradients at all the vertices $k \in \Delta$.
- Let $k \in \Delta$, then

$$\phi(k) = \alpha, \quad \frac{\partial}{\partial x}\phi(k) = \beta, \quad \frac{\partial}{\partial y}\phi(k) = \gamma,$$

  and the other Hermite data being zero, determines uniquely $\phi \in S_2^1(\Delta^*)$ and ϕ is local.
- Generating function vector

$$\Phi = [\phi_1, \phi_2, \phi_3]^T$$

Nested dyadic refinement

- Define the 2×2 dilation matrix

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

- Dyadic refinement:

$$\Delta_j := D^{-j}\Delta, \qquad \Delta_j^* := D^{-j}\Delta^*$$

- Nested spaces:

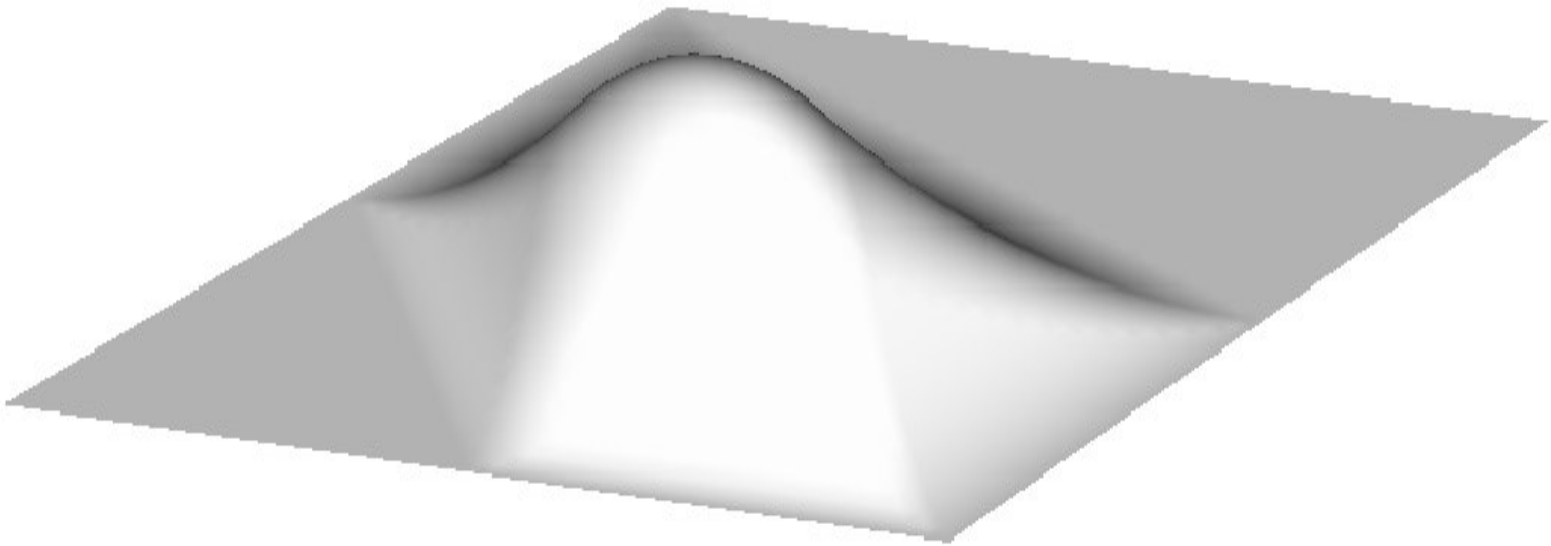
$$S_j := S_2^1(\Delta_j^*) \subset S_{j+1} := S_2^1(\Delta_{j+1}^*)$$

- Generator Φ satisfies

$$\Phi(u) = \sum_{k \in \Delta} \mathbf{A}_k \Phi(Du - k)$$

The generator $\Phi = [\phi_1, \phi_2, \phi_3]^T$

Basis function ϕ_1



The generating functions ϕ_1, ϕ_2 and ϕ_3 are globally invariant by a rotation of $2\pi/3$.

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Our objectives

- Nested structure:

$$S_{j+1} = S_0 \oplus^{\perp_{L_2}} W_0 \oplus^{\perp_{L_2}} W_1 \oplus^{\perp_{L_2}} \cdots \oplus^{\perp_{L_2}} W_j$$

- The basis functions that span the complement spaces W_j are called prewavelets and are generated by $\Psi := [\psi_1, \psi_2, \psi_3]^T$
- We want the prewavelets to have compact support

Theorem based on work by W. Dahmen and P. Oswald

If we can achieve our objectives, then our multilevel basis will be a stable basis for the Sobolev spaces H^s with $-\frac{5}{2} < s < \frac{5}{2}$.

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Step 1: construction of a dual basis

- A dual basis $\{\widetilde{\Phi}_{j,k} \mid k \in \Delta_j\}$:

$$\left\langle \widetilde{\Phi}_{j,k}, \Phi_{j,l}^T \right\rangle = \begin{cases} \mathbf{I} & k = l, \\ 0 & k \neq l \end{cases}$$

- We want local support!
- If we look in the same spline space S_j we won't get a locally supported dual basis.
- Trick: look for a $\{\widetilde{\Phi}_{j,k} \mid k \in \Delta_j\} \subset S_{j+1}$

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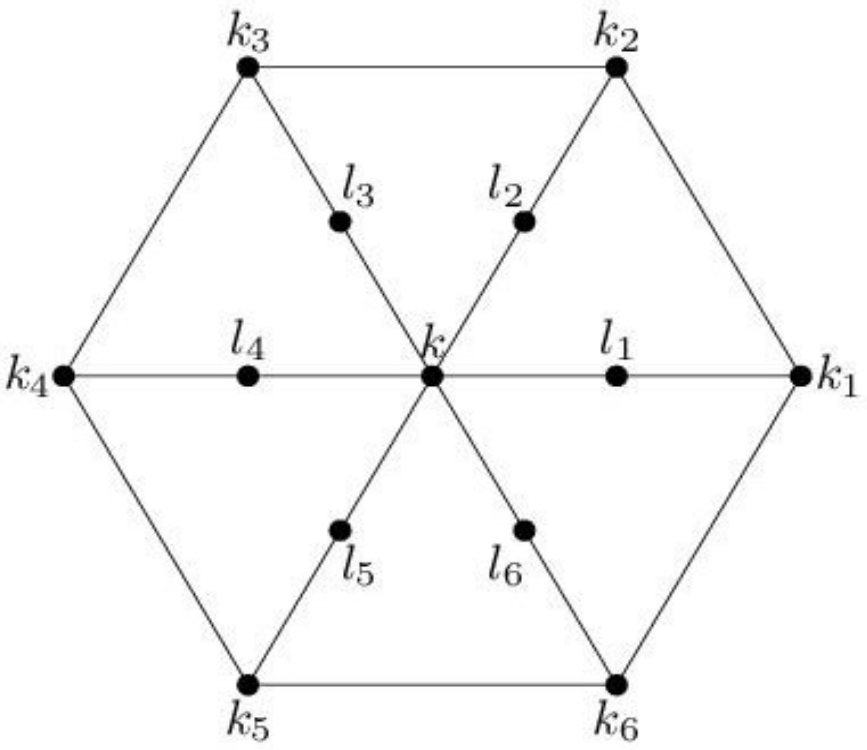
Step 1: construction of a dual basis

$$\widetilde{\Phi}_{j,k} := \widetilde{\mathbf{A}}_0 \Phi_{j+1,k} + \sum_{i=1}^{6} \widetilde{\mathbf{A}}_i \Phi_{j+1,l_i}$$

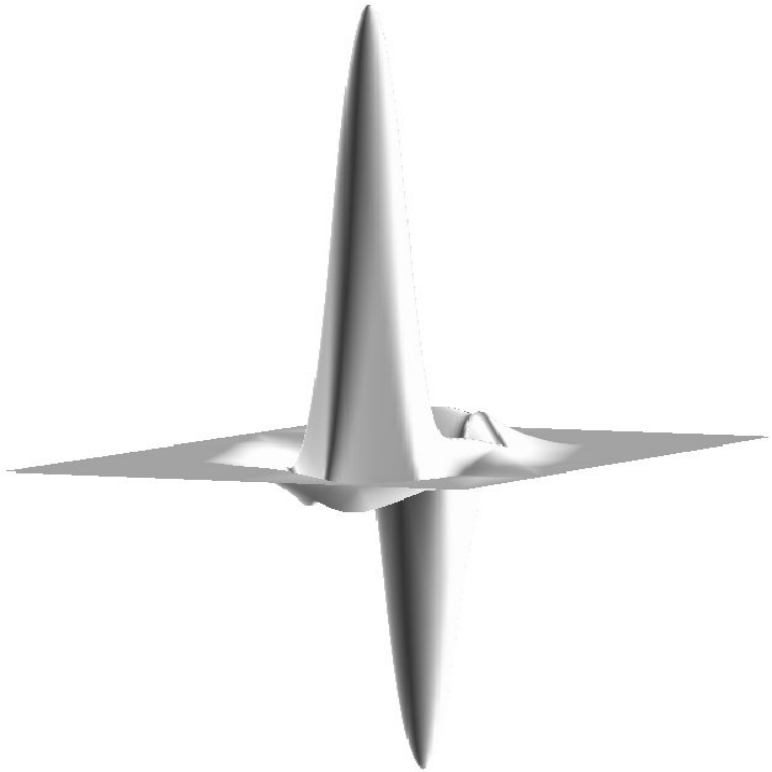


Solve the system

$$\begin{cases} \langle \widetilde{\Phi}_{j,k}, \Phi_{j,k}^T \rangle_{L_2} = \mathbf{I} \\ \langle \widetilde{\Phi}_{j,k}, \Phi_{j,k_i}^T \rangle_{L_2} = 0 \end{cases}$$

$(i = 1, \ldots, 6)$

The dual generator $\widetilde{\Phi} = [\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3]^T$



Dual basis function $\widetilde{\phi}_1$

The dual functions $\widetilde{\phi}_1$, $\widetilde{\phi}_2$ and $\widetilde{\phi}_3$ are globally invariant by a rotation of $2\pi/3$.

Step 2: projection into the orthogonal space

- For $I \in \Delta_{j+1} \setminus \Delta_j$ we define

$$\Psi_{j,I} := \Phi_{j+1,I} - \sum_{k \in \Delta_j} \left\langle \Phi_{j+1,I}, \Phi_{j,k}^T \right\rangle \widetilde{\Phi}_{j,k}$$

- Note that

$$\begin{aligned} \langle \Psi_{j,I}, \Phi_{j,m}^T \rangle &= \langle \Phi_{j+1,I}, \Phi_{j,m}^T \rangle - \sum_{k \in \Delta_j} \left\langle \Phi_{j+1,I}, \Phi_{j,k}^T \right\rangle \langle \widetilde{\Phi}_{j,k}, \Phi_{j,m}^T \rangle \\ &= \langle \Phi_{j+1,I}, \Phi_{j,m}^T \rangle - \langle \Phi_{j+1,I}, \Phi_{j,m}^T \rangle \\ &= 0 \end{aligned}$$

Two theorems

Theorem 1

The set

$$\{\widetilde{\Phi}_{j,k} \mid k \in \Delta_j\} \cup \{\Phi_{j+1,l} \mid l \in \Delta_{j+1} \setminus \Delta_j\}$$

is a Riesz basis for the space S_{j+1}.

Theorem 2

The set

$$\{\Psi_{j,l} \mid l \in \Delta_{j+1} \setminus \Delta_j\}$$

is a Riesz basis for the space W_j.

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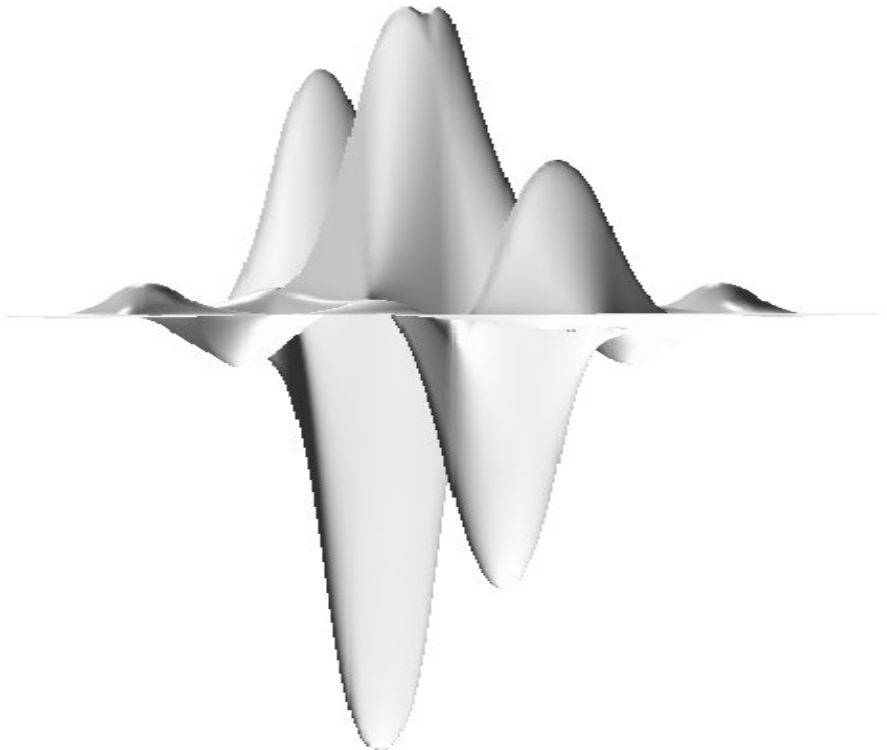
The wavelet generator $\Psi = [\psi_1, \psi_2, \psi_3]^T$



Wavelet function ψ_1

The wavelets ψ_1, ψ_2 and ψ_3 are globally invariant by a rotation of $2\pi/3$.

Some references

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