The Solution of a Two-Parameter Eigenvalue Problem and a Connection with the Implicitly Restarted Arnoldi Method

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Abstract

Consider the eigenvalue problem

$$(A + \lambda B)x = \mu M x \tag{1}$$

where μ is the eigenvalue, λ is a (real) parameter and A, B, and M are real large square matrices. The goal is to find λ for which (1) has purely imaginary μ. Such problems arise in the study of the stability of steady state solutions of dynamical systems. Mathematically, this problem can be written as finding the solutions of the eigenvalue problem

$$(A \otimes M + M \otimes A)u + \lambda(B \otimes M + M \otimes B)u = 0 \ . \tag{2}$$

The problem can be written as the matrix equation problem

$$AUM^T + MUA^T + \lambda(BUM^T + MUB^T) = 0 \tag{3}$$

which we call the Lyapunov eigenvalue problem. The $n \times n$ matrix U is the 'eigenvector'. If, for a given λ, $\pm\mu$ are the only purely imaginary eigenvalues of (1) and are simple, then λ is a double eigenvalue of (2). Then there is a unique (up to a scalar factor) real symmetric 'eigenvector' U associated with λ, which appears to have rank two. This unique matrix can be found by an inverse iteration method on (3) [3]. Imposing symmetry allows to use large scale Lyapunov solvers. The difficulty, however, is that inverse iteration on (3) is infeasible since the iterates are full rank dense matrices. When the method starts to converge, U can be well approximated by a rank two matrix. In the first iterations, however, this is usually not possible, which makes the method expensive. We therefore introduce a projection step that has two important consequences: the iterates now have rank two since they arise from the solution of a small eigenvalue problem of the form (3); and convergence is faster than inverse iteration.

Equation (1) can be solved by the two-parameter Jacobi-Davidson method [1], but this requires good starting values for λ and μ. Where, usually, starting values for λ are available, this is often not the case for μ. One advantage of our method is that a starting value of μ is not required. As a result, our method can be used for finding starting values for [1].

In this talk, we make the somehow straightforward extension to inverse subspace iteration. This extension is important to make the connection with the implicitly restarted Arnoldi method. For the special case $B = M$, and when the Arnoldi method is used as Lyapunov solver [6], we can show a nice interpretation that may help understand the inverse iteration method for (3). In this case, the method is the implicitly restarted Arnoldi method applied to $A^{-1}M$ with a proper selection of exact shifts and one zero shift [4]. In that respect, the proposed method can be viewed as a combination of subspace iteration and projection as the implicitly restarted Arnoldi method [2] and has similar convergence properties.

We finally will present an extension of the method to the solution of eigenvalue problems with a nonlinear parameter λ, as in

$$A(\lambda)x = \mu M x \ .$$

We discuss several ideas that can be used to solve the nonlinear problem using inverse residual iteration and Jacobi-Davidson [5] for the nonlinear Kronecker eigenvalue problem

$$(A(\lambda) \otimes M + M \otimes A(\lambda))u = 0 \ .$$

We show numerical examples to illustrate the methods.

References

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