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AN ALGEBRAIC METHOD TO SOLVE THE MINIMAL PARTIAL REALIZATION PROBLEM FOR MATRIX SEQUENCES

Marc Van Barel and Adhemar Bultheel

Department of Computer Science, Katholieke Universiteit Leuven, Celestijnenlaan 200A, B-3030 Leuven (Heverlee)

This paper is a generalization to matrix sequences of the method given in [7] to solve the minimal partial realization problem for scalar sequences. In part 1 we define the minimal (partial) realization problem for matrix sequences. In part 2 the block Hankel matrix based on the matrix sequence is decomposed and from this decomposition the solution of the minimal realization problem can be constructed which is in the input-output canonical form of Beghelli and Guidorzi. In part 3 we use the decomposition of part 2 to develop an algebraic algorithm to give all solutions of the minimal partial realization problem in input-output canonical form, i.e. with the minimal number of parameters. Part 4 contains some examples.

1. The minimal partial realization problem

Consider an infinite sequence of $p \times m$ matrices $H_1, H_2, H_3, \ldots$.
The rational matrix function $F(z)^{-1}G(z)$ with $F(z)$ a regular $p \times p$ polynomial matrix and $G(z)$ a $p \times m$ polynomial matrix is a *minimal partial realization* of the matrix series $H_1, H_2, H_3, \ldots$ iff

$$F(z)^{-1}G(z) = H_1 z^{-1} + H_2 z^{-2} + \cdots H_N z^{-N} + O(z^{-N-1}), \quad z \to \infty$$

and the degree of the determinant of $F(z)$ is minimal. If N is infinite then we skip the word partial in our terminology.
The minimal partial realization problem for matrix sequences has applications for linear systems with multiple inputs and multiple outputs [6].

2. Decomposition of the block Hankel matrix

In this section a decomposition of the block Hankel matrix is given on which the algebraic algorithm in the next section will be based.
Consider the infinite sequence $\{H_1, H_2, H_3, \ldots\}$ of $p \times m$ matrices.
The block Hankel matrices $H_{p,q}$ connected with this series for $p,q = 1,2,\ldots$ are denoted as :

$$H_{p,q} = [H_{i+j-1}] \quad i = 1, \ldots, p \quad \text{and} \quad j = 1, \ldots, q$$

The l-th row of $H_{\infty,\infty}$ is called the Hankel row h_l (not the l-th block row).
If a Hankel row is linearly independent of the previous Hankel rows we shall indicate it with a $\times$ and if it is linearly dependent as $\bullet$.
Assume that $H_{\infty,\infty}$ has finite rank n with corresponding observability indices $\{\nu_1, \ldots, \nu_p\}$ [5]. Then the first independent rows of $H_{\infty,\infty}$ indicated with a $\times$ are

h_{i+jp} , $j = 0, \ldots, \nu_i - 1$ and $i = 1, \ldots, p$. Then every row of $H_{\infty,\infty}$ can be expressed as a linear combination of the rows denoted by $h_{j_1}, h_{j_2}, \ldots, h_{j_n}$ with $j_i < j_k$, $i < k$. Hence it follows that we can decompose the infinite block Hankel matrix as

$$H_{\infty,\infty} = A_{\infty,n}\, B_{n,\infty} \tag{1}$$

with

$$[a_{j_1,n}^T\ a_{j_2,n}^T\ \cdots\ a_{j_n,n}^T]^T = I_{n,n} \quad \text{and} \quad B_{n,\infty} = [h_{j_1}^T\ \cdots\ h_{j_n}^T]^T$$

and with $a_{j,n}$ for $j_i < j < j_{i+1}$ ($j_0 = 0$, $j_{n+1} = \infty$) of the form

$$a_{j,n} = [a_{j,n}^{(j_1)} a_{j,n}^{(j_2)} \cdots a_{j,n}^{(j_i)}\ 0 \cdots 0] \text{ so that } h_j = \sum_{k=1}^{i} a_{j,n}^{(j_k)} h_{j_k}.$$

Using the z-transform $h_i(z^{-1}) = h_i[z^{-1}I_m\ z^{-2}I_m\ \cdots]^T$ and the previous decomposition (1), we get

$$z^{\nu_i} h_i(z^{-1}) = \sum_{j=1}^{p} a_{i,j}(z) h_j(z^{-1}) + b_i(z) \quad , \quad i = 1, \ldots, p$$

with the degree of the components of $b_i(z) < \nu_i$ and

$$a_{ij}(z) = a_{i+\nu_i p,n}^{(j)} + a_{i+\nu_i p,n}^{(j+p)} z + \cdots + a_{i+\nu_i p,n}^{(j+lp)} z^l$$

with

$$l = \min\{\nu_i, \nu_j - 1\} \text{ if } j < i \text{ and } l = \min\{\nu_i - 1, \nu_j - 1\} \text{ if } j \geqslant i.$$

When $l = -1$, we set $a_{ij}(z) = 0$.
If we set $A(z) = \operatorname{diag}(z^{\nu_1}, z^{\nu_2}, \ldots, z^{\nu_p}) - [a_{i,j}]_{i,j=1}^{p}$ and $B(z) =$ the polynomial part of $A(z)H(z^{-1})$, we have that $A(z)^{-1}B(z)$ is a minimal partial realization of $\{H_1, H_2, \ldots\}$ and it has the input-output canonical form of Beghelli and Guidorzi [1,2,3,4]. I.e. the matrix A_{hr} consisting of the p highest degree coefficients of the polynomial row vectors is unit lower triangular and the similar matrix for the columns A_{hc} is the unit matrix.
The previous construction reveals the tight relation that exists between this canonical form and the linear dependencies of the Hankel rows. This interpretation is new. For more details see [8].
A new algebraic algorithm to solve the minimal realization problem is as follows :

Construct a constant matrix $Q_{\infty,\infty}$ such that

$$H_{\infty,\infty} \cdot Q_{\infty,\infty} = [A'_{\infty,n}\ 0_{\infty,\infty}] \tag{2}$$

with $[a_{j'_1,n}{}^T\ \cdots\ a_{j'_n,n}{}^T] = I_{n,n}$ and $j'_1 < j'_2 < \cdots < j'_n$ as small as possible.

Then we can write (2) as :

$$H_{\infty,\infty} = [A'_{\infty,n}\ 0_{\infty,\infty}] \cdot Q_{\infty,\infty}^{-1} = A'_{\infty,n} \cdot B'_{n,\infty} \text{ with } B'_{n,\infty} = [Q_{\infty,\infty}^{-1}]_{n,\infty}.$$

Because the decomposition $H_{\infty,\infty} = A_{\infty,n} \cdot B_{n,\infty}$ of (1) is unique we get that $A'_{\infty,n} = A_{\infty,n}$ and $B'_{n,\infty} = B_{n,\infty}$ and $j'_i = j_i$.
From the rows $a_{i+\nu_i p,n}$ of $A'_{\infty,n} = A_{\infty,n}$ we can construct the polynomial matrix $A(z)$. The numerator polynomial matrix $B(z)$ can be found as the polynomial part of $A(z)H(z^{-1})$. In the next section we are solving the problem when only the first

N elements of the infinite sequence are known $\{H_1, \ldots, H_N\}$.

3. An algebraic algorithm to solve the minimal partial realization problem

When we only know a partial sequence $\{H_1, H_2, \ldots, H_N\}$ of N $p \times m$ matrices, it is clear from the block Hankel structure that there exists at least one extension sequence $\{H^*_{N+1}, H^*_{N+2}, \cdots\}$ such that the rank of the block Hankel matrix determined by the specified entries is equal to the rank of $H_{\infty,\infty}$ based on the infinite sequence $\{H_1, H_2, \ldots, H_N, H^*_{N+1}, H^*_{N+2}, \cdots\}$.

This implies that the knowledge of the linear dependencies of the block Hankel matrix $H_{N,N}$ is enough to solve the problem. When we use the ● and × notation to indicate the linear dependency, linear independency respectively of rows and columns of the block Hankel matrix $H_{N,N}$, only the unspecificied values of the extension sequence lying at the crossing of a ●-row and a ●-column have to get specific values to maintain the rank of the block Hankel matrix without increasing it. The elements of the extension sequence that are nowhere at the crossing of a ●-row and a ●-column can be chosen freely.

The algebraic algorithm consists of a sequence of elementary column operations performed on the block Hankel matrix $H_{N,N}$

$$H_{N,N} = \begin{bmatrix} H_1 & \cdots & H_{N-1} & H_N \\ H_2 & \cdots & H_N & H^*_{N+1} \\ \vdots & & \vdots & \vdots \\ H_N & \cdots & H^*_{2N-2} & H^*_{2N-1} \end{bmatrix}.$$

The algorithm consists of 3 parts.

Part 1

In the first part, the linear dependencies of the Hankel rows and columns are computed. None of the unspecified elements are needed. So, if we speak of Hankel rows or columns we mean only those parts which contain the specified elements. For these do the following operations :

1. **For** $j = 1,2, \ldots, N\,m$
   - 1.1 **If** there is a nonzero element in column j **then**
     - — Take the first one and mark the corresponding row r_j and the column j by ×.
     - — Divide column j by this element.
     - — Eliminate all the elements on row r_j in columns $j+1, j+2, \cdots$ by subtracting a suitable multiple of column j.
   - 1.2 **If** all the elements in column j are zero **then**
     - — Mark column j by ●.
2. Mark all the rows which are not × rows by ●.
3. The rank of the original block Hankel matrix n equals the number of nonzero columns = the number of × rows = the number of × columns of the matrix

which has been generated in step 1.

4. For $i = 1,2,\ldots,p$, the observability indices ν_i are defined as the smallest integers such that $i + \nu_i p$ is a $\bullet$ row.

5. For $i = 1,2,\ldots,p$, the controllability indices μ_i are defined as the smallest integers such that $i + \mu_i m$ is a $\bullet$ column.

6. The $\times$ rows have the numbers $j_1, j_2, \ldots, j_n$.

Part 2

1. Now the free elements are introduced. Because the transformations used in part 1 don't change the positions of the linear independent rows and columns, the free elements will be found at the same position as in the original block Hankel matrix. Now the unspecified elements in the original block Hankel matrix appear in several places (cfr. the block Hankel structure), but only those elements which are in none of their appearances at the crossing of a $\bullet$-row and a $\bullet$-column will be free, the free elements will be found at the same places in our transformed matrix. We introduce them in rows $i + \nu_i p$, which may require the introduction of extra rows at the bottom of the matrix. More precisely position $(\nu_i p+i,\ lm + j)$ will contain a free element if none of the positions $(\nu_i - k)p+i,\ (l+k)m+j)$, $k = \cdots,-1,0,1,2,\cdots$ is at the crossing of a $\bullet$ row and $\bullet$ column.

2. Consider the $\times$ rows r_j and eliminate the nonzero elements in this row which precede the 1 in column j by subtracting a suitable multiple of column j from the previous columns. These operations are also performed on the free elements just introduced in rows $Np+1, \ldots, (N+1)p$.

Part 3

Permute the columns, so that the n $\times$ columns are the first n columns and the number of initial zeros increases from left to right. In other words rows $j_1, j_2, \ldots, j_n$ are brought into the form $[I_n, 0_{n,Nm-n}[$.
The first n columns form now $A_{(N+1)p,n}$. From this matrix, $A(z)$ can be constructed as described before and $B(z)$ is the polynomial part of $A(z)H(z)$.

Note that the moment where the free elements are introduced is important. The proposed scheme ensures that $A(z)$ will be linear in the free elements.

4. Examples

(a) Consider example 14.1 of [6]. The finite sequence is

$$\left\{\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}2 & 3\\ 0 & 0\end{bmatrix}\right\}.$$

So $N=3$ and $p=m=2$. The block Hankel matrix that gives enough information about the linear dependencies is $H_{3,3}$:

$$H_{3,3} = \begin{matrix} \\ \times \end{matrix} \begin{matrix} \times \\ \left[\begin{matrix} 1 & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & & \\ 0 & 0 & 0 & 0 & & \\ 2 & 3 & & & & \\ 0 & 0 & & & & \end{matrix}\right] \end{matrix}$$

The first element of the first column is different from zero, so we use this element as a pivot element to make the following entries of the first row zero.

$$H_{3,3}Q_1 = \begin{matrix} & \times & \times \\ \times \\ \\ \times \\ \\ \times \\ \end{matrix} \begin{matrix} \left[\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & & \\ 0 & 0 & 0 & 0 & & \\ 2 & 1 & & & & \\ 0 & 0 & & & & \end{matrix}\right] \end{matrix}$$

The 5-th element of the second column is the first element of the 5-th column different from zero. So we are going to use that element as pivot element.

Because the 5-th row is linearly independent of the previous rows also the thirth row has to be a $\times$-row (because of the block Hankel structure of $H_{3,3}$).

$$H_{3,3}Q_1Q_2 = \begin{matrix} & \times & \times & \times \\ \times \\ \\ \times \\ \\ \times \\ \end{matrix} \begin{matrix} \left[\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & & \\ 0 & 0 & 0 & 0 & & \\ 2 & 1 & & & & \\ 0 & 0 & & & & \end{matrix}\right] \end{matrix}$$

The 3-th element of the third column is its first element different from zero, so we are going to use that element as pivot element.

$$H_{3,3}Q_1Q_2Q_3 = \begin{matrix} & \times & \times & \times & \bullet & \bullet & \bullet \\ \times \\ \bullet \\ \times \\ \bullet \\ \times \\ \bullet \end{matrix} \begin{matrix} \left[\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ 2 & 1 & & & & \\ 0 & 0 & & & & \end{matrix}\right] \end{matrix}$$

The other rows and columns are $\bullet$-rows and $\bullet$-columns.

From this result we get the observability indices which are : $\nu_1 = 3$, $\nu_2 = 0$ and the controllability indices $\mu_1 = 2$, $\mu_2 = 1$.

In the second part of the algorithm we extend $H_{3,3}$ and introduce the three free parameters u, v, w.

$$H_{4,3}Q_1Q_2Q_3 = \begin{array}{c} \\ \times \\ \bullet \\ \times \\ \bullet \\ \times \\ \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \begin{array}{cccccc} \times & \times & \times & \bullet & \bullet & \bullet \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ 2 & 1 & & & & \\ 0 & 0 & & & & \\ u & v & w & & & \\ & & & & & \end{array}\right] \end{array}$$

We have to make the (3,1) and (5,1) elements equal to zero.

$$H_{4,3}Q_1Q_2Q_3Q_4 = \begin{array}{c} \\ \times \\ \bullet \\ \times \\ \bullet \\ \times \\ \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \begin{array}{cccccc} \times & & \times & \times & \bullet & \bullet & \bullet \end{array} \\ \left[\begin{array}{cccccc} 1 & & 0 & 0 & 0 & 0 & 0 \\ 0 & & 0 & 0 & 0 & 0 & 0 \\ 0 & & 0 & 1 & 0 & & \\ 0 & & 0 & 0 & 0 & & \\ 0 & & 1 & & & & \\ 0 & & 0 & & & & \\ u-w-2v & & v & w & & & \\ & & & & & & \end{array}\right] \end{array}$$

The third part of the algorithm permutes the first three columns such that the first three elements of the resulting first, third and fifth row form the unit 3×3 matrix :

$$H_{4,3}Q_1Q_2Q_3Q_4P = \begin{array}{c} \\ \times \\ \bullet \\ \times \\ \bullet \\ \times \\ \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \\ \left[\begin{array}{cccccc} 1 & & 0 & 0 & 0 & 0 & 0 \\ 0 & & 0 & 0 & 0 & 0 & 0 \\ 0 & & 1 & 0 & 0 & & \\ 0 & & 0 & 0 & 0 & & \\ 0 & & & 1 & & & \\ 0 & & & 0 & & & \\ u-w-2v & & w & v & & & \\ & & & & & & \end{array}\right] \end{array}$$

The denominator polynomial matrix $A(z)$ has three free parameters u, v, w :

$$A(z) = \begin{bmatrix} z^3-vz^2-wz-(u-2v-w) & 0 \\ 0 & 1 \end{bmatrix}.$$

The numerator polynomial matrix can be found as : $B(z) =$ the polynomial part of $A(z) \cdot H(z^{-1})$ with

$$H(z^{-1}) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} z^{-2} + \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} z^{-3} + \cdots .$$

this results into

$$B(z) = \begin{bmatrix} (2-v-w)+(1-v)z+z^2 & (3-v-w)+(1-v)z+z^2 \\ 0 & 0 \end{bmatrix}.$$

So $A(z)^{-1}B(z)$ represents all minimal partial realizations with three free parameters. This means that for a different choice for u, v, w we get another

rational matrix function $A(z^{-1})B(z)$. The minimal number of states is three = rank of the block Hankel matrix $H_{3,3} = \sum_{i=1}^{2} \nu_i = \sum_{j=1}^{2} \mu_j$.

(b) Consider the scalar finite sequence {1,2,3}. So $N=3$ and $p=m=1$.

The 3×3 Hankel matrix contains all the necessary information.

$$H_{3,3} = \begin{matrix} \\ \times \\ \end{matrix} \begin{matrix} \times \\ \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & \\ 3 & & \end{bmatrix}$$

(1,1) is the pivot element.

$$H_{3,3}Q_1 = \begin{matrix} & \times & \times \\ \times \\ \times \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & \\ 3 & & \end{bmatrix}$$

(2,2) is the pivot element.

$$H_{3,3}Q_1Q_2 = \begin{matrix} & \times & \times & \bullet \\ \times \\ \times \\ \bullet \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & \\ 3 & & \end{bmatrix}$$

So the controllability and observability index is equal to $\mu_1 = \nu_1 = 2$.

In the second part of the algorithm we have to introduce a free parameter u :

$$H_{3,3}Q_1Q_2 = \begin{matrix} & \times & \times & \bullet \\ \times \\ \times \\ \bullet \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & \\ 3 & u & \end{bmatrix}$$

We make the (2,1) element zero.

$$H_{3,3}Q_1Q_2Q_3 = \begin{matrix} & \times & \times & \bullet \\ \times \\ \times \\ \bullet \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & \\ 3-2u & u & \end{bmatrix}$$

All minimal partial realizations of {1,2,3} are given by :

$$\frac{z+(2-u)}{z^2-uz+(2u-3)} = 1z^{-1}+2z^{-2}+3z^{-3}+(-u+6)z^{-4}+(-u^2+9)z^{-5}+\cdots .$$

REFERENCES :

[1] S. Beghelli, R. Guidorzi : Computational and theoretical aspects in the transformations to a canonical input-output form for multivariable systems. Report nr. 30 - October 1975 (Istituto di automatica, Facoltà di Ingegneria - Università di Bologna).

[2] S. Beghelli, R. Guidorzi : Computational and theoretical aspects in the realization of linear multivariable systems. Report nr. 31 - November 1975

(Istituto di automatica, Facoltà di Ingegneria - Università di Bologna).

[3] S. Beghelli, R. Guidorzi : A new realization algorithm for linear multivariable systems. Ricerche di Automatica 7, 164-181 (1976 no. 2-3).

[4] S. Beghelli, R. Guidorzi : A new input-output canonical form for multivariable systems. IEEE Transactions on automatic control AC-21, 692-696 (1976). IEEE Transactions on automatic control AC-23, 93 (1978) (Comments).

[5] Thomas Kailath : Linear Systems. Prentice Hall, Englewood Cliffs, N.J. 1980.

[6] Marc Van Barel, Adhemar Bultheel : A minimal partial realization algorithm for MIMO systems. TW report nr. 79, Dept. of Computer Science, K.U.Leuven. Revised April 1987.

[7] Marc Van Barel, Adhemar Bultheel : An algebraic method to solve the minimal partial realization problem for scalar sequences. To appear in Linear Algebra and Its Applications.

[8] Marc Van Barel, Adhemar Bultheel : An algebraic method to solve the minimal partial realization problem for matrix sequences. TW report nr. 92, Dept. of Computer Science, K.U.Leuven. August, 1987.