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Round-Robin Tournaments Generated by the Circle Method have Maximum Carry-Over *

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Abstract

The Circle Method is widely used in the field of sport scheduling to generate schedules for round-robin tournaments. If in such a tournament, team A played team B in its previous match and is now playing team C, team C is said to receive a carry-over effect from team B. The so-called carry-over effect value is a number that can be associated to each round-robin schedule; it represents a degree of unbalancedness of the schedule with respect to carry-over.

Here, we prove that, for an even number of teams, the Circle Method generates a schedule with maximum carry-over effect value, answering an open question.

1 Introduction

In a round-robin tournament, each pair of teams (or players) meets an equal number of times; the resulting matches are distributed over rounds such that each team plays at most a single match in each round. Organizers of round-robin tournaments face the problem of generating a schedule, i.e., to decide which match takes place in which round. Graph theory is closely connected to this problem: by having a node for each team, a match can be seen as a pair of nodes, and a round can be seen as a a matching (see De Werra [4, 5]). Then, the schedule boils down to a sequence of matchings, thereby covering the edge set of the resulting complete graph.

In 1847, Reverend T. Kirkman [12] published a method that can be used for constructing a schedule for single round-robin competitions. This method is known as the Circle Method (aka the polygon method, or the canonical procedure), and its outcome

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has been referred to as a Kirkman tournament, or the circle design. When viewed in graph-theoretical terms, the Circle Method partitions the edge set of K_n (n even) into $n-1$ specific perfect matchings and arranges these matchings in a certain cyclical order (see Sect. 2.3 for a precise description).

As there are many different issues that can be taken into account when designing a tournament, various ways of generating a schedule exist, each resulting in a schedule with different properties. We refer to Anderson [1], Froncek [3], Januario et al. [9], and Kendall et al. [10] for introductions and (recent) overviews. It is fair to say, however, that the Circle Method is a very popular (if not the most popular) method when it comes to generating schedules for round-robin competitions. Indeed, the use of the Circle Method is well spread through different sports leagues around the world; for instance, Griggs and Rosa [8], followed by Goossens and Spieksma [6], documented extensively the use of the Circle Method throughout soccer leagues in Europe.

The following phenomenon is relevant in any round-robin tournament. Imagine that your team is facing some other team in an upcoming match; we will argue that the opponent of this other team in the previous match is relevant for the upcoming match. Indeed, if the team you're about to face has experienced a heavy loss in its previous match, the team may have a low morale, or be discouraged, and hence perhaps easier to beat. Then your team is receiving a so-called *carry-over effect* from the previous opponent of the team your team is about to face. Of course, the opposite is possible as well: strengthened by having beaten a weak opponent in their previous match, the team your team is about to face is full of morale, and perhaps more difficult to beat.

Thus, in each round of the competition your team receives a carry-over effect from the team that your opponent played against in its previous match (the rounds are viewed cyclically, i.e., in round 1 your team receives a carry-over effect from the team your opponent plays in the last round, see Sect. 2): we can investigate the set of teams from which your team receives a carry-over effect throughout the competition.

In one extreme case, this set of teams consists of all other teams. Then, in a single round-robin tournament, each other team gives once a carry-over effect to your team. Schedules that satisfy this property are called *balanced*, see Russel [14]. A balanced schedule need not exist; Russel [14] shows that balanced schedules exist when the number of participating teams is a power of 2; Anderson [2] exhibits balanced schedules when the number of participating teams equals 20 or 22.

In another extreme case, only very few teams give carry-over effects to your team. This gives rise to schedules that can be perceived as unbalanced or even unfair. Indeed, different cases have been reported where carry-over effects were blamed for distorting the outcome of the competition; we refer to Goossens and Spieksma [7] who describe a case in the 2007 edition of the Norwegian soccer league (Tippeligean), and a case in the 2006-2007 edition of the Belgian soccer league (ProLeague). Thus, measuring the degree of 'unbalancedness' of a schedule is relevant. Russel [14] proposed to do this by considering the square of the deviations from a balanced schedule; this value is called the carry-over effect value (see Sect. 2.2 for a definition).

The Circle Method is known for generating unbalanced schedules; in fact, Miyashiro and Matsui [13] conjecture that the Circle Method generates a schedule with maximum carry-over effect value. Our contribution here is to prove that this conjecture is correct. Even more, we show that any schedule with a maximum carry-over effect value can be generated by the Circle Method.

The paper is organized as follows. In Sect. 2 we introduce our terminology and we state our result (Theorem 2.1), Sect. 3 formulates the building blocks of our proof, Sect. 4

	1	2	3	4	5	6	7
1	2	6	5	3	8	4	7
2	1	3	7	6	4	8	5
3	6	2	4	1	7	5	8
4	8	7	3	5	2	1	6
5	7	8	1	4	6	3	2
6	3	1	8	2	5	7	4
7	5	4	2	8	3	6	1
8	4	5	6	7	1	2	3

	1	2	3	4	5	6	7
1	8	3	5	7	2	4	6
2	7	8	4	6	1	3	5
3	6	1	8	5	7	2	4
4	5	7	2	8	6	1	3
5	4	6	1	3	8	7	2
6	3	5	7	2	4	8	1
7	2	4	6	1	3	5	8
8	1	2	3	4	5	6	7

Figure 1: Two distinct schedules for $n = 8$ teams.

finalizes the proof.

2 Terminology

This section introduces terminology concerning schedules (Sect. 2.1), explains the value of the carry-over effect of a schedule (Sect. 2.2), and describes the Circle Method (Sect. 2.3).

2.1 About Schedules

Let n denote the number of teams participating in a single round-robin tournament (SRR). Throughout this paper, we assume that n is even, and that $n \geq 6$ (since the cases where $n \in \{2,4\}$ are easy to analyze). We use N to denote the set of teams: $N = \{1,2,\ldots,n\}$. We exclusively focus on so-called *compact* schedules, meaning that there are $n-1$ *rounds* in an SRR; each round consists of $\frac{n}{2}$ matches (of course, a match consists of a pair of two distinct teams). A *schedule* for an SRR specifies, for each of the $n-1$ rounds, which pairs of teams are involved in the matches.

Definition 2.1. A schedule is called *feasible* if:

(i) in each round, each team is in one match, and

(ii) after all rounds, each pair of teams has been in a match.

A schedule can be represented in the form of a table. The two tables depicted in Fig. 1 each represent a possible schedule for $n = 8$ teams. The opponent of team $i \in N$ in round r can be found on the i-th row and the r-th column ($1 \leq r \leq n-1$).

2.2 About the Carry-Over Effect

Consider the schedule represented in Fig. 1 on the left. In round 1, team 1 plays team 2, and in round 2, team 1 plays team 6. Thus, team 2 gives a carry-over effect (coe) to team 6 using team 1 as a carrier. Indeed, any pair of consecutive numbers on a row in a schedule indicates a coe. More generally, the opponent of one's opponent in the previous round is the originator of an effect that is passed to one's team. To capture this effect, we use the following definition.

Definition 2.2. [14] Given a feasible schedule, we say that team $i \in N$ *gives a carry-over effect (coe) to team $j \in N$ in round r*, if there exists a team $k \in N$ that plays team

i in round $r-1$, and plays team j in round r, $1 \leq r \leq n-1$. We also say that team j *receives a coe from team i in round r*.

It is important to realize that we view a schedule cyclically: in round 1, each team receives a coe coming from a match in round $n-1$; and in round $n-1$, each team gives a coe to some team playing in round 1. This is motivated by observing that, often, in practice, a double round robin schedule is found by repeating a single round robin schedule. Thus, when dealing with rounds, we compute modulo $n-1$. Indeed, we use freely the phrase $r-1$ or $r+1$ with $r \in \{1, 2, \ldots, n-1\}$ (as we did in Definition 2.2); clearly, if $r = 1$, then $r-1 = n-1$, and if $r = n-1$, then $r+1 = 1$. Concluding: in each round, a team gives a coe to some team, and receives a coe from some team.

We associate to each schedule a matrix, called the *carry-over effect matrix* (the COE matrix).

Definition 2.3. [14] The COE matrix is an $n \times n$ matrix with entries $c_{i,j}$, that represent the number of times that team i gives a coe to team j in a given schedule, $i, j \in N$. The *carry-over effect value* (the COE value) of a feasible schedule is defined as $\sum\limits_{i \in N} \sum\limits_{j \in N} c_{i,j}^2$.

It will be convenient to consider a team's contribution to the COE value. We define:

Definition 2.4. The *contribution* of a team $i \in N$, denoted by $Co(i)$, to the COE value is defined as $Co(i) = \sum\limits_{j \in N} c_{i,j}^2$.

Observe that the COE value of a given schedule equals the sum of the contributions of the teams. The COE matrices, and their COE values, corresponding to the two schedules in Fig. 1, are given in Fig. 2, with the zero entries left blank.

$$\begin{bmatrix} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & \end{bmatrix} \qquad \begin{bmatrix} & 1 & 5 & & & & & 1 \\ & & 1 & 5 & & & & 1 \\ & & & 1 & 5 & & & 1 \\ & & & & 1 & 5 & & 1 \\ & & & & & 1 & 5 & 1 \\ 5 & & & & & & 1 & 1 \\ 1 & 5 & & & & & & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & \end{bmatrix}$$

A COE matrix with COE value 56. A COE matrix with COE value 196.

Figure 2: Corresponding COE matrices.

As mentioned in Sect. 1, a schedule is called balanced if each team receives a coe from each other team exactly once. The corresponding COE matrix has all entries equal to 1, except for zero entries on the main diagonal. We see in Fig. 2 that the first schedule is balanced.

2.3 About the Circle Method

The Circle Method is a method for constructing a feasible schedule for an SRR with any (even) number of teams. An intuitive description is as follows. Select a team, say team n, and place it in the center of a circle. All other teams are placed on the circle. In round 1, the team in the center plays team 1. The neighbors of team 1 play each other, and in fact, their neighbors also play each other. This is repeated until all teams are matched up, and we have constructed the first round. To construct the next round, we “rotate” the matches, that is, team n plays team 2, the neighbors of team 2 play each other, and so on. This is illustrated in Fig. 3 for the first three rounds; the resulting schedule is represented in the right schedule in Fig. 1.

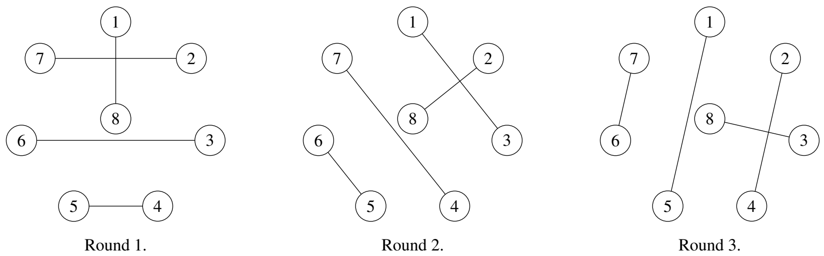

Figure 3: Visual representation of a schedule constructed with the circle method.

A precise description of the Circle Method is as follows. For each round $r \in \{1,\dots,n-1\}$ we have,

- team n plays team r,
- for $i, j \in N \setminus \{r,n\}$: team i plays team j if $i+j \equiv 2r \bmod (n-1)$.

Of course, permuting the teams, and next applying the Circle Method gives other schedules; we will refer to the class of schedules that can be obtained by applying the Circle Method to some permutation of the teams, as the class $\mathscr{C}$.

The COE matrix corresponding to a schedule found by the Circle Method has a specific structure. Consider for example Fig. 2 on the right for the COE matrix where $n = 8$. We use this structure to find an explicit expression for the COE value of a schedule found by the Circle Method; this value is denoted by $CMCOE(n)$ (where n refers to the number of teams), see also Kidd [11].

Observation 2.1.

$$CMCOE(n) = n^3 - 7n^2 + 18n - 12 \text{ , for each even } n \geq 6.$$

Proof. In a schedule found by the Circle Method, teams t and $t+1$ play team n in succession, implying that team t gives a coe to team $t+1$ through team n (for each $t \in N \setminus \{n\}$ where, if $t = n-1, t+1 = 1$). This explains that the diagonal above the main diagonal in the COE matrix consists of 1’s. Further, in each round t ($1 \leq t \leq n-1$), team n (playing team t) receives a coe from the opponent of team t in round $t-1$ (where, if $t = 1$, then $t-1 = n-1$). That is team j such that

$$(t+j) \equiv 2(t-1) \bmod (n-1).$$

This means that $j = (t-2) \bmod (n-1)$, from which we deduce that each team $j \in N \setminus \{n\}$ gives a coe to team n once. This translates into a value of 1 for each entry in the last column. It follows, from an analogous argument, that team n gives a coe to each other team exactly once, resulting in the entries of last row of the COE matrix being equal to 1.

Finally, consider some team $i \in N \setminus \{n\}$, and observe that there are $n-3$ rounds during which the current and next opponent of team i are *both* not team n. Consider one such round, say round r, $1 \leq r \leq n-1$. In round r, the opponent of this team i is $(2r-i)$ mod $(n-1)$, and in the next round the opponent is: $(2r+2-i)$ mod $(n-1)$. In other words, if team j is the current opponent of team i, then its next opponent is team $(j+2)$ mod $(n-1)$. This holds for each team $i \in N \setminus \{n\}$, and for $n-3$ rounds, which explains the $n-3$ values in the diagonal that starts with entry $(1,3)$ in the COE matrix.

The COE value of the COE matrix corresponding to a schedule found by the Circle Method (denoted by $CMCOE(n)$) equals,

$$(n-1)\left((n-3)^2+1+1\right)+(n-1).$$

□ □

We are now well-placed to formulate our main result.

Theorem 2.1. *The Circle Method generates a feasible schedule with maximum COE value.*

3 Building Blocks of the Proof

This section identifies some basic observations in Sect. 3.1, introduces two concepts in Sect. 3.2, characterizes the class of schedules that can be generated by the Circle Method in Sect. 3.3, and proves upper bounds on the contribution of teams with particular properties in Sect. 3.4.

3.1 Basic Observations

The following statements hold for any feasible schedule.

Lemma 3.1. In a feasible schedule

(i) each team gives and receives exactly $n-1$ times a coe;

(ii) a team cannot give, nor receive, a coe to, or from, themselves;

(iii) each team gives a coe to at least three different teams, and each team receives a coe from at least three different teams; and

(iv) each team gives at most $n-3$ times a coe to a single team.

Proof. (i). By definition every team receives and gives a coe in every round. We also know that there are exactly $n-1$ rounds.

(ii). The only way to receive a coe from yourself is when the previous opponent of your current opponent is you, i.e., when two identical duels are played consecutively. That clearly cannot happen in a single round-robin tournament with $n \geq 6$.

(iii). First observe that each team gives a coe to at least two different teams. Consider a team $a \in N$, it is impossible for team a to give a coe to team $b \in N$ in all $n-1$ rounds, because when team a plays team b, team a cannot give a coe to team b.

Suppose now that team a gives a coe only to two distinct teams, teams b and c; we will argue that this leads to a contradiction. Let round r be the round in which team a plays against team b.

Then team a has to give a coe to team c in round r. This implies that team b plays against team c in round $r+1$. Then, team a gives a coe to the opponent of team b in round $r+1$; this opponent of team b in round $r+1$ is, by definition team c. Let round r' ($r' \neq r$) be the round in which team a plays team c (see also Table 1). Then, team a has to give a coe to team b in round $r'+1$. But this means that team b plays team c not only in round $r+1$, but also in round $r'+1$. Clearly, this schedule is not feasible and hence there must exist at least three different teams to which team a gives a coe.

Table 1: Team a cannot give coe to only team b and c.

	r	$r+1$	r'	$r'+1$
a	b		c	
b	a	c		c
c		b	a	b

(iv). This follows directly from statement (i) and (iii). □ □

These statements allow us to determine that the maximum contribution to the COE value of a single team equals $(n-3)^2+1+1$, corresponding to a situation where a team gives $n-3$ times a coe to a single team, and once to two other teams. Clearly, this is the only possible situation where a team gives $n-3$ times a coe to another team. Thus, the second highest possible contribution of a team is $(n-4)^2+2^2+1$, corresponding to giving $n-4$ times a coe to a team, twice to another team, and once to yet another team.

Definition 3.1. Given a feasible schedule, team $i \in N$ is called a *maximally giving* team (an mg team) if there exists a team j receiving $n-3$ times a coe from team i, i.e., if there exists a j with $c_{i,j} = n-3$. Alternatively, we call such a team j a *maximally receiving* team.

When considering the COE matrix on the right in Fig. 2, we see that seven out of the eight teams are mg teams. It is natural to wonder whether it is possible to have a large COE value when there are no mg teams. We can find the following lower bound on the number of mg teams that need to be present in a feasible schedule with a COE value at least $CMCOE(n)$.

Lemma 3.2. Any feasible schedule with a COE value greater or equal than $CMCOE(n)$, has at least $\lceil \frac{n}{2} + \frac{n-6}{n-5} \rceil$ mg teams.

Proof. Consider a feasible schedule with a mg teams ($0 \leq a \leq n$). The remaining $n-a$ teams have a contribution that is bounded by the second highest possible contribution: $(n-4)^2+5$. Thus, the COE value is bounded by

$$a((n-3)^2+2)+(n-a)((n-4)^2+5) = n^3-8n^2+(21+2a)n-10a. \quad (1)$$

If we demand that this value exceeds $CMCOE(n)$ (see Observation 2.1), we find a lowerbound for a:

$$a \geq \frac{(n^2-3n-12)}{2(n-5)} = \frac{n}{2} + \frac{n-6}{n-5}, \quad (2)$$

from which the result follows. □ □

Thus, the COE value of a schedule is bounded from above by that of the Circle Method if no more than half of the teams are maximally giving teams.

3.2 Basic Concepts: k-Chains and Bridge Teams

The existence of an mg team in a schedule has quite some impact on the structure of that schedule. Let us investigate this structure in more detail. Suppose, as an example, that team 1 gives maximally to team 2. Thus, there are $n-3$ rounds during which team 1 gives a coe to team 2. Let us further assume team 1 plays team 2 in round 2. It is then clear that in this round, team 2 does not receive a coe from team 1. Also, in the next round, team 2 does not receive a coe from team 1. But this means that in each of the remaining $n-3$ rounds, team 2 has to receive a coe from team 1. Thus, the opponent of team 1 in round r, and the opponent of team 2 in round $r+1$ are the same team for each round $r \in \{3,4,\dots,n-1\}$. This means that the schedule has the form that is depicted in Table 2 where we use letters as names for the other teams. Notice

Table 2: Partial schedule in case team 1 gives maximally to team 2.

	1	2	3	4	5	6	...	$n-2$	$n-1$
1		2	a	b	c	d	...	y	z
2	z	1		a	b	c	...	x	y

that both teams 1 and 2 still have only a single empty round; in fact, it is easily seen that the opponent of team 1 in round 1 must be the same team as the opponent of team 2 in round 3. As we have the freedom to reindex the remaining teams, let us say that the empty round is taken by team n, and let us number the other teams by considering the round in which such a team plays team 1. This gives rise to Table 3. We emphasize

Table 3: The structure when a team 1 is an mg team giving maximally to team 2.

	1	2	3	4	5	6	...	$n-2$	$n-1$
1	n	2	3	4	5	6	...	$n-2$	$n-1$
2	$n-1$	1	n	3	4	5	...	$n-3$	$n-2$

that the only crucial assumption we made for constructing Table 3 is the assumption that team 1 gives maximally to team 2.

Let us further elaborate on this example by considering what happens if team 2 is an mg team as well, giving maximally to some other team. Notice that, given the structure displayed in Table 3, there can be only one specific team to which team 2 gives maximally, namely team 3 (the team that plays team 1 in round 3). We leave verification of this claim to the reader.

The resulting implications are described in Table 4; notice the special role of team n. We will come back to the role of team n extensively. We now define the concept of

Table 4: The structure when team 1 is an mg team giving maximally to team 2, giving maximally to team 3.

	1	2	3	4	5	6	...	$n-2$	$n-1$
1	n	2	3	4	5	6	...	$n-2$	$n-1$
2	$n-1$	1	n	3	4	5	...	$n-3$	$n-2$
3	$n-2$	$n-1$	1	2	n	4	...	$n-4$	$n-3$

a k-chain, allowing us to capture the essence of the example discussed in the preceding paragraphs.

Definition 3.2. Given a feasible schedule, we define a *k-chain* as a list consisting of $k+1$ teams, where the first k teams are mg teams, giving maximally to the next team in the list. A k-chain in which the first and the last team are the same, is called a *closed k-chain*, otherwise the k-chain is called *open*.

The value k of a k-chain represents the number of maximally giving teams in the chain, and is called the *length* of the k-chain. For instance, the partial schedule depicted in Table 4 exhibits a 2-chain.

Definition 3.3. Suppose that, in some feasible schedule, team $i \in N$ is an mg team, giving maximally to team $j \in N$. Suppose further that team i plays team j in round r. The opponent of team i in round $r-1$ is called the *bridge team* for team i.

For instance, in the partial schedule depicted in Table 4, team n is the bridge team for team 1, and team n is also the bridge team for team 2. Notice that when investigating a k-chain, we can freely re-index the teams. We use this freedom to choose team $n \in N$ as the bridge team for the first team in the k-chain.

Lemma 3.3. Given is a feasible schedule containing a k-chain. Let team $i \in N$ be an mg team in the k-chain, giving maximally to team $j \in N$, and let team $n \in N$ be the bridge team for team i.

(i) If team i plays team j in round r, then team j plays team n in round $r+1$.

(ii) Team j receives exactly one coe from team n.

Proof. (i). Since team i is an mg team giving maximally to team j, it must give $n-3$ times a coe to team j. Observe that team i does not give a coe to team j in round r (since, by Definition 3.3, team i plays team n in round $r-1$, and team j does not play team n in round r). Observe also that team i does not give a coe to team j in round $r+1$ (since teams i and j play against each other in round r). Hence, in each other round r' ($r' \neq r, r' \neq r+1$), team i gives a coe to team j, or, in other words, the opponent of team j in round r' ($r' \neq r, r' \neq r+1$) is the opponent of team i in round $r'-1$. Thus, the only possible round for the match j versus n is round $r+1$.

(ii). We know team i is an mg team, giving maximally to team j. Let the match i versus j be played in round r. It follows from Definition 3.3 that the match i versus n takes place in round $r-1$. Hence team n gives a single coe to team j in round r. Since team i gives maximally to team j, team j receives $n-3$ times a coe from team i in rounds $r+2, r+3, \ldots, n, 1, \ldots, r-1$. Thus, the only remaining round in which team j could potentially receive a coe from team n is round $r+1$. However, as proven in (i), teams n and j play each other in that round. It follows that team j receives a single coe from team n. □ □

In Table 4, team n is the bridge team for both mg-teams in the 2-chain. This is not a coincidence.

Lemma 3.4. Consider a feasible schedule containing a k-chain. There is a unique team that is a bridge team for each mg team in that k-chain.

Proof. We prove this statement by induction on the number of teams in the k-chain. Without loss of generality, we assume that the k-chain consists of teams $1, \ldots, k+1$ (with teams $1, \ldots, k$ being mg teams), and with team n being the bridge team for team 1. The statement is true by definition for a 1-chain. Let us now assume that the statement holds for the first $t-1$ teams ($2 \leq t \leq k$), i.e. team n is a bridge team for teams 1, ..., $t-1$. We now show that team n is also a bridge team for team t that gives maximally to team $t+1$.

First we show by contradiction that teams $t-1$ and $t+1$ cannot be the same teams, i.e. suppose that team $t-1$ gives maximally to team t and team t gives maximally to team $t-1$. Let round r be the round in which team $t-1$ plays team t. The opponent of team t in round $r-1$ is some team, say team $n-1$; Definition 3.3 implies that team $n-1$ is the bridge team for team t. Then, according to Lemma 3.3(i), team $t-1$ plays team $n-1$ in round $r+1$. However, team $t-1$ giving maximally to team t implies that team $t-1$ gives a coe to team t in round $r-1$; this implies that team $t-1$ plays team $n-1$ in round $r-2$. These two implications cannot be reconciled, since $n \geq 6$, and it follows that team $t-1$ cannot be team $t+1$.

Hence we know that teams $t-1$ and $t+1$ are different teams. Next, suppose that team t plays team $t+1$ in round r', $1 \leq r' \leq n-1$. Since team $t-1$ gives maximally to team t, we know that team $t-1$ plays team $t+1$ in round $r'-1$. Further, suppose that team $t-1$ plays team t in round r. Again, since team t gives maximally to team $t+1$, we know that team $t-1$ plays team $t+1$ in round $r+1$, see Table 5. It follows that $r+1 = r'-1$.

Table 5: Team $t-1$ and team t are mg teams.

	r	$r+1$	$r'-1$	r'
$t-1$	t	$t+1$	$t+1$	
t	$t-1$			$t+1$
$t+1$		$t-1$	$t-1$	t

Thus, we now know that team t plays team $t+1$ in round $r' = r+2$. The induction hypothesis tells us that team n is a bridge team for team $t-1$, and hence, Lemma 3.3(i) tells us that the match t versus team n is played in round $r+1$. Hence, by Definition 3.3, team n is a bridge team for team t. □ □

We have collected now quite a bit of information concerning a k-chain, and its impact on a feasible schedule. More specifically, consider the first team in a k-chain, team 1, and the round where team 1 plays bridge team n. In each next round, team 1 plays the next team from the k-chain. Each team i that is not in the k-chain, and that is not a bridge team, forms "diagonals" starting in row 1 and round i, thereby completely specifying the schedule for teams in the k-chain. We record this information in the following property:

Property 3.1. Given a feasible schedule containing a k-chain $\{1,2,\ldots,k+1\}$, each team $i \in \{1,2,\ldots,k+1\}$ plays bridge team n in round $(2i-1) \bmod (n-1)$; further, the opponent of team i in each other round r is team $(r-i+1) \bmod (n-1)$.

Lemma 3.5. In any feasible schedule, the only closed k-chains that can occur are those of length $n-1$.

Proof. Let us consider a k-chain, with its last team being team $k+1$. We have a closed $(k+1)$-chain, if team $k+1$ gives maximally to team 1. By Lemma 3.4, we know that team n will be the bridge player for each of the teams in the k-chain. When team $k+1$ plays team 1 or team n it does not give a coe to team 1. Hence team $k+1$ has to give a coe to team 1 in every other round. Consider a round r; in this round, team 1 plays team r (Property 3.1). In order for team $k+1$ to give a coe to team 1, it has to play team r in round $r-1$. In round $r-1$ we know from Property 3.1 that team $k+1$ plays team $(r-1-k) \bmod (n-1)$, which is equal to r only if $k=n-2$.

Hence team $k+1$ can only give maximally to team 1 if $k=n-2$, which implies that we have a closed $(n-1)$-chain. □ □

3.3 Characterizing the class $\mathscr{C}$

In this section, we characterize when a schedule is in the class $\mathscr{C}$, i.e., when a schedule can be constructed by the Circle Method. Also, we give two properties, and we show that a schedule having one of these properties is in the class $\mathscr{C}$.

Lemma 3.6. A schedule can be generated with the Circle Method (i.e., is in the class $\mathscr{C}$) if and only if it contains a closed $(n-1)$-chain.

Proof. To show that applying the Circle Method leads to a schedule containing a closed $(n-1)$-chain, we refer to the discussion in Sect. 2.3 prior to Observation 2.1.

Consider a feasible schedule containing a closed $(n-1)$-chain. Let us denote the bridge team in this schedule by team n. Property 3.1 tells us that each team $t \in \{1,\dots,n-1\}$ plays the bridge team n in round $(2t-1) \bmod (n-1)$, and in each other round r, the opponent of team t is team $(r-t+1) \bmod (n-1)$. In other words, teams $i,j \in N \setminus \{n\}$ play in round $(i+j-1) \bmod (n-1)$.

To show that this schedule can be generated by the Circle Method, let us rename the teams as follows. Team n remains team n, each other team t will become team $(2t-1) \bmod (n-1)$. Notice that $\{(2j+2) \bmod (n-1) \ : j \in \{1,\dots,n-1\}\} = \{1,\dots,n-1\}$. For each round $(2r-1) \bmod (n-1)$, $r \in \{1,\dots,n-1\}$ we have,

- Team n plays team $(2r-1) \bmod (n-1)$,
- Team $(2i-1) \bmod (n-1)$ plays team $(2j-1) \bmod (n-1)$ if $2(i+j-1) \equiv 2(2r-1) \bmod (n-1)$, for $i,j \in N \setminus \{n\}$.

This satisfies the description of the Circle Method. □ □

It is interesting to observe that if the schedule contains a k-chain that is "long enough", feasibility of the schedule allows us to argue that the schedule must, in fact, contain a closed $(n-1)$-chain, and hence be a schedule in the class $\mathscr{C}$.

As example we give in Table 6 a partial schedule where the number of teams equals $n=10$, and where we have a 3-chain consisting of teams 1, 2, 3, and 4 with bridge team 10. A crucial question is to find a round for the match team 5 versus team 9. Careful analysis shows that there is only one round available for this particular match, namely round 4. Repeating this analysis allows us to fix all matches of team 5, and next to fix all remaining matches. We can argue this in a more general fashion.

Lemma 3.7. A feasible schedule that contains a k-chain of length at least $\frac{n}{2}-2$ is a schedule that can be generated by the Circle Method.

Proof. We claim that a schedule containing a k-chain with $\frac{n}{2}-2 \le k \le n-3$, also contains a $(k+1)$-chain. When starting out with a schedule containing a $(\frac{n}{2}-2)$-chain, and applying this claim $\frac{n}{2}$ times, we get a $(n-2)$-chain; and moreover, we find a schedule for team $n-1$. It will follow that team $n-1$ gives maximally to team 1, i.e. we have a closed $(n-1)$-chain. Together with Lemma 3.6 this gives us that a schedule

Table 6: Partial schedule of a $(\frac{n}{2}-2)$-chain for $n = 10$

	1	2	3	4	5	6	7	8	9
1	10	2	3	4	5	6	7	8	9
2	9	1	10	3	4	5	6	7	8
3	8	9	1	2	10	4	5	6	7
4	7	8	9	1	2	3	10	5	6
5				□	1	2	3	4	
6						1	2	3	4
7	4						1	2	3
8	3	4						1	2
9	2	3	4	□					1
10	1		2		3		4		

with a chain of length at least $\frac{n}{2}-2$ is a schedule that can be generated by the Circle Method.

Let us first prove the claim. Consider a k-chain, with $\frac{n}{2}-2 \leq k \leq n-3$. Recall that the schedule of teams 1 up to $k+1$ is specified as follows (Property 3.1). Every team $t \in \{1,\ldots,k+1\}$ plays against the bridge team in round $(2t-1) \bmod (n-1)$, and for each other round r, the opponent of team t is team $(r-t+1) \bmod (n-1)$. Hence, when $r = t-1$, team t plays team $n-1$ in rounds $(t-1) \bmod (n-1)$ for $t \in \{1,\ldots,k+1\}$.

Now we can derive the schedule of team $k+2 \leq n-1$, and show that team $k+2$ receives maximally from team $k+1$. Clearly, team $k+2$ plays team $t \in \{1,\ldots,k+1\}$ in round $((k+2)+t-1) \bmod (n-1)$. Further, team $n-1$ plays team $t \in \{1,\ldots,k+1\}$ in rounds $(t-1) \bmod (n-1)$ respectively. Since $k \geq \frac{n}{2}-2$, there is only one round available where team $k+2$ can play team $n-1$: round $k+1$. Likewise, team $n-2$ plays team 1 up to team $k+1$ in rounds $n-2$ up to $k-1$ respectively. Again, with $k \geq \frac{n}{2}-2$ this implies that team $k+2$ and team $n-2$ can only play in round k. Repeating this argument for teams $n-3$ down to and including team $k+3$ leaves us with a single round that is available where team $k+2$ can play team n. This round is the round directly after the round where team $k+2$ plays team $k+1$, namely round $2k+3$.

Thus, to summarize the schedule of team $k+2$, in round $(2(k+2)-1) \bmod (n-1)$ it plays team n, and in every other round r it plays team $(r-(k+2)+2) \bmod (n-1)$. It follows that team $k+1$ gives maximally to team $k+2$, and we have found a $(k+1)$-chain.

We have shown that the schedule must contain an $(n-2)$-chain. Suppose we have a schedule with an $(n-2)$-chain. Let us now prove that team $n-1$ gives maximally to team 1. We know that team $n-1$ plays team t in round $t-1 \bmod (n-1)$ for $t \in \{2,\ldots,n-2\}$. Recall that team 1 plays team t in round $t \bmod (n-1)$ for $t \in \{2,\ldots,n-2\}$. So indeed team $n-1$ gives maximally to team 1.

□ □

We can also show that when a team is a bridge team for "enough" teams, feasibility of the schedule allows us to argue that the schedule must contain a k-chain of length at least $\frac{n}{2}-2$, and hence be a schedule in the class $\mathscr{C}$.

Lemma 3.8. If, in a given schedule, a team is a bridge team for at least $n-3$ other teams, then that schedule can be generated by the Circle Method.

Proof. Suppose we have a schedule in which there exists a team that is a bridge team for at least $n-3$ other teams.

If the schedule contains a closed k-chain, then Lemmas 3.5 and 3.6 imply that the schedule can be generated by the Circle Method.

Thus, let us assume that each k-chain is open. It follows that each k-chain contains one maximally receiving, but not maximally giving team (the last team in the k-chain). Since N apparently contains at least one bridge team and $n-3$ mg teams, there are only two teams remaining that could be the last team of a k-chain. This means that $n-3$ mg teams have to be divided over at most two k-chains. The pigeonhole principle gives us that there is at least one k-chain of length at least $\frac{n}{2}-2$. Lemma 3.7 tells us that this is a schedule that can be generated by the Circle Method. □ □

3.4 Upper Bounds on the Contribution of Bridge Teams

We now prove two lemmas that specify an upper bound on the contribution of a bridge team to the COE value.

Lemma 3.9. Consider a feasible schedule. Let team $b \in N$ be a bridge team for ℓ distinct mg teams. Then, the contribution of team b is bounded by

$$Co(b) \leq (n-2-\ell)^2 + \ell + 1.$$

Proof. Let $MG \subseteq N$ denote the set of teams for which team b is a bridge team, and let $MR \subseteq N$ denote the corresponding set of maximally receiving teams. Clearly $|MG| = |MR| = \ell$.

Recall that Lemma 3.3(ii) states that a bridge team b gives exactly one coe to each team in MR. Since $|MR| = \ell$, it follows immediately that no team can receive more than $(n-1-\ell)$ times a coe from team b, and hence, $Co(b) \leq (n-1-\ell)^2 + \ell$. We will tighten this upper bound by showing that any team $a \notin MR \cup \{b\}$, cannot receive more than $n-2-\ell$ times a coe from team b. Observe that if no such team a exists, it follows that $MG = MR = N \setminus \{b\}$, and the schedule is a schedule generated by the Circle Method for which the bound applies.

$a \notin MG$. Let R be the set of rounds in which a team from MG plays its corresponding team from MR; clearly $|R| = \ell$. Notice that team a cannot receive a coe from bridge team b in each of the rounds in R (since, in such a round, bridge team b gives a coe to a team from MR). We now show that there is at least one more round in which team a cannot receive a coe from bridge team b.

Let r be the round where team b plays team a. Obviously, team a does not receive a coe from team b in round $r+1$. Round $r+1$, however, is not in R, since otherwise team b plays a team from MG in round r, and we know that team b plays team a in round r. This implies that in this case there are at most $n-1-\ell-1$ rounds in which team a can receive a coe from team b.

$a \in MG$. Each time that team b gives a coe to team a, there is a team which they play consecutively. We refer to this team as a carrier of a coe. Since team b plays $n-1$ different teams, there are $n-1$ distinct teams that serve as a carrier exactly once. We will show that there exist $\ell+1$ teams that cannot act as carrier of a coe that team b gives to team a, and hence team a cannot receive more than $n-1-(\ell+1)$. Each time team b gives a coe to a team in MR a team in MG is its carrier. Thus, we know $|MG| = \ell$ teams that cannot act as carrier of a coe that team b gives to team a. We now show that there is at least one more team that cannot act as carrier of a coe that team b gives to team a

Since we know that $a \notin MR \cup \{b\}$, it follows that the case where $a \in MG$ implies that team a is the first team in an open k-chain. Team b is bridge for team a and hence team b is bridge for every team in that chain. Hence we know that all the teams in this k-chain are in MG and that $k \leq \ell$. As mentioned before, team b cannot give coe to team a with a team in MG as carrier. Note however that team $k+1$ is not in MG. Hence it is sufficient to show that team b does not give a coe to team a with team $k+1$ as carrier, i.e. team b doesn't play team $k+1$ the round before team a plays team $k+1$.

Let us, without loss of generality, set the k-chain equal to $\{1,2,\ldots,k+1\}$ (hence $a=1$) and bridge team b equal to team n. Property 3.1 gives us that team $a(=1)$ plays team $k+1$ in round $k+1$ and team $b(=n)$ plays team $k+1$ in round $r = 2k+1 \bmod (n-1)$. We distinguish two cases.

- If $k < \frac{n}{2} - 2$, we know that rounds $k+1$ and $(2k+1) \bmod (n-1)$ are not consecutive rounds. Hence, in this case, there are at most $n-\ell-2$ rounds where a bridge team b can give a coe to a team a.
- If $k \geq \frac{n}{2} - 2$, then Lemma 3.7 tells us that in fact we have a closed $n-1$-chain and the schedule can be generated by the Circle Method. That implies that $Co(b) = n-1 \leq (n-2-\ell)^2 + \ell + 1$, for $1 \leq \ell \leq n-1$.

□ □

In the special case where $\ell = 2$, we can improve the bound derived in the previous lemma. We prove the following statement.

Lemma 3.10. Consider a feasible schedule. Let team $b \in N$ be a bridge team for two distinct mg teams. Then, the contribution of team b is bounded by

$$Co(b) \leq (n-5)^2 + 6.$$

Proof. We distinguish two cases, one where the two mg teams form a 2-chain, and one where they do not.

Case 1: the two mg teams form a 2-chain. Let us call the two mg teams team 1 and team 2, and since they form a 2-chain, we know that team 1 gives maximally to team 2, and let us say that team 2 gives maximally to team 3. The corresponding partial schedule is depicted in Table 7.

We will show that the bridge team b does not give more than $n-5$ times a coe to any team in N. To show this, we will argue that there are at least four rounds for each team $i \in N \setminus \{b\}$ during which team i does not receive a coe from team b. We first consider teams 2 and 3, then team 1, team 4, and finally, any other team in $N \setminus \{1,2,3,4,b\}$.

Obviously, since teams 2 and 3 are maximally receiving teams (from teams 1 and 2 respectively), team b gives each of them only a single coe (Lemma 3.3(ii)). Since team 1 plays team b in round 1, it follows that team 1 does not receive a coe from team b in rounds 1 and 2. Also in round 3, team 1 does not receive a coe from the bridge team b, since team 1 receives a coe through team 3 in that round, while the bridge team b plays team 3 in round 5 (see Table 7). Finally, we can deduce from Table 7 that team 1 does not receive a coe from team b in rounds 4 and 6, implying that team b does not give more than $n-5$ times a coe to team 1.

Table 7: Partial schedule involving bridge team b, and two corresponding mg teams forming a 2-chain

	1	2	3	4	5	6
1	b	2	3	4		
2		1	b	3	4	
3			1	2	b	4
4				1	2	3
b	1		2		3	

Consider now team 4. Team 4 cannot receive a coe from team b in rounds 2, 4 and 5 (see Table 7). Let us consider the round in which team 4 plays bridge team b. If it is any other round other than round 2, 4 or 5, then we know that team 4 cannot receive a coe from team b in four distinct rounds, and hence receives at most $n-5$ times a coe from team b. Consider these rounds 2, 4 and 5. Team 4 does not play team b in round 4 or 5, but team 4 might play team b in round 2. In that case, however, team 4 does not receive a coe from team b in round 3, yielding again at least four rounds during which team 4 does not receive a coe from team b. Hence, team 4 receives at most $n-5$ times a coe from team b.

Finally, consider now a team $i \in N \setminus \{1,2,3,4,b\}$. This team plays teams 1, 2 and 3 in succession (since team 1 and team 2 are mg teams giving maximally to respectively team 2 and team 3). In these rounds, team i will receive a coe from the previous opponent of teams 1, 2 and 3. Of these three consecutive rounds, at most one can be equal to round 2, 4 or 6, because team i does not play team 1 in rounds 2, 3 or 4. Hence, there are 4 rounds in which team $i \in N \setminus \{1,2,3,4,b\}$ cannot receive a coe from team b, implying that it receives at most $n-5$ times a coe from team b. Since teams 2 and 3 each receive a single coe from team b, it follows that the contribution of team b is bounded by $(n-5)^2+1+1+2^2$.

Case 2: the two mg teams each form a 1-chain. Let teams 1 and x be the two distinct mg teams giving maximally to teams 2 and y respectively. Suppose that team 1 plays team 2 in round 2, and team x plays team y in round $r+1$. Notice that teams 2 and x are different teams, otherwise we are in Case 1. Also, since $x \neq 1$, and $y \neq 1$, it follows that rounds 2, $r+1$ and $r+3$ are distinct rounds. Let team z be the team that plays team x in round $r+2$ and team y in round $r+3$. The corresponding partial schedule is depicted below in Table 8.

We are going to argue that in any feasible schedule there is no team receiving more than $n-5$ times a coe from team b. We do this by first considering teams 2, 1 and 3, then we use symmetry for teams y, x, and z, and finally, we consider some team in $N \setminus \{1,2,3,x,y,z,b\}$.

As before, we know that bridge team b gives a single coe to team 2 in round 2, and a single coe to team y in round $r+1$. Notice that bridge team b also gives a single COE to team z in round $r+3$.

Consider team 1. Team 1 cannot receive a coe from team b in round 1, because team 1 plays team b in that round. Neither can team 1 receive a coe from team b in rounds 2, $r+1$ and $r+3$, because teams 2, y and z already do so. Hence, team b can give at most $n-5$ times a coe to team 1. Symmetry implies that team b can give at

Table 8: Partial schedule involving bridge team b, and two corresponding teams (1 and x) each forming a 1-chain

	1	2	3	4	$\cdots$	r	$r+1$	$r+2$	$r+3\cdots$
1	b	2	3						
2		1	b	3					
3			1	2					
x						b	y	z	
y							x	b	z
z								x	y
b	1		2			x		y	

most $n-5$ times a coe to team x.

Consider now team 3. Team 3 cannot receive a coe from team b in rounds 2 and $r+1$, because teams 2 and y already do so. Neither can team 3 receive a coe from team b in round 3, because team 3 plays team 1 in that round, while team b plays team 1 in round 1.

We now show that team 3 plays team x in a round unequal to 2, 3, $r+1$, while team b plays team x in round r. Hence there is a fourth round in which team 3 does not receive a coe from team b. Team x gives maximally to team y, hence team 3 plays team x and y consecutively. That implies that team 3 cannot play team x in round 2, because in round 3 it doesn't play team y. Team 3 cannot play team x in round 3, because it then plays team 1. Neither can team 3 play team x in round $r+1$, because then team x play team y. Hence, there is a fourth round unequal to 2, 3, $r+1$ in which team 3 does not receive a coe from team b. Similarly, we know that team b gives at most $n-5$ times a coe to team z.

For any other team not mentioned so far, i.e. for any team in $N \setminus \{1,2,3,x,y,z,b\}$, we know it cannot receive a coe from team b in rounds 2, 4, $r+1$ and $r+3$. We know these are distinct rounds because $2 \neq x$. Hence, this team can receive at most $n-5$ times a coe from team b.

We conclude that, also in this case, there is no feasible schedule where the bridge team gives more than $n-5$ times a coe to the same team. Moreover, since the bridge team gives exactly one coe to two distinct teams (team 2 and team y), the maximal contribution of the bridge team is given by $(n-5)^2+1+1+2^2$. □ □

4 Proving the Theorem

In this section, we 'assemble' the building blocks proven in Sect. 3, in order to prove Theorem 2.1. First, we deal with the case $n \in \{6,8\}$; next, we partition in Sect. 4.1 the set of teams N into different types of subsets. In Sect. 4.2, we show how to bound the average contribution of teams in a subset, culminating in the final proof.

Lemma 4.1. The Circle Method generates a feasible schedule with maximum COE value when $n \in \{6,8\}$.

Proof. Consider a feasible schedule with a COE value greater than or equal to $CMCO(6)$ ($CMCO(8)$). Lemma 3.2 implies that this schedule has at least 3 (5) mg teams. This means that, in case $n = 6$, the schedule must contain a 1-chain, and in case $n = 8$, the schedule must contain a 2-chain (indeed, a schedule for $n = 8$ featuring 5 mg teams without a 2-chain, has the 5 mg teams each giving maximally to 5 distinct other teams, a clear impossibility). Thus, the schedule satisfies the conditions of Lemma 3.7, and hence can be generated by the Circle Method. □ □

4.1 Identifying Subsets of Teams

In Sect. 3, we introduced two possible properties of a team: given a schedule, a team can be a maximally giving team, and a team can be a bridge team. Consider a team, say team $b \in N$, that is both maximally giving, as well as a bridge team. (One might wonder whether this is possible; however, there exist schedules containing mg bridge teams). This team has very specific properties as witnessed by the following lemma.

Lemma 4.2. Let team $b \in N$ be a maximally giving bridge team, giving maximally to team $a \in N$ and a bridge team for team 1. Then: (i) team 1, being an mg team, is a bridge team for team b and (ii) team a is neither an mg team, nor a bridge team.

Proof. Before proving the lemma, we first discuss the structure of a schedule in which team $b \in N$ is a maximally giving bridge team.

Since team b is a bridge team, there exist two teams, say teams 1 and 2, such that team 1 gives maximally to team 2 with team b as bridge team (see Table 9, where we placed, without loss of generality, the match between team 1 and team 2 in round 2). Also, since team b is an mg team, there exist two teams, say team a and p such that team b gives $n-3$ times a coe to team a with team p as bridge team (see Table 10, where we placed, without loss of generality, the match between team b and team a in round $r+1$).

Table 9: b being a bridge team.

	1	2	3
1	b	2	
2		1	b
b	1		2

Table 10: b being an mg team.

	r	$r+1$	$r+2$
b	p	a	
a		b	p
p	b	c	a

Team b gives coe's to three different teams, namely $n-3$ times a coe to team a, once to the bridge team p and once, through team p, to another team called team c (see also Table 10). Since team b is the bridge team for team 1, Lemma 3.3(ii) implies that team 2 receives a single coe from team b, see also Table 9. This means that team 2 is either team p or team c. We now argue, using contradiction, that team 2 is, in fact, identical to team c.

Suppose that team 2 equals team p, the bridge team for team b. Since the match between teams 2 and b is in round 3, the match between teams 2 and a is in round 5 (see Table 11). However, since team 2 receives $n-3$ times a coe from team 1, it follows that team 1 plays team a in round 4 (this is indicated by the box in Table 11). This gives a contradiction, since team a already plays team b in round 4 (and of course teams b and 1 are different teams). Hence, team 2 cannot be equal to team p, and must be equal to be team c, as illustrated in Table 12.

Table 11: Team 2 is team p.

	1	2	3	4	5
1	b	2		$\square$	
2		1	b		a
b	1		2	a	
a				b	2

Table 12: Team 2 is team c

	r	$r+1$	$r+2$
1			
2		p	
b	p	a	
a		b	p
p	b	2	a

Let us now prove the lemma.

(i) To prove that team 1 is the bridge team for team b, we need to show that team 1 is equal to team p. Team 2 receives $n-3$ times a coe from team 1, once from the bridge team b, and once from another team, through the bridge team b. Thus, team 2 receives a single coe from team b in round $r+1$, as shown in the last row in Table 12. This means two things: firstly, that team p is team 1, and secondly that $r=1$ (compare Table 9 and Table 12). Note that indeed team 1 is a maximally giving bridge team, as it is bridge for maximally giving bridge team b; proving one of the two claims (see Table 13).

Table 13: Implications

	1	2	3	4
1	b	2	a	
2		1	b	a
b	1	a	2	
a		b	1	2

(ii) We now prove our claim that team a is neither an mg team, nor a bridge team. We argue by contradiction.

First, suppose team a gives maximally. Then teams b and a are part of a k-chain ($k \geq 2$) with team 1 as a bridge team. Thus, there exist at least two teams for which team 1 acts as a bridge team. Then, Lemma 3.9 implies that team 1 cannot be an mg team. This is a contradiction, and hence team a is not an mg team.

Second, suppose team a is a bridge team. Then there exists a maximally giving team, say team x, for which team a is the bridge team. It follows that team a receives a single coe from team x. However, we already know that team a receives $n-3$ times a coe from team b, and once from both teams 1 and 2. Notice that team x can not be team 1, since team 1 already has team b as bridge. Suppose that team x is equal to team 2. Then team 2 is an mg team, and it follows that team 1 and team 2 are part of a 2-chain with team b as a bridge. But this implies (Lemma 3.9) that team b is not an mg team, which is a contradiction. Thus, team a is not a bridge team.

It follows that for every maximally giving bridge team there exists a unique team that is not a bridge team and does not give maximally.

□ □

The team pairs (b,a) and $(1,2)$, where teams b and 1 are mg teams as well as bridge teams, and where teams a and 2 receive maximally from teams b and 1 respectively, are of interest to us. We now define three different types of subsets of teams.

Definition 4.1. Given a feasible schedule, a type 1 subset is a pair of teams (b,a) where team b is an mg, bridge team, and team a receives maximally from team b.

We use $T_1 \subseteq N$ to denote the set of teams that are in sets of Type 1.

Teams that are maximally giving bridge teams are contained in sets of type 1. Let us now discuss the remaining bridge teams. Recall that for every bridge team $b \in N$, there is a set of mg teams for which team b is a bridge team.

Definition 4.2. Given a feasible schedule, a type 2 subset is a set of teams that consists of one non-maximally giving bridge team b, and the mg teams for which team b is a bridge team.

We use $T_2 \subseteq N$ to denote the set of teams that are in sets of Type 2. It follows from Lemma 4.2 that these mg teams cannot be bridge teams themselves. If such an mg team would be bridge team, then team b would be mg team as well. From this it is clear that $T_1 \cap T_2 = \emptyset$.

Each team that gives maximally or is a bridge team (or both) is now classified in a subset of type 1 or 2. And some of the non-maximally giving, non-bridge teams are classified as well. All remaining teams will form a single set, called a set of type 3.

Definition 4.3. The type 3 subset contains all teams that are not part of any set of type 1 or of type 2.

We use $T_3 \subseteq N$ to denote the set of teams that are in the type 3 set. The definitions 4.1, 4.2 and 4.3 imply that the sets T_1, T_2, and T_3 form a partition of N.

4.2 Proving Theorem 2.1

We show how to obtain an upper bound on the average contribution of a team in a subset of a particular type. Then we compare these averages with the average contribution of a team in a schedule found by the Circle Method. Clearly, the average contribution of a team in a schedule created by the Circle Method is:

$$\frac{CMCOE(n)}{n} = \frac{((n-3)^2+3)(n-1)}{n} = n^2 - 7n + 18 - \frac{12}{n}. \tag{3}$$

We will show that, for $n \geq 10$, the average contribution of a team in a schedule not in $\mathscr{C}$, is less than the average contribution of a team in a schedule that is in $\mathscr{C}$.

Lemma 4.3. In any feasible schedule not in $\mathscr{C}$, we have, for each subset T_j ($j \in \{1,2,3\}$) and for each $n \geq 10$:

$$\frac{\sum_{i \in T_j} Co(i)}{|T_j|} < \frac{CMCOE(n)}{n}.$$

Proof. We prove the above inequality for T_1, T_2 and T_3 separately.

Proof for T_1.

Consider an arbitrary subset of type 1 consisting of a maximally giving bridge team b, contributing $(n-3)^2+2$, and a non-maximally giving (non-bridge) team a, contributing at most $(n-4)^2+5$. Hence, the average contribution of the two teams in a set of type 1 is at most:

$$\frac{Co(b)+Co(a)}{2} \leq \frac{((n-3)^2+2)+((n-4)^2+5)}{2} = n^2-7n+16. \tag{4}$$

Hypothesizing that $\frac{Co(b)+Co(a)}{2} < \frac{CMCOE(n)}{n}$, is equivalent to (using (3) and (4)):

$$\begin{aligned} n^2-7n+16 &< n^2-7n+18-\frac{12}{n} \iff \\ \frac{12}{n} &< 2. \end{aligned}$$

Since this last inequality holds for $n \geq 10$, we have shown that the average contribution of teams in a subset of type 1 is strictly less than $\frac{CMCOE(n)}{n}$.

Proof for T_2.

We distinguish subsets of Type 2 based on their cardinality. Let ℓ denote the number of mg teams in the set (thus, the cardinality of a set of Type 2 equals $\ell+1$). We consider three cases: $\ell \geq 3$ (Case 1), $\ell = 2$ (Case 2), and $\ell = 1$ (Case 3).

Case 1: A type 2 subset containing at least 3 mg teams

Lemma 3.9 states that the bridge team b contributes at most $(n-2-\ell)^2+\ell+1$. It follows that the average contribution of the teams in such a subset is bounded by:

$$\frac{\ell\Big((n-3)^2+2\Big)+\Big((n-2-\ell)^2+\ell+1\Big)}{\ell+1} \text{ with } \ell \geq 3. \tag{5}$$

Since we assume that the schedule is not in $\mathscr{C}$, it follows from Lemma 3.8 that $\ell < n-3$. We compare (5) with the average contribution of a team in a schedule in $\mathscr{C}$ (given by (3)):

$$\frac{\ell((n-3)^2+2)+((n-2-\ell)^2+\ell+1)}{\ell+1} < n^2-7n+18-\frac{12}{n}. \tag{6}$$

Straightforward manipulations imply that this inequality is equivalent to:

$$\frac{(\ell-3)n^2-\big(\ell^2-2\ell-13\big)n-12\ell-12}{(\ell+1)n} > 0. \tag{7}$$

We claim that this strict inequality holds if $3 \leq \ell \leq n-3$; we refer to the appendix for a proof of this claim.

Case 2: A type 2 subset containing two mg teams

From Lemma 3.10, we deduce that the average contribution of the teams in such a subset is bounded by:

$$\frac{2((n-3)^2+2)+((n-5)^2+6)}{3} = n^2-\frac{22}{3}n+\frac{53}{3}. \tag{8}$$

We compare (8) with the average contribution of a team in a schedule in $\mathscr{C}$ (given by (3)):

$$\begin{aligned} n^2 - \frac{22}{3}n + \frac{53}{3} &< n^2 - 7n + 18 - \frac{12}{n} \iff \\ 0 &< \frac{1}{3}n + \frac{1}{3} - \frac{12}{n}. \end{aligned}$$

This holds when $n \geq 10$.

Case 3: A type 2 *subset containing at exactly one mg team.*

These subsets contain one bridge team that does not give maximally, and one maximally giving team. The contribution of the bridge team is at most $(n-4)^2+5$. Hence, the average contribution of the teams in this subset is bounded by

$$\frac{((n-3)^2+2)+((n-4)^2+5)}{2}. \tag{9}$$

This bound is the same as (4), for which we already proved that it is smaller than the average contribution of a team in a schedule generated by the Circle Method.

It follows that the bound holds in each of the three cases.

Proof for T_3.

The contribution of each team in this subset is bounded by:

$$(n-4)^2+5 = n^2-8n+21. \tag{10}$$

We compare (10) with the average contribution of a team in a schedule in $\mathscr{C}$ (given by (3)).

$$\begin{aligned} n^2 - 8n + 21 &< n^2 - 7n + 18 - \frac{12}{n} \iff \\ 0 &< n - 3 - \frac{12}{n} \iff \\ 0 &< n^2 - 3n - 12. \end{aligned}$$

This inequality holds when $n \geq 10$.

We now have proven that the inequality holds for T_1, T_2 and T_3. □ □

It is now easy to see that Lemma 4.3 implies Theorem 2.1: Since the sets T_1, T_2 and T_3 form a partition of N, the COE value of a schedule not in $\mathscr{C}$ is smaller than $CMCOE(n)$, the COE value found by the Circle Method. This means that the Circle Method maximizes the COE value.

Due to the fact that the average contribution of teams in every type of set is less than that of the Circle Method, the reverse holds as well. This means that a schedule that has a maximum COE value can be generated by the Circle Method.

A Appendix: Verification of the bound

We prove the claim that

$$\frac{(\ell-3)n^2-\big(\ell^2-2\ell-13\big)n-12\ell-12}{(\ell+1)n}>0 \tag{11}$$

when ℓ lies between 3 and $n-3$. We use the assumption that $n \geq 10$ as well.

Since n and ℓ are positive, it is sufficient to check that the numerator is positive:

$$(\ell-3)n^2-\big(\ell^2-2\ell-13\big)n-12\ell-12>0.$$

The left hand side is a quadratic polynomial in both n and ℓ. Given a value for n we want to know for which values of ℓ this expression is positive. So we rewrite the previous expression ordered by powers of ℓ:

$$-n\ell^2+(n^2+2n-12)\ell+(-3n^2+13n-12).$$

This quadratic polynomial has a strictly negative leading coefficient, so it is positive between its zero points which are given by

$$\ell_1=\frac{n^2+2n-\sqrt{n^4-8n^3+32n^2-96n+144}-12}{2n},$$
$$\ell_2=\frac{n^2+2n+\sqrt{n^4-8n^3+32n^2-96n+144}-12}{2n}.$$

Now the claim can be formulated as follows: the interval over ℓ where expression (11) is positive, contains the interval $[3,n-3]$, or equivalently, we have: $\ell_1<3$ and $\ell_2>n-3$. To prove the first inequality for all values of n, it is sufficient to show that (i) ℓ_1 is strictly increasing, and (ii) has as limit 3 when n goes to infinity. To calculate this limit, we multiply numerator and denominator by $n^2+2n-12+\sqrt{n^4-8n^3+32n^2-96n+144}$, and get

$$\begin{aligned}\ell_1 &= \frac{12n^3-52n^2+48n}{2n(n^2+2n+\sqrt{n^4-8n^3+32n^2-96n+144}-12)},\\ &= \frac{6n^2-26n+24}{n^2+2n+\sqrt{n^4-8n^3+32n^2-96n+144}-12}.\end{aligned}$$

Clearly, $\lim_{n\to\infty}\ell_1=3$.

To prove that ℓ_1 is strictly increasing, we consider the derivative of ℓ_1 with respect to n:

$$\ell_1'=-\frac{n^4-4n^3-\big(n^2+12\big)\sqrt{n^4-8n^3+32n^2-96n+144}+48n-144}{2n^2\sqrt{n^4-8n^3+32n^2-96n+144}}.$$

Notice that $n^4-8n^3+32n^2-96n+144$ is always positive (indeed, this polynomial has four imaginary roots). So it is sufficient to show that the numerator is positive:

$$-n^4+4n^3+(n^2+12)\sqrt{n^4-8n^3+32n^2-96n+144}-48n+144>0. \tag{12}$$

We obtain the following series of equivalent inequalities.

$$
\begin{aligned}
(n^2+12)\sqrt{n^4-8n^3+32n^2-96n+144} &> n^4-4n^3+48n-144 \iff \\
(n^2+12)\sqrt{n^4-8n^3+32n^2-96n+144} &> (n^2+12)(n^2-4n-12) \iff \\
\sqrt{n^4-8n^3+32n^2-96n+144} &> n^2-4n-12 \iff \\
n^4-8n^3+32n^2-96n+144 &> (n^2-4n-12)^2 \iff \\
8n(5n-24)+(n^2-4n-12)^2 &> (n^2-4n-12)^2 \iff \\
8n(5n-24) &> 0
\end{aligned}
$$

This is true since we assume that $n \geq 10$; it follows that $\ell_1 < 3$ if $n \geq 10$.

For the other inequality $\ell_2 > n-3$, it is sufficient to prove that (i) this inequality holds for $n = 8$, and (ii) that the derivative of ℓ_2 with respect to n is greater than, or equal to, 1. This condition on the derivative implies that if n increases with 1 that ℓ_2 increases with at least 1.

One can deduce that for $n = 8$ the value of $\ell_2 \approx 8.71058907144937 > n-3$.

Next, the derivative of ℓ_2 equals

$$
\frac{n^4-4n^3+\left(n^2+12\right)\sqrt{n^4-8n^3+32n^2-96n+144}+48n-144}{2n^2\sqrt{n^4-8n^3+32n^2-96n+144}},
$$

and this has to be greater than, or equal to, 1. Thus, the following inequality has to hold

$$
n^4-4n^3-\left(n^2-12\right)\sqrt{n^4-8n^3+32n^2-96n+144}+48n-144 \geq 0.
$$

The calculation to show this is analogous to the previous one. Indeed, we have:

$$
\begin{aligned}
(n^2-12)\sqrt{n^4-8n^3+32n^2-96n+144} &\leq n^4-4n^3+48n-144 \iff \\
(n^2-12)\sqrt{n^4-8n^3+32n^2-96n+144} &\leq (n^2-12)(n^2-4n+12) \iff \\
\sqrt{n^4-8n^3+32n^2-96n+144} &\leq n^2-4n+12 \iff \\
n^4-8n^3+32n^2-96n+144 &\leq (n^2-4n+12)^2 \iff \\
n^4-8n^3+32n^2-96n+144 &\leq n^4-8n^3+40n^2-96n+144 \iff \\
0 &\leq 8n^2
\end{aligned}
$$

So the claim that $\ell_2 > n-3$, holds for $n \geq 10$.

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