Model Reduction for a Class of Nonlinear Delay Differential Equations with Time-varying Delays

Nathan van de Wouw, Wim Michiels, and Bart Besselink

***Abstract*— In this paper, a structure-preserving model reduction approach for a class of nonlinear delay differential equations with time-varying delays is proposed. Benefits of this approach are, firstly, the fact that the delay nature of the system is preserved after reduction, secondly, that input-output stability properties are preserved and, thirdly, that a computable error bound reflecting the accuracy of the reduction is provided. These results are also applicable to large-scale linear delay differential equations with constant delays. The effectiveness of the results is evidenced by means of an illustrative example involving the nonlinear delayed dynamics of the turning process.**

I. Introduction

Complex dynamical system models in terms of delay differential equations appear naturally in a wide variety of problems in for example engineering, biology and control theory [1], [14], [26], [31]. In support of the dynamic analysis, optimization or controller design for such systems, we often desire to reduce model complexity. Model order reduction is a tool for the order reduction of high-order dynamical systems in pursuit of complexity reduction.

For linear delay differential equations (DDEs) different approaches for model reduction are available, albeit to a more limited extent than for ordinary differential equations. Methods for the finite-dimensional approximation of delay systems through rational approximations have been proposed in [22], [23], see also [13]. Recently, a technique based on the dominant pole algorithm has been proposed to obtain a rational approximation of an input-output transfer function representing second-order delay differential equations [28]. A Krylov-based model reduction approach leading to finite-dimensional (delay-free) model approximations has been proposed in [25]. In [16], Krylov methods for infinite-dimensional systems, applicable to delay systems, have been proposed also leading to finite-dimensional approximations. The above methods have the common property that the resulting models are of a finite-dimensional nature; hence the inherent delay nature of the original system is lost.

In this paper, we aim at constructing reduced-order models which preserve the delay nature of the system dynamics (i.e. the reduced-order model is also a delay differential equation, though of a reduced order). The desire to preserve the delay nature in the reduced-order model is motivated by, firstly, the fact that, for a given order of the reduced model, a reduced model in the form of a delay differential equation is in general more accurate than a reduced model in the form of a delay free system, see e.g. [28], and, secondly, the fact that by preserving the delay nature also related system properties (such as e.g. the infinite-dimensional system character and the infinite number of eigenvalues in the linear case) are preserved. Such structure-preserving model reduction techniques for delay differential equations, yielding reduced-order delay models, are needed as, on the one hand, powerful simulation and controller synthesis techniques for such systems have become available in the recent past [4], [14], [26], [29], while, on the other hand, the main bottleneck of these methods is that in most cases they require the order of the delay differential equation to be moderate. In [3], interpolatory projection methods based have been proposed, which are also applicable to delay systems and preserve the delay nature in the reduced-order model. In [19], a structure preserving model reduction technique for delay differential equations has been proposed, which extends the notion of position balancing from second-order systems to time-delay systems and relies on solving delay Lyapunov equations.

In this paper, we propose a structure-preserving model order reduction strategy for *nonlinear* delay differential equations, based on balancing techniques, which, firstly, preserves the delay nature of the model, secondly, guarantees the preservation of both internal and input-output stability properties and, thirdly, comes with a computable error bound on the reduced-order model. We note that the latter two aspects (stability preservation and an error bound) are lacking in the existing results in the literature mentioned above. Error bounds have been proposed for finite-dimensional rational approximations, see [13]. Moreover, error bounds and the preservation of stability is also guaranteed in the works [20], [34], in which an H_∞ model reduction approach for *linear* time-delay systems has been proposed.

In the current paper, we propose a model reduction approach for a class of *nonlinear* delay differential equations (with time-varying delays), also applicable to linear time-delay systems. An additional benefits of the approach proposed in the current paper in comparison with the approach in [20], [34] (which is only applicable to linear systems) are twofold. Firstly, by the grace of the fact that we employ balancing-type techniques as a basis, which use the solution to two algebraic Lyapunov equations, the approach proposed

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here is applicable to systems up to order $O(10^3)$ using standard (Bartels-Stewart) algorithms and to systems up to order $O(10^6)$ using tailored algorithms, see e.g. [5]. On the other hand, the approach in [20], [34] of reformulating the model reduction problem as a H_∞-norm minimization problem of the 'error system', induced by the reduction, leads to an (non-convex) optimization problem constrained by a set of matrix inequalities. The latter fact makes such an approach more computationally complex and hence obstructs applicability to systems of high order. Secondly, we propose a natural approach of decomposing the delay system dynamics in terms a feedback interconnection between a finite-dimensional linear part and a delay-operator part. This approach is natural in many applications, in which the delay only affects certain outputs, see e.g. models for high-speed milling processes [1], [7], [18] and drilling processes [10], [11], in which also nonlinearities in the delay-related terms may arise. Moreover, such a decomposition allows to employ incremental $\mathcal{L}_2$-gain properties of the systems in the feedback interconnection to guarantee the preservation of stability and to provide an error bound. We provide such an expression for an a priori error bound depending on 1) the properties of the high-order system, 2) the delay, 3) the properties of the nonlinear terms and 4) the order of the reduced-order system.

The structure of the paper is as follows. Section II specifies in detail the problem formulation and the class of delay systems considered. Next, in Section III the model reduction approach is introduced. Section IV presents the results on the preservation of stability properties and a bound on the reduction error. Finally, Section V presents an illustrative example involving the nonlinear delayed dynamics of the turning process. and Section VI presents concluding remarks.

Notation. The field of real numbers is denoted by $\mathbb{R}$. For a vector $x \in \mathbb{R}^n$, $|x|^2 = x^T x$. The space $\mathcal{L}_2^n$ consists of all functions $x : [0, \infty) \to \mathbb{R}^n$ which are bounded using the norm $\|x\|_2^2 := \int_0^\infty |x(t)|^2 dt$.

II. Problem Formulation

Consider a class of nonlinear systems with point-wise, time-varying (and potentially uncertain) delays of the following form

$$\Sigma : \begin{cases} \dot{x}(t) = A_0 x(t) + B_v f\big(z(t) - z(t - \tau - \delta\tau(t))\big) \\ \qquad\quad + B_u u(t) \\ z(t) = C_z x(t) \\ y(t) = C_y x(t) + D_{yu} u(t) \end{cases} \tag{1}$$

with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^q$, $f : \mathbb{R}^q \to \mathbb{R}^q$, with $f(0) = 0$, $y \in \mathbb{R}^m$ and $u \in \mathbb{R}^p$, and typically $q \ll n$. We assume that the time-varying delay $\tau + \delta\tau(t)$ is a measurable function and satisfies the condition $-\tau \leq -\mu \leq \delta\tau(t) \leq \mu$ for some $\mu \geq 0$ and for all $t \geq 0$. Assumptions on the function f in (1) will be made explicit in Section III.

Systems of the form (1) are common in application fields such as high-speed milling [1], [7], [18] and deep drilling [10], [11] and (without the nonlinearity) also in the scope of networked control systems. Moreover, we emphasize that all developments in this paper are also applicable to generic linear delay differential equations with point-wise delays, see Remark 1 below.

Let us explicate what we mean by model reduction for a delay differential equation as in (1). Hereto, we recall the fact that the model in (1) is infinite-dimensional, i.e. the initial condition for system (1) is the function segment $\phi \in \mathcal{C}([-\tau-\mu, 0], \mathbb{R}^n)$ with $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ the Banach space of continuous functions mapping the interval $[-\tau - \mu, 0]$ to $\mathbb{R}^n$. In fact, we aim to preserve the infinite-dimensional nature of the system in the model reduction approach to be proposed. Still, we can speak of the order of the delay differential equation (1) in terms of the number of equations in the first equality in (1), which in this case is n. Now, we aim at constructing a reduced-order model in terms of a linear delay differential equation of order $\hat{n}$ (i.e. with 'state' $\hat{x}(t) \in \mathbb{R}^{\hat{n}}$) such that,

- the reduced-order model is also a delay differential equation similar in form to (1), i.e. the delay-nature of the system is preserved;
- $\hat{n} < n$, i.e. model (order) reduction is achieved;
- if the origin of (1) is asymptotically stable (for $u = 0$) and (1) is finite $\mathcal{L}_2$-gain stable with respect to the input/output pair (u, y), then the origin of the reduced-order model is also asymptotically stable asymptotically stable (for $u = 0$) and the reduced-order model is $\mathcal{L}_2$-gain stable with respect to the same input/output pair $(u, \hat{y})$, where $\hat{y}$ is the output of the reduced-order system;
- there exists a computable error bound reflecting the accuracy of the reduction.

Clearly, in the above problem statement we aim at the preservation of asymptotic stability for zero inputs[1] and finite $\mathcal{L}_2$-gain stability with respect to the input/output pair (u, y), the latter of which is defined below (see also [9]).

Definition 1: System (1) is called finite $\mathcal{L}_2$-gain stable with respect to the input/output pair (u, y) with finite gain γ if for solutions of (1) corresponding to the zero initial condition ($\phi = 0$) it holds that $\|y\|_2 \leq \gamma \|u\|_2$.

Remark 1: We care to stress that the results presented for systems of the form (1) are also applicable to a generic class of linear delay differential equations. Namely, consider a generic class of linear delay differential equations (with constant delays) of the form:

$$\begin{aligned} \dot{x}(t) &= \bar{A}_0 x(t) + \bar{A}_1 x(t - \tau) + B u(t), \\ y(t) &= C_y x(t) + D_{yu} u(t) \end{aligned} \tag{2}$$

with $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$ and $u(t) \in \mathbb{R}^p$. Alternatively, the dynamics in (2) can be written in the following form:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 (x(t) - x(t - \tau)) + B u(t), \\ y(t) &= C_y x(t) + D_{yu} u(t) \end{aligned} \tag{3}$$

[1]For a definition of asymptotic stability for functional differential equations, we refer to [14], [15].

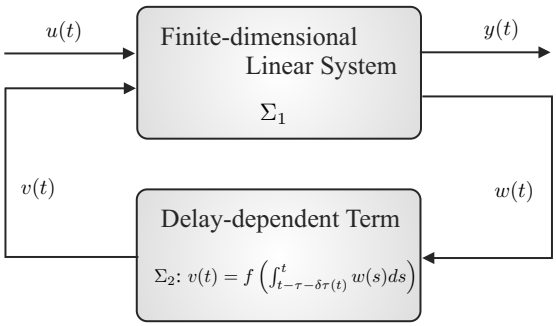

Fig. 1: Schematic of system decomposition in (4), (5).

with $A_0 = \bar{A}_0 + \bar{A}_1$ and $A_1 = -\bar{A}_1$. If we employ a (rank revealing) decomposition of the matrix A_1 in (3) in the form $A_1 = B_v C_z$, we note that (3) is of the form (1), with $f(z(t) - z(t-\tau-\delta\tau(t))) = z(t) - z(t-\tau-\delta\tau(t))$.

III. Model Reduction Approach

In support of the pursuit of the model reduction of system Σ in (1), let us transform this system into a feedback interconnection of a finite-dimensional linear system Σ_1 and an operator Σ_2 related to the delay (we will denote this feedback interconnection by (Σ_1, Σ_2)):

$$\Sigma_1 : \begin{cases} \dot{x}(t) = A_0 x(t) + B_v v(t) + B_u u(t), \\ w(t) = C_w x(t) + D_{wv} v(t) + D_{wu} u(t), \\ y(t) = C_y x(t) + D_{yu} u(t), \end{cases} \tag{4}$$

$$\Sigma_2 : v(t) = f\left(\int_{t-\tau-\delta\tau(t)}^{t} w(s) ds \right). \tag{5}$$

where $v(t)$, $w(t) \in \mathbb{R}^q$. Moreover, in (4) we defined $C_w := C_z A_0$, $D_{wv} := C_z B_v$ and $D_{wu} := C_z B_u$. In interpreting how (4), (5) represents (3), it helps to realize that $v(t) = f(C_z(x(t) - x(t-\tau-\delta\tau(t))))$ and $w(t) = \dot{z}(t)$ with $z(t) = C_z x(t)$.

The system decomposition as a feedback interconnection of a finite-dimensional linear system and a delay-dependent term, see (4), (5), is schematically depicted in Figure 1. Clearly, with such decomposition we pursue a delay-dependent approach towards the analysis of the delay system involved, see e.g. [14]. Moreover, if $q \ll n$, the form of the system decomposition in (4), (5) naturally supports a model reduction strategy in which the order of Σ_1 is reduced, while Σ_2 is left unchanged. In this way, we meet the objectives, as put forward in Section II, of achieving order reduction while preserving the delay nature of the system.

Let us adopt the following assumption on system (4).

Assumption 1: Σ_1 is asymptotically stable (i.e. A_0 is Hurwitz).

Remark 2: Note that, due to the asymptotic stability of Σ_1 (Assumption 1), there exist input-output operators $\mathcal{F}_y : \mathcal{L}_2^p \times \mathcal{L}_2^q \to \mathcal{L}_2^m$ and $\mathcal{F}_w : \mathcal{L}_2^p \times \mathcal{L}_2^q \to \mathcal{L}_2^q$ defined as $y = \mathcal{F}_y(u, v)$ and $w = \mathcal{F}_w(u, v)$, respectively. These operators generate the outputs y and w of the finite-dimensional linear system Σ_1 for given inputs u and v and zero initial condition $x(0) = 0$. Linearity and asymptotic stability of Σ_1 together imply a bounded incremental $\mathcal{L}_2$ gain property, such that the above input-output operators satisfy

$$\begin{aligned} \|\mathcal{F}_i(u_1, v_1) - \mathcal{F}_i(u_2, v_2)\|_2 \leq & \gamma_{iu} \|u_1 - u_2\|_2 \\ & + \gamma_{iv} \|v_1 - v_2\|_2, \end{aligned} \tag{6}$$

for all u_1, $u_2 \in \mathcal{L}_2^p$, v_1, $v_2 \in \mathcal{L}_2^q$, and some bounded γ_{iu}, $\gamma_{iv} \geq 0$ with $i \in \{y, w\}$. Due to linearity, the incremental $\mathcal{L}_2$ gain is equivalent to the (non-incremental) $\mathcal{L}_2$ gain, such that the gains γ_{ij} in (6) can be chosen as the H_∞-norm of the corresponding transfer functions.

Let us now formulate the following assumptions on the nonlinearity and the (uncertain) time-varying delays characterizing Σ_2.

Assumption 2: The following statements hold:

- The function f is globally Lipschitz with Lipschitz constant L;
- the time-varying delay $\tau + \delta\tau(t)$ is a measurable function and satisfies the condition $-\tau \leq -\mu \leq \delta\tau(t) \leq \mu$ for some $\mu \geq 0$ and for all $t \geq 0$.

Later, we will use the following lemma on an incremental gain property of the nonlinear delay operator Σ_2.

Lemma 1: Under Assumption 2, the operator $\Sigma_{2,nl}$ satisfies the following incremental gain property:

$$\|v_2 - v_1\|_2 \leq L\sigma \|w_2 - w_1\|_2, \tag{7}$$

for all w_1, w_2, where $\sigma := \left(\sqrt{\frac{7}{4}}\mu + \tau \right)$.

Proof: Let us start by considering the non-incremental version of the $\mathcal{L}_2$-gain property in (7). Based on the definition of the operator Σ_2 and by using Assumption 2, we can write $\|v\|_2 \leq L \left\| \int_{t-\tau-\delta\tau(t)}^{t} w(s) ds \right\|_2$. Let us now combine the facts that

- $\| \int_{t-\tau-\delta\tau(t)}^{t} w(s) ds \|_2 \leq \| \int_{t-\tau-\delta\tau(t)}^{t-\tau} w(s) ds \|_2 + \| \int_{t-\tau}^{t} w(s) ds \|_2$;
- $\| \int_{t-\tau}^{t} w(s) ds \|_2 \leq \tau \|w\|_2$, see [9], [24];
- $\| \int_{t-\tau-\delta\tau(t)}^{t-\tau} w(s) ds \|_2 \leq \sqrt{\frac{7}{4}}\mu \|w\|_2$, based on Assumption 2 and Lemma 1 in [24] (see also [30]),

to obtain the following $\mathcal{L}_2$-gain property for Σ_2:

$$\|v\|_2 \leq L \left(\sqrt{\frac{7}{4}}\mu + \tau \right) \|w\|_2. \tag{8}$$

Due to the fact that the global Lipschitz property of f, see Assumption 2, implies that $\|f(\rho_1) - f(\rho_2)\|_2 \leq L\|\rho_1 - \rho_2\|_2$, for all ρ_1, ρ_2, and the linearity of the operator $\int_{t-\tau-\delta\tau(t)}^{t} w(s) ds$, the $\mathcal{L}_2$-gain property in (8) also implies the validity of the *incremental* $\mathcal{L}_2$-gain property in (7). ■

Let us now adopt the following assumption on the feedback interconnection (Σ_1, Σ_2) given by (4), (5).

Assumption 3: The feedback interconnection (Σ_1, Σ_2) satisfies the small-gain condition

$$\gamma_{wv} L \sigma < 1. \tag{9}$$

Remark 3: Due to the asymptotic stability of Σ_1 (Assumption 1), γ_{wv} always exists (i.e. is bounded) and hence

Assumption 3 can always be satisfied for small enough τ and μ (i.e. small enough delay), since $\sigma = (\sqrt{\frac{7}{4}}\mu + \tau)$.

The following result provides a sufficient condition under which system (4), (5) exhibits certain stability properties, which we subsequently desire to preserve under model reduction.

Lemma 2: Consider system (4), (5) satifying Assumptions 1, 2 and 3. Then the feedback interconnection (Σ_1, Σ_2) is 1) $\mathcal{L}_2$-gain stable with respect to the input/output pair (u, y) and 2) its origin is asymptotically stable for $u = 0$.

Proof: Under Assumption 1, there exist bounded γ_{wu} and γ_{wv} such that $\|w\|_2 \leq \gamma_{wu}\|u\|_2 + \gamma_{wv}\|v\|_2$. Using (9) and the non-incremental version of Lemma 1, we conclude that

$$\|w\|_2 \leq \frac{\gamma_{wu}}{1 - \gamma_{wv} L\sigma}\|u\|_2. \tag{10}$$

Using (10) and the non-incremental version of Lemma 1 in $\|y\|_2 \leq \gamma_{yu}\|u\|_2 + \gamma_{yv}\|v\|_2$ gives

$$\begin{aligned} \|y\|_2 &\leq \gamma_{yu}\|u\|_2 + \gamma_{yv} L\sigma \|w\|_2 \\ &\leq \left(\gamma_{yu} + \frac{\gamma_{yv} L\sigma \gamma_{wv}}{1 - \gamma_{wv} L\sigma}\right)\|u\|_2, \end{aligned} \tag{11}$$

which shows that (Σ_1, Σ_2) is $\mathcal{L}_2$-gain stable with respect to the input/output pair (u, y). Now, using the fact that system Σ_1 is an asymptotically stable linear time-invariant system, Σ_2 has no internal dynamics, i.e. output v in (5) is solely determined by input w, and the feedback interconnection (Σ_1, Σ_2) satisfies a small gain condition, we can conclude that (Σ_1, Σ_2) is also asymptotically stable for $u = 0$ (see also [17], [32]). This completes the proof. ∎

In pursuing model reduction of (4), (5), we construct a reduced-order model $\hat{\Sigma}_1$ for the linear finite-dimensional system Σ_1 in the following form:

$$\hat{\Sigma}_1 : \begin{cases} \dot{\hat{x}}(t) = \hat{A}_0\hat{x}(t) + \hat{B}_v\hat{v}(t) + \hat{B}_u u(t), \\ \hat{w}(t) = \hat{C}_w\hat{x}(t) + \hat{D}_{wv}\hat{v}(t) + \hat{D}_{wu}u(t), \\ \hat{y}(t) = \hat{C}_y\hat{x}(t) + \hat{D}_{yv}\hat{v}(t) + \hat{D}_{yu}u(t) \end{cases} \tag{12}$$

with $\hat{x}(t) \in \mathbb{R}^{\hat{n}}$ and $\hat{n} < n$. The reduced-order model $\hat{\Sigma}$ is now given by the feedback interconnection of $\hat{\Sigma}_1$ and Σ_2, denoted by $(\hat{\Sigma}_1, \Sigma_2)$, where Σ_2 relates $\hat{v}$ to $\hat{w}$ according to $\hat{v}(t) = f\left(\int_{t-\tau-\delta\tau(t)}^{t} \hat{w}(s)ds\right)$, i.e. the dynamics of $\hat{\Sigma}$ is characterized by (12) and

$$\Sigma_2 : \hat{v}(t) = f\left(\int_{t-\tau-\delta\tau(t)}^{t} \hat{w}(s)ds\right). \tag{13}$$

Figure 2 depicts a schematic of this reduced-order system decomposition.

For an efficient reduction of the system in (4) to the system in (12), the number of inputs and outputs should be small. For approaches based on balanced truncation, this can be understood from the fact that in such a case the decay rate of the Hankel singular values is fast [2]. In (4), the number of inputs is determined by the dimension of $u(t)$ and the dimension of $v(t)$, the latter of which stems from a feedback interconnection interpretation of the delayed term, see Figure 1. Hence, it is important to keep the size of $v(t)$ (and $w(t)$) as small as possible. In many engineering applications in which models are formulated as delay differential equations, such as e.g. models for high-speed milling processes [1], [7], [18] and drilling processes [10], [11], the variables z involved in the delay terms are indeed of much smaller dimension than the state x. Namely, in such models the high-order x-related dynamics typically corresponds to models of the structural dynamics of the spindle-tool dynamics in high speed milling or the drill-string dynamics in drilling, while the delay-related terms relate to localized cutting processes depending on low-dimensional variables z. Similarly, in the context of boundary control of partial differential equations, feedback delays affect control inputs localized at the boundary also lead to z being of significantly caller dimension than x.

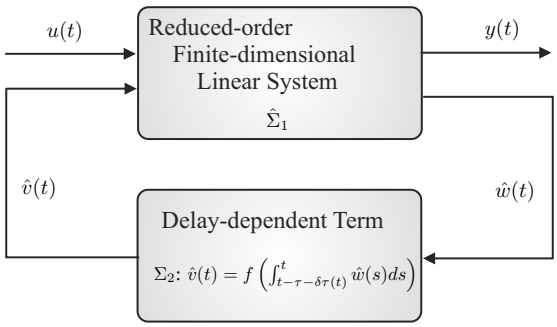

Fig. 2: Schematic of reduced-order system decomposition in (12), (13).

Let us adopt the following assumption on the reduced-order linear system $\hat{\Sigma}_1$.

Assumption 4: The following statements hold:

- $\hat{\Sigma}_1$ is asymptotically stable;
- An (incremental) error bound on reduction of the linear subsystem exists of the form

$$\begin{aligned} \|\mathcal{E}_i(u_1, v_1) - \mathcal{E}_i(u_2, v_2)\|_2 \leq \epsilon_{iu}\|u_1 - u_2\|_2 \\ + \epsilon_{iv}\|v_1 - v_2\|_2, \end{aligned} \tag{14}$$

  for all $u_1, u_2 \in \mathcal{L}_2^p$, $v_1, v_2 \in \mathcal{L}_2^q$, with ϵ_{iu}, $\epsilon_{iv} \geq 0$ and $i \in \{y, w\}$. In (14), $\mathcal{E}_i := \mathcal{F}_i - \hat{\mathcal{F}}_i$, $i \in \{y, w\}$, denotes the error operator with $\hat{\mathcal{F}}_i : \mathcal{L}_2^p \times \mathcal{L}_2^q \to \mathcal{L}_2^{\{m,q\}}$ the input-output operators of the reduced-order linear subsystem $\hat{\Sigma}_1$ for zero initial condition, which exist by the grace of asymptotic stability and linearity.

If we employ balanced truncation [27], optimal Hankel norm approximation [12], or balanced residualization[2], then the resulting reduced-order linear system is of the form $\hat{\Sigma}_1$ and satisfies Assumption 4. Note in this respect that the incremental error bound in (14) is, due to linearity, directly implied by an ordinary (i.e. non-incremental) error bound.

It can be shown that if balanced residualization is used to reduce Σ_1, then the *delay-structure of the original system* (1) *is preserved* in the reduced-order system [33]. More

[2]By balanced residualization, we indicate the singular perturbation approximation of balanced realizations as proposed in [8], [21].

precisely, in this case there exist a matrix $\hat{C}_z$ such that $\hat{w}(t) = \hat{C}_z\dot{\hat{x}}(t)$, and (12) and (13) can be reformulated as

$$\begin{aligned}\dot{\hat{x}}(t) &= \hat{A}_0\hat{x}(t) + \hat{B}_v f(\hat{z}(t) - \hat{z}(t-\tau-\delta\tau(t))) + \hat{B}_u u,\\ \hat{y}(t) &= \hat{C}_y\hat{x} + \hat{D}_{yv} f(\hat{z}(t) - \hat{z}(t-\tau-\delta\tau(t))) + \hat{D}_{yu}u\end{aligned} \tag{15}$$

with $\hat{z} = \hat{C}_z\hat{x}(t)$.

IV. Stability Analysis and Error bound

The following result provides conditions under which, firstly, the reduced-order system inherits certain stability properties from the original system, and, secondly, an error bound can be computed reflecting the accuracy of the reduction.

Theorem 1: Suppose the system (4), (5) satisfies Assumptions 1 and 2. Let $\hat{\Sigma}_1$ in (12) be a reduced-order linear system satisfying Assumption 4. Then, the following statements hold:

1) The reduced-order system $(\hat{\Sigma}_1, \Sigma_2)$ given by (12), (13) is $\mathcal{L}_2$ stable with respect to the input/output pair (u, y) and asymptotically stable for $u = 0$ if

$$L\sigma(\gamma_{wv} + \epsilon_{wv}) < 1; \tag{16}$$

2) Suppose (16) is satisfied. Then, the output error $\delta y := y - \hat{y}$ is bounded as $\|\delta y\|_2 \leq \epsilon\|u\|_2$ with

$$\begin{aligned}\epsilon = \epsilon_{yu} &+ \frac{\epsilon_{yv}L\sigma\gamma_{wu}}{1-\gamma_{wv}L\sigma}\\ &+ \frac{(\gamma_{yv}+\epsilon_{yv})L\sigma}{1-(\gamma_{wv}+\epsilon_{wv})L\sigma}\left(\epsilon_{wu} + \frac{\epsilon_{wv}L\sigma\gamma_{wu}}{1-\gamma_{wv}L\sigma}\right).\end{aligned} \tag{17}$$

Proof: Inspired by the work in [6], statements (1) and (2) are proven separately below.

Statement (1): Lemma 2 can be employed to show that if $\hat{\gamma}_{wv}L\sigma < 1$, then statement (1) of the theorem is valid. Note that $\hat{\gamma}_{wv}$ denotes the $\mathcal{L}_2$-gain of system $\hat{\Sigma}_1$ from input w to output v, which is bounded by the grace of asymptotic stability of $\hat{\Sigma}_1$ (Assumption 4). However, the gain $\hat{\gamma}_{wv}$ is not known a priori (i.e. before actually performing the reduction). Still, we can obtain an upper bound for $\hat{\gamma}_{wv}$ as follows. By the triangle inequality, we have that $\|\hat{w}\|_2 \leq \|w\|_2 + \|w - \hat{w}\|_2$, which implies that $\|\hat{w}\|_2 \leq \gamma_{wv}\|v\|_2 + \gamma_{wu}\|u\|_2 + \epsilon_{wv}\|v\|_2 + \epsilon_{wu}\|u\|_2 \Rightarrow \|\hat{w}\|_2 \leq (\gamma_{wv} + \epsilon_{wv})\|v\|_2 + (\epsilon_{wu} + \gamma_{wu})\|u\|_2$, where we used (14) for $i = w$. Clearly, $(\gamma_{wv} + \epsilon_{wv})$ provides an upper bound on $\hat{\gamma}_{wv}$ and, consequently, (16) implies that $\hat{\gamma}_{wv}L\sigma < 1$, which proves, using Lemma 2, that system $(\hat{\Sigma}_1, \Sigma_2)$ is finite $\mathcal{L}_2$-gain stable with respect to the input/output pair (u, y). Now, using the fact that system $\hat{\Sigma}_1$ is an asymptotically stable linear time-invariant system, Σ_2 has no internal dynamics, and the feedback interconnection $(\hat{\Sigma}_1, \Sigma_2)$ satisfies a small gain condition, we can conclude that $(\hat{\Sigma}_1, \Sigma_2)$ is also asymptotically stable for $u = 0$ (see also [17], [32]).

Statement (2): By using the fact that (16) implies the satisfaction of Assumption 3 (note that $\epsilon_{wv} \geq 0$), we can employ (10) in the proof of Lemma 2 to formulate a bound on $\|w\|_2$. Subsequently using (10) and Lemma 1, we can construct the following bound on $\|v\|_2$:

$$\|v\|_2 \leq \frac{L\sigma\gamma_{wu}}{1-\gamma_{wv}L\sigma}\|u\|_2. \tag{18}$$

The reduction error on w, defined by $\delta w := w - \hat{w}$, satisfies $\delta w = \mathcal{F}_w(u, v) - \hat{\mathcal{F}}_w(u, \hat{v}) = \mathcal{F}_w(u, v) - \hat{\mathcal{F}}_w(u, v) + \hat{\mathcal{F}}_w(u, v) - \hat{\mathcal{F}}_w(u, \hat{v})$, such that δw can be bounded as follows:

$$\begin{aligned}\|\delta w\|_2 \leq \|\mathcal{F}_w(u, v) - \hat{\mathcal{F}}_w(u, v)\|_2\\ + \|\hat{\mathcal{F}}_w(u, v) - \hat{\mathcal{F}}_w(u, \hat{v})\|_2.\end{aligned} \tag{19}$$

Herein, we have that

$$\begin{aligned}\|\mathcal{F}_w(u, v) - \hat{\mathcal{F}}_w(u, v)\|_2 &= \|\mathcal{E}_w(u, v)\|_2\\ &\leq \epsilon_{wu}\|u\|_2 + \epsilon_{wv}\|v\|_2,\end{aligned} \tag{20}$$

which follows from (14). Moreover, we have that

$$\|\hat{\mathcal{F}}_w(u, v) - \hat{\mathcal{F}}_w(u, \hat{v})\|_2 \leq \hat{\gamma}_{wv}\|v - \hat{v}\|_2 = \hat{\gamma}_{wv}\|\delta v\|_2 \tag{21}$$

with $\delta v := v - \hat{v}$. Using (20) and (21) in (19) yields

$$\|\delta w\|_2 \leq \epsilon_{wu}\|u\|_2 + \epsilon_{wv}\|v\|_2 + \hat{\gamma}_{wv}\|\delta v\|_2. \tag{22}$$

As shown in the proof of statement (1) of the theorem, we have that $\hat{\gamma}_{wv} \leq \gamma_{wv} + \epsilon_{wv}$. Moreover, Lemma 1 implies that $\|\delta v\|_2 \leq L\sigma\|\delta w\|_2$. Exploiting these two facts in (22) gives

$$\|\delta w\|_2 \leq \frac{1}{1-(\gamma_{wv}+\epsilon_{wv})L\sigma}(\epsilon_{wu}\|u\|_2 + \epsilon_{wv}\|v\|_2), \tag{23}$$

where the small-gain condition in (16) guarantees the existence of the latter bound. Substituting (18) in (23) yields

$$\|\delta w\|_2 \leq \frac{1}{1-(\gamma_{wv}+\epsilon_{wv})L\sigma}\left(\epsilon_{wu} + \frac{\epsilon_{wv}L\sigma\gamma_{wu}}{1-\gamma_{wv}L\sigma}\right)\|u\|_2. \tag{24}$$

We employ Lemma 1 once again to obtain a bound on $\|\delta v\|_2$:

$$\|\delta v\|_2 \leq \frac{L\sigma}{1-(\gamma_{wv}+\epsilon_{wv})L\sigma}\left(\epsilon_{wu} + \frac{\epsilon_{wv}L\sigma\gamma_{wu}}{1-\gamma_{wv}L\sigma}\right)\|u\|_2. \tag{25}$$

The above bound on δv will be exploited to obtain the final error bound on the output y. Hereto, the output error $\delta y := y - \hat{y}$ is considered: $\delta y = \mathcal{F}_y(u, v) - \hat{\mathcal{F}}_y(u, \hat{v}) = \mathcal{F}_y(u, v) - \hat{\mathcal{F}}_y(u, v) + \hat{\mathcal{F}}_y(u, v) - \hat{\mathcal{F}}_y(u, \hat{v})$, such that δy can be bounded as follows: $\|\delta y\|_2 \leq \|\mathcal{F}_y(u, v) - \hat{\mathcal{F}}_y(u, v)\|_2 + \|\hat{\mathcal{F}}_y(u, v) - \hat{\mathcal{F}}_y(u, \hat{v})\|_2$. Using Assumption 4, the latter inequality yields

$$\|\delta y\|_2 \leq \epsilon_{yu}\|u\|_2 + \epsilon_{yv}\|v\|_2 + \hat{\gamma}_{yv}\|\delta v\|_2. \tag{26}$$

Combining (26) with (25), using (18) and the fact that $\hat{\gamma}_{yv} \leq \gamma_{yv} + \epsilon_{yv}$ gives

$$\begin{aligned}\|\delta y\|_2 \leq \epsilon_{yu}\|u\|_2 &+ \epsilon_{yv}\frac{L\sigma\gamma_{wu}}{1-\gamma_{wv}L\sigma}\|u\|_2\\ &+ (\gamma_{yv}+\epsilon_{yv})\frac{L\sigma}{1-(\gamma_{wv}+\epsilon_{wv})L\sigma}\\ &\times\left(\epsilon_{wu} + \frac{\epsilon_{wv}L\sigma\gamma_{wu}}{1-\gamma_{wv}L\sigma}\right)\|u\|_2,\end{aligned} \tag{27}$$

which confirms the validity of the error bound in (17). ■

Theorem 1 employs knowledge on the error bounds ϵ_{ij}, $i \in \{y, w\}$, $j \in \{u, v\}$, for the linear reduced-order system $\hat{\Sigma}_1$, providing bounds on all relevant input-output pairs. However, existing model reduction techniques for linear systems generally provide a single error bound ϵ_{lin}, uniform for all input-output pairs. When this error bound is exploited as $\epsilon_{ij} \leq \epsilon_{lin}$ for $i \in \{y, w\}$, $j \in \{u, v\}$, the error bound (17) reduces to

$$\epsilon = \epsilon_{lin}\left(1 + \frac{L\sigma\gamma_{wu}}{1 - \gamma_{wv}L\sigma}\right)\left(1 + \frac{(\gamma_{yv} + \epsilon_{lin})L\sigma}{1 - (\gamma_{wv} + \epsilon_{lin})L\sigma}\right). \quad (28)$$

The small-gain condition in (16) and the error bound (17) only require knowledge on, firstly, properties of the high-order system Σ_1, secondly, the error bound on the linear reduced-order system $\hat{\Sigma}_1$ and, thirdly, the nonlinearity and the delay and can therefore be evaluated a priori (i.e. without actually performing the reduction first). However, a tighter error bound can be obtained when the gains $\hat{\gamma}_{wv}$ and $\hat{\gamma}_{yv}$ of the reduced-order linear subsystem are computed a posteriori (i.e. after the reduction has been employed). These gains can directly be used in (22) and (26), respectively, instead of using their bounds $\gamma_{iv} + \epsilon_{iv}$, $i \in \{y, w\}$. Moreover, the knowledge on $\hat{\gamma}_{wv}$ can be used for the direct evaluation of the small-gain condition via $\hat{\gamma}_{wv}L\sigma < 1$ instead of via (16), leading to less conservative results.

Remark 4: In case the only available information on the original system consists of the system matrices and a delay interval $[\tau-\mu,\ \tau+\mu]$, one can still construct a reduced model by using the nominal delay τ in the system Σ_2 in (13) of the reduced-order system, instead of using the time-varying delay $\tau+\delta\tau(t)$ therein. In such a case, the stability condition in (16) of Theorem 1 remains valid. The error bound (17) is not valid anymore since its derivation, in particular the estimate $\|\delta v\|_2 \leq L\sigma\|\delta w\|_2$, relies on the assumption that the feedback operator Σ_2 does not change. An error bound can still be derived using the same principles. This is however beyond the scope of the paper.

V. Illustrative Example

High-speed milling and turning processes are widely used for the manufacturing of high-tech machine components. These processes are known to exhibit undesired vibrational phenomena, called regenerative chatter. To study such vibrational aspects, models in terms of nonlinear delay differential equations have been proposed in the literature [1], [7], [18]. The high-speed turning process, as schematically depicted in Figure 3, is considered in order to illustrate the results of this paper. In such processes, the cutting force (perpendicular to the cutting direction) associated to the removal of material is generally modelled as a nonlinear function of the form $F(d) := kd^c$ (see [18]), where d represents the chip thickness and k and c are constants. Given a nominal chip thickness d_0, deviations $\delta := d_0 - d$ are dependent on the current position of the tool as well as the material profile left behind by the tool at the previous rotation of the work-piece, such that $\delta(t) = z(t) - z(t-\tau)$. Here, z is the perturbed position of the tool (relative to the nominal feed motion of the tool

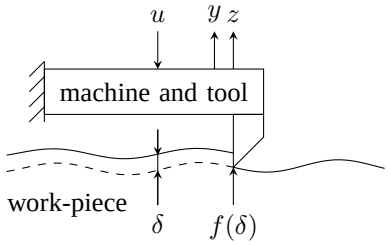

Fig. 3: Schematic illustration of the turning process.

associated to the nominal depth-of-cut d_0) as indicated in Figure 3 and τ is the time it takes the work-piece to complete one full rotation.

The combination of this cutting model with a model of the structural dynamics of the tool holder and machine leads to nonlinear delayed dynamics of the following form:

$$\begin{aligned} M\ddot{q} + D\dot{q} + Kq &= b_u u + b_v f\big(z(t) - z(t-\tau)\big), \\ y &= c_y q, \\ z &= c_z q, \end{aligned} \quad (29)$$

where M, D, and K denote the mass, damping, and stiffness matrices characterizing the structural dynamics with nodal coordinates $q \in \mathbb{R}^N$ (with $N = 10$) of a finite-element model. Moreover, u represents an external input (e.g. available for the control of machine vibrations), whereas y is a measurement of the (perturbed) displacement of the tool position. Finally, the nonlinearity f reads $f(\delta) := F(d_0 - \delta) - F(d_0)$ and its Lipschitz constant can be obtained by considering perturbations δ such that $|\delta| \leq 0.2d_0$. Then, for $k = 1100$, $c = 0.8$, and $d_0 = 50 \cdot 10^{-6}$m, the Lipschitz constant in Assumption 2 reads $L = 6669$. Next, after rewriting (29) in first-order state-space form, it is clear that (29) is of the form (1).

The frequency response function of the finite-dimensional linear part Σ_1 of Σ is depicted in Figure 4 for input $v = f\big(z(t) - z(t-\tau)\big)$ and output $w = \dot{z}$, from which the gain γ_{wv} is obtained as $\gamma_{wv} = 1.02 \cdot 10^{-5}$. We note that Σ_1 corresponds to the dynamics of the tool holder and machine and is asymptotically stable in practice due to structural damping effects (i.e. Assumption 1 holds). Moreover, it is readily checked that Assumption 3 is satisfied when compact sets are considered for which the Lipschitz constant as defined above holds and the (constant) delay τ (i.e. $\delta\tau(t) = 0$) is chosen as $\tau = 12 \cdot 10^{-3}$ s.

Reduction of the model order is highly beneficial for such turning models as it supports the computational burden of stability analysis and controller design (for chatter mitigation). The application of balanced residualization to Σ_1 leads to a reduced-order finite-dimensional system $\hat{\Sigma}_1$ of order $\hat{n} = 2$, which is depicted in Figure 4. This approximation captures the first resonance peak of Σ_1 and satisfies Assumption 4 with $\epsilon_{wv} = 1.63 \cdot 10^{-6}$. It is then readily checked that $\mathcal{L}_2$ stability of the reduced-order system $(\hat{\Sigma}_1, \Sigma_2)$ is guaranteed through condition (16) in Theorem 1, whereas the error bound (17) is obtained as $\epsilon = 4.16 \cdot 10^{-5}$. Also, we remark

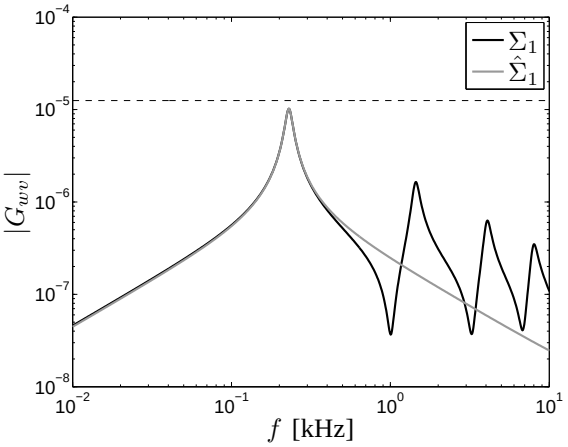

Fig. 4: Frequency response function G_{wv} of the high-order finite-dimensional system Σ_1 and reduced-order approximation $\hat{\Sigma}_1$ with respect to the input v and output w. The dashed line represents the value $(L\sigma)^{-1}$ and allows for the visual evaluation of the small-gain condition (9).

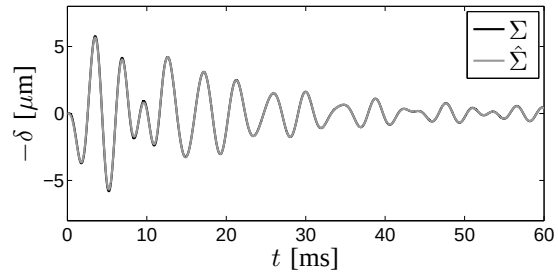

Fig. 5: Deviations δ from the nominal chip thickness for the high-order system Σ and reduced-order system $\hat{\Sigma}$ for delay $\tau = 12$ [ms], input function $u(t) = 10\sin(2\pi 335t)$, and zero initial conditions.

that the use of balanced residualization ensures that the reduced-order system can be written in the form (15).

Finally, a comparison between the high-order system Σ and reduced-order approximation $\hat{\Sigma}$ is given in Figure 5, which depicts the perturbation from the nominal chip thickness. Clearly, an accurate approximation of the time-domain behavior is obtained by the reduced-order nonlinear time-delay system.

VI. Conclusions

We have proposed a structure-preserving model reduction approach for a class of nonlinear delay differential equations with time-varying delays. In this approach, a finite-dimensional part of the system is separated from the nonlinear and delay characteristics and the former part is reduced through balancing-type techniques. Benefits of this approach are, firstly, the fact that the delay nature of the system is preserved after reduction, secondly, that input-output stability properties are preserved and, thirdly, that a computable error bound reflecting the accuracy of the reduction is provided. These results are also applicable to large-scale linear delay differential equations with constant delays. The effectiveness of the results is evidenced by means of an illustrative example involving the nonlinear delayed dynamics in the turning process.

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