Construction of Fully Symmetric Cubature Formulae of Degree 4k-3 for Fully Symmetric Planar Regions.

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ABSTRACT

A new method is described for the construction of cubature formulae of degree 4k-3 for two dimensional symmetric regions. This method is a generalisation of the T-method. Some formulae of degree 5, 9, 13 and 17 for the square, the circle and the entire plane are constructed.

Construction of fully symmetric cubature formulae of degree 4k-3 for fully symmetric planar regions.

1. Introduction

We are concerned with determining the knots and weights in a cubature formula of the form

$$\iint_R w(x,y)\ f(x,y)\ dxdy \simeq \sum_{i=1}^{N} w_i\ f(x_i,y_i)$$

where R is a region in the two-dimensional Euclidean space and $w(x,y) \geq 0$.

The vectorspace of all polynomials in x and y of degree $\leq n$ is denoted by P_n. A cubature formula which is exact for all $P \in P_m$ but not for all $P \in P_{m+1}$ is said to have degree m.

If we speak of a region R, we intend this to include the associated weight function w. A region is called symmetric when
$(x,y) \in R \Rightarrow (\pm x,\pm y) \in R$ and $w(x,y) = w(-x,y) = w(x,-y) = w(-x,-y)$.
A region is called fully symmetric if it is symmetric and when
$(x,y) \in R \Rightarrow (y,x) \in R$ and $w(x,y) = w(y,x)$.

A polynomial P is orthogonal to a polynomial Q if

$$\iint_R w(x,y)\ P(x,y)\ Q(x,y)\ dxdy = 0.$$

A polynomial P of degree n which is orthogonal to all polynomials $Q \in P_{n-1}$ is called an orthogonal polynomial.
The orthogonal polynomials

$$P^{\ell-k,k} = x^{\ell-k}\ y^{k} + S_{\ell k} \qquad k = 0(1)\ell \qquad \ell = 0(1)n$$

where $S_{\ell k} \in P_{\ell-1}$, form a basis for the vectorspace P_n.

We are specially interested in the following fully symmetric regions :

C_2 : the square $\{(x,y) : -1 \leq x,y \leq 1\}$ with weight function $w(x,y) = 1$,

S_2 : the circle $\{(x,y) : x^2 + y^2 \leq 1\}$ with weight function $w(x,y) = 1$,

$E_2^{r^2}$: the entire two-dimensional space with weight function $w(x,y) = \exp(-x^2-y^2)$,

E_2^r : the entire two-dimensional space with weight function $w(x,y) = \exp(-(x^2+y^2)^{1/2})$.

In table I we give orthogonal polynomials of degree 2, 4, 6 and 8 for this regions. Note that because of the symmetry of the region, each polynomial has only even or odd exponents in x and y. Morrow and Patterson (1978) and Schmid (1979) used the connection between the knots of a cubature formula for product regions, and ideal theory in the so called T-method. In the general case this method gives no results for degreeshigher than 7 because there are too many unknowns and the nonlinear system cannot be solved. Morrow and Patterson (1978) imposed the condition of symmetry on the formulae so that some unknows become zero. Then they were able to reconstruct symmetric formulae of degree 9 and 11.

We adapted this T-method to what we called the S-method to construct symmetric formulae for symmetric regions. In this report we give only fully symmetric formulae of degree 5, 9, 13 and 17.

2. Deduction of the S-method

Definition 1

A polynomial ideal A is a set of polynomials such that if $f,g \in A$ and a,b are polynomials then $af + bg \in A$.

Definition 2

If A is a polynomial ideal, then the set of polynomials $g_1,\dots,g_k \in A$ form a basis for A if each $f \in A$ may be written in the form

$$f = \sum_{i=1}^{k} a_i g_i \text{ where } a_i \text{ are polynomials.}$$

Hilbert's theorem (Gröbner, 1949).
For any polynomial ideal there exist a finit basis.

Definition 3

The basis $g_1,\dots,g_k$ of the polynomial ideal A is canonical if each polynomial $f \in A$ of degree d may be written as

$$f = \sum_{i=1}^{k} b_i g_i \text{ where } g_i \text{ is of degree } \mu_i \text{ and } b_i \text{ of degree at most } d - \mu_i.$$

Theorem 1 (Möller,1973)

For any polynomial ideal there exist a canonical basis, but not every basis is canonical.

Theorem 2 (Möller,1973)

If $g_1,\ldots,g_k$ is a canonical basis of a polynomial ideal A and if the set of common zeros of $g_1,\ldots,g_k$ (denoted as $NG(A)$) is finite and nonempty then the following two statements are equivalent.

1) There is a cubature formula of degree m which has as knots the common zeros of $g_1,\ldots,g_k$. These zeros may be multiple, leading to the use of function derivates in the cubature formula.
2) g_i is of degree t_i and g_i is orthogonal to all polynomials of degree $\leqslant m - t_i$ $i = 1(1)k$.

Corollary 3

Every polynomial P of degree $d \leqslant m$ that becomes zero at the knots of a cubature formula of degree m, is orthogonal to all polynomials Q of degree $\leqslant m - d$; because degree $P.Q \leqslant m$.

Theorem 4 (Möller,1976)

Let A_j be the set of polynomials of degree $\leqslant j$ that become zero at the N knots of a cubature formula of degree $2k - 1 = m$ for an integral over a symmetric region. Then :

1) $\dim A_k \leqslant k+1 - \left[\frac{k}{2}\right]$

   where $[x]$ = greatest integer less or equal to x

2) $N = \dim P_{m+1} - \dim A_{m+1}$

   $N \geqslant N_{min} = \frac{k(k+1)}{2} + \left[\frac{k}{2}\right]$

3) $3 \dim A_k - k+2\left[\frac{k}{2}\right] \leqslant \dim A_{k+1} \leqslant \dim A_k + k+2$

Corollary 5

If $N = N_{min} + p$,

$\dim A_k = k+1 - \left[\frac{k}{2}\right] - a \ (a \geqslant 0)$ and

$$\dim A_{k+1} - \dim A_k = k+2-b \quad (b \geqslant 0)$$

then : 1°) $b \leqslant 2a$

2°) $a + b \leqslant p$

3°) $a = p$ if and only if $b = 0$

Proof

1°) From theorem 4-3) we know that

$$3 \dim A_k - k+2\left[\frac{k}{2}\right] \leqslant \dim A_{k+1}$$

or

$$2 \dim A_k - k+2\left[\frac{k}{2}\right] \leqslant \dim A_{k+1} - \dim A_k$$

this is the same as

$$2(k+1-\left[\frac{k}{2}\right]-a)-k + 2\left[\frac{k}{2}\right] \leqslant k+2-b$$

or $b \leqslant 2a$ □

2°) From theorem 4-2) we know that

$$N - N_{min} = p$$

$$\Leftrightarrow \dim P_{m+1} - \dim A_{m+1} - \frac{k(k+1)}{2} - \left[\frac{k}{2}\right] = p$$

$$\Leftrightarrow (\dim P_{m+1} - \dim P_{k-1}) - \dim A_{m+1} - \left[\frac{k}{2}\right] = p$$

$$\Leftrightarrow \sum_{\ell=0}^{m-k+1} (k+\ell+1) - \sum_{\ell=1}^{m-k+1} (\dim A_{\ell+k} - \dim A_{k+\ell-1})$$

$$- \dim A_k - \left[\frac{k}{2}\right] = p$$

$$\Leftrightarrow k+1 - \dim A_k - \left[\frac{k}{2}\right] + \sum_{\ell=1}^{m-k+1} [k+\ell+1 - (\dim A_{\ell+k} - \dim A_{k+\ell-1})] = p$$

$$\Leftrightarrow a + b + \sum_{\ell=2}^{m-k+1} [k+\ell+1 - (\dim A_{\ell+k} - \dim A_{\ell+k-1})] = p \qquad (1)$$

$\sum_{\ell=2}^{m-k+1} [k+\ell+1 - (\dim A_{\ell+k} - \dim A_{\ell+k-1})] \geqslant 0$ because

$$\dim A_{k+\ell} - \dim A_{k+\ell-1} \leqslant k+\ell-1$$

Therefore, we become $N - N_{min} = p \Leftrightarrow a+b \leqslant p$ □

3°) Note that if $\dim A_{\ell} - \dim A_{\ell-1} = \ell+1$

then $\dim A_{\ell+1} - \dim A_{\ell} = \ell+2$

Therefore, if $b = 0$ then $\sum_{\ell=2}^{m-k+1} [k+\ell+1 - (\dim A_{\ell+k} - \dim A_{\ell+k-1})] = 0$

and (1) becomes $a = p$.
Otherwise, if $a = p$ then $b = 0$ because $a+b \leqslant p$ □

From corollary 5 it follows that :

1) if $N = N_{min}$ then $a = b = 0$

2) if $N = N_{min} + 1$ then $a = 1$, $b = 0$

3) if $N = N_{min} + 2$ then $a = 1$, $b = 1$ or $a = 2$, $b = 0$

This means that there always exist k+2 polynomials

$$R_i = P^{k+1-i,i} + \sum_{j=0}^{k-1} d_{ij} P^{k-1-j,j} \qquad i = 0(1)k+1 \tag{2}$$

that become zero at the knots of a N_{min} or $(N_{min}+1)$ - point cubature formula.

If the cubature formula is symmetric, $d_{ij} = 0$ for $i+j$ odd. The R_i's can be divided into two groups :

$$A = \{R_i | i \text{ even}\} \text{ and } B = \{R_i | i \text{ odd}\}$$

From $A \cup B$ it is possible to construct a canonical basis and then use theorem 2 to construct cubature formulae with N_{min} or $N_{min}+1$ knots. However, there is no garantee that such formulae exist. (T-method).

Instead of demanding that $NG(A \cup B)$ can be used as knots of a cubature formula, we demand that $NG(A)$ or $NG(B)$ can be used. Then we are certain to find all cubature formulae with N_{min}, $N_{min} + 1$ or $N_{min} + 2$ knots, if they exist, and we can find rules with more knots. (S-method).

We define $S_i = y^2 R_i - x^2 R_{i+2} \quad i = 0(1)k-1.$ (3)

S_i is a polynomial of degree k+3 that becomes zero at the elements of NG(C), where C = A or C = B, and S_i is orthogonal to all $P \in P_{k-4}$. From (2) and (3) it follows that S_i can be written as

$$S_i = y^2 . x^{k+1-i} . y^i - x^2 . x^{k-1-i} . y^{i+2} + \sum_{j=0}^{k+1} a_{ij} P^{k+1-j,j} + \sum_{j=0}^{k-1} b_{ij} P^{k-1-j,j}$$

$$+ \sum_{j=0}^{k-3} c_{ij} P^{k-3-j,j}$$

where a_{ij}, b_{ij} and c_{ij} are linear combinations of d_{ij}'s. Because the highest degree terms disappaer, degree S_i = k+1.

From corollary 3 follows : $c_{ij} = 0$. (4)

This is a set of linear equations in the unknowns : d_{ij}.

We also know that $S_i \in$ vct {C}. This means that $S_i - \sum_{\ell=0}^{[\frac{k+1-q}{2}]} a_{i,2\ell+q} R_{2\ell+q} \equiv 0$

where q = 0 if C = A or q = 1 if C = B.

This condition can only be fullfilled if

$$b_{ij} - \sum_{\ell=0}^{[\frac{k+1-q}{2}]} a_{i,2\ell+q} \cdot d_{2\ell+q,j} = 0 \qquad i,j = 0(1)k-1 \tag{5}$$

Because the a_{ij} and b_{ij} are linear combinations of the d_{ij}'s, (5) is a set of quadratic equations in the unknowns d_{ij}.

At the moment we are only interested in fully-symmetric cubature formulae. For fully symmetric regions this means that in (2)

$d_{ij} = 0$ if i+j odd

and

$$d_{ij} = d_{k+1-i,k-1-j} \; . \tag{6}$$

If k is even and NG(A) can be used as knots of a cubature formula, then from (6) it follows that NG(A) = NG(B) = NG(A ∪ B). In this case the method reduces to the T-method and we are only sure to find cubature formulae with N_{min} or N_{min} + 1 points, if they exist. Therefore we only consider the case where k is odd.

With (4) and (6) we can reduce the number of unknowns in the quadratic equations (5). When C = A, we can further reduce the number of unknowns by demanding that (0,0) is a knot of the cubature formula.

The number of linear independent quadratic equations (eqn) and unknowns (un) is given in the following table.

m	k	A		A + (0,0)		B	
		#eqn	#un	#eqn	#un	#eqn	#un
5	3	1	2	0	0	0	1
9	5	2	3	1	1	1	2
13	7	4	4	2	1	2	3
17	9	—	—	—	—	4	4

The number of elements N in NG(A) is always

$$N_{min} \leqslant N \leqslant \frac{(k+1)(k+3)}{2} = N_{min} + k + 2 .$$

NG(B) contains always x = 0 and y = 0. This means that the knots on the axis are still indetermined. They can be found by solving a 1-dimensional quadrature problem.

3. Results

A. Degree 5

a) C = B

The integral $\iint_R w(x,y)\, x^i y^j \,dx\, dy$ will be denoted by μ_{ij}.

$R_1 = P^{31} + d_{11} P^{11}$ $\qquad P^{31} = xy(x^2-b)$

$R_3 = P^{13} + d_{31} P^{11}$ $\qquad P^{11} = xy$

The only linear equation is : $d_{11} = d_{31} = d$. There are no quadratic equations. So, there is an infinite number of 9-point formulae with knots $(\pm r,\pm r)$ and weights w

$(\pm a,0)$ and weights α

$(0,\pm a)$ and weights α

$(0,0)$ and weight β

$$r^2 = b-d \qquad w = \frac{\mu_{22}}{4(b-d)^2}$$

$$a^2 = \frac{(\mu_{40}-\mu_{22})(b-d)}{\mu_{20}(b-d)-\mu_{22}} \qquad \alpha = \frac{(\mu_{20}(b-d)-\mu_{22})^2}{2(b-d)^2(\mu_{40}-\mu_{22})}$$

$$\beta = \frac{\mu_{00}(b-d)^2(\mu_{40}-\mu_{22}) - 2(\mu_{20}(b-d)-\mu_{22})^2 - \mu_{22}(\mu_{40}-\mu_{22})}{(b-d)(\mu_{40}-\mu_{22})}$$

$$d \neq b \qquad d \neq b - \frac{\mu_{22}}{\mu_{20}}$$

The cubature formula has only 8 knots if $\beta = 0$ or

$$0 = d^2(\mu_{00}\,\mu_{40} - \mu_{00}\,\mu_{20} - 2\,\mu_{20}^2)$$

$$+ d\,(-2b\,\mu_{00}\,\mu_{40} + 2b\,\mu_{00}\,\mu_{22} + 4b\,\mu_{20}^2 - 4\,\mu_{20}\,\mu_{22})$$

$$+ (b^2\,\mu_{00}\,\mu_{40} - b^2\,\mu_{00}\,\mu_{22} - 2\,\mu_{20}^2\,b^2 - 2\,\mu_{22}^2 - \mu_{22}\,\mu_{40} + \mu_{22}^2).$$

So, there can be at most two formulae with 8 knots. In the following table we give the solutions that lead to 8-point formulae with real knots.

region	number of solutions	solutions	reference
C_2	1	$d = -8/45$	Stroud C_n[5-4]
S_2	2	$d = \frac{-3+4\sqrt{3}}{24}$	Table III S_2(5-1)
		$d = \frac{-3-4\sqrt{3}}{24}$	Stroud S_n[5-3]
$E_2^{r^2}$	0	———	———
E_2^r	0	———	———

b) C = A

The linear and quadratic equations can be found in table II.5.A. The nonlinear system has an infinite number of solutions leading to fully symmetric formulae with 12, 9 and 8 knots. The 8-point formulae are the same as for C = B. From the equations it follows that there is only one 9-point formula for each region when C = A. These were already known by Stroud (1971). We give the references in the following table :

region	reference
C_2	Stroud C_n[5-7]
S_2	Stroud S_n[5-5]
$E_2^{r^2}$	Stroud $E_n^{r^2}$ [5-5]
E_2^r	Stroud E_n^r [5-4]

B. Degree 9

a) C = A

The linear and quadratic equations can be found in table II.9.A. The nonlinear system has an infinite number of solutions, leading to an infinite number of 24-point formulae and a finite number of 21 and 20-point formulae. In table III we only give the formulae with 20 or 21 real knots. The references to the formulae are given in the following table :

region	number of knots	reference
C_2	20	Table III C_2(9-1)
	21	Stroud C_2[9-2]
S_2	20	Table III S_2(9-1)
	20	Table III S_2(9-2)
	21	Table III S_2(9-3)
$E_2^{r^2}$	20	Stroud $E_2^{r^2}$ [9-1]
	21	Table III $E_2^{r^2}$ (9-1)
E_2^r	20	Table III E_2^r(9-1)
	21	Table III E_2^r(9-2)

b) C = B

The linear and quadratic equations can be found in table II.9.B. The nonlinear system has an infinite number of solutions, leading to an infinite number of 21-point formulae and a finite number of 20 point formulae. In table III we only give the formulae with 20 real knots. The references to the formulae are given in the following table :

region	reference
C_2	Stroud C_2[9-1]
S_2	Stroud S_2[9-2]
	Table III S_2(9-4)
	Table III S_2(9-5)
	Table III S_2(9-6)
$E_2^{r^2}$	Table III $E_2^{r^2}$ (9-2)
E_2^r	Stroud E_2^r[9-1]

C. Degree 13

a) C = A

The linear and quadratic equations can be found in table II.13.A. From this equations it follows that, for C_2, S_2 and $E_2^{r^2}$, there doesn't exist cubature formulae with C = A and the origin as a knot. The non-linear system has only a finite number of solutions, leading to formulae with 40 knots. For S_2 and $E_2^{r^2}$ we didn't found a formula with 40 real knots. The references to the real formulae are given in the following table :

region	reference
C_2	Table III C_2(13-1)
	Table III C_2(13-2)

b) C = B

The linear and quadratic equations can be found in table II.13.B. The nonlinear systems have an infinite number of solutions leading to an infinite number of 37-point formulae and one 36-point formula for each region.
In table III, we only give the 36-point formula. The references to the formulae are given in the following table :

region	reference
C_2	Table III C_2(13-3)
S_2	Table III S_2(13-1)
$E_2^{r^2}$	Table III $E_2^{r^2}$ (13-1)
E_2^r	Table III E_2^r(13-1)

D. Degree 17

C = B

The linear and quadratic equations can be found in table II.17.B. We found only a few real solutions of the nonlinear systems, leading to formulae with 57 knots.

For $E_2^{r^2}$ and S_2 we didn't found a formula with 57 real knots. The references to the real formulae are given in the following table :

region	reference
C_2	Table III C_2(17-1) Table III C_2(17-2)

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Table I :

Orthogonal polynomials
of degree 2,4,6 and 8
for $C_2, S_2, E_2^{r^2}$ and E_2^r

Table I.C_2

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REGION: C2
----------

P20 = ( 3*X**2 - 1) / 3

P11 = X*Y

P20 = ( 3*Y**2 - 1) / 3

P40 = ( 35*X**4 - 30*X**2 + 3 ) / 35

P31 = X*Y * ( 5*X**2 - 3 ) / 5

P22 = ( 9*X**2*Y**2 - 3*X**2 - 3*Y**2 + 1 ) / 9

P13 = X*Y * ( 5*Y**2 - 3 ) / 5

P04 = ( 35*Y**4 - 30*Y**2 + 3 ) / 35

P60 = ( 231*X**6 - 315*X**4 + 105*X**2 - 5 ) / 231

P51 = X*Y * (63*X**4 - 70*X**2 + 15 ) / 63

P42 = ( 105*X**4*Y**2 - 35*X**4 - 90*X**2*Y**2 + 30*X**2 + 9*Y**2 - 3 ) / 105

P33 = X*Y * (25*X**2*Y**2 - 15*X**2 - 15*Y**2 + 9 ) / 25

P24 = ( 105*X**2*Y**4 - 90*X**2*Y**2 - 35*Y**4 + 9*X**2 + 30*Y**2 - 3 ) / 105

P15 = X*Y * (63*Y**4 - 70*Y**2 + 15 ) / 63

P06 = ( 231*Y**6 - 315*Y**4 + 105*Y**2 - 5 ) / 231

P80 = ( 6435*X**8 - 12012*X**6 + 6930*X**4 - 1260*X**2 + 35 ) / 6435

P71 = X*Y * ( 429*X**6 - 693*X**4 + 315*X**2 - 35 ) / 429

P62 = ( 693*X**6*Y**2 - 231*X**6 - 945*X**4*Y**2 + 315*X**4 + 315*X**2*Y**2 - 105*X**2 - 15*Y**2 + 5 ) / 693

P53 = X*Y * ( 315*X**4*Y**2 - 189*X**4 - 350*X**2*Y**2 + 210*X**2 + 75*Y**2 - 45 ) / 315

P44 = ( 980*X**4*Y**4 - 840*X**4*Y**2 + 84*X**4 - 840*X**2*Y**4 + 720*X**2*Y**2 - 72*X**2 + 84*Y**4 - 72*Y**2 + 735 ) / 980

P35 = X*Y * ( 315*X**2*Y**4 - 350*X**2*Y**2 - 189*Y**4 + 75*X**2 + 210*Y**2 - 45 ) / 315

P26 = ( 693*X**2*Y**6 - 945*X**2*Y**4 - 231*Y**6 + 315*X**2*Y**2 + 315*Y**4 - 15*X**2 - 105*Y**2 + 5 ) / 693

P17 = X*Y * ( 429*Y**6 - 693*Y**4 + 315*Y**2 - 35 ) / 429

P08 = ( 6435*Y**8 - 12012*Y**6 +  6930*Y**4 - 1260*Y**2 + 35 ) / 6435
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Table I.S_2

```
REGION: S2
----------

P20 = ( 4*X**2 - 1 ) / 4

P11 = X*Y

P02 = ( 4*Y**2 - 1 ) / 4

P40 = ( 16*X**4 - 12*X**2 + 1 ) / 16

P31 =  X*Y * ( 8*X**2 - 3 ) / 8

P22 = ( 48*X**2*Y**2 - 6*X**2 - 6*Y**2 + 1 ) / 48

P13 =  X*Y * ( 8*Y**2 - 3 ) / 8

P04 = ( 16*Y**4 - 12*Y**2 + 1 ) / 16

P60 = ( 64*X**6 - 80*X**4 + 24*X**2 - 1 ) / 64

P51 = X*Y * ( 24*X**4 - 20*X**2 + 3 ) / 24

P42 = ( 960*X**4*Y**2 - 80*X**4 - 480*X**2*Y**2 + 48*X**2 + 24*Y**2 - 3 ) / 960

P33 = X*Y * ( 40*X**2*Y**2 - 10*X**2 - 10*Y**2 + 3 ) / 40

P24 = ( 960*X**2*Y**4 - 480*X**2*Y**2 + 24*X**2 - 80*Y**4 + 48*Y**2 - 3 ) / 960

P15 = X*Y * ( 24*Y**4 - 20*Y**2 + 3 ) / 24

P06 = ( 64*Y**6 - 80*Y**4 + 24*Y**2 - 1 ) / 64

P80 = ( 256*X**8 - 448*X**6 + 240*X**4,- 40*X**2 + 1 ) / 256

P71 = X*Y * ( 128*X**6 - 168*X**4 + 60*X**2 - 5 ) / 128

P62 = ( 1692*X**6*Y**2 - 112*X**6 - 1680*X**4*Y**2 + 120*X**4 + 360*X**2*Y**2 - 30*X**2 - 10*Y**2 + 1 ) / 1792

P53 = X*Y * ( 896*X**4*Y**2 - 168*X**4 - 560*X**2*Y**2 + 120*X**2 + 60*Y**2 - 15 ) / 896

P44 = ( 8960*X**4*Y**4 - 3360*X**4*Y**2 + 120*X**4 - 3360*X**2*Y**4 + 1440*X**2*Y**2 - 60*X**2 + 120*Y**4 - 60*Y**2 + 3 ) / 8960

P35 = X*Y * ( 896*X**2*Y**4 - 560*X**2*Y**2 + 60*X**2 - 168*Y**4 + 120*Y**2 - 15 ) / 896

P26 = ( 1792*X**2*Y**6 - 1680*X**2*Y**4 + 360*X**2*Y**2 - 10*X**2 - 112*Y**6 + 120*Y**4 - 30*Y**2 + 1 ) / 1792

P17 = X*Y * (128*Y**6 - 168*Y**4 + 60*Y**2 - 5 ) / 128

P08 = ( 256*Y**8 - 448*Y**6 + 240*Y**4 - 40*Y**2 + 1 ) / 256
```

Table I. $E_2^{r^2}$

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REGION: E2R2
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P20 = ( 2*X**2 - 1 ) / 2
P11 = X*Y
P02 = ( 2*Y**2 - 1 ) / 2
P40 = ( 4*X**4 - 12*X**2 + 3 ) / 4
P31 = X*Y * ( 2*X**2 - 3 ) / 2
P22 = ( 4*X**2*Y**2 - 2*X**2 - 2*Y**2 + 1 ) / 4
P13 = X*Y * ( 2*Y**2 - 3 ) / 2
P04 = ( 4*Y**4 - 12*Y**2 + 3 ) / 4
P60 = ( 8*X**6 - 60*X**4 + 90*X**2 - 15 ) / 8
P51 = X*Y * ( 4*X**4 - 20*X**2 + 15 ) / 4
P42 = ( 8*X**4*Y**2 - 4*X**4 - 24*X**2*Y**2 + 12*X**2 + 6*Y**2 - 3 ) /  8
P33 = X*Y * ( 4*X**2*Y**2 - 6*X**2 - 6*Y**2 + 9 ) / 4
P24 = ( 8*X**2*Y**4 - 4*Y**4 - 24*X**2*Y**2 + 6*X**2 + 12*Y**2 - 3 ) / 8
P15 = X*Y * ( 4*Y**4 - 20*Y**2 + 15 ) / 4
P06 = ( 8*Y**6 - 60*Y**4 + 90*Y**2 - 15 ) / 8
P80 = ( 16*X**8 - 224*X**6 + 840*X**4 - 840*X**2 + 105 ) / 16
P71 = X*Y * ( 8*X**6 - 84*X**4 + 210*X**2 - 105 ) / 8
P62 = ( 16*X**6*Y**2 - 8*X**6 - 120*X**4*Y**2 + 60*X**4 + 180*X**2*Y**2 - 90*X**2 - 30*Y**2 + 15 ) / 16
P53 = X*Y * ( 8*X**4*Y**2 - 12*X**4 - 40*X**2*Y**2 + 60*X**2 + 30*Y**2 - 45 ) / 8
P44 = ( 16*X**4*Y**4 - 48*X**4*Y**2 - 48*X**2*Y**4 + 12*X**4 + 144*X**2*Y**2 + 12*Y**4 - 36*X**2 - 36*Y**2 + 9 ) / 16
P35 = X*Y * ( 8*X**2*Y**4 - 40*X**2*Y**2 - 12*Y**4 + 30*X**2 + 60*X**2 - 45 )/ 8
P26 = ( 16*X**2*Y**6 - 120*X**2*Y**4 - 8*Y**6 + 180*X**2*Y**2 + 60*Y**4 - 30*X**2 - 90*Y**2 + 15 ) / 16
P17 = X*Y * ( 8*Y**6 - 84*Y**4 + 210*Y**2 - 105 ) / 8
P08 = ( 16*Y**8 - 224*Y**6 + 840*Y**4 - 840*Y**2 + 105 ) / 16
```

```
REGION: E2R
-----------

P20 = X**2 - 3

P11 = X*Y

P02 = Y**2 - 3

P40 = ( 7*X**4 - 282*X**2 + 12*Y**2 + 495 ) / 7

P31 = X*Y * ( X**2 - 21 )

P22 = ( 7*X**2*Y**2 - 45*X**2 - 45*Y**2 + 165 ) / 7

P13 = X*Y * ( Y**2 - 21 )

P04 = ( 7*Y**4 + 12*X**2 - 282*Y**2 + 495 ) / 7

P60 = ( 307*X**6 - 4 6683*X**4 + 4260*X**2*Y**2 + 288*Y**4 + 103 8681*X**2 - 6 3756*Y**2 - 138 4425 )/ 307

P51 = X*Y * ( X**4 - 106*X**2 + 4*Y**2 + 1197 )

P42 = ( 1535*X**4*Y**2 - 1 4141*X**4 - 9 7050*X**2*Y**2 + 1516*Y**4 + 67 1202*X**2 + 30 3723*Y**2 - 138 4425 ) / 1535

P33 = X*Y * ( 5*X**2*Y**2 - 153*X**2 - 153*Y**2 + 3591 ) / 5

P24 = ( 1535*X**2*Y**4 + 1516*X**4 - 9 7050*X**2*Y**2 - 1 4141*Y**4 + 30 3723*X**2 + 67 1202*Y**2 - 138 4425 ) / 1535

P15 = X*Y * ( Y**4 + 4*X**2 - 106*Y**2 + 1197 )

P06 = ( 307*Y**6 + 288*X**4 + 4260*X**2*Y**2 - 4 6683*Y**4 -6 3756*X**2 + 103 8681*Y**2 - 138 4425 ) / 307

P80 = ( 1 3422 9127*X**8 - 507 4872 0820*X**6 + 64 5841 3640*X**4*Y**2 + 9 3657 0880*X**2*Y**4 + 1 0566 9760*Y**6 +
      4 1543 8722 4890*X**4 - 6127 6157 2080*X**2*Y**2 - 352 2190 7280 *Y**4 - 67 1032 6723 1700*X**2 + 4 8004 7701 1400*Y**2 +
      74 8518 5511 8775 ) / 1 3422 9127

P71 = X*Y* ( 4853*X**6 - 143 4475*X**4 + 9 9820*X**2*Y**2 + 8063 8635*X**2 + 5600*Y**4 - 490 1820*Y**2 - 6 4991 2725 ) / 4853

P62 = ( 1 3422 9127*X**6*Y**2 - 15 8179 6685*X**6 - 286 0072 4375*X**4*Y**2 + 12 0641 1860*X**2*Y**4 + 3722 2880*Y**6 +
      2748 5760 2775*X**4 + 9264 4696 6065*X**2*Y**2 - 244 0019 2380*Y**4 - 7 0181 9016 7275*X**2 - 1 8822 0843 5625*Y**2 +
      10 6931 2215 9825 ) / 1 3422 9127

P51 = X*Y * (4853*X**4*Y**2 - 19 0665*X**4 - 68 5170*X**2*Y**2 + 2233 9350*X **2 + 1 5060*Y**4 + 1011 9285*Y**2 - 2 7853 4025 )/4853

P44 = ( 1 3422 9127*X**4*Y**4 + 1 0564 2636*X**6 - 110 6336  7642*X**4*Y**2 - 110 6336 7642*X**2*Y**4 + 1 0564 2636*Y**6 +
      413 3774 5335*X**4 + 7586 6504 6340*X**2*Y**2 + 413 3774 5335*Y**4 - 2 6701 1958 0870*X**2 - 2 6701 1958 0870*Y**2 +
      6 4158 7329 5895 ) /  1 3422 9127

P35 = X*Y * ( 4853*X**2*Y**4 + 1 5060*X**4 - 68 5170*X**2*Y**2 + 1011 9285*X**2 - 19 0665*Y**4 + 2233 9350*Y**2 - 2 7853 4025 )/4853

P26 = ( 1 3422 9127*X**2*Y**6 + 3722 2880*X**6 + 12 0641 1860*X**4 *Y**2 - 286 0072 4375*X**2*Y**4 - 15 8179 6685*Y**6 -
      244 0019 2380*X**4 + 9264 4696 6065*X**2*Y**2 + 2748 5760 2775*Y**4 - 1 8822 0843 5625*X**2 - 7 0181 9016 7275*Y**2 +
      10 6931 2215 9825 ) / 1 3422 9127

P17 = X*Y * ( 4853*Y**6 + 5600*X**4 + 9 9820*X**2*Y**2 - 143 4475*Y**4 - 490  1820*X**2 + 8063 8635*Y**2 - 6 4991 2725 ) / 4853

P08 = ( 1 3422 9127*Y**8 + 1 0566 9760*X**6 + 9 3657 0880*X**4*Y**2 + 64 5841 3640*X**2*Y**4 - 507 4872 0820*Y**6 -
      352 2190 7280*X**4 - 6127 6157 2080*X**2*Y**2+4 1543 8722 4890*Y**4 + 4 8004 7701 1400*X**2 - 67 1032 6723 1700*Y**2 +
      74 8518 5511 8775 ) / 1 3422 9127
```

Table II :

Solutions of the linear equations and
the quadratic equations

Table II.5.A.C_2

```
REGION: C2                DEGREE = 5              C = A
----------                ----------              -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:
D(0,2) = D(2,0)
D(2,2) = D(2,0)
D(4,0) = D(2,0)
D(4,2) = D(0,0)

THE QUADRATIC EQUATION IS:
> 315*D(0,0)*D(2,0) - 315*D(2,0)**2 - 60*D(2,0) + 14 = 0

IF THE ORIGIN IS A KNOT OF THE CUBATURE FORMULA ...
THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:
D(0,0) = 19 / 210
D(2,0) = 1 / 6
THE QUADRATIC EQUATION IS IDENTICAL ZERO.
```

Table II.5.A.S_2

```
REGION: S2              DEGREE = 5              C = A
----------              ----------              -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:
D(0,0) = 3*D(0,2) - 2*D(2,0)
D(2,2) = D(2,0)
D(4,0) = D(0,2)
D(4,2) = 3*D(0,2) - 2*D(2,0)

THE QUADRATIC EQUATION IS:
> 192*D(0,2)**2 - 192*D(2,0)**2 - 48*D(2,0) + 1 = 0

IF THE ORIGIN IS A KNOT OF THE CUBATURE FORMULA ...
THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:
D(0,2) = 1 / 12
D(2,0) = 1 / 24
THE QUADRATIC EQUATION IS IDENTICAL ZERO.
```

Table II.5.A.$E_2^{r^2}$

```
REGION: E2R2                DEGREE = 5            C = A
------------                ----------            -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:
D(0,2) = D(2,0)
D(2,2) = D(2,0)
D(4,0) = D(2,0)
D(4,2) = D(0,0)

THE QUADRATIC EQUATION IS:
> 4*D(0,0)*D(2,0) - 4*D(2,0)**2 - 8*D(2,0) + 1 = 0

IF THE ORIGIN IS A KNOT OF THE CUBATURE FORMULA ...
THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:
D(0,0) = 5 / 4
D(2,0) = 1 / 4
THE QUADRATIC EQUATION IS IDENTICAL ZERO.
```

Table II.5.A.E_2^r

```
REGION: E2R              DEGREE = 5              C = A
-----------              ----------              -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(0,0) = - 6*D(0,2) + 7*D(2,0)

D(2,2) = D(2,0)

D(4,0) = D(0,2)

D(4,2) = - 6*D(0,2) + 7*D(2,0)

THE QUADRATIC EQUATION IS:

> 49*D(0,2)**2 + 168*D(0,2) - 49*D(2,0)**2 + 336*D(2,0) - 726 = 0


IF THE ORIGIN IS A KNOT OF THE CUBATURE FORMULA ...

THE SOLUTIONS OF THE LINEAR EQUATION IS:

D(0,2) = 11 / 14

D(2,0) = 55 / 14

THE QUADRATIC EQUATION IS IDENTICAL ZERO.
```

Table II.9.A.C_2

```
REGION: C2                  DEGREE = 9                  C = A
----------                  ----------                  -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(0,2) = 36*D(2,0) / 49

D(0,4) = D(4,0)

D(2,2) = 36*D(4,0) / 49

D(2,4) = D(4,0)

D(4,2) = 36*D(4,0) / 49

D(4,4) = D(2,0)

D(6,0) = D(4,0)

D(6,2) = 36*D(2,0) / 49

D(6,4) = D(0,0)

THE QUADRATIC EQUATIONS ARE:

> 9 7020*D(2,0)**2 + 16 8300*D(4,0)**2 - 9 7020*D(0,0)*D(4,0) - 16 8300*D(2,0)*D(4,0) +
  1 8480*D(2,0) - 1680*D(4,0) - 8624  = 0

> 1 4256*D(2,0)**2 - 1 4256*D(4,0)**2 + 3 8808*D(0,0)*D(4,0) - 3 8808*D(2,0)*D(4,0) - 6720*D(4,0)  =  0


IF THE ORIGIN IS A KNOT OF THE CUBATURE FORMULA ...

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(0,0) = ( - 1 2507*D(2,0) + 2324 ) / 2 8413

D(4,0) = ( - 21*D(2,0) + 7 ) / 41

THE QUADRATIC EQUATION IS:

> 37 3860*D(2,0)**2 - 4 9980*D(2,0) - 9751 = 0
```

Table II.9.A.S_2

```
REGION: S2              DEGREE = 9              C = A
----------              ----------              -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:
D(0,0) = ( 140*D(0,4) - 153*D(2,2) + 250*D(2,4) ) / 12
D(0,2) = ( 20*D(0,4) - 15*D(2,2) + 22*D(2,4) ) / 2
D(2,0) = ( 17*D(2,2) - 30*D(2,4) ) / 12
D(4,0) = D(2,4)
D(4,2) = D(2,2)
D(4,4) = ( 17*D(2,2) - 30*D(2,4) ) / 12
D(6,0) = D(0,4)
D(6,2) = ( 20*D(0,4) - 15*D(2,2) + 22*D(2,4) ) / 2
D(6,4) = ( 140*D(0,4) - 153*D(2,2) + 250*D(2,4) ) / 12

THE QUADRATIC EQUATIONS ARE:
> 112*D(0,4)**2 + 96*D(0,4)*D(2,4) - 102*D(2,2)**2 + 184*D(2,2)*D(2,4)  -
  3*D(2,2) - 40*D(2,4)**2 - 2*D(2,4) = 0
> 400*D(0,4)**2 + 1440*D(0,4)*D(2,4) - 425*D(2,2)**2- 140*D(2,2)*D(2,4 ) -
  55*D(2,2) + 1100*D(2,4)**2 + 70*D(2,4) + 1 = 0


IF THE ORIGIN IS A KNOT OF THE CUBATURE FORMULA ...
THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:
D(0,4) = - ( 50*D(2,4) - 3 ) / 70
D(2,2) = ( 30*D(2,4) + 1 ) / 35

THE QUADRATIC EQUATION IS:
> 2560*D(2,4)**2 - 576*D(2,4) + 3 = 0
```

Table II.9.A.$E_2^{r^2}$

```
REGION: E2R2              DEGREE = 9          C = A
-------------             ----------          -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:
D(0,2) = 6*D(2,0)
D(0,4) = D(4,0)
D(2,2) = 6*D(4,0)
D(2,4) = D(4,0)
D(4,2) = 6*D(4,0)
D(4,4) = D(2,0)
D(6,0)= D(4,0)
D(6,2) = 6*D(2,0)
D(6,4) = D(0,0)

THE QUADRATIC EQUATIONS ARE:
> 2*D(2,0)**2 + 14*D(4,0)**2 - 2*D(0,0)*D(4,0) - 14*D(2,0)*D(4,0) + 4*D(2,0) + 4*D(4,0) - 1 = 0

> 3*D(2,0)**2 - 3*D(4,0)**2 + D(0,0)*D(4,0) - D(2,0)*D(4,0) - 4*D(4,0)  = 0

IF THE ORIGIN IS A KNOT OF THE CUBATURE FORMULA ...
THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:
D(0,0) = ( - 5*D(2,0) + 7 ) / 3
D(4,0) = ( - 2*D(2,0) + 1 ) / 6
THE QUADRATIC EQUATION IS:
> 128*D(2,0)**2 + 16*D(2,0) - 13 = 0
```

Table II.9.A.E_2^r

```
REGION: E2R                 DEGREE = 9              C = A
-----------                 ----------              -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(0,0) = ( 5 0112*D(0,4) - 4 0669*D(2,2) + 77 7000*D(2,4) ) / 378

D(0,2) = -2 * ( 2700*D(0,4) - 1874*D(2,2) + 3 4755*D(2,4) ) / 63

D(2,0) = - ( 67*D(0,4) - 1416*D(2,4) ) / 54

D(4,0) = D(2,4)

D(4,2) = D(2,2)

D(4,4) = - ( 67*D(0,4) - 1416*D(2,4) ) / 54

D(6,0) = D(0,4)

D(6,2) = -2 * ( 2700*D(0,4) - 1874*D(2,2) + 3 4755*D(2,4) ) / 63

D(6,4) = ( 5 0112*D(0,4) - 4 0669*D(2,2) + 77 7000*D(2,4) ) / 378

THE QUADRATIC EQUATIONS ARE:

> 206 1225 6300*D(0,4)*D(2,4) + 203 5712 0880*D(0,4) - 4 3554 8575*D(2,2)**2  - 29 0428 1400*
  D(2,2)*D(2,4) - 83 8813 6440*D(2,2) + 2184 1404 4800*D(2,4)**2 + 1218 2044 8960*D(2,4) +
  316 3868 1648  = 0

> 2656 6908 1200*D(0,4)**2 - 1717 6880.2500*D(0,4)*D(2,4) + 3288 1136 7600*D(0,4) -
  1479 2144 9275*D(2,2)**2 + 5 6544 3105 5400*D(2,2)*D(2,4) - 4091 6858 0760*D(2,2)- 53 9118 2288 5200*
  D(2,4)**2 + 7 4945 4775 7760*D(2,4) - 2661 4028 3472  = 0


IF THE ORIGIN IS A KNOT OF THE CUBATURE FORMULA ...

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(0,4) = 3 * (50655*D(2,4) - 62702 ) / 16885

D(2,2) = 6 * (16885*D(2,4) - 7911 ) / 3377

THE QUADRATIC EQUATION IS:

> 8 2939 1200*D(2,4)**2 - 25 5771 5240*D(2,4) + 18 0105 6919  = 0
```

Table II.9.B.C_2

```
DEGREE: C2               DEGREE = 9               C = B
----------               ----------               -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(1,3) = D(3,1)

D(3,3) = D(3,1)

D(5,1) = D(3,1)

D(5,3) = D(1,1)

THE QUADRATIC EQUATIONS IS:

> - 3150*D(3,1)**2 + 3150*D(1,1)*D(3,1) + 280*D(3,1) + 108 = 0
```

Table II.9.B.S_2

```
REGION: S2                DEGREE = 9                C = B
----------                ----------                -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(1,1) = ( 5*D(1,3) - 2*D(3,3) ) / 3

D(3,1) = D(3,3)

D(5,1) = D(1,3)

D(5,3) = ( 5*D(1,3) - 2*D(3,3) ) / 3

THE QUADRATIC EQUATION IS:

> 80*D(1,3)**2 - 80*D(3,3)**2 - 20*D(3,3) + 1 = 0
```

Table II.9.B.$E_2^{r^2}$

```
REGION: E2R2                DEGREE = 9             C = B
------------                ----------             -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(1,3) = D(3,1)

D(3,3) = D(3,1)

D(5,1) = D(3,1)

D(5,3) = D(1,1)

THE QUADRATIC EQUATION IS:

> 4*D(1,1)*D(3,1) - 4*D(3,1)**2 - 8*D(3,1) + 3 = 0
```

Table II.9.B.E_2^r

```
REGION: E2R                DEGREE = 9            C = B
-----------                ----------            -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(1,1) = - ( 4*D(1,3) - 5*D(3,3) )

D(3,1) = D(3,3)

D(5,1) = D(1,3)

D(5,3) = - ( 4*D(1,3) - 5*D(3,3) )

THE QUADRATIC EQUATION IS:

> 25*D(1,3)**2 - 25*D(3,3)**2 + 200*D(1,3) + 480*D(3,3) - 5054 = 0
```

Table II.13.A.C_2

```
REGION: C2                DEGREE = 13             C = A
----------                -----------             -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:
D(0,2) = 1125*D(2,0) / 1573
D(0,4) = 1125*D(4,0) / 1573
D(0,6) = 1573*D(4,2) / 1125
D(2,2) = 1125*D(4,0) / 1573
D(2,4) = D(4,2)
D(2,6) = 1573*D(4,2) / 1125
D(4,4) = D(4,2)
D(4,6) = D(4,0)
D(6,0) = 1573*D(4,2) / 1125
D(6,2) = D(4,2)
D(6,4) = 1125*D(4,0) / 1573
D(6,6) = D(2,0)
D(8,0) = 1573*D(4,2) / 1125
D(8,2) = 1125*D(4,0) / 1573
D(8,4) = 1125*D(2,0) / 1573
D(8,6) = D(0,0)

THE QUADRATIC EQUATIONS ARE:
> 115 6155*D(2,0)*D(4,2) - 82 6875*D(4,0)**2 + 82 6875*D(4,0)*D(4,2) - 115 6155*D(4,2)**2 + 2 0020*D(4,2) +3 7752 = 0

> 139 3579 6875*D(0,0)*D(4,0) - 139 3579 6875*D(2,0)**2 + 139 3579 6875*D(2,0)*D(4,2) - 26 5443 7500*D(2,0) -
  99 6679 6875*D(4,0)**2 + 334 2113 7750*D(4,0)*D(4,2) + 4826 2500*D(4,0) - 467 3017 7494*D(4,2)**2 + 12 3873 7500 = 0

> 3711 4935*D(0,0)*D(4,2) + 1898 4375*D(2,0)*D(4,0) - 3711 4935*D(2,0)*D(4,2) - 2654 4375*D(4,0)*D(4,2) - 629 8292*D(4,2) = 0

> 8660 1515*D(0,0)*D(4,2) - 6193 6875*D(2,0)*D(4,0) + 1 4853 8390*D(2,0)*D(4,2) + 4429 6875*D(4,0)**2 - 1 4853 8390*
  D(4,0)*D(4,2) - 1072 5000*D(4,0) - 8660 1515*D(4,2)**2 + 179 9512*D(4,2) = 0
```

```
IF THE ORIGIN IS A KNOT OF THE CUBATURE FORMULA ...
THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:
D(0,0) = ( 2588383*D(4,2) - 2046225 ) / 2091375
D(2,0) = - ( 172232*D(4,2) + 1874625 ) / 117000
D(4,0) = - 33 * ( 8*D(4,2) - 105 ) / 200

THE QUADTRATIC EQUATIONS ARE:
> 480 5081 8816*D(4,2)**2 - 1255 8566 7000*D(4,2) + 1 7879 7653 4375 = 0

> 26 8815 7696*D(4,2)**2 + 4033 5048 9000*D(4,2) - 1 8031 6099 6875 = 0
```

Table II.13.A.S_2

```
REGION: S2                DEGREE = 13               C = A
----------                -----------               -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(0,0) = ( 1 0395*D(0,6) - 20 4120*D(2,6) + 7140*D(4,0) + 3 9278*D(4,2) ) / 225
D(0,2) =( 945*D(0,6) - 1 5120*D(2,6) + 330*D(4,0) + 2926*D(4,2) ) / 15
D(0,4) = 3 * ( 7*D(0,6) - 72*D(2,6) + 14*D(4,2) )
D(2,0) = ( 10 3950*D(2,6) - 8295*D(4,0) - 1 9639*D(4,2) ) / 2250
D(2,2) = ( 9450*D(2,6) + 105*D(4,0) - 1859*D(4,2) ) / 150
D(2,4) = ( 525*D(2,6) + 60*D(4,0) - 98*D(4,2) ) / 25
D(4,4) = D(4,2)
D(4,6) = D(4,0)
D(6,0) = D(2,6)
D(6,2) = ( 525*D(2,6) + 60*D(4,0) - 98*D(4,2) ) /25
D(6,4) = ( 9450*D(2,6) + 105*D(4,0) - 1859*D(4,2) ) / 150
D(6,6) = ( 10 3950*D(2,6) - 8295*D(4,0) - 1 9639*D(4,2) ) / 2250
D(8,0) = D(0,6)
D(8,2) = 3 * (7*D(0,6) - 72*D(2,6) + 14*D(4,2) )
D(8,4) = ( 945*D(0,6) - 1 5120*D(2,6) + 330*D(4,0) + 2926*D(4,2) ) / 15
D(8,6) = ( 1 0395*D(0,6) - 20 4120*D(2,6) + 7140*D(4,0) + 3 9278*D(4,2) ) / 225

THE QUADRATIC EQUATIONS ARE:

> 2 2680 0000*D(0,6)*D(2,6) - 7560 0000*D(0,6)*D(4,0) + 46 2672 0000*D(2,6)**2 + 13 4769 6000*D(2,6)*D(4,0) -
  22 8439 6800*D(2,6)*D(4,2) + 1890 0000*D(2,6) - 2832 4800*D(4,0)**2 - 2 5304 8320*D(4,0)*D(4,2) - 189 0000*D(4,0) +
  2 6396 6528*D(4,2)**2 - 478 8000*D(4,2) + 8 4375  = 0

> 1 9800*D(0,6)**2 + 2 7000*D(0,6)*D(2,6) + 9000*D(0,6)*D(4,0) - 471 2400*D(2,6)**2 - 8 4840*D(2,6)*D(4,0) +
  172 3472*D(2,6)*D(4,2) - 2250*D(2,6) + 8400*D(4,0)**2 + 1 8920*D(4,0)*D(4,2) - 157112*D(4,2)**2 + 375*D(4,2) = 0

> 283 5000*D(0,6)**2 + 1417 5000*D(0,6)*D(2,6) + 1012 5000*D(0,6)*D(4,0) - 10 8297 0000*D(2,6)**2 - 1 1812 5000*
  D(2,6)*D(4,0) + 3 6945 0000*D(2,6)*D(4,2) - 253 1250*D(2,6) + 335 1600*D(4,0)**2 + 2389 0440*D(4,0)*D(4,2) +
  6 7500*D(4,0) - 3095 2376*D(4,2)**2 + 48 0375*D(4,2) = 0
```

```
> 2 1168 0000*D(0,6)**2 + 22 6800 0000*D(0,6)*D(2,6) + 31 7520 0000*D(0,6)*D(4,0) - 1926 2880 0000*D(2,6)**2 -
  400 9219 2000*D(2,6)*D(4,0) + 674 9487 3600*D(2,6)*D(4,2) - 7 9380 0000*D(2,6) + 10 3652 6400*D(4,0)**2 +
  78 4061 3760*D(4,0)*D(4,2) + 4347 0000*D(4,0) - 58 4750 7904*D(4,2)**2 + 1 5107 4000*D(4,2) + 25 3125 = 0

  IF THE ORIGIN IS A KNOT OF THE CUBATURE FORMULA ...
  THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:
  D(0,6) = - ( 245*D(4,2) - 27 ) / 1575
  D(2,6) = ( 4480*D(4,2) + 27 ) / 2 5200
  D(4,0) = - ( 56*D(4,2) - 3 ) / 280

  THE QUADRATIC EQUATIONS ARE:
> 12 5440*D(4,2)**2 + 4 2336*D(4,2) - 243 = 0

> 12 5440*D(4,2)**2 + 4 2336*D(4,2) - 3267 = 0
```

Table II.13.A.$E_2^{r^2}$

```
REGION: E2R2              DEGREE = 13              C = A
------------              -----------              -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(0,2) = 15*D(2,0)

D(0,4) = 15*D(4,0)

D(0,6) = D(4,2) / 15

D(2,2) = 15*D(4,0)

D(2,4) = D(4,2)

D(2,6) = D(4,2) / 15

D(4,4) = D(4,2)

D(4,6) = D(4,0)

D(6,0) = D(4,2) / 15

D(6,2) = D(4,2)

D(6,4) = 15*D(4,0)

D(6,6) = D(2,0)

D(8,0) = D(4,2) / 15

D(8,2) = 15*D(4,0)

D(8,4) = 15*D(2,0)

D(8,6) = D(0,0)

THE QUADRATIC EQUATIONS ARE:

> 2*D(2,0)*D(4,2) - 30*D(4,0)**2 + 30*D(4,0)*D(4,2) - 2*D(4,2)**2 - 4*D(4,2) + 3 = 0

> D(0,0)*D(4,2) + 225*D(2,0)*D(4,0) - D(2,0)*D(4,2) - 15*D(4,0)*D(4,2) - 6*D(4,2) = 0

> D(0,0)*D(4,2) - 15*D(2,0)*D(4,0) + 16*D(2,0)*D(4,2) + 225*D(4,0)**2 - 16*D(4,0)*D(4,2) - 60*D(4,0) - D(4,2)**2 - 4*D(4,2) = 0

> 450*D(0,0)*D(4,0) - 450*D(2,0)**2 + 450*D(2,0)*D(4,2) - 900*D(2,0) - 6750*D(4,0)**2 + 480*D(4,0)*D(4,2) -
  900*D(4,0) - 32*D(4,2)**2 + 225 = 0
```

```
IF THE ORIGIN IS A KNOT OF THE CUBATURE FORMULA ...
THE SOLUTIONS OF THE THE LINEAR EQUATIONS ARE:
D(0,0) = - ( 14*D(4,2) - 87 ) / 30
D(2,0) = ( 20*D(4,2) + 3 ) / 60
D(4,0) = - ( 4*D(4,2) - 3 ) / 20

THE QUADRATIC EQUATIONS ARE:
> 1024*D(4,2)**2 - 288*D(4,2) - 351 = 0

> 1024*D(4,2)**2 - 288*D(4,2) - 243 = 0
```

Table II.13.B.C_2

```
REGION: C2                 DEGREE = 13               C = B
----------                 -----------               -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(1,1) = D(7,5)

D(1,3) = 2500*D(3,1) / 2673

D(1,5) = D(5,1)

D(3,3) = 2500*D(5,1) / 2673

D(3,5) = D(5,1)

D(5,3) = 2500*D(5,1) / 2673

D(5,5) = D(3,1)

D(7,1) = D(5,1)

D(7,3) = 2500*D(3,1) / 2673

THE QUADRATIC EQUATIONS ARE:

> 608 1075*D(3,1)**2 - 1176 8575*D(3,1)*D(5,1) - 54 0540*D(3,1) + 1176 8575*D(5,1)**2 - 608 1075*D(5,1)*D(7,5) -
  4 1580*D(5,1) - 41 6988 = 0

> 8 1250*D(3,1)**2 - 17 3745*D(3,1)*D(5,1) - 8 1250*D(5,1)**2 + 17 3745*D(5,1)*D(7,5) + 1 6632*D(5,1) = 0
```

Table II.13.B.S_2

```
REGION: S2                 DEGREE = 13                C = B
----------                 -----------                -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(1,1) = ( 1134*D(1,5) - 1221*D(3,3) + 1666*D(3,5) ) / 270

D(1,3) = 2 * ( 63*D(1,5) - 42*D(3,3) + 52*D(3,5) ) / 27

D(3,1) = ( 111*D(3,3) - 140*D(3,5) ) / 90

D(5,1) = D(3,5)

D(5,3) = D(3,3)

D(5,5) = ( 111*D(3,3) - 140*D(3,5) ) / 90

D(7,1) = D(1,5)

D(7,3) = 2 * ( 63*D(1,5) - 42*D(3,3) + 52*D(3,5) ) / 27

D(7,5) = ( 1134*D(1,5) - 1221*D(3,3) + 1666*D(3,5) ) / 270

THE QUADRATIC EQUATIONS ARE:

> 1161 2160*D(1,5)*D(3,5) - 104 4288*D(3,3)**2 - 728 4480*D(3,3)*D(3,5) - 89 2080*D(3,3) + 796 5440*D(3,5)**2 +
  126 0000*D(3,5) + 5 4675 = 0

> 2916*D(1,5)**2 + 3240*D(1,5)*D(3,5) - 2664*D(3,3)**2 + 1968*D(3,3)*D(3,5) - 135*D(3,3) + 1204*D(3,5)**2 - 180*D(3,5) = 0
```

Table II.13.B.$E_2^{r^2}$

```
REGION: E2R2              DEGREE = 13          C = B
------------              -----------          -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(1,1) = D(7,5)

D(1,3) = 10*D(3,1) / 3

D(1,5) = D(5,1)

D(3,3) = 10*D(5,1) / 3

D(3,5) = D(5,1)

D(5,3) = 10*D(5,1) / 3

D(5,5) = D(3,1)

D(7,1) = D(5,1)

D(7,3) = 10*D(3,1) / 3

THE QUADRATIC EQUATIONS ARE:

> 6*D(3,1)**2 - 26*D(3,1)*D(5,1) + 12*D(3,1) + 26*D(5,1)**2 - 6*D(5,1)*D(7,5) + 12*D(5,1) - 9  = 0

> 5*D(3,1)**2 - 3*D(3,1)*D(5,1) - 5*D(5,1)**2 + 3*D(5,1)*D(7,5) - 12*D(5,1) = 0
```

Table II.13.B.E_2^r

```
REGION: E2R              DEGREE = 13              C = B
-----------              -----------              -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(1,1) = ( 652 0688*D(1,5) - 452 6457*D(3,3) + 3013 5612*D(3,5) ) /  25 0190

D(1,3) = - 2 * ( 21 6154*D(1,5) - 10 4061*D(3,3) + 52 0191*D(3,5) ) / 2 5019

D(3,1) = - ( 1257*D(3,3) - 1 1380*D(3,5) ) / 1270

D(5,1) = D(3,5)

D(5,3) = D(3,3)

D(5,5) = - ( 1257*D(3,3) - 1 1380*D(3,5) ) / 1270

D(7,1) = D(1,5)

D(7,3) = - 2 * ( 21 6154*D(1,5) - 10 4061*D(3,3) + 52 0191*D(3,5) ) / 2 5019

D(7,5) = ( 652 0688*D(1,5) - 452 6457*D(3,3) + 3013 5612*D(3,5) ) / 25 0190

THE QUADRATIC EQUATIONS ARE:

> 296 2938 4321 7580*D(1,5)*D(3,5) + 919 4694 5783 1600*D(1,5) - 23 2821 7870 4659*D(3,3)**2 + 78 2213 9417 9560 *
  D(3,3)*D(3,5) - 60 0541 8324 4200*D(3,3) + 580 6800 8668 4720*D(3,5)**2 - 3358 6721 1738 9600*D(3,5) +
  1 5528 4748 7847 2000 = 0

> 9751 8660 8087 3992*D(1,5)**2 - 6465 2836 0456 8220*D(1,5)*D(3,5) + 2442 5569 7939 2400*D(1,5) - 3080 6606 0807 5293 *
  D(3,3)**2 + 3 6516 8080 0231 3016*D(3,3)*D(3,5) - 3 5984 5120 8515 2440*D(3,3) - 9 7704 3286 9360 7752*D(3,5)**2 +
  15 8386 9231 7786 9600*D(3,5) - 16 8754 0287 2352 5600 = 0
```

Table II.17.B.C_2

```
REGION: C2                DEGREE = 17              C = B
----------                -----------              -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(1,1) = D(9,7)
D(3,1) = 1859*D(9,5) / 1715
D(5,1) = 1859*D(9,3) / 1715
D(7,1) = D(9,1)
D(1,3) = 1715*D(3,1) / 1859
D(1,5) = 1715*D(5,1) / 1859
D(1,7) = D(7,1)
D(3,3) = 1715*D(5,1) / 1859
D(3,5) = 1715*D(7,1) / 1859
D(3,7) = D(9,1)
D(5,3) = 1715*D(7,1) / 1859
D(5,5) = 1715*D(9,1) / 1859
D(5,7) = 1859*D(9,3) / 1715
D(7,3) = 1715*D(9,1) / 1859
D(7,5) = D(9,3)
D(7,7) = 1859*D(9,5) / 1715

THE QUADRATIC EQUATIONS ARE:

> 4085 8027 0875*D(9,1)**2 - 4428 8671 9275*D(9,1)*D(9,3) - 4800 7370 9115*D(9,1)*D(9,5) - 30 2828 5260*
  D(9,1) + 5203 8310 5099*D(9,3)**2 - 153 9437 9000 = 0

> 37 9015*D(9,1)*D(9,3) + 41 0839*D(9,1)*D(9,5) - 37 9015*D(9,1)*D(9,7) - 37044*D(9,1) - 41 0839*D(9,3)*D(9,5) = 0

> 7717 6273 3875*D(9,1)**2 + 17433 6969 8070*D(9,1)*D(9,3) - 1 7433 6969 8070*D(9,1)*D(9,5) - 8365 6380 3075*
  D(9,1)*D(9,7) - 74 0247 5080*D(9,1) - 9068 0589 4995*D(9,3)**2 + 9829 4586 5187*D(9,3)*D(9,5) - 868 0535 0632*D(9,3) = 0

> 1 6083 2653 6950*D(9,1)**2 - 1 7433 6969 8070*D(9,1)*D(9,3) - 8365 6380 3075*D(9,1)*D(9,5) + 9068 0589 4995*
  D(9,3)**2 - 9068 0589 4995*D(9,3)*D(9,7) - 18 2364 1820*D(9,3) + 9829 4586 5187*D(9,5)**2 - 806 0496 8444*D(9,5) -
  573 6437 5068 = 0
```

Table II.17.B.$E_2^{r^2}$

```
REGION: E2R2                DEGREE = 17             C = B
------------                -----------             -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(1,1) = D(9,7)
D(1,3)= 7*D(3,1)
D(1,5)= 7*D(5,1)
D(1,7)= D(7,1)
D(3,1) = D(9,5) / 7
D(3,3)= 7*D(5,1)
D(3,5)= 7*D(7,1)
D(3,7)= D(9,1)
D(5,1) = D(9,3) / 7
D(5,3)= 7*D(7,1)
D(5,5)= 7*D(9,1)
D(5,7)= D(9,3) / 7
D(7,1) = D(9,1)
D(7,3)= 7*D(9,1)
D(7,5)= D(9,3)
D(7,7)= D(9,5) / 7

THE QUADRATIC EQUATIONS ARE:

> 686*D(9,1)**2 - 98*D(9,1)*D(9,3) - 14*D(9,1)*D(9,5) + 196*D(9,1) + 2*D(9,3)**2 - 35 = 0

> 7*D(9,1)*D(9,3) + D(9,1)*D(9,5) - 7*D(9,1)*D(9,7) + 42*D(9,1) - D(9,3)*D(9,5) = 0

> 343*D(9,1)**2 + 56*D(9,1)*D(9,3) - 56*D(9,1)*D(9,5) - 49*D(9,1)*D(9,7) + 196*D(9,1) - 7*D(9,3)**2 +
  D(9,3)*D(9,5) + 28*D(9,3) = 0

> 784*D(9,1)**2 - 112*D(9,1)*D(9,3) - 98*D(9,1)*D(9,5) + 14*D(9,3)**2 - 14*D(9,3)*D(9,7) + 28*D(9,3) +
  2*D(9,5)**2 + 28*D(9,5) - 147 = 0
```

Table II.17.B.S_2

```
REGION: S2                DEGREE = 17             C = B
----------                -----------             -----

THE SOLUTIONS OF THE LINEAR EQUATIONS ARE:

D(1,1) = ( 2 1021*D(1,7) - 15 5232*D(3,7) - 3 7170*D(5,1) + 5 1704*D(5,3) ) / 1715

D(1,3) = ( 4851*D(1,7) - 2 8224*D(3,7) - 8197*D(5,1) + 9675*D(5,3) ) / 245

D(1,5) = ( 441*D(1,7) - 1568*D(3,7) - 567*D(5,1) + 555*D(5,3) ) / 49

D(3,1) = ( 2 1021*D(3,7) + 2331*D(5,1) - 6463*D(5,3) ) / 1715

D(3,3) = ( 4851*D(3,7) + 1386*D(5,1) - 1618*D(5,3) ) / 245

D(3,5) = ( 441*D(3,7) + 175*D(5,1) - 135*D(5,3) ) / 49

D(5,5) = D(5,3)

D(5,7) = D(5,1)

D(7,1) = D(3,7)

D(7,3) = ( 441*D(3,7) + 175*D(5,1) - 135*D(5,3) ) / 49

D(7,5) = ( 4851*D(3,7) + 1386*D(5,1) - 1618*D(5,3) ) / 245

D(7,7) = ( 21021*D(3,7) + 2331*D(5,1) - 6463*D(5,3) ) / 1715

D(9,1) = D(1,7)

D(9,3) = ( 441*D(1,7) - 1568*D(3,7) - 567*D(5,1) + 555*D(5,3) ) / 49

D(9,5) = ( 4851*D(1,7) - 2 8224*D(3,7) - 8197*D(5,1) + 9675*D(5,3) ) / 245

D(9,7) = ( 2 1021*D(1,7) - 15 5232*D(3,7) - 3 7170*D(5,1) + 5 1704*D(5,3) ) / 1715

THE QUADRATIC EQUATIONS ARE:

> 1 4117 8800*D(1,7)*D(3,7) - 17 3044 8720*D(1,7)*D(5,1) + 224 8096 6368*D(3,7)**2 + 206 4106 4976*D(3,7)*D(5,1) -
151 6184 8128*D(3,7)*D(5,3) + 3 4608 9744*D(3,7) + 41 1931 3968*D(   5,1)**2 - 69 6796 0608*D(5,1)*D(5,3) +
2955 3909*D(5,1) + 25 3701 1872*D(5,3)**2 - 1 1170 1037*D(5,3) - 152 9437 = 0

> 549 3488*D(1,7)**2 + 887 4096*D(1,7)*D(3,7) + 403 3680*D(1,7)*D(5,1) - 2 3072 6496*D(3,7)**2 - 1 5158 4048*
D(3,7)*D(5,1) + 1 3443 0912*D(3,7)*D(5,3) - 80 6736*D(3,7) - 2364 4656*D(5,1)**2 + 4537 7472*D(5,1)*D(5,3) -
19 4481*D(5,1) - 1913 0480*D(5,3)**2 + 19 0365*D(5,3) = 0
```

```
> 54 3312*D(1,7)**2 + 172 8720*D(1,7)*D(3,7) + 134 4560*D(1,7)*D(5,1) - 3440 9760*D(3,7)**2 - 2388 4560*D(3,7)*
  D(5,1) + 1902 3424*D(3,7)*D(5,3) - 26 8912*D(3,7) - 404 3088*D(5,1)**2 + 688 8000*D(5,1)*D(5,3) - 7 0315*D(5,1) -
  250 0560*D(5,3)**2 + 8 3055*D(5,3) = 0

> 518 6160*D(1,7)**2 + 2823 5760*D(1,7)*D(3,7) + 3630 3120*D(1,7)*D(5,1) - 6 1611 5808*D(3,7)**2 - 4 7282 9616*D(3,7)*
  D(5,1) + 3 4961 0688*D(3,7)*D(5,3) - 726 0624*D(3,7) - 8670 4128*D(5,1)**2 + 1 4057 5008*D(5,1)*D(5,3) -
  146 2209*D(5,1) - 4774 5792*D(5,3)**2 + 229 7757*D(5,3) + 1 6807 = 0
```

Table III :

Knots and weights of cubature formulae

Knots and weights of cubature formulae for C_2

Formula $C_2(9-1)$: 20 - point formula of degree 9.

$\pm x_i$	$\pm y_i$	$w_i/4$
0.3448 7202 5364	0.9186 2044 1057	0.3611 3055 8151(-1)
0.9186 2044 1057	0.3448 7202 5364	0.3611 3055 8151(-1)
0.6908 8055 0486	0.6908 8055 0486	0.5355 0090 2317(-1)
0.9396 5525 8097	0.9396 5525 8097	0.1068 2807 9664(-1)
0.4889 2685 6974	0.0	0.1135 4099 0172
0.0	0.4889 2685 6974	0.1135 4099 0172

d(0,0) = 0.1164 8338 8856

d(2,0) = 0.1067 9458 5388(-1)

d(4,0) = -0.1839 7857 4749

Formula C_2(13-1): 40-point formula of degree 13.

$\pm x_i$	$\pm y_i$	$w_i/4$
0.2666 7673 8695(-1)	0.3777 2431 2590	0.3488 1879 0231(-1)
0.3777 2431 2590	0.2666 7673 8695(-1)	0.3488 1879 0231(-1)
0.2359 8833 2487	0.7933 9617 1109	0.3449 9849 6602(-1)
0.7933 9617 1109	0.2359 8833 2487	0.3449 9849 6602(-1)
0.2654 8656 0241	0.9787 6174 7825	0.9874 4194 6914(-2)
0.9787 6174 7825	0.2654 8656 0241	0.9874 4194 6914(-2)
0.7021 4159 8362	0.9139 0945 7030	0.2034 9080 5188(-1)
0.9139 0945 7030	0.7021 4159 8362	0.2034 9080 5188(-1)
0.5514 7328 0570	0.5514 7328 0570	0.4753 2502 9082(-1)
0.9683 4072 0218	0.9683 4072 0218	0.3257 0397 4952(-2)

d(0,0) = 0.2956 5444 5279(-1)

d(2,0) = 0.1677 1182 1716

d(4,0) = -0.1724 7447 9777

d(4,2) = -0.8019 7670 7806(-1)

Formula C_2(13-2): 40-point formula of degree 13.

$\pm x_i$	$\pm y_i$	$w_i/4$
0.1682 3494 7696	0.9147 9446 3441	0.1973 8632 1888(-1)
0.9147 9446 3441	0.1682 3494 7696	0.1973 8632 1888(-1)
0.2526 6697 6106	0.5912 9437 8163	0.4843 6316 6325(-1)
0.5912 9437 8163	0.2526 6697 6106	0.4843 6316 6325(-1)
0.5840 4770 6043	0.1021 3969 5463(+1)	0.4212 8189 9422(-2)
0.1021 3969 5463(+1)	0.5840 4770 6043	0.4212 8189 9422(-2)
0.5867 1301 4973	0.8260 8170 9475	0.2872 5596 8895(-1)
0.8260 8170 9475	0.5867 1301 4973	0.2872 5596 8895(-1)
0.1788 9868 9064	0.1788 9868 9064	0.3610 6143 4781(-1)
0.9141 9795 6909	0.9141 9795 6909	0.1166 7127 1121(-1)

d(0,0) = -0.2744 8524 7116

d(2,0) = -0.1190 0778 7762(-1)

d(4,0) = 0.2125 1374 3389

d(4,2) = -0.2214 0959 5071(-2)

Formula $C_2(13-3)$: 36-point formula of degree 13.

$\pm x_i$	$\pm y_i$	$w_i/4$
0.3780 7179 6016	0.6999 9884 7123	0.4945 6653 0423(-1)
0.6999 9884 7123	0.3780 7179 6016	0.4945 6653 0423(-1)
0.5500 7949 6058	0.9708 9906 7992	0.1273 0007 4570(-1)
0.9708 9906 7992	0.5500 7949 6058	0.1273 0007 4570(-1)
0.8230 7135 5549	0.8230 7135 5549	0.2334 2321 4409(-1)
0.9682 3622 5230	0.9682 3622 5230	0.3271 6205 6943(-2)
0.3786 1734 4780	0.0	0.6973 0776 4166(-1)
0.9165 9709 5501	0.0	0.2928 2393 8505(-1)
0.1468 5570 9191(+1)	0.0	-0.4332 7605 9377(-6)
0.0	0.3786 1734 4780	0.6973 0776 4166(-1)
0.0	0.9165 9709 5501	0.2928 2393 8505(-1)
0.0	0.1468 5570 9191(+1)	-0.4332 7605 9377(-6)

d(3,1) = 0.7690 5404 2336(-1)

d(5,1) = -0.1716 4187 1970

Formula $C_2(17-1)$: 57-point formula of degree 17.

$\pm x_i$	$\pm y_i$	$w_i/4$
0.3088 6458 1390	0.8045 7485 5885	0.2724 4836 1374(-1)
0.8045 7485 5885	0.3088 6458 1390	0.2724 4836 1374(-1)
0.4503 1930 5992	0.9818 4311 8407	0.6946 1725 6964(-2)
0.9818 4311 8407	0.4503 1930 5992	0.6946 1725 6964(-2)
0.7050 0845 3650	0.8889 7447 3176	0.1548 7333 0479(-1)
0.8889 7447 3176	0.7050 0845 3650	0.1548 7333 0479(-1)
0.8879 0121 8626	0.9823 5439 3735	0.3463 8837 2572(-2)
0.9823 5439 3735	0.8879 0121 8626	0.3463 8837 2572(-2)
0.3259 4835 5163	0.3259 4835 5163	0.4133 2891 1864(-1)
0.5924 0053 7052	0.5924 0053 7052	0.3079 8812 8622(-1)
0.6131 7056 3119(-1)	0.0	0.7938 8127 2540(-1)
0.5856 0628 6805	0.0	0.3952 1148 4339(-1)
0.9457 5900 1219	0.0	0.1533 1156 8384(-1)
0.1618 1166 4947(+1)	0.0	-0.2687 1302 5397(-8)
0.0	0.6131 7056 3119(-1)	0.7938 8127 2540(-1)
0.0	0.5856 0628 6805	0.3952 1148 4339(-1)
0.0	0.9457 5900 1219	0.1533 1156 8384(-1)
0.0	0.1618 1166 4947(+1)	-0.2687 1302 5397(-8)
0.0	0.0	-0.2506 2633 9396

d(9,1) = 0.1700 6947 1871

d(9,3) = -0.6937 6030 0985(-1)

d(9,5) = 0.4456 2608 5911(-1)

d(9,7) = -0.9910 4617 6789(-1)

Formula $C_2(17-2)$: 57-point formula of degree 17.

$\pm x_i$	$\pm y_i$	$w_i/4$
0.3055 1768 0072	0.5835 0583 6600	0.3753 6600 9201(-1)
0.5835 0583 6600	0.3055 1768 0072	0.3753 6600 9201(-1)
0.3413 0047 5379	0.9348 4129 0568	0.1482 5612 4836(-1)
0.9348 4129 0568	0.3413 0047 5379	0.1482 5612 4836(-1)
0.5920 7502 7734	0.7952 2915 0385	0.2154 4850 5851(-1)
0.7952 2915 0385	0.5920 7502 7734	0.2154 4850 5851(-1)
0.7030 7119 7729	0.9807 9764 0771	0.5641 0158 2412(-2)
0.9807 9764 0771	0.7030 7119 7729	0.5641 0158 2412(-2)
0.8833 4020 5930	0.8833 4020 5930	0.1004 3053 3978(-1)
0.9775 9595 2140	0.9775 9595 2140	0.1468 8621 6209(-2)
0.3063 1216 7965	0.0	0.4613 8400 6259(-1)
0.8055 7752 3569	0.0	0.2872 3053 1547(-1)
0.9955 6506 3481	0.0	0.4545 7344 9597(-2)
0.1150 9936 5446(+1)	0.0	-0.6087 0605 1255(-5)
0.0	0.3063 1216 7965	0.4613 8400 6259(-1)
0.0	0.8055 7752 3569	0.2872 3053 1547(-1)
0.0	0.9955 650.03481	0.4545 7344 9597(-2)
0.0	0.1150 9936 5446(+1)	-0.6087 0605 1255(-5)
0.0	0.0	-0.3670 5607 0868(-4)

d(9,1) = -0.1625 1234 8025

d(9,3) = 0.4007 2805 9112(-1)

d(9,5) = 0.5020 1738 6838(-2)

d(9,7) = -0.5088 1228 0736(-1)

Knots and weights of cubature formulae for S_2

Formula $S_2(5-1)$: 8-point formula of degree 5.

$\pm x_i$	$\pm y_i$	w_i/π
0.4597 0084 3381	0.4597 0084 3381	0.2332 5317 5473
0.1255 9260 6040(+1)	0.0	0.1674 6824 5269(-1)
0.0	0.1255 9260 6040(+1)	0.1674 6824 5269(-1)

d(1,1) = 0.1636 7513 4595

Formula $S_2(9-1)$: 20-point formula of degree 9.

$\pm x_i$	$\pm y_i$	w_i/π
0.2432 4419 1752	0.8094 5826 0086	0.5672 0960 1536(-1)
0.8094 5826 0086	0.2432 4419 1752	0.5672 0960 1536(-1)
0.3022 1738 6264	0.3022 1738 6264	0.1099 4886 6164
0.6643 4134 8594	0.6643 4134 8594	0.2619 0019 2462(-1)
0.1342 7908 0737(+1)	0.0	0.4191 9428 2996(-3)
0.0	0.1342 7908 0737(+1)	0.4191 9428 2996(-3)

d(0,4) = -0.4302 0128 2044(-1)

d(2,2) = 0.3154 9150 0404(-1)

d(2,4) = -0.2121 3797 8477(-1)

Formula $S_2(9-2)$: 20-point formula of degree 9.

$\pm x_i$	$\pm y_i$	w_i/π
0.3438 5534 5294	0.9447 7801 7142	0.1234 4769 6401(-1)
0.9447 7801 7142	0.3438 5534 5294	0.1234 4769 6401(-1)
0.2774 9650 0297	0.2774 9650 0297	0.9327 1963 3554(-1)
0.5923 5538 7396	0.5923 5538 7396	0.5894 9678 3783(-1)
0.7786 1081 9923	0.0	0.7308 8818 9861(-1)
0.0	0.7786 1081 9923	0.7308 8818 9861(-1)

d(0,4) = -0.7984 5095 0059(-1)

d(2,2) = 0.7206 8526 3871(-1)

d(2,4) = 0.7988 7355 9647(-1)

Formula S_2(13-1): 36-point formula of degree 13.

$\pm x_i$	$\pm y_i$	w_i/π
0.3357 6760 0830	0.9386 9458 3835	0.5793 0240 9093(-2)
0.9386 9458 3835	0.3357 6760 0830	0.5793 0240 9093(-2)
0.3797 1701 1170	0.7520 4277 6804	0.3048 7945 6062(-1)
0.7520 4277 6804	0.3797 1701 1170	0.3048 7945 6062(-1)
0.6773 5510 6028	0.6773 5510 6028	0.1380 7195 4802(-1)
0.4126 7221 6763	0.4126 7221 6763	0.4781 9163 9417(-1)
0.2834 0283 2349	0.0	0.4973 2695 1853(-1)
0.6490 0723 0578	0.0	0.4397 3209 0591(-1)
0.9205 7547 4036	0.0	0.2210 5796 9395(-1)
0.0	0.2834 0283 2349	0.4973 2695 1853(-1)
0.0	0.6490 0723 0578	0.4397 3209 0591(-1)
0.0	0.9205 7547 4036	0.2210 5796 9395(-1)

d(1,5) = 0.4623 9116 4930(-2)

d(3,3) = 0.2706 8361 2957(-1)

d(3,5) = -0.3580 0571 0956(-1)

Knots and weights of cubature formulae for $E_2^{r^2}$

Formula $E_2^{r^2}$ (9-1): 21-point formula of degree 9.

$\pm x_i$	$\pm y_i$	w_i/π
0.4309 1398 2288	0.1040 3183 8026(+1)	0.7775 1058 4910(-1)
0.1040 3183 8026(+1)	0.4309 1398 2288	0.7775 1058 4910(-1)
0.5409 3734 8258	0.2106 9972 9303(+1)	0.3634 5339 6223(-2)
0.2106 9972 9303(+1)	0.5409 3734 8258	0.3634 5339 6223(-2)
0.1538 1890 0132(+1)	0.1538 1890 0132(+1)	0.3895 4817 6017(-2)
0.0	0.0	0.3333 3333 3333

d(2,0) = 0.2622 5952 6419

Formula $E_2^{r^2}$ (9-2): 20 point formula of degree 9.

$\pm x_i$	$\pm y_i$	w_i/π
0.1160 3820 5568(+1)	0.2526 9674 1024(+1)	0.4294 0575 7101(-3)
0.2526 9674 1024(+1)	0.1160 3820 5568(+1)	0.4294 0575 7101(-3)
0.1087 6638 7358(+1)	0.1087 6638 7358(+1)	0.3938 2009 8682(-1)
0.6812 5003 8633	0.0	0.1889 2584 5284
0.1732 0508 0757(+1)	0.0	0.2083 3333 3333(-1)
0.0	0.6812 5003 8633	0.1889 2584 5284
0.0	0.1732 0508 0757(+1)	0.2083 3333 3333(-1)

d(1,1) = -0.2573 5571 5851(+1)

d(3,1) = 0.1584 9364 9054

Formula $E_2^{r^2}$ (13-1): 36-point formula of degree 13.

$\pm x_i$	$\pm y_i$	w_i/π
0.9318 2402 2776	0.1779 2467 3891(+1)	0.5445 4723 6359(-2)
0.1779 2467 3891(+1)	0.9318 2402 2776	0.5445 4723 6359(-2)
0.9962 5342 6173	0.3030 4360 6537(+1)	0.2411 0284 9399(-4)
0.3030 4360 6537(+1)	0.9962 5342 6173	0.2411 0284 9399(-4)
0.8462 4998 8448	0.8462 4998 8448	0.5772 5718 2690(-1)
0.1985 5313 3992(+1)	0.1985 5313 3992(+1)	0.1620 5905 0315(-3)
0.5460 8441 5259	0.0	0.1418 0369 8741
0.1376 4866 7970(+1)	0.0	0.3807 8501 7590(-1)
0.2358 9322 6198(+1)	0.0	0.1290 8568 8395(-2)
0.0	0.5460 8441 5259	0.1418 0369 8741
0.0	0.1376 4866 7970(+1)	0.3807 8501 7590(-1)
0.0	0.2358 9322 6198(+1)	0.1290 8568 8395(-2)

d(3,1) = 0.5275 6166 4423

d(5,1) = 0.6403 6172 8604(-1)

Knots and weights of cubature formulae for E_2^r

Formula $E_2^r(9-1)$: 20-point formula of degree 9.

$\pm x_i$	$\pm y_i$	$w_i/(2\pi)$
0.5711 3957 9277(+1)	0.1517 8129 6516(+2)	0.4823 3030 8379(-5)
0.1517 8129 6516(+2)	0.5711 3957 9277(+1)	0.4823 3030 8379(-5)
0.1344 4581 1725(+1)	0.1344 4581 1725(+1)	0.2335 1239 5414
0.4824 4899 8918(+1)	0.4824 4899 8918(+1)	0.5379 8002 3105(-2)
0.6024 8261 8922(+1)	0.0	0.1109 8157 7487(-1)
0.0	0.6024 8261 8922(+1)	0.1109 8157 7487(-1)

d(0,4) = 0.1441 9885 5006(+2)

d(2,2) = 0.2164 1171 7095(+2)

d(2,4) = 0.1546 9430 6452

Formula $E_2^r(9-2)$: 21-point formula of degree 9.

$\pm x_i$	$\pm y_i$	$w_i/(2\pi)$
0.1397 0707 1070(+1)	0.3372 8270 5735(+1)	0.5110 2949 9996(-1)
0.3372 8270 5735(+1)	0.1397 0707 1070(+1)	0.5110 2949 9996(-1)
0.2351 4485 7485(+1)	0.8967 4772 3404(+1)	0.5280 5375 2564(-3)
0.8967 4772 3404(+1)	0.2351 4485 7485(+1)	0.5280 5375 2564(-3)
0.6555 3397 4494(+1)	0.6555 3397 4494(+1)	0.5475 1630 5222(-3)
0.0	0.0	0.5847 6190 4762

d(2,4) = 0.1088 0658 7087(+1)

Formula E_2^r(13-1): 36-point formula of degree 13.

$\pm x_i$	$\pm y_i$	$w_i/(2\pi)$
0.4408 6875 0107(+1)	0.8461 3744 1898(+1)	0.3495 7615 9382(-3)
0.8461 3744 1898(+1)	0.4408 6875 0107(+1)	0.3495 7615 9382(-3)
0.6128 7581 3778(+1)	0.1841 7239 6974(+2)	0.8118 1788 0180(-7)
0.1841 7239 6974(+2)	0.6128 7581 3778(+1)	0.8118 1788 0180(-7)
0.3335 7112 7362(+1)	0.3335 7112 7362(+1)	0.2216 3954 1537(-1)
0.1070 6962 3452(+2)	0.1070 6962 3452(+2)	0.2352 3774 3266(-5)
0.1646 0483 8529(+1)	0.0	0.2153 2967 1891
0.5473 7334 6063(+1)	0.0	0.1175 4592 4716(-1)
0.1173 4521 3439(+2)	0.0	0.5011 4423 6799(-4)
0.0	0.1646 0483 8529(+1)	0.2153 2967 1891
0.0	0.5473 7334 6063(+1)	0.1175 4592 4716(-1)
0.0	0.1173 4521 3439(+2)	0.5011 4423 6799(-4)

d(1,5) = -0.5091 9900 7295(+1)

d(3,3) = 0.1415 7879 0845(+2)

d(3,5) = 0.2095 7722 3074(+1)