On the Time Value of Ruin

Hans U. Gerber* and Elias S.W. Shiu†

Abstract

This paper studies the joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin. The time of ruin is analyzed in terms of its Laplace transforms, which can naturally be interpreted as discounting. Hence the classical risk theory model is generalized by discounting with respect to the time of ruin. We show how to calculate an expected discounted penalty, which is due at ruin and may depend on the deficit at ruin and on the surplus immediately before ruin. The expected discounted penalty, considered as a function of the initial surplus, satisfies a certain renewal equation, which has a probabilistic interpretation. Explicit answers are obtained for zero initial surplus, very large initial surplus, and arbitrary initial surplus if the claim amount distribution is exponential or a mixture of exponentials. We generalize Dickson's formula, which expresses the joint distribution of the surplus immediately prior to and at ruin in terms of the probability of ultimate ruin. Explicit results are obtained when dividends are paid out to the stockholders according to a constant barrier strategy.

1. Introduction

Two particular questions of interest in classical ruin theory are (a) the *deficit at ruin* and (b) the *time of ruin*, both of which have been treated separately in the literature. In this paper certain answers to both questions are given at the same time. From a mathematical point of view, a crucial role is played by the amount of surplus immediately before ruin occurs. Hence we study the joint distribution of three random variables: the surplus immediately before ruin, the deficit at ruin, and the time of ruin. The time of ruin is analyzed in terms of its Laplace transforms, which can naturally be interpreted as discounting. The study of the joint distribution is embedded in the study of an expected discounted *penalty*, which is due at ruin and depends on the deficit at ruin and on the surplus immediately prior to ruin. The expected discounted penalty, considered as a function of the initial surplus, satisfies a certain renewal equation. We find a natural probabilistic interpretation for the renewal equation by considering the first time the surplus falls below the initial level and whether or not ruin takes place at that time.

Explicit answers are obtained for zero initial surplus, large initial surplus, and arbitrary initial surplus if the claim amount distribution is exponential or a mixture of exponentials. Additional insight is obtained from a pair of exponential martingales. We generalize Dickson's (1992) formula, which expresses the joint distribution of the surpluses immediately prior to and at ruin in terms of the probability of ultimate ruin. We also obtain explicit results in the situation in which dividends are paid out to the stockholders according to a constant barrier strategy.

The paper generalizes and adds to a better understanding of classical ruin theory, which can be retrieved by setting the force of interest (Laplace transform variable) equal to zero. For example, in the classical model, the *adjustment coefficient* is the solution of an implicit equation, which has 0 as the other solution. If the interest rate is positive, the situation is suddenly symmetric: the corresponding equation, called *Lundberg's fundamental equation*, has a positive solution and a negative solution. Both solutions are important and are used to construct exponential martingales.

This paper was originally motivated by the problem of pricing *American options*. The classical model uses the geometric Brownian motion to model the stock price process. Such a process has continuous sample paths, which facilitate the analysis of an American

*Hans U. Gerber, A.S.A., Ph.D., is Professor of Actuarial Science at the Ecole des HEC (Business School), University of Lausanne, CH-1015 Lausanne, Switzerland.

†Elias S. W. Shiu, A.S.A., Ph.D., is Principal Financial Group Foundation Professor of Actuarial Science at the University of Iowa, Iowa City, Iowa 52242.

option: the option is exercised as soon as the stock price arrives on the optimal exercise boundary, and the price of the option is the expected discounted payoff. On the other hand, we would like to price an American option in a perhaps more realistic model in which the stock price may have jumps. The resulting mathematical problem is more intricate, because now, at the time of the exercise, the stock price is not *on* but *beyond* the optimal exercise boundary. If the logarithm of the stock price is modeled by a shifted compound Poisson process, this leads to the type of problems discussed in this paper. Evidently "penalty at ruin" has to be replaced by "payoff at exercise." Thus the paper lays the mathematical bases for a financial application, which is explained in Gerber and Shiu (1997b, 1998).

2. WHEN AND HOW DOES RUIN OCCUR?

We follow the notation in Chapter 13 of *Actuarial Mathematics* (Bowers et al. 1997). Thus $u \geq 0$ is the insurer's initial surplus. The premiums are received continuously at a constant rate, c, per unit time. The aggregate claims constitute a compound Poisson process, $\{S(t)\}$, given by the Poisson parameter λ and individual claim amount distribution function $P(x)$ with $P(0)=0$. That is,

$$S(t) = \sum_{j=1}^{N(t)} X_j, \tag{2.1}$$

where $\{N(t)\}$ is a Poisson process with mean per unit time λ and $\{X_j\}$ are independent random variables with common distribution $P(x)$. Then

$$U(t) = u + ct - S(t) \tag{2.2}$$

is the surplus at time t, $t \geq 0$. For simplicity we assume that $P(x)$ is differentiable, with

$$P'(x) = p(x)$$

being the individual claim amount probability density function.

Let T denote the *time of ruin*,

$$T = \inf\{t | U(t) < 0\} \tag{2.3}$$

($T=\infty$ if ruin does not occur). We consider the probability of ultimate ruin as a function of the initial surplus $U(0)=u \geq 0$,

$$\psi(u) = \Pr[T < \infty | U(0) = u]. \tag{2.4}$$

Let p_1 denote the mean of the individual claim amount distribution,

$$p_1 = \int_0^\infty x\, p(x)\, dx = E(X_j).$$

We assume

$$c > \lambda p_1 \tag{2.5}$$

to ensure that $\{U(t)\}$ has a positive drift; hence

$$\lim_{t \to \infty} U(t) = \infty \tag{2.6}$$

with certainty, and

$$\psi(u) < 1. \tag{2.7}$$

We also consider the random variables $U(T\text{-})$, the surplus immediately before ruin, and $U(T)$, the surplus at ruin. See Figure 1. For given $U(0)=u \geq 0$, let $f(x, y, t|u)$ denote the joint probability density function of $U(T\text{-})$, $|U(T)|$ and T. Then

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x, y, t|u)\, dx\, dy\, dt = \Pr[T < \infty | U(0) = u] = \psi(u). \tag{2.8}$$

Because of (2.7), $f(x, y, t|u)$ is a *defective* probability density function. We remark that, for $x > u + ct$,

$$f(x, y, t|u) = 0,$$

and that

$$f(u + ct, y, t|u)\, dx\, dy\, dt = e^{-\lambda t} \lambda p(u + ct + y)\, dy\, dt.$$

It is easier to analyze the following function, the study of which is a central theme in this paper. For $\delta \geq 0$, define

$$f(x, y|u) = \int_0^\infty e^{-\delta t} f(x, y, t|u)\, dt. \tag{2.9}$$

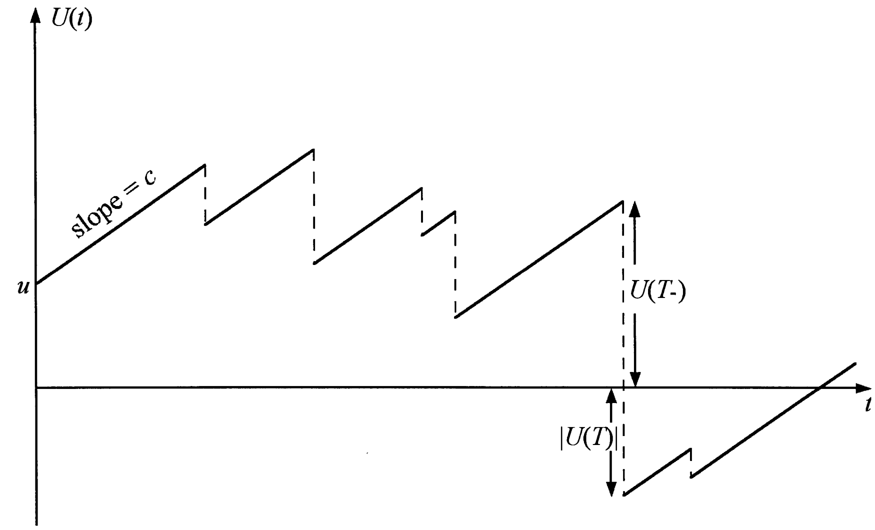
Here δ can be interpreted as a force of interest or, in the context of Laplace transforms, as a dummy variable. For notational simplicity, the symbol $f(x, y|u)$ does not exhibit the dependence on δ. If $\delta=0$, (2.9) is the defective joint probability density function of $U(T\text{-})$ and $|U(T)|$, given $U(0)=u$. Also, if $\delta>0$, then

$$e^{-\delta T} = e^{-\delta T} I(T < \infty),$$

where I denotes the indicator function, that is, $I(A)=1$ if A is true and $I(A)=0$ if A is false.

Let $w(x, y)$ be a nonnegative function of $x>0$ and $y>0$. We consider, for $u \geq 0$, the function $\phi(u)$ defined as

FIGURE 1
THE SURPLUS IMMEDIATELY BEFORE AND AT RUIN



$$\phi(u) = \mathrm{E}[w(U(T\text{-}), |U(T)|)\, e^{-\delta T}\, I(T < \infty)|U(0) = u] \tag{2.10}$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty w(x, y)\, e^{-\delta t} f(x, y, t|u)\, dt\, dx\, dy \tag{2.11}$$

$$= \int_0^\infty \int_0^\infty w(x, y)\, f(x, y|u)\, dx\, dy. \tag{2.12}$$

Note that the symbol $\phi(u)$ does not exhibit the dependence on the parameter δ and the function $w(x, y)$. [With $\delta=0$ and $w(x, y)=w(-y)$, $\phi(u)$ is denoted as $\psi(u; w)$ in the proof of Theorem 13.5.1 of *Actuarial Mathematics*.] For $x_0>0$ and $y_0>0$, if $w(x, y)$ is a "generalized" density function with mass 1 for $(x, y)=(x_0, y_0)$ and 0 for other values of (x, y), then

$$\phi(u) = f(x_0, y_0|u). \tag{2.13}$$

Hence the analysis of the function $f(x, y|u)$ is included in the analysis of the function $\phi(u)$.

If we interpret δ as a force of interest and w as some kind of penalty when ruin occurs, then $\phi(u)$ is the expectation of the discounted penalty. If w is interpreted as the benefit amount of an insurance (or reinsurance) payable at the time of ruin, then $\phi(u)$ is the single premium of the insurance. We should clarify that, while it can be very helpful to consider δ as a force of interest in this paper, we are dealing with the classical model in which the surplus does not earn any interest.

An interesting example of a penalty function is

$$w(x, y) = \frac{1 - e^{-\rho y}}{\delta}, \tag{2.14}$$

where ρ is the positive solution of Lundberg's fundamental equation (discussed later in this section). Then $\phi(u)$ is the expected present value of a deferred continuous annuity at a rate of 1 per unit time, starting at the time of ruin and ending as soon as the surplus rises to zero. This example is discussed in Remark (v) of Section 6. Another interesting example arises in the context of option pricing, where penalty at ruin is replaced by payoff at exercise. The payoff function considered by Gerber and Shiu (1997b) is

$$w(x, y) = \max(K - e^{a-y}, 0),$$

where K is the exercise price of a perpetual American put option, and e^a is the value of an option-exercise boundary.

Our immediate goal is to derive a functional equation for $\phi(u)$ by applying the law of iterated expectations to the right-hand side of (2.10). For $h>0$, consider the time interval $(0, h)$, and condition on the time t and the amount x of the first claim in this time interval. Note that the probability that there is no claim up to time h is $e^{-\lambda h}$, the probability that the first claim occurs between time t and time $t+dt$ is $e^{-\lambda t}\lambda dt$, and

$$x > u + ct$$

means that ruin has occurred with the first claim. Hence

$$\phi(u) = e^{-(\delta+\lambda)h}\,\phi(u+ch) + \int_0^h \left[\int_0^{u+ct} \phi(u+ct-x)p(x)dx\right] e^{-(\delta+\lambda)t}\,\lambda dt + \int_0^h \left[\int_{u+ct}^{\infty} w(u+ct, x-u-ct)p(x)dx\right] e^{-(\delta+\lambda)t}\,\lambda dt. \tag{2.15}$$

Differentiating (2.15) with respect to h and setting $h=0$, we obtain

$$\begin{aligned} 0 &= -(\delta+\lambda)\phi(u) + c\phi'(u) \\ &\quad + \lambda\int_0^u \phi(u-x)p(x)dx \\ &\quad + \lambda\int_u^{\infty} w(u, x-u)p(x)dx \\ &= -(\delta+\lambda)\phi(u) + c\phi'(u) \\ &\quad + \lambda\int_0^u \phi(u-x)p(x)\,dx + \lambda\omega(u), \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} \omega(u) &= \int_u^{\infty} w(u, x-u)\,p(x)\,dx \\ &= \int_0^{\infty} w(u, y)\,p(u+y)\,dy. \end{aligned} \tag{2.17}$$

For further analysis, we use the technique of *integrating factors*. Let

$$\phi_\rho(u) = e^{-\rho u}\,\phi(u), \tag{2.18}$$

where ρ is a nonnegative number to be specified later. Multiplying (2.16) with $e^{-\rho u}$, applying the product rule for differentiation, and rearranging yields

$$c\phi_\rho'(u) = (\delta+\lambda-c\rho)\phi_\rho(u) - \lambda\int_0^u \phi_\rho(u-x)e^{-\rho x}p(x)dx - \lambda e^{-\rho u}\omega(u). \tag{2.19}$$

Define

$$\ell(\xi) = \delta + \lambda - c\xi; \tag{2.20}$$

hence the coefficient of $\phi_\rho(u)$ in (2.19) is $\ell(\rho)$. In this paper we let $\hat{f}$ denote the Laplace transform of a function f,

$$\hat{f}(\xi) = \int_0^{\infty} e^{-\xi x}\,f(x)\,dx. \tag{2.21}$$

The Laplace transform of p, $\hat{p}(\xi)$, is defined for all nonnegative numbers ξ and is a decreasing convex function because

$$\hat{p}'(\xi) = -\int_0^{\infty} e^{-\xi x}\,x\,p(x)\,dx < 0$$

and

$$\hat{p}''(\xi) = \int_0^{\infty} e^{-\xi x}\,x^2\,p(x)\,dx > 0.$$

Consider the equation

$$\ell(\xi) = \lambda\hat{p}(\xi). \tag{2.22}$$

Since

$$\ell(0) = \delta + \lambda \geq \lambda = \lambda\hat{p}(0)$$

and the slope of the line $\ell(\xi)$ is negative, Equation (2.22) has a unique nonnegative root, say ξ_1. See Figure 2.

It is obvious from Figure 2 that ξ_1 is an increasing function of δ, with $\xi_1=0$ when $\delta=0$. Furthermore, if the individual claim amount density function, p, is sufficiently regular, Equation (2.22) has one more root, say ξ_2, which is negative. This negative root, which is denoted as $-R$, plays an important role later. As shown in Section 5, both roots are related to the construction of exponential martingales. When $\delta=0$, (2.22) is equivalent to (13.4.3) in *Actuarial Mathematics* and R is the *adjustment coefficient*. Because Lundberg (1932, p. 144) pointed out that (2.22) is "fundamental to the whole of collective risk theory," we refer to this equation as *Lundberg's fundamental equation*.

The trick for solving (2.19) is to choose

$$\rho = \xi_1, \tag{2.23}$$

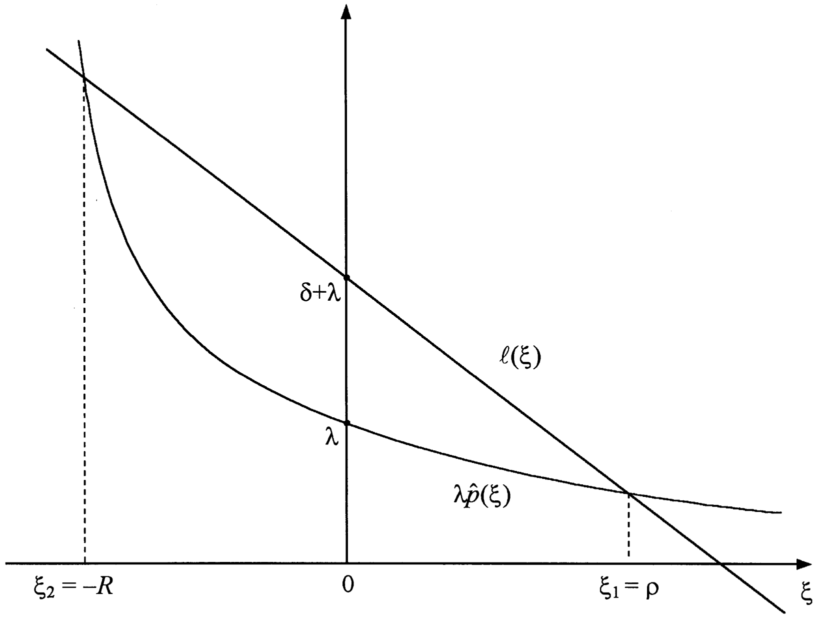
so that (2.19) becomes

$$\begin{aligned} c\phi_\rho'(u) &= \lambda\hat{p}(\rho)\phi_\rho(u) \\ &\quad - \lambda\int_0^u \phi_\rho(u-x)e^{-\rho x}\,p(x)dx - \lambda e^{-\rho u}\,\omega(u) \\ &= \lambda\left[\hat{p}(\rho)\phi_\rho(u) - \int_0^u \phi_\rho(x)e^{-\rho(u-x)}\,p(u-x)dx \right. \\ &\qquad \left. - e^{-\rho u}\,\omega(u)\right]. \end{aligned} \tag{2.24}$$

For $z>0$, we integrate (2.24) from $u=0$ to $u=z$. After a division by λ, the resulting equation is

$$\begin{aligned} &\lambda^{-1}c[\phi_\rho(z) - \phi_\rho(0)] \\ &\quad = \hat{p}(\rho)\int_0^z \phi_\rho(u)du \\ &\qquad - \int_0^z \left[\int_0^u \phi_\rho(x)e^{-\rho(u-x)}\,p(u-x)dx\right]du \\ &\qquad - \int_0^z e^{-\rho u}\,\omega(u)du \end{aligned}$$

FIGURE 2
THE TWO ROOTS OF LUNDBERG'S FUNDAMENTAL EQUATION



$$= \hat{p}(\rho) \int_0^z \phi_\rho(u)du$$

$$- \int_0^z \left[\int_x^z e^{-\rho(u-x)}\, p(u-x)du\right] \phi_\rho(x)dx$$

$$- \int_0^z e^{-\rho u}\, \omega(u)du$$

$$= \int_0^z \phi_\rho(x) \left[\int_{z-x}^\infty e^{-\rho y}\, p(y)dy\right]dx$$

$$- \int_0^z e^{-\rho u}\, \omega(u)du. \qquad (2.25)$$

For $z \to \infty$, the first terms on both sides of (2.25) vanish, which shows that

$$\phi_\rho(0) = \frac{\lambda}{c} \int_0^\infty e^{-\rho u}\, \omega(u)du = \frac{\lambda\, \hat{\omega}(\rho)}{c}. \qquad (2.26)$$

Finally, substituting (2.26) in (2.25) and simplifying yields

$$\phi_\rho(z) = \frac{\lambda}{c}\left\{\int_0^z \phi_\rho(x) \left[\int_{z-x}^\infty e^{-\rho y}\, p(y)dy\right]dx + \int_z^\infty e^{-\rho u}\, \omega(u)du\right\}, \quad z \ge 0. \qquad (2.27)$$

Multiplying (2.27) with $e^{\rho z}$ and applying (2.18), we have

$$\phi(z) = \frac{\lambda}{c}\left\{\int_0^z \phi(x) \left[\int_{z-x}^\infty e^{\rho(z-x-y)}\, p(y)dy\right] dx + \int_z^\infty e^{\rho(z-u)}\, \omega(u)du\right\}. \qquad (2.28)$$

For two integrable functions f_1 and f_2 defined on $[0, \infty)$, the *convolution* of f_1 and f_2 is the function

$$(f_1 * f_2)(x) = \int_0^x f_1(y)\, f_2(x-y)\, dy, \quad x \ge 0. \qquad (2.29)$$

Note that $f_1 * f_2 = f_2 * f_1$. With the definitions

$$g(x) = \frac{\lambda}{c} \int_x^\infty e^{-\rho(y-x)}\, p(y)\, dy \qquad (2.30)$$

$$= \frac{\lambda}{c} \int_0^\infty e^{-\rho z}\, p(x+z)\, dz, \quad x \ge 0, \qquad (2.31)$$

and

$$h(x) = \frac{\lambda}{c} \int_x^\infty e^{-\rho(u-x)}\, \omega(u)\, du \qquad (2.32)$$

$$= \frac{\lambda}{c} \int_x^\infty \int_0^\infty e^{-\rho(u-x)}\, w(u, y)\, p(u+y)\, dy\, du, \quad x \ge 0, \qquad (2.33)$$

Equation (2.28) can be written more concisely as

$$\phi = \phi * g + h. \tag{2.34}$$

In the literature of integral equations, (2.34) is classified as a *Volterra equation of the second kind*. The function g is a nonnegative function on $[0, \infty)$ and hence may be interpreted as a (not necessarily proper) probability density function; in probability theory, (2.34) is known as a *renewal equation* for the function ϕ.

The solution of (2.34) can be expressed as an infinite series of functions, sometimes called a *Neumann series*,

$$\phi = h + g*h + g*g*h + g*g*g*h + g*g*g*g*h + \cdots. \tag{2.35}$$

Equation (2.35) can be obtained from (2.34) by the method of successive substitution.

Remarks

(i) With $\delta=0$ and hence $\rho=0$, it is well known (*Actuarial Mathematics*, Theorem 13.5.1) that the differential

$$g(y)\,dy = \lambda c^{-1}[1 - P(y)]dy \tag{2.36}$$

can be interpreted as the probability that the surplus will ever fall below its initial u and will be between $u-y$ and $u-y-dy$ when it happens for the first time. Furthermore, with $\delta=0$ and $w\equiv 1$, we have

$$h(x) = \int_x^\infty g(y)dy,$$

which is the probability that the surplus will ever fall below its initial level u and will be below $u-x$ when it happens for the first time. The renewal equation (2.34) generalizes Exercise 13.11 of *Actuarial Mathematics*; see also (6.44) below.

(ii) It follows from the conditional probability formula,

$$\Pr(A \cap B) = \Pr(A)\Pr(B|A),$$

that the joint probability density function of $U(T\text{-})$, $|U(T)|$, and T at the point (x, y, t) is the joint probability density function of $U(T\text{-})$ and T at the point (x, t) multiplied by the conditional probability density function of $|U(T)|$ at y, given that $U(T\text{-})=x$ and $T=t$. The latter does not depend on t and is

$$\frac{p(x+y)}{\int_0^\infty p(x+y)\,dy} = \frac{p(x+y)}{1-P(x)}, \quad y \geq 0.$$

Hence

$$f(x, y, t|u) = \left[\int_0^\infty f(x, z, t|u)\,dz\right]\frac{p(x+y)}{1-P(x)}. \tag{2.37}$$

With the definition

$$f(x|u) = \int_0^\infty f(x, y|u)\,dy \tag{2.38}$$

$$= \int_0^\infty \int_0^\infty e^{-\delta t} f(x, y, t|u)\,dt\,dy, \tag{2.39}$$

multiplying (2.37) with $e^{-\delta t}$ and then integrating with respect to t yields

$$f(x, y|u) = f(x|u)\frac{p(x+y)}{1-P(x)}. \tag{2.40}$$

With $\delta=0$, (2.40) was pointed out by Dufresne and Gerber (1988, Eq. 3); another proof can be found in Dickson and Egídio dos Reis (1994). Also, it follows from (2.11), (2.37), (2.39), and (2.17) that

$$\begin{aligned}\phi(u) &= \int_0^\infty \int_0^\infty \int_0^\infty w(x, y)\, e^{-\delta t} f(x, y, t|u)\,dt\,dx\,dy \\ &= \int_0^\infty \int_0^\infty \int_0^\infty w(x, y)\, e^{-\delta t}\left[\int_0^\infty f(x, z, t|u)\,dz\right] \frac{p(x+y)}{1-P(x)}\,dt\,dx\,dy \\ &= \int_0^\infty \int_0^\infty w(x, y) f(x|u)\frac{p(x+y)}{1-P(x)}\,dx\,dy \\ &= \int_0^\infty \omega(x)\frac{f(x|u)}{1-P(x)}\,dx.\end{aligned} \tag{2.41}$$

(iii) It follows from an integration by parts that

$$\hat{p}(\xi) = 1 - \xi\int_0^\infty e^{-\xi x}[1 - P(x)]\,dx, \tag{2.42}$$

with which we can rewrite Lundberg's fundamental equation (2.22) as

$$\delta = c\xi - \lambda[1 - \hat{p}(\xi)] \tag{2.43}$$

$$= \xi\{c - \lambda\int_0^\infty e^{-\xi x}[1 - P(x)]\,dx\}. \tag{2.44}$$

Hence

$$\begin{aligned}\lim_{\delta\to 0}\frac{\delta}{\rho} &= c - \lambda\int_0^\infty [1 - P(x)]\,dx \\ &= c - \lambda p_1,\end{aligned} \tag{2.45}$$

which is the drift of $\{U(t)\}$. With $\delta=0$, the negative root $\xi=\xi_2$ of (2.44) or (2.22) is determined by the equation

$$\lambda \int_0^\infty e^{-\xi x} [1 - P(x)]\, dx = c; \tag{2.46}$$

this condition is discussed in Exercise 13.9 of *Actuarial Mathematics*.

(iv) Equation (2.34) can be solved by the method of Laplace transforms (Spiegel 1965). Taking Laplace transforms, we have

$$\hat{\phi}(\xi) = \hat{\phi}(\xi)\hat{g}(\xi) + \hat{h}(\xi), \tag{2.47}$$

or

$$\hat{\phi}(\xi) = \frac{\hat{h}(\xi)}{1 - \hat{g}(\xi)}. \tag{2.48}$$

Hence ϕ can obtained by inverting or identifying the right-hand side of (2.48). If we expand the right-hand side of (2.48) as a geometric series, we obtain

$$\hat{\phi}(\xi) = \sum_{n=0}^{\infty} \hat{g}(\xi)^n\, \hat{h}(\xi), \tag{2.49}$$

which is the Laplace transform of (2.35).

(v) From (2.30) and by changing of the order of integration, we see that

$$\begin{aligned}\hat{g}(\xi) &= \frac{\lambda}{c} \int_0^\infty e^{-\xi x} \left[\int_x^\infty e^{-\rho(y-x)}\, p(y)\, dy\right] dx \\ &= \frac{\lambda}{c(\rho - \xi)} \int_0^\infty [e^{(\rho-\xi)y} - 1]\, e^{-\rho y}\, p(y)\, dy \\ &= \frac{\lambda}{c(\rho - \xi)} [\hat{p}(\xi) - \hat{p}(\rho)]. \end{aligned} \tag{2.50}$$

Because ρ satisfies Lundberg's fundamental equation (2.22), it follows that

$$\hat{g}(\xi) = \frac{\lambda\hat{p}(\xi) + c\rho - \delta - \lambda}{c(\rho - \xi)}, \tag{2.51}$$

or

$$1 - \hat{g}(\xi) = \frac{\lambda[1 - \hat{p}(\xi)] + \delta - c\xi}{c(\rho - \xi)}. \tag{2.52}$$

We note that (2.22) is the condition that the numerator on the right-hand side of (2.52) vanishes. Hence the negative root ξ_2 of (2.22) is determined by the condition that

$$\hat{g}(\xi) = 1. \tag{2.53}$$

(vi) It follows from (2.50) that

$$\hat{g}'(\xi) = \frac{\hat{g}(\xi) + \lambda c^{-1}\hat{p}'(\xi)}{\rho - \xi}. \tag{2.54}$$

Since the negative root ξ_2 satisfies (2.53), a particular case of (2.54) is

$$\hat{g}'(\xi_2) = \frac{1 + \lambda c^{-1}\hat{p}'(\xi_2)}{\rho - \xi_2}. \tag{2.55}$$

This result and (2.56) below are used to derive an asymptotic formula for $\phi(u)$; see (4.9), (4.8), and (4.10) below.

(vii) From (2.32) and by changing the order of integration, we get

$$\begin{aligned}\hat{h}(\xi) &= \frac{\lambda}{c} \int_0^\infty e^{-\xi x} \left[\int_x^\infty e^{-\rho(u-x)}\, \omega(u)\, du\right] dx \\ &= \frac{\lambda}{c(\rho - \xi)} \int_0^\infty [e^{(\rho-\xi)u} - 1]\, e^{-\rho u}\, \omega(u)\, du \\ &= \frac{\lambda}{c(\rho - \xi)} \int_0^\infty \int_0^\infty (e^{-\xi u} - e^{-\rho u})\, w(u, y)\, p(u + y)\, du\, dy. \end{aligned} \tag{2.56}$$

Consider the special case with the penalty function $w(x, y) \equiv 1$. Then (2.56) becomes

$$\begin{aligned}\hat{h}(\xi) &= \frac{\lambda}{c(\rho - \xi)} \int_0^\infty \int_0^\infty (e^{-\xi u} - e^{-\rho u})\, p(u + y)\, du\, dy \\ &= \frac{\lambda}{c(\rho - \xi)} \int_0^\infty (e^{-\xi u} - e^{-\rho u})\, [1 - P(u)]\, du \\ &= \frac{1}{c(\rho - \xi)} \left\{\lambda \int_0^\infty e^{-\xi u} [1 - P(u)]\, du + \frac{\delta}{\rho} - c\right\}, \end{aligned} \tag{2.57}$$

because ρ satisfies (2.44). Applying (2.42) yields

$$\hat{h}(\xi) = \frac{1}{c(\rho - \xi)} \left\{\frac{\lambda}{\xi} [1 - \hat{p}(\xi)] + \frac{\delta}{\rho} - c\right\}. \tag{2.58}$$

Hence, with $w(x, y) \equiv 1$,

$$\begin{aligned}\hat{\phi}(\xi) &= \frac{\hat{h}(\xi)}{1 - \hat{g}(\xi)} \\ &= \frac{\lambda\rho[1 - \hat{p}(\xi)] + \xi(\delta - c\rho)}{\xi\rho\{\lambda[1 - \hat{p}(\xi)] + \delta - c\xi\}} \end{aligned} \tag{2.59}$$

by (2.58) and (2.52). In deriving (2.59), we assume that δ and hence ρ are positive. The case where $\delta=\rho=0$ and hence $\phi=\psi$ is best treated as a limiting case: From (2.59) and (2.45), we obtain

$$\hat{\psi}(\xi) = \frac{\lambda[1 - p_1\xi - \hat{p}(\xi)]}{\xi\{\lambda[1 - \hat{p}(\xi)] - c\xi\}}, \tag{2.60}$$

which can be reconciled with (13.6.9) in *Actuarial Mathematics* by the formula

$$\int_0^\infty e^{-\xi u}\, \psi'(u)\, du = -\psi(0) + \xi\, \hat{\psi}(\xi). \quad (2.61)$$

(viii) Let

$$\psi(u, t) = \Pr[T \le t | U(0) = u] \quad (2.62)$$

be the probability of ruin by time t, $t \ge 0$. Then

$$\psi(u, t) = \int_0^t \left[\int_0^\infty \int_0^\infty f(x, y, s|u)\, dxdy\right] ds, \quad (2.63)$$

from which it follows that

$$\frac{\partial}{\partial t}\psi(u, t) = \int_0^\infty \int_0^\infty f(x, y, t|u)\, dxdy. \quad (2.64)$$

Hence

$$\begin{aligned} \mathrm{E}[e^{-\delta T}\, I(T < \infty)|U(0) = u] &= \int_0^\infty e^{-\delta t}\frac{\partial}{\partial t}\psi(u, t)\, dt \\ &= \delta \int_0^\infty e^{-\delta t}\, \psi(u, t)\, dt \quad (2.65) \end{aligned}$$

after an integration by parts.

3. THE FIRST SURPLUS BELOW THE INITIAL LEVEL

In the first part of this section we study functions such as $f(x|0)$ [defined by (2.38)] and $f(x, y|0)$ [defined by (2.9)]. With initial surplus $U(0)=u=0$, some very explicit results can be obtained. Since ϕ satisfies the renewal equation (2.34), it follows that

$$\phi(0) = h(0). \quad (3.1)$$

From (2.12) and (2.33) we obtain

$$\begin{aligned} &\int_0^\infty \int_0^\infty w(x, y)\, f(x, y|0)\, dx\, dy \\ &= \frac{\lambda}{c}\int_0^\infty \int_0^\infty e^{-\rho x}\, w(x, y)\, p(x + y)\, dx\, dy. \quad (3.2) \end{aligned}$$

Because this identity holds for an arbitrary function w, it follows that

$$f(x, y|0) = \frac{\lambda}{c} e^{-\rho x}\, p(x + y), \quad x > 0, y > 0. \quad (3.3)$$

This formula plays a central role; an alternative proof and additional insight are given in Section 5. Some immediate consequences can be obtained by integrating over x, y, and both:

$$\begin{aligned} \int_0^\infty f(x, y|0)\, dx &= \frac{\lambda}{c}\int_0^\infty e^{-\rho x}\, p(x + y)\, dx \\ &= g(y), \quad (3.4) \end{aligned}$$

as defined by (2.31);

$$\begin{aligned} f(x|0) &= \int_0^\infty f(x, y|0)\, dy \\ &= \frac{\lambda}{c} e^{-\rho x}\int_0^\infty p(x + y)\, dy \\ &= \frac{\lambda}{c} e^{-\rho x}\, [1 - P(x)]; \quad (3.5) \end{aligned}$$

$$\begin{aligned} \mathrm{E}[e^{-\delta T}\, I(T < \infty)|U(0) = 0] &= \int_0^\infty \int_0^\infty f(x, y|0)\, dy\, dx \\ &= \frac{\lambda}{c}\int_0^\infty e^{-\rho x}\, [1 - P(x)]\, dx. \quad (3.6) \end{aligned}$$

As a check, note that (3.3) and (3.5) satisfy (2.40) with $u=0$.

With $\delta=0$, and hence $\rho=0$, (3.3) reduces to a result of Dufresne and Gerber (1988, Eq. 10). In particular,

$$f(x, y|0) = f(y, x|0). \quad (3.7)$$

Dickson (1992) has pointed out that this symmetry can be explained in terms of "duality." Further discussion can be found in Dickson and Egídio dos Reis (1994) and in Section 6 below. For $\delta>0$, Formula (3.7) does not hold any longer.

For $\delta=0$, (3.6) reduces to the famous formula

$$\psi(0) = \frac{\lambda}{c}\int_0^\infty [1 - P(x)]\, dx = \frac{\lambda\, p_1}{c}. \quad (3.8)$$

For $\delta>0$, we can use (3.6) and the fact that ρ is a solution of (2.44) to see that

$$\begin{aligned} \mathrm{E}[e^{-\delta T}|U(0) = 0] &= \mathrm{E}[e^{-\delta T}\, I(T < \infty)|U(0) = 0] \\ &= 1 - \frac{\delta}{c\rho}. \quad (3.9) \end{aligned}$$

Formula (3.8) can be obtained as a limiting case of (3.9) because of (2.45).

Example

Let us look at the case of an exponential individual claim amount distribution,

$$p(x) = \beta e^{-\beta x}, \quad x \ge 0, \quad (3.10)$$

with $\beta>0$ and $c>\lambda p_1=\lambda/\beta$. The number ρ is ξ_1, the nonnegative solution of (2.22), which is

$$\delta + \lambda - c\xi = \frac{\lambda\beta}{\beta + \xi},$$

or

$$c\xi^2 + (c\beta - \delta - \lambda)\xi - \beta\delta = 0. \qquad (3.11)$$

Hence

$$\rho = \xi_1 = \frac{\lambda + \delta - c\beta + \sqrt{(c\beta - \delta - \lambda)^2 + 4c\beta\delta}}{2c}. \qquad (3.12)$$

(Note that, if $\delta=0$, then $\rho=\xi_1=0$.) Then

$$f(x, y|0) = \lambda\beta c^{-1}e^{-(\rho+\beta)x-\beta y} = \lambda c^{-1}e^{-(\rho+\beta)x}\, p(y);$$

$$g(y) = \frac{\lambda\beta}{c(\beta + \rho)}\, e^{-\beta y} = \frac{\lambda}{c(\beta + \rho)}\, p(y); \qquad (3.13)$$

$$f(x|0) = \lambda c^{-1}e^{-(\rho+\beta)x};$$

$$\begin{aligned} \mathrm{E}[e^{-\delta T}\, I(T < \infty)|U(0) = 0] &= \int_0^\infty \int_0^\infty f(x, y|0)\, dy\, dx \\ &= \frac{\lambda}{c(\beta + \rho)} \qquad (3.14) \\ &= \frac{2\lambda}{c\beta + \delta + \lambda + \sqrt{(c\beta - \delta - \lambda)^2 + 4c\beta\delta}}. \qquad (3.15) \end{aligned}$$

An alternative to (3.14) and (3.15) is Formula (3.9), which is simple and general at the same time. In Section 5 we show that

$$\mathrm{E}[e^{-\delta T}\, I(T < \infty)|U(0) = u] = \mathrm{E}[e^{-\delta T}\, I(T < \infty)|U(0) = 0]\, e^{\xi_2 u}, \qquad (3.16)$$

where ξ_2 is the negative root of (3.11); see (5.38) and (5.43). Hence it follows from (2.65), (3.16), and (3.14) that

$$\int_0^\infty e^{-\delta t}\, \psi(u, t)\, dt = \frac{\lambda}{\delta c(\beta + \rho)}\, e^{\xi_2 u}. \qquad (3.17)$$

On the other hand, using (3.9) instead of (3.14) yields

$$\int_0^\infty e^{-\delta t}\, \psi(u, t)\, dt = \left(\frac{1}{\delta} - \frac{1}{c\rho}\right) e^{\xi_2 u}. \qquad (3.18)$$

Finally, we note that (2.40) can be simplified to

$$f(x, y|u) = f(x|u)\, p(y), \quad u \geq 0,\ x > 0,\ y > 0. \qquad (3.19)\ \square$$

Results concerning "ruin" for zero initial surplus can be translated into results that are related to when the surplus falls below the initial level for the first time. We can use (3.3) and (3.4) to derive the renewal equation (2.34) by probabilistic reasoning. We condition on the first time when the surplus falls below the initial level. For given initial surplus $U(0)=u\geq 0$, the probability that this event occurs between time t and time $t+dt$, with

$$u + x \leq U(t\text{-}) \leq u + x + dx$$

and

$$u - y - dy \leq U(t) \leq u - y,$$

is

$$f(x, y, t|0)\, dx\, dy\, dt. \qquad (3.20)$$

Furthermore, the occurrence

$$y > u$$

means that ruin also takes place with this claim. Thus

$$\begin{aligned} \phi(u) &= \int_0^u \int_0^\infty \int_0^\infty e^{-\delta t}\, \phi(u - y)\, f(x, y, t|0)\, dt\, dx\, dy \\ &\quad + \int_u^\infty \int_0^\infty \int_0^\infty e^{-\delta t}\, w(x + u, y - u)\, f(x, y, t|0) dt\, dx\, dy \\ &= \int_0^u \int_0^\infty \phi(u - y)\, f(x, y|0)\, dx\, dy \\ &\quad + \int_u^\infty \int_0^\infty w(x + u, y - u)\, f(x, y|0) dx\, dy. \qquad (3.21) \end{aligned}$$

Applying (3.4) and (3.3) to the right-hand side of (3.21) yields

$$\begin{aligned} \phi(u) &= \int_0^u \phi(u - y)g(y)dy \\ &\quad + \frac{\lambda}{c}\int_u^\infty \int_0^\infty w\,(x + u, y - u)e^{-\rho x}p(x + y)dx\, dy \\ &= (\phi * g)(u) + h(u) \end{aligned}$$

by (2.33). This is the probabilistic proof of (2.34).

Remarks

(i) If we consider $f(x|u)$ and $f(x, y|u)$ as functions of u, they satisfy renewal equations similar to (2.34). By distinguishing whether or not ruin occurs at the first time when the surplus falls below the initial value u, we see that

$$f(x, y|u) = \int_0^u f(x, y|u - z)g(z)dz + f(x - u, y + u|0), \quad 0 \leq u < x, \qquad (3.22)$$

and

$$f(x, y|u) = \int_0^u f(x, y|u - z)g(z)dz, \quad 0 < x \le u. \quad (3.23)$$

By (3.3),

$$f(x - u, y + u|0) = \lambda c^{-1} e^{-\rho(x-u)} p(x + y). \quad (3.24)$$

Hence, for $u \ge 0$, $x > 0$, $y > 0$,

$$f(x, y|u) = \int_0^u f(x, y|u - z)g(z)dz + \lambda c^{-1}e^{-\rho(x-u)}p(x + y)I(x > u). \quad (3.25)$$

Integrating (3.25) with respect to y, we obtain by (2.38)

$$f(x|u) = \int_0^u f(x|u - z)g(z)dz + \lambda c^{-1}e^{-\rho(x-u)}[1 - P(x)]I(x > u). \quad (3.26)$$

(ii) As a function of x, $f(x|u)$ has a discontinuity of amount

$$\lambda c^{-1}[1 - P(u)] \quad (3.27)$$

at $x=u$. Surprisingly, it does not depend on δ.

(iii) With appropriate choices of $w(x, y)$, (3.25) and (3.26) can be obtained as special cases of (2.34). See (2.13).

4. ASYMPTOTIC FORMULAS

Since the function $\phi(u)$ is the solution of a renewal equation, its asymptotic behavior is best examined by renewal theory. Let $f(x)$ and $z(x)$ be two nonnegative functions on $[0, \infty)$. Consider the integral equation

$$Z(x) = (f*Z)(x) + z(x), \quad x \ge 0, \quad (4.1)$$

which is a renewal equation for $Z(x)$. Seek a real number R such that

$$\hat{f}(-R) = \int_0^\infty e^{Rx}f(x)dx = 1. \quad (4.2)$$

The number R is unique because

$$\frac{d}{d\xi}\hat{f}(-\xi) = \int_0^\infty e^{\xi x}\, x\, f(x)\, dx > 0.$$

If $\hat{f}(0)<1$ (f is a defective density), then $R>0$; if $\hat{f}(0)=1$ (f is a proper density), then $R=0$; if $\hat{f}(0)>1$ (f is an excessive density), then $R<0$. The *key renewal theorem* (Feller 1971, Resnick 1992) states that, if the function z is sufficiently regular, then

$$\lim_{x\to\infty} e^{Rx}Z(x) = \frac{\int_0^\infty e^{Ry}\, z(y)\, dy}{\int_0^\infty y\, e^{Ry}\, f(y)\, dy} = \frac{\hat{z}(-R)}{-\hat{f}'(-R)}. \quad (4.3)$$

Let $f_1(x)$ and $f_2(x)$ be two functions; we write

$$f_1(x) \sim f_2(x) \quad \text{for } x \to \infty \quad (4.4)$$

if

$$\lim_{x\to\infty} \frac{f_1(x)}{f_2(x)} = 1.$$

Then (4.3) can be restated as

$$Z(x) \sim \frac{\hat{z}(-R)}{-\hat{f}'(-R)}\, e^{-Rx} \quad \text{for } x \to \infty. \quad (4.5)$$

We now apply the key renewal theorem to the function ϕ, which satisfies (2.34),

$$\phi = \phi * g + h,$$

where g and h are defined by (2.31) and (2.32), respectively. Since $\rho \ge 0$, we have

$$\hat{g}(0) \le \lambda c^{-1}\, p_1 < 1,$$

which means that (2.34) is a defective renewal equation. Thus we seek $R>0$ such that

$$1 = \hat{g}(-R) = \int_0^\infty e^{Rx}\, g(x)\, dx, \quad (4.6)$$

which is equation (2.53). Hence

$$R = -\xi_2,$$

where ξ_2 is the negative root of Lundberg's fundamental equation (2.22). Note that both ρ (or ξ_1) and R (or $|\xi_2|$) are increasing functions of δ and do not depend on the penalty function w. When confusion may arise, we write $\rho(\delta)$ for ρ and $R(\delta)$ for R. We observe that $\rho(0)=0$, and $R(0)$ is the adjustment coefficient in classical risk theory.

It follows from the key renewal theorem that

$$\phi(u) \sim \frac{\hat{h}(-R)}{-\hat{g}'(-R)}\, e^{-Ru} \quad \text{for } u \to \infty. \quad (4.7)$$

By (2.56),

$$\hat{h}(-R) = \frac{\lambda}{c(\rho + R)} \int_0^\infty \int_0^\infty (e^{Ru} - e^{-\rho u})\, w(u, y)\, p(u + y)\, dy\, du. \quad (4.8)$$

By (2.55),

$$\hat{g}'(-R) = \frac{1}{\rho + R}\left[1 + \frac{\lambda}{c}\hat{p}'(-R)\right]. \qquad (4.9)$$

Hence

$$\phi(u) \sim \frac{\lambda \int_0^\infty \int_0^\infty w(x, y)\,(e^{Rx} - e^{-\rho x})\, p(x + y)\, dx\, dy}{-\lambda\hat{p}'(-R) - c}\, e^{-Ru}$$

for $u \to \infty$. (4.10)

Now, consider the special case where $w(x, y)\equiv 1$ and $\delta=0$. Then $\phi=\psi$, and the renewal equation (2.34) is

$$\psi = \psi * g + h, \qquad (4.11)$$

with

$$g(x) = \lambda c^{-1}[1 - P(x)] \qquad (4.12)$$

and

$$h(x) = \lambda c^{-1}\int_x^\infty [1 - P(y)]dy$$

$$= \int_x^\infty g(y)\, dy. \qquad (4.13)$$

Equation (4.11) is the same as Exercise 13.11 in *Actuarial Mathematics*. Because $-R$ is the solution of (2.46), we have

$$\int_0^\infty \int_0^\infty (e^{Rx} - 1)\, p(x + y)\, dx\, dy$$

$$= \int_0^\infty (e^{Rx} - 1)\,[1 - P(x)]\, dx$$

$$= c\lambda^{-1} - \int_0^\infty [1 - P(x)]\, dx$$

$$= c\lambda^{-1} - p_1. \qquad (4.14)$$

Hence (4.10) simplifies as

$$\psi(u) \sim \frac{c - \lambda p_1}{-\lambda\hat{p}'(-R) - c}\, e^{-Ru} \quad \text{for } u \to \infty, \qquad (4.15)$$

which is usually called *Lundberg's asymptotic formula*.

We can also obtain an asymptotic formula ($u \to \infty$) for $\int_0^\infty e^{-\delta t}\,\psi(u, t)\, dt$, $\delta>0$, the single Laplace transform of the finite-time ruin function. By (2.65),

$$\int_0^\infty e^{-\delta t}\,\psi(u, t)\, dt = \frac{\phi(u)}{\delta}, \qquad (4.16)$$

where $w(x, y)\equiv 1$. It follows from (2.58) and (2.43) that

$$\hat{h}(-R) = \frac{1}{c(R + \rho)}\left\{\frac{\delta}{R} + c + \frac{\delta}{\rho} - c\right\}.$$

$$= \frac{\delta}{c(R + \rho)}\left(\frac{1}{R} + \frac{1}{\rho}\right). \qquad (4.17)$$

Hence

$$\phi(u) \sim \frac{\delta}{-\lambda\hat{p}'(-R) - c}\left(\frac{1}{R} + \frac{1}{\rho}\right) e^{-Ru}$$

for $u \to \infty$ (4.18)

[which can also be derived by applying (2.44) to (4.10)]. Substituting (4.18) in (4.16) yields

$$\int_0^\infty e^{-\delta t}\,\psi(u, t)\, dt \sim \frac{1}{-\lambda\hat{p}'(-R) - c}\left(\frac{1}{R} + \frac{1}{\rho}\right) e^{-Ru}$$

for $u \to \infty$. (4.19)

Lundberg's asymptotic formula (4.15) is, of course, a special case of (4.18):

$$\psi(u) = \lim_{\delta\to 0} \phi(u)$$

$$\sim \lim_{\delta\to 0} \frac{1}{-\lambda\hat{p}'(-R(\delta)) - c}\left[\frac{\delta}{R(\delta)} + \frac{\delta}{\rho(\delta)}\right] e^{-R(\delta)u},$$

for $u \to \infty$, (4.20)

which, by (2.45), is (4.15).

We note that (4.10) is for an arbitrary function $w(x, y)$. By comparing this formula with (2.12) we gather that

$$f(x, y|u) \sim \frac{\lambda(e^{Rx} - e^{-\rho x})\, p(x + y)}{-\lambda\hat{p}'(-R) - c}\, e^{-Ru}$$

for $u \to \infty$, (4.21)

which generalizes Dufresne and Gerber (1988, Eq. 24). Because (3.25) is a renewal equation for $f(x, y|u)$ (as a function of u), it can also be used to derive (4.21); here

$$h(u) = f(x, y|0)\, e^{\rho u} I(x > u), \qquad (4.22)$$

and hence

$$\hat{h}(-R) = f(x, y|0)\int_0^\infty e^{Ru}\, e^{\rho u} I(x > u)\, du$$

$$= f(x, y|0)\,\frac{e^{(R+\rho)x} - 1}{R + \rho}$$

$$= \frac{\lambda(e^{Rx} - e^{-\rho x})p(x + y)}{c(R + \rho)}. \qquad (4.23)$$

Example

As in Section 3, let us consider the case of an exponential individual claim distribution. The negative solution of (3.11) is

$$-R = \xi_2 = \frac{\lambda + \delta - c\beta - \sqrt{(c\beta - \delta - \lambda)^2 + 4c\beta\delta}}{2c}, \tag{4.24}$$

and the adjustment coefficient is

$$R(0) = \beta - \lambda c^{-1}. \tag{4.25}$$

From (3.13)

$$\hat{g}(\xi) = \frac{\lambda\beta}{c(\beta + \rho)(\beta + \xi)}, \tag{4.26}$$

and hence

$$-\hat{g}'(-R) = \frac{\lambda\beta}{c(\beta + \rho)(\beta - R)^2}. \tag{4.27}$$

Now, let us consider the particular case where $w(x, y) = w(y)$, a function not depending on x. Then

$$\begin{aligned}\omega(x) &= \int_0^\infty w(y)\, p(x + y)\, dy \\ &= \beta e^{-\beta x} \int_0^\infty w(y)\, e^{-\beta y}\, dy \\ &= \beta e^{-\beta x}\, \hat{w}(\beta),\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}h(x) &= \frac{\lambda}{c} \int_0^\infty e^{-\rho z}\, \omega(u + z)\, dz \\ &= \frac{\lambda}{c}\, \beta \hat{w}(\beta) \int_0^\infty e^{-\rho z}\, e^{-\beta(x+z)}\, dz \\ &= \frac{\lambda\, \beta\, \hat{w}(\beta)\, e^{-\beta x}}{c(\beta + \rho)}.\end{aligned} \tag{4.29}$$

Hence

$$\hat{h}(\xi) = \frac{\lambda\, \beta\, \hat{w}(\beta)}{c(\beta + \rho)(\beta + \xi)}. \tag{4.30}$$

It follows from (4.30) and (4.27) that

$$\frac{\hat{h}(-R)}{-\hat{g}'(-R)} = \hat{w}(\beta)\, (\beta - R). \tag{4.31}$$

Thus, with $w(x, y) = w(y)$, and $P(x) = 1 - e^{-\beta x}$, we have

$$\phi(u) \sim \hat{w}(\beta)\, (\beta - R)\, e^{-Ru} \quad \text{for } u \to \infty. \tag{4.32}$$

In the next section, we see that (4.32) is in fact an equality valid for all $u \geq 0$.

Furthermore, (4.21) is

$$f(x, y|u) \sim \frac{\lambda(e^{Rx} - e^{-\rho x})\, \beta e^{-\beta(x+y)}}{\lambda\beta(\beta - R)^{-2} - c}\, e^{-Ru} \quad \text{for } u \to \infty. \tag{4.33}$$

Because ρ and $-R$ are the roots of (3.11), we have

$$\begin{aligned}c(\beta + \rho)(\beta - R) &= c(-\beta)^2 + (c\beta - \delta - \lambda)(-\beta) - \beta\delta \\ &= \lambda\beta.\end{aligned} \tag{4.34}$$

It follows from (4.34) and some algebra that (4.33) can be rewritten as

$$f(x, y|u) \sim \frac{\lambda\beta(\beta - R)}{c(R + \rho)}\, [e^{Rx} - e^{-\rho x}]\, e^{-\beta(x+y)}\, e^{-Ru} \quad \text{for } u \to \infty. \tag{4.35}$$

Applying (3.19) to (4.35) yields

$$f(x|u) \sim \frac{\lambda(\beta - R)}{c(R + \rho)}\, [e^{Rx} - e^{-\rho x}]\, e^{-\beta x}\, e^{-Ru} \quad \text{for } u \to \infty. \tag{4.36}$$

It turns out that (4.35) and (4.36) are exact for $0 < x \leq u$; see (6.40) below. □

To conclude this section, we look at the Laplace transform of T, given that ruin occurs:

$$\begin{aligned}\mathrm{E}[e^{-\delta T}|T < \infty, U(0) = u] &= \frac{\mathrm{E}[e^{-\delta T} I(T < \infty)|U(0) = u]}{\mathrm{E}[I(T < \infty)|U(0) = u]} \\ &= \frac{\phi(u)}{\psi(u)},\end{aligned} \tag{4.37}$$

where $w \equiv 1$. Consider

$$\frac{\hat{h}(-R)}{-\hat{g}'(-R)}$$

as a function of δ, and write it as $C(\delta)$. It follows from (4.7) that, for $u \to \infty$,

$$\frac{\phi(u)}{\psi(u)} \sim \frac{C(\delta)e^{-R(\delta)u}}{C(0)e^{-R(0)u}} = \frac{C(\delta)}{C(0)}\, e^{-[R(\delta)-R(0)]u}. \tag{4.38}$$

If $\delta > 0$, then $R(\delta) > R(0)$, and hence

$$\lim_{u\to\infty} e^{-[R(\delta)-R(0)]u} = 0,$$

which means that

$$\lim_{u\to\infty} \mathrm{E}[e^{-\delta T}|T < \infty, U(0) = u] = 0. \tag{4.39}$$

Thus, for each $t > 0$,

$$\lim_{u\to\infty} \Pr[T \leq t|T < \infty, U(0) = u] = 0, \tag{4.40}$$

which implies that, for a large initial surplus u and given that ruin occurs, it occurs late. The result (4.40) is compatible with the observation that the conditional expectation

$$\mathrm{E}[T|T < \infty, U(0) = u]$$

is essentially a linear function in u in some cases; see (5.46) and (5.47) below, Gerber (1979, p. 138, Example 3.2), and Seal (1969, p. 114).

5. MARTINGALES

Further insight can be provided to the reader who has some familiarity with *martingales*. Let ξ be a number. Because $\{U(t)\}_{t\geq 0}$ is a stochastic process with stationary and independent increments, a process of the form

$$\{e^{-\delta t+\xi U(t)}\}_{t\geq 0} \tag{5.1}$$

is a martingale if and only if, for each $t>0$, its expectation at time t is equal to its initial value, that is, if and only if

$$\mathrm{E}[e^{-\delta t+\xi U(t)}|U(0) = u] = e^{\xi u}. \tag{5.2}$$

Since

$$\mathrm{E}[e^{-\delta t+\xi U(t)}|U(0) = u] = \exp(-\delta t + \xi u + \xi ct + \lambda t[\hat{p}(\xi) - 1]),$$

the martingale condition is that

$$-\delta + c\xi + \lambda[\hat{p}(\xi) - 1] = 0,$$

which is again Lundberg's fundamental equation (2.22). Thus, for (5.1) to be a martingale, the coefficient of $U(t)$ in (5.1) is either $\xi_1=\rho\geq 0$ or $\xi_2=-R<0$.

With such a ξ, (5.2) holds for each fixed t, $t\geq 0$. However, if we replace t by a *stopping time*, which is a random variable, then there is no guarantee that (5.2) will hold. Fortunately, it holds in two important cases, as we see in this and the next paragraph. If the stopping time is T, we consider the martingale (5.1) with $\xi=-R$. For $0\leq t<T$,

$$\delta t + RU(t) \geq 0,$$

and hence

$$0 < e^{-\delta t-RU(t)} \leq 1.$$

With $\{e^{-\delta t-RU(t)}\}_{0\leq t<T}$ being bounded, the *optional sampling theorem* is applicable and we obtain

$$\mathrm{E}[e^{-\delta T-RU(T)}|U(0) = u] = e^{-Ru}. \tag{5.3}$$

Furthermore, it follows from (2.6) that, even if $\delta=0$,

$$\mathrm{E}[e^{-\delta T-RU(T)}I(T = \infty)|U(0) = u] = 0.$$

Consequently, we can rewrite (5.3) as

$$e^{-Ru} = \mathrm{E}[e^{-\delta T-RU(T)}I(T < \infty)|U(0) = u], \quad \delta \geq 0,\ u \geq 0. \tag{5.4}$$

The above is a proof by martingale theory of a generalization of Theorem 13.4.1 in *Actuarial Mathematics*.

We now show that the quantity $e^{-\rho(x-u)}$, which appears throughout this paper (usually with $u=0$), has a probabilistic interpretation. For $x>U(0)=u$, let

$$T_x = \min \{t|U(t) = x\} \tag{5.5}$$

be the first time when the surplus reaches the level x. We can use equality to define the stopping time T_x because the process $\{U(t)\}$ is skip-free (jump-free) upward. Then, for $0\leq t\leq T_x$,

$$e^{-\delta t+\rho U(t)} \leq e^{\rho x}. \tag{5.6}$$

Hence we can apply the optional sampling theorem to the martingale $\{e^{-\delta t+\rho U(t)}\}$ to obtain

$$\begin{aligned} e^{\rho u} &= \mathrm{E}[e^{-\delta T_x+\rho U(T_x)}|U(0) = u] \\ &= \mathrm{E}[e^{-\delta T_x}|U(0) = u]\, e^{\rho x}, \end{aligned}$$

or

$$e^{-\rho(x-u)} = \mathrm{E}[e^{-\delta T_x}|U(0) = u]. \tag{5.7}$$

Formula (5.7) was probably first given by Kendall (1957, Eq. 14), although he did not provide a complete proof.

With δ interpreted as a force of interest, the quantity $e^{-\rho(x-u)}$ is the expected discounted value of a payment of 1 due at the time when $U(t)=x$ for the first time. We note that (5.7) remains valid even if u is negative. The required condition is $x>u$. The condition $u\geq 0$ is not needed anywhere in the derivation.

Formula (5.7) can be used to give an alternative proof of the important formula (3.3). For $x>u=U(0)$, let $\pi_1(x, t|u)$, $t>0$, denote the probability density function of the random variable T_x. Hence (5.7) states that

$$\int_0^\infty e^{-\delta t}\, \pi_1(x, t|u)\, dt = e^{-\rho(x-u)}. \tag{5.8}$$

The differential $\pi_1(x, t|u)dt$ is the probability that the surplus process upcrosses level x between t and $t+dt$ and that then this happens for the first time. We remark that the surplus cannot reach the level x before time $t=(x-u)/c$, and that it may reach x before the first claim occurs. Hence, for $t<(x-u)/c$, $\pi_1(x, t|u)=0$, and the distribution of T_x has a point mass at $t=(x-u)/c$ so that

$$\pi_1\, x, \frac{x-u}{c} \Big| u\ dt = \exp \frac{-\lambda(x-u)}{c}.$$

For $U(0)=u\geq 0$, $x>0$, let $\pi_2(x, t|u)$, $t>0$, be the function defined by the condition that $\pi_2(x, t|u)dt$ is the probability that ruin does not occur by time t and that there is an upcrossing of the surplus process at level x between t and $t+dt$. It can be proved by duality, a notion discussed in the next section, that

$$\pi_1(x, t|0) = \pi_2(x, t|0), \quad x > 0, t > 0. \qquad (5.9)$$

Now, $f(x, y, t|u)dtdxdy$ can be interpreted as the probability of the event that "ruin" does not take place by time t, that the surplus process upcrosses through level x between time t and time $t+dt$, but does not attain level $x+dx$, that is, that there is a claim within $c^{-1}dx$ time units after T_x, and that the size of this claim is between $x+y$ and $x+y+dy$. Thus

$$\begin{aligned} &f(x, y, t|u)\ dt\ dx\ dy \\ &\quad = [\pi_2(x, t|u)\ dt]\ [\lambda c^{-1}dx]\ [p(x+y)\ dy], \qquad (5.10)\end{aligned}$$

from which it follows that

$$f(x, y, t|u) = \lambda c^{-1}\, p(x+y)\, \pi_2(x, t|u). \qquad (5.11)$$

This formula is particularly useful if $u=0$; then it follows from (5.9) that

$$f(x, y, t|0) = \lambda c^{-1}\, p(x+y)\, \pi_1(x, t|0). \qquad (5.12)$$

If we multiply (5.12) by $e^{-\delta t}$, integrate from $t=0$ to $t=\infty$, and apply (5.8) with $u=0$, we obtain (3.3) once again.

Remarks

(i) For $x>u=U(0)\geq 0$, the functions $\pi_1(x, t|u)$ and $\pi_2(x, t|0)$ can be expressed in terms of $\pi_3(x, t|u)$, the *passage time density* of the surplus process at the level x. The differential $\pi_3(x, t|u)dt$ is the probability that the surplus process upcrosses level x between t and $t+dt$. This is the same as the probability that the surplus at time t is between $x-dx$ and x with $dx=c\ dt$. Hence, we have

$$\pi_3(x, t|u) = c\, f_{S(t)}(u+ct-x), \qquad (5.13)$$

where

$$f_{S(t)}(s) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}\, p*^n(s) \qquad (5.14)$$

is the probability density function of $S(t)$, the aggregate claims up to time t. The following version of the *ballot theorem*,

$$\pi_1(x, t|0) = \frac{x}{ct}\, \pi_3(x, t|0), \quad x > 0, t > 0, \qquad (5.15)$$

was first given by Kendall (1957, Eq. 17). For $x>u$ and $t>0$, because

$$\pi_1(x, t|u) = \pi_1(x-u, t|0) \qquad (5.16)$$

and

$$\pi_3(x, t|u) = \pi_3(x-u, t|0), \qquad (5.17)$$

we have

$$\pi_1(x, t|u) = \frac{x-u}{ct}\, \pi_3(x, t|u). \qquad (5.18)$$

Gerber (1988, Theorem 2) has given a proof of

$$\pi_1(x, t|0) = \frac{x}{ct}\, \pi_3(x, t|0) = \pi_2(x, t|0) \qquad (5.19)$$

by martingales. The second equality of (5.19) is equivalent to Equation (2.1) on page 112 of Gerber (1979).

(ii) The differential $\pi_2(x, t|u)dt$ can be interpreted as the probability that ruin does not occur by time t and that the surplus at time t is between $x-dx$ and x, where $dx=cdt$. Consider the *nonruin function*

$$\begin{aligned}\sigma(u, t) &= \Pr[T > t|U(0) = u] \\ &= 1 - \psi(u, t). \qquad (5.20)\end{aligned}$$

Then

$$\sigma(u, t) = \int_0^{\infty} \pi_2(x, t|u)\ c^{-1}dx. \qquad (5.21)$$

Hence, it follows from the second equality of (5.19) and from (5.13) that

$$\begin{aligned}\sigma(0, t) &= \int_0^{\infty} \frac{x}{ct}\, \pi_3(x, t|0)\ c^{-1}dx \\ &= \frac{1}{ct} \int_0^{\infty} x\, f_{S(t)}(ct-x)\ dx. \qquad (5.22)\end{aligned}$$

This result was first given by Prabhu (1961, Eq. 4.6) and is sometimes called the *Prabhu formula* (De Vylder 1996, p. 132). Let

$$F_{S(t)}(s) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}\, P^{*n}(s) \qquad (5.23)$$

be the probability distribution function of $S(t)$. Then integrating the right-hand side of (5.22) by parts and noting that $F_{S(t)}(s)=0$ for $s<0$ yields

$$\begin{aligned}\sigma(0, t) &= 0 - 0 + \frac{1}{ct}\int_0^{\infty} F_{S(t)}(ct - x)\, dx \\ &= \frac{1}{ct}\int_0^{ct} F_{S(t)}\,(ct - x)\, dx \\ &= \frac{1}{ct}\int_0^{ct} F_{S(t)}\,(s)\, ds. \end{aligned} \tag{5.24}$$

(iii) By (5.21), (5.9), and (5.8), the Laplace transform of the function $\sigma(0, t)$ is

$$\begin{aligned}\int_0^{\infty} e^{-\delta t}\sigma(0, t)\, dt &= \int_0^{\infty} e^{-\delta t}\left[\int_0^{\infty}\pi_2(x, t|0)\, c^{-1}dx\right] dt \\ &= \frac{1}{c}\int_0^{\infty}\int_0^{\infty} e^{-\delta t}\pi_1(x, t|0)\, dt\, dx \\ &= \frac{1}{c}\int_0^{\infty} e^{-\rho x}\, dx \\ &= \frac{1}{c\rho}. \end{aligned} \tag{5.25}$$

This elegant formula can also be obtained from the *initial value theorem* of Laplace transforms (Spiegel 1965, p. 5, Theorem 1-16) which states that, for a sufficiently regular function f,

$$\lim_{u\to 0} f(u) = \lim_{\xi\to\infty} \xi\,\hat{f}(\xi).$$

Hence

$$\int_0^{\infty} e^{-\delta t}\sigma(0, t)\, dt = \lim_{\xi\to\infty}\xi\,\hat{\sigma}(\xi, \delta), \tag{5.26}$$

where $\hat{\sigma}(\xi, \delta)$ is the double Laplace transform of $\sigma(u, t)$,

$$\hat{\sigma}(\xi, \delta) = \int_0^{\infty}\int_0^{\infty} e^{-\xi u - \delta t}\sigma(u, t)\, dt\, du, \qquad \xi > 0,\ \delta > 0. \tag{5.27}$$

It follows from (5.20) that

$$\hat{\sigma}(\xi, \delta) = \frac{1}{\xi\delta} - \hat{\psi}(\xi, \delta), \tag{5.28}$$

where $\hat{\psi}(\xi, \delta)$ is the double Laplace transform of $\psi(u, t)$. Because the left-hand side of (2.65) is $\phi(u)$ with $w(x, y)\equiv 1$, we have

$$\begin{aligned}\hat{\psi}(\xi, \delta) &= \hat{\phi}(\xi)/\delta \\ &= \frac{\lambda\rho[1 - \hat{p}(\xi)] + \xi(\delta - c\rho)}{\delta\xi\rho\{\lambda[1 - \hat{p}(\xi)] + \delta - c\xi\}} \end{aligned} \tag{5.29}$$

by (2.59). From (5.29) and (5.28) we obtain the following result, which is of an independent interest:

$$\hat{\sigma}(\xi, \delta) = \frac{\xi^{-1} - \rho^{-1}}{\lambda[1 - \hat{p}(\xi)] + \delta - c\xi}. \tag{5.30}$$

Note that the denominator on the right-hand side of (5.30) is the difference of the two sides in Lundberg's fundamental equation (2.22). Substituting (5.30) in the right-hand side of (5.26) and taking the limit as ξ tends to ∞ yields (5.25).

(iv) For $u\ge 0$, $t>0$, consider the probability

$$\Pr[U(t) \ge 0|U(0) = u] = F_{S(t)}(u + ct).$$

By conditioning on whether or not ruin occurs before time t and distinguishing according to the time τ when the surplus process upcrosses the level 0 for the last time, we have the following equation for the nonruin function $\sigma(u, t)$,

$$F_{S(t)}(u + ct) = \sigma(u, t) + \int_0^{t}\pi_3(0, \tau|u)\,\sigma(0, t - \tau)\, d\tau, \tag{5.31}$$

which, in the context of risk theory, was first given by Prabhu (1961, Eq. 3.3). As $t\to\infty$, (5.31) becomes

$$1 = \sigma(u, \infty) + \int_0^{\infty}\pi_3(0, \tau|u)\,\sigma(0, \infty)\, d\tau,$$

or

$$\psi(u) = [1 - \psi(0)]\int_0^{\infty}\pi_3(0, \tau|u)\, d\tau \tag{5.32}$$

$$= \frac{c - \lambda p_1}{c}\int_0^{\infty}\pi_3(0, \tau|u)\, d\tau, \quad u \ge 0. \tag{5.33}$$

Seah (1990, p. 426) has pointed out that (5.33) "is not practical for computing."

(v) For $x\le u=U(0)$, we have

$$\pi_3(x, \tau|u) = \pi_3(0, \tau|u - x),$$

from which and (5.32) it follows that

$$\int_0^{\infty}\pi_3(x, \tau|u)\, d\tau = \frac{\psi(u - x)}{1 - \psi(0)}. \tag{5.34}$$

For $x>u=U(0)$, because $\{U(t)\}$ has a positive drift by (2.5), the surplus will reach level x with probability 1. Hence, for $x>u$,

$$\int_0^{\infty}\pi_3(x, \tau|u)\, d\tau = 1 + \int_0^{\infty}\pi_3(x, \tau|x)\, d\tau \tag{5.35}$$

$$\begin{aligned} &= 1 + \frac{\psi(x - x)}{1 - \psi(0)} \\ &= \frac{1}{1 - \psi(0)} \end{aligned} \tag{5.36}$$

by (5.34). Furthermore, (5.35) can be generalized as

$$\int_0^\infty \pi_3(x, \tau|u)\, e^{-\delta\tau}\, d\tau$$

$$= e^{-\rho(x-u)}\left[1 + \int_0^\infty \pi_3(x, \tau|x)\, e^{-\delta\tau}\, d\tau\right]$$

$$= e^{-\rho(x-u)}\left[1 + \int_0^\infty \pi_3(0, \tau|0)\, e^{-\delta\tau}\, d\tau\right],$$

$$x > u,\ \delta \geq 0. \quad (5.37)$$

Further results are given in Remark (vi) of Section 6.

Example

Again, consider the case in which the individual claim amount distribution is exponential, $p(x)=\beta e^{-\beta x}$. Then R is given by (4.24). Applying (3.19) to (5.4) yields

$$e^{-Ru} = \left[\int_0^\infty e^{Ry} p(y)dy\right] \mathrm{E}[e^{-\delta T}\, I(T < \infty)|U(0) = u]$$

$$= \frac{\beta}{\beta - R}\, \mathrm{E}[e^{-\delta T}\, I(T < \infty)|U(0) = u].$$

Hence

$$\mathrm{E}[e^{-\delta T} I(T < \infty)|U(0) = u] = \frac{\beta - R}{\beta}\, e^{-Ru}. \quad (5.38)$$

Formula (5.38) should be compared with the first line of (13.4.8) in *Actuarial Mathematics*, which is only for $\delta=0$. To reconcile (5.38) for $u=0$ with (3.9), we need to show that, for $\delta>0$,

$$\frac{R}{\beta} = \frac{\delta}{c\rho}. \quad (5.39)$$

Equation (5.39) holds because the product of the two roots of the quadratic equation (3.11) is $-\beta\delta/c$. As a further check, we want to see that (5.38) with $u=0$ is consistent with (3.14); here we need the identity

$$\frac{\beta - R}{\beta} = \frac{\lambda}{c(\beta + \rho)}, \quad (5.40)$$

which is true because of (4.34). It follows from (3.13) and (5.40) that

$$g(y) = (\beta - R)e^{-\beta y} = \frac{p(y)}{\hat{p}(-R)}. \quad (5.41)$$

In the particular case in which $w(x, y)=w(y)$, a function not depending on x, we can apply (3.19) and (5.38) to obtain an explicit expression for $\phi(u)$:

$$\phi(u) = \mathrm{E}[e^{-\delta T}\, w(|U(T)|)\, I(T < \infty)|U(0) = u]$$

$$= \left[\int_0^\infty w(y)p(y)dy\right] \mathrm{E}[e^{-\delta T}\, I(T < \infty)|U(0) = u]$$

$$= \left[\int_0^\infty w(y)e^{-\beta y}dy\right] (\beta - R)e^{-Ru}$$

$$= \hat{w}(\beta)(\beta - R)e^{-Ru}. \quad (5.42)$$

This shows that the asymptotic formula (4.32) is actually an exact formula, and

$$\phi(u) = \phi(0)e^{-Ru}. \quad (5.43)$$

Furthermore, for $\delta>0$,

$$\mathrm{E}[e^{-\delta T} w(|U(T)|)|T < \infty,\ U(0) = u]$$

$$= \frac{\mathrm{E}[e^{-\delta T}\, w(|U(T)|)\, I(T < \infty)|U(0) = u]}{\mathrm{E}[I(T < \infty)|U(0) = u]}$$

$$= \frac{\phi(u)}{\psi(u)}$$

$$= \frac{\beta\hat{w}(\beta)[\beta - R(\delta)]}{\beta - R(0)}\, e^{-[R(\delta)-R(0)]u}. \quad (5.44)$$

Because of (5.40) [or (4.34)], we can rewrite (5.44) in terms of $\rho(\delta)$ and $\rho(0)$. Noting that $\rho(0)=0$, we have

$$\mathrm{E}[e^{-\delta T} w(|U(T)|)|T < \infty,\ U(0) = u]$$

$$= \frac{\beta^2\hat{w}(\beta)}{\beta + \rho(\delta)} \exp\left(\frac{\lambda}{c}\left[\frac{\beta}{\beta + \rho(\delta)} - 1\right]u\right). \quad (5.45)$$

Differentiating (5.44) with respect to δ and then setting $\delta=0$ yields

$$\mathrm{E}[T\ w(|U(T)|)|T < \infty,\ U(0) = u]$$

$$= \beta\hat{w}(\beta)R'(0)\left[\frac{1}{\beta - R(0)} + u\right], \quad (5.46)$$

which is a linear function in u. By (4.25)

$$\beta - R(0) = \lambda c^{-1}.$$

From (4.24),

$$R'(0) = \frac{\lambda}{c(c\beta - \lambda)}.$$

Hence

$$\mathrm{E}[T\ w(|U(T)|)|T < \infty,\ U(0) = u]$$

$$= \frac{\lambda\beta\hat{w}(\beta)}{c(c\beta - \lambda)}\left[\frac{c}{\lambda} + u\right]. \quad (5.47)\ \square$$

6. GENERALIZATION OF DICKSON'S FORMULA

For the case $\delta=0$, Dickson (1992) has found the following astonishing result:

$$f(x|u) = \begin{cases} f(x|0)\,\dfrac{1-\psi(u)}{1-\psi(0)}, & x > u \geq 0, \quad (6.1) \\ f(x|0)\,\dfrac{\psi(u-x)-\psi(u)}{1-\psi(0)}, & 0 < x \leq u. \quad (6.2) \end{cases}$$

Here,

$$f(x|0) = \lambda c^{-1}[1 - P(x)]; \qquad (6.3)$$

see (3.5). The purpose of this section is to generalize (6.1) and (6.2) to the case in which $\delta \geq 0$.

A first question is how to extend the definition of $\psi(u)$ for $\delta>0$. It turns out that the appropriate definition is

$$\psi(u) = \mathrm{E}[e^{-\delta T+\rho U(T)}\, I(T<\infty)|U(0)=u], \quad u \geq 0. \qquad (6.4)$$

Thus $\psi(u)=\phi(u)$ with $w(x, y)=e^{-\rho y}$; see (2.10). [Compare the expressions on the right-hand sides of (6.4) with (5.4).] Then Dickson's formula can be generalized as

$$f(x|u) = \begin{cases} f(x|0)\,\dfrac{e^{\rho u}-\psi(u)}{1-\psi(0)}, & x > u \geq 0, \quad (6.5) \\ f(x|0)\,\dfrac{e^{\rho x}\psi(u-x)-\psi(u)}{1-\psi(0)}, & 0 < x \leq u, \quad (6.6) \end{cases}$$

with

$$f(x|0) = \lambda c^{-1} e^{-\rho x}[1 - P(x)] \qquad (6.7)$$

according to (3.5). Hence, as a function of x, $f(x|u)$ has a discontinuity of amount

$$f(u|0)\, e^{\rho u} = \lambda c^{-1}[1 - P(u)]$$

at $x=u$. This is the same result as (3.27).

To prove (6.5) and (6.6), we need some more concepts. We begin by extending the definition of the stopping time T_x as given by (5.5). For a real number x, we now let T_x denote the time of the first upcrossing of the surplus through the level x; we set $T_x=\infty$ if the surplus never upcrosses through the level x. For $x>U(0)$, this is the same as (5.5). For $x<U(0)$, the surplus has to drop below the level x before it can ever upcross through x. We call the stopping time T_0 the *time of recovery*; it is the first time the surplus reaches zero after ruin. It follows from (5.7) that, for $a<b$,

$$\mathrm{E}[e^{-\delta(T_b-T_a)}|T_a < T_b] = e^{-\rho(b-a)}. \qquad (6.8)$$

Hence

$$\mathrm{E}[e^{-\delta(T_0-T)}|T < \infty] = e^{\rho U(T)}, \qquad (6.9)$$

from which and the law of iterated expectations it follows that

$$\begin{aligned} \mathrm{E}[e^{-\delta T_0}\, I(T<\infty)|U(0)=u] &= \mathrm{E}[e^{-\delta(T_0-T)}\, e^{-\delta T}\, I(T<\infty)|U(0)=u] \\ &= \mathrm{E}[e^{\rho U(T)}\, e^{-\delta T}\, I(T<\infty)|U(0)=u] \\ &= \psi(u). \end{aligned} \qquad (6.10)$$

This formula shows that the generalized $\psi(u)$ can be interpreted as the expected present value of a payment of 1 that is made at the time of recovery, if ruin takes place.

For $a \leq u < b$,

$$\Pr[T_a < \infty|U(0) = u] < 1 \qquad (6.11)$$

and

$$\Pr[T_b < \infty|U(0) = u] = 1 \qquad (6.12)$$

because the surplus process $\{U(t)\}$ has a positive drift. We define the stopping time

$$T_{a,b} = \min(T_a, T_b), \qquad (6.13)$$

and consider the functions

$$\begin{aligned} A(a, b|u) &= \mathrm{E}[e^{-\delta T_{a,b}}\, I(U(T_{a,b})=a)|U(0)=u] \\ &= \mathrm{E}[e^{-\delta T_a}\, I(T_a < T_b)|U(0) = u], \end{aligned} \qquad (6.14)$$

and

$$\begin{aligned} B(a, b|u) &= \mathrm{E}[e^{-\delta T_{a,b}}\, I(U(T_{a,b})=b)|U(0)=u] \\ &= \mathrm{E}[e^{-\delta T_b}\, I(T_a > T_b)|U(0)=u]. \end{aligned} \qquad (6.15)$$

With δ interpreted as a force of interest, $A(a, b|u)$ is the expected present value of *a* payment of 1 that is made when the surplus upcrosses the level a for the first time, provided that the surplus has not reached the level b in the meantime. Similarly, $B(a, b|u)$ is the expected present value of a payment of 1 that is made when the surplus reaches the level b for the first time, provided that the surplus has not dropped below the level a in the meantime. Note that, for each constant k,

$$A(a, b|u) = A(a + k, b + k|u + k) \qquad (6.16)$$

and

$$B(a, b|u) = B(a + k, b + k|u + k). \quad (6.17)$$

It follows from (6.10) that, for $u \geq a$,

$$\begin{aligned} A(a, \infty|u) &= \lim_{b\to\infty} A(a, b|u) \\ &= \lim_{b\to\infty} A(0, b - a|u - a) \\ &= \mathrm{E}[e^{-\delta T_0} I(T_0 < \infty)|U(0) = u - a] \\ &= \psi(u - a). \quad (6.18) \end{aligned}$$

Similarly, it follows from (6.12) and (5.7) that, for $u < b$,

$$\begin{aligned} B(-\infty, b|u) &= \lim_{a\to-\infty} B(a, b|u) \\ &= e^{-\rho(b-u)}. \quad (6.19) \end{aligned}$$

Note that, with $\delta=0$ and $0 \leq u < b$, $A(0, b|u)$ is the probability of ruin from an initial surplus u in the presence of an absorbing upper barrier at b.

For $a' < a \leq u < b < b'$, by considering whether $T_a < T_b$ or $T_a > T_b$, we obtain the identities

$$A(a, b'|u) = A(a, b|u) + B(a, b|u)A(a, b'|b) \quad (6.20)$$

and

$$B(a', b|u) = A(a, b|u)B(a', b|a) + B(a, b|u). \quad (6.21)$$

With $a=0$, $b=x$, $b'=\infty$ and because of (6.18), (6.20) becomes

$$\psi(u) = A(0, x|u) + B(0, x|u)\psi(x). \quad (6.22)$$

With $a'=-\infty$, $a=0$, $b=x$, $b'=\infty$ and because of (6.19), (6.21) becomes

$$e^{-\rho(x-u)} = A(0, x|u)e^{-\rho x} + B(0, x|u). \quad (6.23)$$

For $0 \leq u < x$, formulas (6.22) and (6.23) are two linear equations for $A(0, x|u)$ and $B(0, x|u)$; their solution is

$$A(0, x|u) = \frac{e^{\rho x}\psi(u) - e^{\rho u}\psi(x)}{e^{\rho x} - \psi(x)} \quad (6.24)$$

and

$$B(0, x|u) = \frac{e^{\rho u} - \psi(u)}{e^{\rho x} - \psi(x)}. \quad (6.25)$$

With $\delta=0$, Segerdahl (1970) denotes $A(0, x|u)$ and $B(0, x|u)$ as $\xi(u, x)$ and $\chi(u, x)$, respectively. Formulas (6.24) and (6.25) extend Dickson's (1992) formulas (1.3) and (1.4) to the general case of $\delta \geq 0$.

With (6.25), we can now prove (6.5). Let $0 \leq u < x$. If ruin should occur with $U(0)=0$ such that the surplus immediately before ruin is x, then the surplus must attain the level u prior to ruin. Hence

$$f(x|0) = B(0, u|0)\, f(x|u), \quad (6.26)$$

or

$$\begin{aligned} f(x|u) &= \frac{f(x|0)}{B(0, u|0)} \\ &= f(x|0)\,\frac{e^{\rho u} - \psi(u)}{1 - \psi(0)}, \quad (6.27) \end{aligned}$$

which is (6.5).

Formula (6.6) is more intricate because the condition $U(0)=u \geq x=U(T\text{-})>0$ means that the surplus is to drop below the level x some time before ruin occurs. Its proof is based on the notion of *duality*, which, as pointed out by Feller (1971, p. 395), enables us "to prove in an elementary way theorems that would otherwise require deep analytic methods." Formula (6.6) follows from the identity

$$B(0, u|0)\, f(x|u) \int_0^\infty e^{-\rho y}\, \frac{p(x + y)}{1 - P(x)}\, dy = g(x)\, A(-u, 0|-x)\, e^{-\rho u}, \quad (6.28)$$

valid for $0 < x \leq u$. Solving for $f(x|u)$ and using (3.4) and (6.16), we get

$$f(x|u) = \lambda c^{-1}[1 - P(x)]e^{-\rho u}\, \frac{A(0, u|u - x)}{B(0, u|0)}. \quad (6.29)$$

Applying (6.24) and (6.25) to (6.29) yields

$$f(x|u) = \lambda c^{-1}[1 - P(x)]\, \frac{\psi(u - x) - e^{-\rho x}\psi(u)}{1 - \psi(0)}, \quad (6.30)$$

which is indeed (6.6).

It remains to prove the identity (6.28). We multiply it by dx. Then the expression on the left-hand side can be interpreted as

$$\mathrm{E}[e^{-\delta T_0} I(T_u < T < \infty,\ x \leq U(T\text{-}) \leq x + dx)\, |U(0) = 0], \quad (6.31)$$

the expected present value of a payment of 1 that is due at the time of recovery, provided that the surplus has been above the level u prior to ruin and between x and $x+dx$ immediately before ruin. The expression on the right-hand side is

$$\mathrm{E}[e^{-\delta T_0} I(-x-dx \leq U(T) \leq -x,\ \min_{T<t<T_0} U(t) < -u)\, |U(0) = 0], \quad (6.32)$$

the expected present value of 1 that is due at the time of recovery, provided that the surplus has been between $-x-dx$ and $-x$ at the time of ruin and below $-u$ prior to recovery. Finally, the equality of (6.31) and (6.32) can be explained by duality.

A dual process $\{U^*(t)\}$ of the process $\{U(t)\}$ with $U(0)=0$ is defined as follows: If $T=\infty$, we set $U^*(t)=U(t)$, and if $T<\infty$, we set

$$U^*(t) = \begin{cases} -U(T_0 - t) & \text{for } 0 \le t \le T_0 \\ U(t) & \text{for } t > T_0 \end{cases}. \qquad (6.33)$$

See Figures 3 and 4. In other words, suppose that $\{U(t)\}$ has n jumps before the time of recovery T_0 and that the jump of size X_i occurs at time t_i, $t_i<T_0$, $i=1, \ldots, n$. Then $\{U^*(t)\}$ has the same n jumps before time T_0, except that the jump of size X_i occurs at time T_0-t_i, $i=1, \ldots, n$. This is a measure-preserving correspondence, and hence the process $\{U^*(t)\}$ follows the same probability law as the process $\{U(t)\}$. That is, if a certain event in terms of $\{U(t)\}$ is translated as an event that is formulated in terms of $\{U^*(t)\}$, the probabilities, or, as in the case of (6.31) and (6.32), the contingent expectations, are identical. (Incidentally, this duality also explains the symmetric Formula (3.7), which is for the case $\delta=0$.) This completes the proof of (6.6).

Using (2.40), we obtain from (6.5) and (6.6) the formula

$$f(x, y|u) =$$

$$\begin{cases} f(x, y|0)\,\dfrac{e^{\rho u} - \psi(u)}{1 - \psi(0)}, & x > u \ge 0, \qquad (6.34) \\[2ex] f(x, y|0)\,\dfrac{e^{\rho x}\psi(u - x) - \psi(u)}{1 - \psi(0)}, & 0 < x \le u, \qquad (6.35) \end{cases}$$

with

$$f(x, y|0) = \lambda c^{-1} e^{-\rho x}\, p(x + y) \qquad (6.36)$$

according to (3.3).

Example

One consequence of (6.34) and (6.35) is that there is an explicit formula for $f(x, y|u)$ whenever there is an explicit expression for the function $\psi(u)$. This is the case for an exponential claim amount distribution,

$$p(x) = \beta e^{-\beta x}, \quad x \ge 0.$$

Here we have, for $u\ge 0$,

$$\psi(u) = \mathrm{E}[e^{-\delta T+\rho U(T)}\, I(T < \infty)|U(0) = u]$$
$$= \frac{\beta - R}{\beta + \rho}\, e^{-Ru} \qquad (6.37)$$

according to (5.42) [with $w(y)=e^{-\rho y}$]. Then

$$\psi(u - x) = e^{Rx}\psi(u). \qquad (6.38)$$

Hence, by (6.5) and (6.6) we obtain

$$f(x|u) =$$

$$\begin{cases} \dfrac{\lambda}{c(R + \rho)}\, e^{-(\rho+\beta)x}\, [(\beta + \rho)e^{\rho u} - (\beta - R)e^{-Ru}], \\ \qquad x > u \ge 0, \qquad (6.39) \\[2ex] \dfrac{\lambda(\beta - R)}{c(R + \rho)}\, e^{-(\rho+\beta)x}\, [e^{(R+\rho)x} - 1]\, e^{-Ru}, \\ \qquad 0 < x \le u. \qquad (6.40) \end{cases}$$

To determine $f(x, y|u)$, we apply (3.19). We may use the formula

$$\int_0^\infty f(x|u)\, dx = \mathrm{E}[e^{-\delta T}\, I(T < \infty)|U(0) = u]$$

as a check for the validity of (6.39) and (6.40). After some calculation the integral on the left-hand side simplifies as

$$\frac{\lambda}{c(\beta + \rho)}\, e^{-Ru}, \qquad (6.41)$$

while the right-hand side is

$$\frac{\beta - R}{\beta}\, e^{-Ru} \qquad (6.42)$$

by (5.38). These two terms are the same because of (5.40). It is amusing to note that the integral of expression (6.40) from $x=0$ to $x=\infty$ is also (6.41). □

Remarks

(i) With $w(x, y)=e^{-\rho y}$,

$$h(u) = \int_u^\infty e^{-\rho(z-u)}\, g(z)\, dz \qquad (6.43)$$

by (2.32), (2.17), and (3.4). It follows from (2.34) [with $\phi(u)=\psi(u)$] that

$$\psi(u) = (\psi * g)(u) + h(u)$$

$$= \int_0^u \psi(u - z)\, g(z)\, dz + \int_u^\infty e^{-\rho(z-u)}\, g(z)\, dz, \qquad (6.44)$$

FIGURE 3

A SAMPLE PATH OF THE PROCESS $\{U(t)\}$ THAT CONTRIBUTES TO EXPRESSION (6.31)

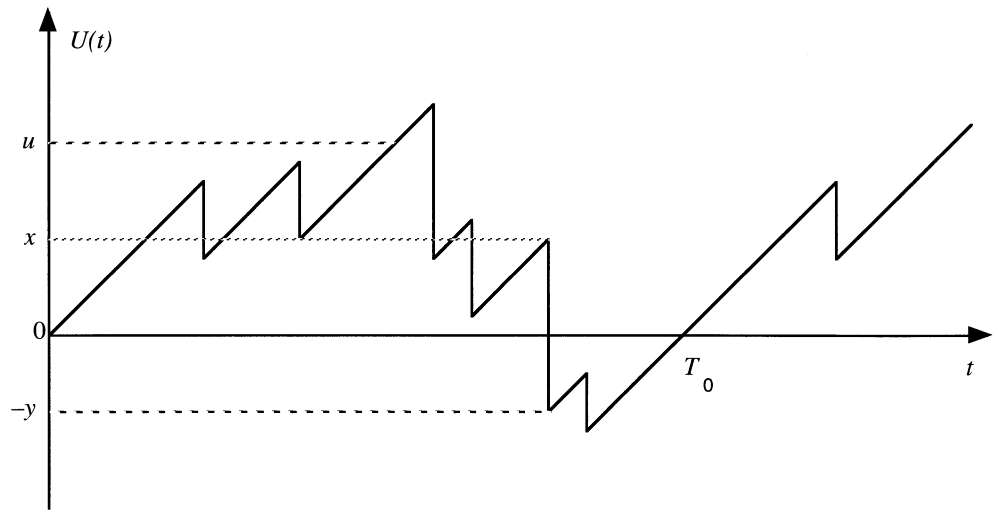
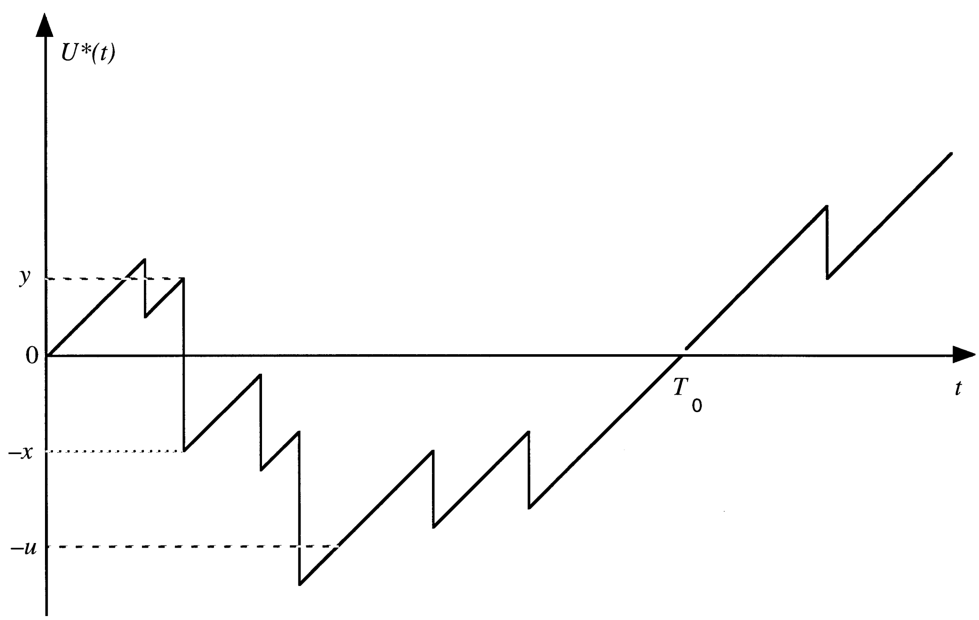


FIGURE 4

THE DUAL SAMPLE PATH THAT CONTRIBUTES TO EXPRESSION (6.32)



which generalizes Exercise 13.11 of *Actuarial Mathematics*. With $u=0$, (6.44) becomes

$$\psi(0) = \hat{g}(\rho); \tag{6.45}$$

recall that $\hat{g}(-R)=1$.

(ii) As an alternative proof, we would like to show that (6.5) and (6.6) satisfy the renewal equation (3.26), or equivalently, with the definition

$$\varphi(u) = \begin{cases} e^{\rho u}, & x > u \geq 0, \\ e^{\rho x}\psi(u - x), & 0 < x \leq u, \end{cases} \tag{6.46}$$

that

$$\varphi(u) - \psi(u) = [(\varphi - \psi)*g](u) + [1 - \psi(0)]\, e^{\rho u}\, I(x > u) \tag{6.47}$$

holds. A direct verification of (6.47) seems difficult. However, we can confirm its validity by means of Laplace transforms. Taking Laplace transforms of (6.44) yields

$$\hat{\psi}(\xi) = \hat{\psi}(\xi)\hat{g}(\xi) + \hat{h}(\xi), \tag{6.48}$$

where

$$\hat{h}(\xi) = \int_0^\infty e^{-\xi u} \left[\int_u^\infty e^{-\rho(z-u)}\, g(z)\, dz\right] du = \frac{\hat{g}(\xi) - \hat{g}(\rho)}{\rho - \xi} \tag{6.49}$$

by changing the order of integration. Hence

$$\hat{\psi}(\xi) = \frac{\hat{h}(\xi)}{1 - \hat{g}(\xi)} = \frac{\hat{g}(\xi) - \hat{g}(\rho)}{[1 - \hat{g}(\xi)](\rho - \xi)}. \tag{6.50}$$

From (6.46)

$$\hat{\varphi}(\xi) = \frac{e^{(\rho-\xi)x} - 1}{\rho - \xi} + e^{(\rho-\xi)x}\, \hat{\psi}(\xi). \tag{6.51}$$

Thus

$$\hat{\varphi}(\xi) - \hat{\psi}(\xi) = \frac{e^{(\rho-\xi)x} - 1}{\rho - \xi}\, \frac{1 - \hat{g}(\rho)}{1 - \hat{g}(\xi)} = \frac{e^{(\rho-\xi)x} - 1}{\rho - \xi}\, \frac{1 - \psi(0)}{1 - \hat{g}(\xi)} \tag{6.52}$$

by (6.45). We now see that (6.47) holds.

(iii) As we pointed out earlier, it follows from our generalization of Dickson's formula that there is an explicit formula for $f(x|u)$ [and $f(x, y|u)$] whenever there is an explicit expression for the function $\psi(u)$. If $\hat{\psi}(\xi)$ is a rational function, then by locating its poles (singularities), we can determine $\psi(u)$. It follows from (6.50) that $\hat{\psi}(\xi)$ is a rational function if and only if $\hat{g}(\xi)$ is a rational function; by (2.50) $\hat{g}(\xi)$ is a rational function if and only if $\hat{p}(\xi)$ is a rational function. It also follows from (6.50) that the singularities of $\hat{\psi}(\xi)$ are exactly the roots of the equation

$$\hat{g}(\xi) = 1. \tag{6.53}$$

We should clarify that here the functions $\hat{\psi}(\xi)$, $\hat{g}(\xi)$, and $\hat{p}(\xi)$ are defined on the whole complex plane by analytic continuation. Consider the example where $p(x)=\beta e^{-\beta x}$; although the integral

$$\int_0^\infty e^{-\xi x}\ p(x)dx$$

is not defined for complex numbers ξ with $Re(\xi)\leq-\beta$, the rational function $\beta/(\beta+\xi)$ is. Consequently, while (2.53) has at most one solution, (6.53) can have multiple solutions. Now, let $-r_1, -r_2, \ldots, -r_m$ be the distinct roots of (6.53) and $n_1, n_2, \ldots, n_m$ be their multiplicities. Then it follows from *Heaviside's expansion formula* (Spiegel 1965, p. 73) that

$$\psi(u) = \sum_{k=1}^{m} \frac{1}{(n_k - 1)!} \lim_{\xi\to -r_k} \frac{d^{n_k-1}}{d\xi^{n_k-1}} [(\xi + r_k)^{n_k}\, \hat{\psi}(\xi)\, e^{\xi u}], \tag{6.54}$$

where $\hat{\psi}(\xi)$ is given by (6.50). In the special case where all poles of $\hat{\psi}(\xi)$ are simple, that is, $n_1=n_2=\ldots=n_m=1$, then (6.54) simplifies as

$$\psi(u) = \sum_{k=1}^{m} \lim_{\xi\to -r_k} [(\xi + r_k)\, \hat{\psi}(\xi)\, e^{\xi u}] = \sum_{k=1}^{m} \frac{\hat{h}(-r_k)}{-\hat{g}'(-r_k)}\, e^{-r_k u} = \sum_{k=1}^{m} \frac{\hat{g}(-r_k) - \hat{g}(\rho)}{-\hat{g}'(-r_k)(\rho + r_k)}\, e^{-r_k u}. \tag{6.55}$$

By (6.53) and (6.45),

$$\hat{g}(-r_k) - \hat{g}(\rho) = 1 - \psi(0).$$

Similar to (2.55), we have

$$-\hat{g}'(-r_k)(\rho + r_k) = -(\lambda/c)\hat{p}'(-r_k) - 1.$$

Hence (6.55) simplifies as

$$\psi(u) = [1 - \psi(0)] \sum_{k=1}^{m} \frac{e^{-r_k u}}{-(\lambda/c)\hat{p}'(-r_k) - 1}. \tag{6.56}$$

Putting $u=0$ in (6.56) and rearranging, we obtain

$$1 - \psi(0) = \frac{1}{1 + \sum_{k=1}^{m} \frac{1}{1-(\lambda/c)\hat{p}'(-r_k) - 1}}, \quad (6.57)$$

which can be substituted in (6.56) yielding

$$\psi(u) = \frac{c}{1 + \sum_{k=1}^{m} \frac{c}{1-\lambda\hat{p}'(-r_k) - c}} \sum_{k=1}^{m} \frac{e^{-r_k u}}{-\lambda\hat{p}'(-r_k) - c}. \quad (6.58)$$

Consider the case in which $p(x)$ is a mixture of exponential distributions,

$$p(x) = \sum_{j=1}^{n} A_j \beta_j e^{-\beta_j x}, \quad x \geq 0, \quad (6.59)$$

where

$$0 < \beta_1 < \beta_2 < \ldots < \beta_n,$$

and $\sum_{j=1}^{n} A_j=1$. Then

$$\hat{p}(\xi) = \sum_{j=1}^{n} \frac{A_j \beta_j}{\beta_j + \xi}, \quad (6.60)$$

and Lundberg's fundamental equation (2.22) becomes

$$\delta + \lambda - c\xi = \lambda \sum_{j=1}^{n} \frac{A_j \beta_j}{\beta_j + \xi}. \quad (6.61)$$

The nonnegative solution of (6.61) is ρ and the negative solutions are the poles of $\hat{\psi}(\xi)$. We now impose the condition that $A_j>0$, $j=1, 2, \ldots, n$. Then (6.61) has n distinct negative roots $\{-r_k\}$ with

$$0 < r_1 = R < \beta_1 < r_2 < \beta_2 < \ldots < r_n < \beta_n.$$

[Inequalities (13.6.15) of *Actuarial Mathematics* are for the case $\delta=0$; see also Figure 13.6.2 of *Actuarial Mathematics*.] It follows from

$$\hat{p}'(\xi) = -\sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j + \xi)^2} \quad (6.62)$$

and (6.58) that, given the roots $\{-r_k\}$, we have an explicit formula for $\psi(u)$ [and hence explicit formulas for $f(x|u)$ and $f(x, y|u)$]. On the other hand, by (2.50) and (6.60),

$$\hat{g}(\xi) = \frac{\lambda}{c(\rho - \xi)} [\hat{p}(\xi) - \hat{p}(\rho)]$$
$$= \frac{\lambda}{c} \sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j + \xi)(\beta_j + \rho)}, \quad (6.63)$$

from which we obtain

$$h(\xi) = \frac{\hat{g}(\xi) - \hat{g}(\rho)}{\rho - \xi} = \frac{\lambda}{c} \sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j + \xi)(\beta_j + \rho)^2} \quad (6.64)$$

and

$$-\hat{g}'(\xi) = \frac{\lambda}{c} \sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j + \xi)^2(\beta_j + \rho}. \quad (6.65)$$

It follows from (6.55), (6.64), and (6.65) that we have the following alternative formula for $\psi(u)$,

$$\psi(u) = \sum_{k=1}^{n} \frac{\sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j - r_k)(\beta_j + \rho)^2}}{\sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j - r_k)^2(\beta_j + \rho)}} e^{-r_k u}. \quad (6.66)$$

Note that in the special case $n=1$, we obtain again (6.37).

(iv) Substituting the asymptotic expression of $\psi(u)$,

$$\psi(u) \sim Ce^{-Ru} \quad \text{for } u \to \infty, \quad (6.67)$$

in (6.35) yields

$$f(x, y|u) \sim f(x, y|0) \frac{C}{1 - \psi(0)} [e^{\rho x} e^{-R(u-x)} - e^{-Ru}]$$
$$= \frac{\lambda C}{c[1 - \psi(0)]} (e^{Rx} - e^{-\rho x})\, p\,(x + y)\, e^{-Ru}$$
$$\text{for } u \to \infty. \quad (6.68)$$

Because

$$C = \frac{\hat{g}(-R) - \hat{g}(\rho)}{-\hat{g}'(-R)(\rho + R)}$$
$$= \frac{1 - \psi(0)}{-\lambda c^{-1}\hat{p}'(-R) - 1}, \quad (6.69)$$

the asymptotic formulas (6.68) and (4.21) are the same.

(v) The expression

$$\mathrm{E}[v^T\, \bar{a}_{\overline{T_0-T}|}\, I(T < \infty)|U(0) = u] \quad (6.70)$$

is the expected present value of a continuous annuity at a rate of 1 per unit time between the time of ruin and the time of recovery for a given initial surplus u. Because

$$v^T\, \bar{a}_{\overline{T_0-T}|} = \frac{e^{-\delta T} - e^{-\delta T_0}}{\delta}, \quad (6.71)$$

(6.70) is

$$\frac{\int_0^\infty f(x|u)\, dx - \psi(u)}{\delta}. \quad (6.72)$$

Alternatively, (6.70) is

$$\mathrm{E}\left[v^T \frac{1 - e^{\rho U(T)}}{\delta} I(T < \infty)\middle|U(0) = u\right] \quad (6.73)$$

by (6.9). That is, it is $\phi(u)$ with the penalty function w given by (2.14). We note that

$$\lim_{\delta\to 0} \mathrm{E}[v^T \bar{a}_{\overline{T_0-T}|}\, I(T < \infty)|U(0) = u]$$
$$= \mathrm{E}[(T_0 - T)\, I(T < \infty)|U(0) = u]. \quad (6.74)$$

It follows from (6.74), (6.73), and (2.45) that

$$\mathrm{E}[(T_0 - T)\, I(T < \infty)|U(0) = u]$$
$$= \lim_{\delta\to 0} \frac{\rho}{\delta} \mathrm{E}[|U(T)| I(T < \infty)|U(0) = u]$$
$$= \frac{1}{c - \lambda p_1} \mathrm{E}[|U(T)| I(T < \infty)|U(0) = u]. \quad (6.75)$$

Formula (6.75) is intuitively clear because $c-\lambda p_1$ is the drift of $\{U(t)\}$. For related results see Egídio dos Reis (1993).

(vi) Recall the function $\pi_3(x, t|u)$, the passage time density of the surplus process at the level x, an explicit formula for which is given by (5.13) and (5.14). Similarly to (5.37), we have, for $x\le u$ and $\delta\ge 0$,

$$\int_0^\infty \pi_3(x, \tau|u)\, e^{-\delta\tau}\, d\tau$$
$$= \psi(x - u) \left[1 + \int_0^\infty \pi_3(x, \tau|x)\, e^{-\delta\tau}\, d\tau\right]$$
$$= \psi(x - u) \left[1 + \int_0^\infty \pi_3(0, \tau|0)\, e^{-\delta\tau}\, d\tau\right]. \quad (6.76)$$

Putting $x=u=0$ in (6.76) and solving for the integral yields

$$\int_0^\infty \pi_3(0, \tau|0)\, e^{-\delta\tau}\, d\tau = \frac{\psi(0)}{1 - \psi(0)}. \quad (6.77)$$

Applying (6.77) to (5.37) and (6.76), we obtain

$$\int_0^\infty \pi_3(x, \tau|u)\, e^{-\delta\tau}\, d\tau = \begin{cases} \dfrac{e^{-\rho(x-u)}}{1 - \psi(0)}, & x > u \\[2ex] \dfrac{\psi(u - x)}{1 - \psi(0)}, & x \le u \end{cases}. \quad (6.78)$$

The right-hand side of (6.78) can be written as a pair of infinite series using the geometric series formula

$$\frac{1}{1 - \psi(0)} = \sum_{n=0}^\infty \psi(0)^n;$$

the j-th term of either series represents the contribution of the j-th upcrossing at the level x to the integral on the left-hand side of (6.78).

7. OPTIMAL DIVIDEND STRATEGIES

We now consider a problem that is due to Bruno de Finetti, has been treated by Karl Borch and others, and can be found in the textbooks of Bühlmann (1970, Section 6.4) and Gerber (1979, Section 10.1). Here the surplus model is modified in that dividends are paid to the shareholders of the insurance company. We assume that the dividends are paid according to a *barrier strategy* corresponding to a barrier at the level b. Thus whenever the surplus is on the barrier b, dividends are paid continuously, at a rate of c so that the surplus stays on the barrier, until the next claim occurs and the surplus falls below b. If the surplus is below b, no dividends are being paid. Evidently, ruin will occur with certainty in this model. For $0\le u\le b$, let $V(u, b)$ denote the expected present value of the dividend payments until ruin.

Since no dividends are paid unless the surplus reaches the level b before ruin occurs, we have, for $0\le u\le b$,

$$V(u, b) = B(0, b|u)\, V(b, b), \quad (7.1)$$

or, by (6.25)

$$V(u, b) = \frac{e^{\rho u} - \psi(u)}{e^{\rho b} - \psi(b)} V(b, b). \quad (7.2)$$

To determine $V(b, b)$, we need a boundary condition at $u=b$. To obtain it, we compare two situations at time 0: one with initial surplus b, and the other with initial surplus $u=b-h$, $0<h\le b$. Then h/c is the time the surplus reaches the barrier in the second case, provided that there is no claim by then. By conditioning on the time t and the amount x of the first claim in the time interval $(0, h/c)$ and noting that the dividend payments start immediately in the first case, we see that

$$V(b, b) - V(b - h, b)$$
$$= e^{-\lambda h/c}\, c\, \bar{a}_{\overline{h/c}|} + \int_0^{h/c} \lambda e^{-\lambda t}\, c\, \bar{a}_{\overline{t}|}\, dt$$
$$+ \int_0^{h/c} \lambda e^{-\lambda t} e^{-\delta t} \left[\int_0^b V(b - x, b) p(x) dx - \int_0^{b-h+ct} V(b - h + ct - x, b) p(x) dx\right] dt. \quad (7.3)$$

Differentiating (7.3) with respect to h and then setting $h=0$, we obtain the condition

$$\left.\frac{\partial V(u, b)}{\partial u}\right|_{u=b} = 1. \tag{7.4}$$

Now, differentiating (7.2) with respect to u, setting $u=b$, and applying (7.4) yields

$$1 = \frac{\rho e^{\rho b} - \psi'(b)}{e^{\rho b} - \psi(b)} V(b, b).$$

Hence

$$V(u, b) = \frac{e^{\rho u} - \psi(u)}{\rho e^{\rho b} - \psi'(b)}, \quad 0 \le u \le b. \tag{7.5}$$

This formula should be compared with (1.13) in Chapter 10 of Gerber (1979). In Section 10.1 of Gerber (1979), the function $B(0, b|u)$ is denoted as $W(u, b)$.

Let $\tilde{b}$ be the optimal barrier, that is, $\tilde{b}$ is the value of b that maximizes the expected present value of the dividends. In view of (7.5), $\tilde{b}$ is the value that minimizes the denominator, that is, $\tilde{b}$ satisfies

$$\rho^2 e^{\rho b} - \psi''(b) = 0. \tag{7.6}$$

An equivalent condition is that

$$\left.\frac{\partial^2 V(u, \tilde{b})}{\partial u^2}\right|_{u=\tilde{b}} = 0; \tag{7.7}$$

this follows from the explicit form of (7.5).

Example

In the case of an exponential claim amount distribution, there is an explicit expression for $\psi(u)$. Substituting (6.37) into (7.5) yields

$$V(u, b) = \frac{(\beta + \rho)e^{\rho u} - (\beta - R)e^{-Ru}}{\rho(\beta + \rho)e^{\rho b} + R(\beta - R)e^{-Rb}}. \tag{7.8}$$

The optimal value $\tilde{b}$ is obtained from the condition that

$$\rho^2(\beta + \rho)e^{\rho b} - R^2(\beta - R)e^{-Rb} = 0. \tag{7.9}$$

Thus

$$\tilde{b} = \frac{1}{\rho + R} \ln \frac{R^2(\beta - R)}{\rho^2(\beta + \rho)} \tag{7.10}$$

is the optimal barrier. □

8. CONCLUDING REMARKS

This paper studies the joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin. The time of ruin is studied in terms of its Laplace transforms, which can naturally be interpreted as discounting. New results are derived, many of which have a probabilistic interpretation, and additional insight is gained for existing results in the classical model.

Formulas (2.34) and (3.3) are the mathematical keys. They are equivalent formulas. Formula (2.34) is derived analytically in Section 2. Section 3 shows that (3.3) is an immediate consequence of (2.34) and conversely that (2.34) can be obtained from (3.3) by probabilistic reasoning. In Section 5, we derive (3.3) by a probabilistic argument.

The results presented in this paper can be generalized in various directions. For example, several formulas can be extended to the case in which the compound Poisson process is replaced by a more general process with positive, independent and stationary increments, such as a *gamma process* or an *inverse Gaussian process*.

ACKNOWLEDGMENTS

We thank the anonymous referees for their comments. Elias Shiu gratefully acknowledges the support from the Principal Financial Group Foundation.

REFERENCES

BOWERS, N.L., JR., GERBER, H.U., HICKMAN, J.C., JONES, D.A., AND NESBITT, C.J. 1997. *Actuarial Mathematics*, 2nd ed. Schaumburg, Ill.: Society of Actuaries.

BÜHLMANN, H. 1970. *Mathematical Methods in Risk Theory*. New York: Springer-Verlag.

DE VYLDER, F.E. 1996. *Advanced Risk Theory: A Self-Contained Introduction*. Brussels: Editions de l'Universite de Bruxelles.

DICKSON, D.C.M. 1992. "On the Distribution of Surplus prior to Ruin," *Insurance: Mathematics and Economics* 11:191–207.

DICKSON, D.C.M., AND EGÍDIO DOS REIS, A.D. 1994. "Ruin Problems and Dual Events," *Insurance: Mathematics and Economics* 14:51–60.

DUFRESNE, F., AND GERBER, H.U. 1988. "The Surpluses Immediately before and at Ruin, and the Amount of the Claim Causing Ruin," *Insurance: Mathematics and Economics* 7:193–99.

EGÍDIO DOS REIS, A.D. 1993. "How Long is the Surplus Below Zero?" *Insurance: Mathematics and Economics* 12:23–38.

FELLER, W. 1971. *An Introduction to Probability Theory and Its Applications*, Vol. 2, 2nd ed. New York: Wiley.

GERBER, H.U. 1979. "An Introduction to Mathematical Risk Theory." *S.S. Huebner Foundation Monograph Series No. 8*. Homewood, Ill.: Irwin.

GERBER, H.U. 1988. "Mathematical Fun with Ruin Theory," *Insurance: Mathematics and Economics* 7:15–23.

GERBER, H.U., AND SHIU, E.S.W. 1997a. "On the Time Value of Ruin," *Actuarial Research Clearing House* 1997.1:145–99.

GERBER, H.U., AND SHIU, E.S.W. 1997b. "From Ruin Theory to Option Pricing," *Joint Day Proceedings Volume of the XXVIIIth International ASTIN Colloquium/7th International AFIR Colloquium*, Cairns, Australia, pp. 157–76.

GERBER, H.U., AND SHIU, E.S.W. 1997c. "The Joint Distribution of the Time of Ruin, the Surplus Immediately before Ruin, and the Deficit at Ruin," *Insurance: Mathematics and Economics* 21:129–37.

GERBER, H.U., AND SHIU, E.S.W. 1998. "Pricing Perpetual Options for Jump Processes," unpublished manuscript.

KENDALL, D.G. 1957. "Some Problems in the Theory of Dams," *Journal of the Royal Statistical Society* Series B 19:207–12.

LUNDBERG, F. 1932. "Some Supplementary Researches on the Collective Risk Theory," *Skandinavisk Aktuarietidskrift* 15:137–58.

PRABHU, N.U. 1961. "On the Ruin Problem of Collective Risk Theory," *Annals of Mathematical Statistics* 32:757–64.

RESNICK, S.I. 1992. *Adventures in Stochastic Processes*. Boston: Birkhäuser.

SEAH, E.S. 1990. "Computing the Probability of Eventual Ruin," *Transactions of the Society of Actuaries*, XLII:421–46.

SEAL, H.L. 1969. *Stochastic Theory of a Risk Business*. New York: Wiley.

SEGERDAHL, C.-O. 1970. "On Some Distributions in Time Connected with the Collective Theory of Risk," *Skandinavisk Aktuarietidskrift* 53:167–92.

SPIEGEL, M.R. 1965. *Schaum's Outline of Theory and Problems of Laplace Transforms*. New York: Schaum.

DISCUSSIONS

F. ETIENNE DE VYLDER* AND MARC J. GOOVAERTS†

The paper is a new chapter and a summit of classical actuarial ruin theory. The primary goal of ruin theory is the evaluation of the ruin probability $P(T<\infty)$. In recent years actuaries have dissected the risk process and considered more components such as the time of ruin T, the deficit at ruin $-U(T)$, and the risk reserve just before ruin $U(T\text{-})$. At each stage, more insight has been gained. The novelty of this paper is the introduction of a discounting factor at ruin $e^{-\delta T}$. This allows one to solve an optimal dividend strategies problem (Section 7), and it lays the bases for a model of pricing American options. We will show that the number of the claim provoking ruin also merits consideration.

The density of results of this paper is amazing. Almost all classical results—results by Dufresne and Gerber (1988), by Gerber (1988), by Dixon (1992), and by Egídio dos Reis (1993)—are obtained as particular cases for $\delta=0$, and almost all the formulas are extended to the case $\delta>0$.

Another characteristic of the paper is its richness of tools and techniques: integrating factors, Laplace transforms, convolutions, renewal equations and key renewal theorem, martingales and optional sampling theorem, probabilistic duality, and Heaviside's expansion.

It is remarkable that the paper is not based on any other results of classical ruin theory. Our first idea when we saw the paper was that it would be an excellent last chapter of any book on the subject. Our second opinion is that due to its self-containedness, it should better be a first chapter.

The rest of this discussion focuses on the central simple but deep relation (3.3) of the paper, that is,

$$f(x, y|0) = (\lambda/c)e^{-\rho x}\, p(x + y),\ (x > 0,\ y > 0). \quad (1)$$

The difficult part of the proof of this relation is the insertion of the root $\rho\geq 0$ of Lundberg's fundamental equation (2.22) for ξ,

$$\delta + \lambda - c\xi = \lambda\hat{p}(\xi). \quad (2)$$

The first proof of (1) is based on renewal equation (2.34) in which ρ is introduced by the ingenious trick of *integrating factors*. The proof of (2.34) is a masterpiece. In the most elegant second proof of (1) in Section 5, ρ is inserted via the optional sampling theorem of martingale theory. We guess that (1) has been discovered by the martingale proof.

We now develop a complete third proof of (1) based on the consideration of the claim number provoking ruin and on Lagrange's expansion of a function of a root of an equation. Hereafter, the classical risk process is considered with $U(0)=0$ only. The condition $U(0)=0$ is understood everywhere. The claim instants are $T_1<T_2<\ldots$ and the corresponding claim amounts

*F. Etienne De Vylder, Ph.D., is a former professor of actuarial science at a Belgian university. He can be reached at Broederlijke Weversplein 32, 9000 Gent, Belgium.

†Marc J. Goovaerts, Ph.D., is a professor at the Katholeike Universiteit Leuven, Ninderbroederstraat 5, B-3000 Leuven, Belgium.

are $X_1, X_2, \ldots$. For $x, y, n \geq 0$, let $f_n(x,y|0)$ and $f_n(x|0)$ be defined by the relations

$$f_n(x,y|0)dxdy = \mathrm{E}\{I[T = T_{n+1},\ x < U(T\text{-}) < x+dx,\ y < |U(T)| < y+dy]e^{-\delta T}\}, \quad (3)$$

$$f_n(x|0)dx = \mathrm{E}\{I[\text{no ruin in } [0, T_{n+1}),\ x < U(T_{n+1}\text{-}) < x+dx]e^{-\delta T_{n+1}}\}. \quad (4)$$

Then

$$f_n(x,y|0) = f_n(x|0)\ p(x+y), \quad (5)$$

$$f(x, y|0) = \sum_{n\geq 0} f_n(x,y|0) = \left[\sum_{n\geq 0} f_n(x|0)\right] p(x+y), \quad (6)$$

and thus it is enough to prove that

$$\sum_{n\geq 0} f_n(x|0) = \frac{\lambda}{c}\, e^{-\rho x}. \quad (7)$$

The expectation of the indicator function of an event equals the probability of that event. Hence, $f_n(x,y|0)dxdy$ and $f_n(x|0)dx$ are probabilities if $\delta=0$.

In order to evaluate $f_n(x|0)$, we observe that the density of $(X_1, \ldots, X_n, T_1, \ldots, T_n, T_{n+1})$ is

$$f(x_1, \ldots, x_n, t_1, \ldots, t_n, t_{n+1}) = p(x_1)\ldots p(x_n)\ \lambda^{n+1}\ e^{-\lambda t_{n+1}},$$

on the subset of R^{2n+1} defined by the relations $x_1 \geq 0, \ldots, x_n \geq 0$ and $0<t_1<\ldots<t_n<t_{n+1}<\infty$. Hence,

$$f_n(x|0)dx = \lambda^{n+1} \int \ldots \int_I p(x_1)\ldots p(x_n)dx_1\ldots dx_n \left[\int \ldots \int_J e^{-\lambda t_{n+1}}\ e^{-\delta t_{n+1}}\ dt_1\ldots dt_n dt_{n+1}\right], \quad (8)$$

where I is the integration domain defined by the relations $x_1 \geq 0, \ldots, x_n \geq 0$ and for fixed $x_1, \ldots, x_n$, J is the integration domain defined by the relations $0<t_1<\ldots<t_n<t_{n+1}$ and

$$x_1 < ct_1,\ x_1 + x_2 < ct_2, \ldots,\ x_1 + x_2 + \ldots + x_n < ct_n,$$

$$x < ct_{n+1} - (x_1 + \ldots + x_n) < x + dx. \quad (9)$$

In fact, no integration with respect to t_{n+1} must be performed and $t_{n+1}=(x_1+\ldots+x_n+x)/c$, $dt_{n+1}=dx/c$ by (9). Then by (8),

$$f_n(x|0) = (\lambda/c)\ \lambda^n \int \ldots \int_I p(x_1)\ldots p(x_n)e^{-a(x_1+\ldots+x_n+x)}\ dx_1\ldots dx_n \left[\int \ldots \int_K dt_1\ldots dt_n\right], \quad (10)$$

where for fixed $x_1, \ldots, x_n$, K is the integration domain defined by the relations

$$0 < t_1 < \ldots < t_n < (x_1 + \ldots + x_n + x)/c,$$
$$t_1 > x_1/c,\ t_2 > (x_1 + x_2)/c, \ldots,$$
$$t_n > (x_1 + \ldots + x_n)/c, \quad (11)$$

and $a=(\lambda+\delta)/c$. In case $n=0$, (10) must be understood as $f_0(x|0)=(\lambda/c)e^{-ax}$. The multiple interior integral of (10) is a polynomial in $x_1, \ldots, x_n$. By the argument of De Vylder (1996), pp. 132–135, it can be replaced by the polynomial

$$(t^n/n!)(1-s_n/t)=t^{n-1}\ \ (t-s_n)/n!,$$

where

$$t = \frac{x_1 + \ldots + x_n + x}{c}$$

and

$$s_n = \frac{x_1 + \ldots + x_n}{c}.$$

Hence,

$$f_n(x|0) = \frac{(\lambda x/c)(\lambda/c)^n}{n!} \int \ldots \int_I (x_1 + \ldots + x_n + x)^{n-1}\ e^{-a(x_1+\ldots+x_n+x)}\ p(x_1)\ldots p(x_n)dx_1\ldots dx_n$$

$$= \frac{(\lambda x/c)(\lambda/c)^n}{n!}\ \mathrm{E}[(X_1 + \ldots + X_n + x)^{n-1}\ e^{-a(X_1+\ldots+X_n+x)}], \quad (12)$$

which is $(\lambda x/c)e^{-ax}$ if $n=0$.

Let us regard a as a variable. Then the expectation in the last member of (12) can be displayed as

$$(-1)^{n-1}\left(\frac{\partial}{\partial a}\right)^{n-1} \mathrm{E}[e^{-a(X_1+\ldots+X_n+x)}] = (-1)^{n-1}\left(\frac{\partial}{\partial a}\right)^{n-1} [e^{-ax}\hat{p}^n(a)],$$

where the differentiations under the E operator are easily justified. Then

$$f_n(x|0) = \frac{(\lambda/c)(-x)(-\lambda/c)^n}{n!} \left(\frac{\partial}{\partial a}\right)^{n-1} [e^{-ax}\hat{p}^n(a)]. \quad (13)$$

Let us now recall Lagrange's development. It says that under regularity conditions on the functions f and g,

$$f(\xi) = f(a) + \sum_{n\geq 1} \left(\frac{t^n}{n!}\right) \left(\frac{\partial}{\partial a}\right)^{n-1} [f'(a)g^n(a)] \quad (14)$$

if ξ is the root of the equation $\xi = a + tg(\xi)$, which tends to a as t tends to 0. Lundberg's fundamental equation (2) can be displayed as

$$\xi = a + (-\lambda/c)\hat{p}(\xi).$$

If $(-\lambda/c) = t \rightarrow 0$, then $\xi \rightarrow a > 0$. Hence, the positive root ρ of Lundberg's fundamental equation is involved in Lagrange's development. By Lagrange's development (14) with the functions $g(z) = \hat{p}(z)$ and $f(z) = e^{-xz}$ (x fixed), relation (7) results from (13).

DAVID C.M. DICKSON*

Professors Gerber and Shiu have produced a very comprehensive paper. They have introduced some elegant mathematics and provided a framework from which many results in classical ruin theory can be derived.

I would like to comment on a particular part of the paper, namely the derivation of $\phi(u)$ in Section 2. Throughout the paper the authors mention different approaches to solving a given problem. I was therefore somewhat surprised that they did not consider solving for $\phi(u)$ through its Laplace transform, especially as special cases of this transform are mentioned later in the paper.

Starting from Equation (2.16)

$$0 = -(\delta + \lambda)\phi(u) + c\phi'(u) + \lambda \int_0^u \phi(u - x)p(x)dx + \lambda\omega(u),$$

we have

$$0 = -(\delta + \lambda)\hat{\phi}(\xi) + c\,[\xi\hat{\phi}(\xi) - \phi(0)] + \lambda\hat{\phi}(\xi)\hat{p}(\xi) + \lambda\hat{\omega}(\xi),$$

so that

$$\hat{\phi}(\xi) = \frac{\lambda\hat{\omega}(\xi) - c\phi(0)}{\delta + \lambda - c\xi - \lambda\hat{p}(\xi)} = \frac{\lambda\hat{\omega}(\xi) - c\phi(0)}{\ell(\xi) - \lambda\hat{p}(\xi)}.$$

For reasons given in the paper, there is a unique positive number ρ such that $\ell(\rho) = \lambda\hat{p}(\rho)$. Since $\phi(u) > 0$ unless $w(x, y) = 0$ for all $x, y > 0$, it follows that $\lambda\hat{\omega}(\rho) = c\phi(0)$ and so

$$\hat{\phi}(\xi) = \frac{\lambda[\hat{\omega}(\xi) - \hat{\omega}(\rho)]}{\ell(\xi) - \lambda\hat{p}(\xi)}.$$

For example, if $w(x, y) = 1$ for all x and y, then

$$\omega(u) = 1 - P(u)$$

and so

$$\hat{\omega}(\xi) = \xi^{-1}\,[1 - \hat{p}(\xi)].$$

Hence

$$\hat{\phi}(\xi) = \frac{\lambda\,\{\xi^{-1}\,[1 - \hat{p}(\xi)] - \rho^{-1}\,[1 - \hat{p}(\rho)\}}{\lambda\,[1 - \hat{p}(\xi)] + \delta - c\xi},$$

and since $\lambda\hat{p}(\rho) = \delta + \lambda - c\rho$,

$$\hat{\phi}(\xi) = \frac{\lambda\rho\,[1 - \hat{p}(\xi)] + \xi\,(\delta - c\rho)}{\rho\xi\,(\lambda\,[1 - \hat{p}(\xi)] + \delta - c\xi)},$$

which is the authors' formula (2.59).

For certain forms of $p(x)$ (especially those mentioned in the paper) and $w(x, y)$ (say exponential functions), it should be easy to invert $\hat{\phi}(\xi)$ to find $\phi(u)$. Although it could be argued that solutions via Laplace transforms deprive us of the insight gained through other approaches, there is no doubt that inversion of Laplace transforms, particularly with the aid of the powerful mathematical software available nowadays, is a useful way of solving many problems.

VLADIMIR KALASHNIKOV*

The authors, well-known experts in actuarial science, present a paper giving a new insight into the ruin problem. In most papers on this topic, the probability of ultimate ruin is considered. It can be interpreted as the probability that the time T of ruin is finite. Using renewal arguments or the martingale approach,

*David C.M. Dickson, F.F.A., F.I.A.A., Ph.D., is Associate Professor in the Centre for Actuarial Studies, The University of Melbourne, Parkville, Victoria, 3052, Australia.

*Vladimir Kalashnikov, Ph.D., is a professor in the Department of Probability Theory, Faculty of Mechanics and Mathematics, Moscow State University, Vorob'evy Gory, 119899 Moscow, Russia.

one can investigate the limiting behavior of the ruin probability.

In this paper, the authors make a very important observation: if one equips the time T of ruin with other characteristics of the surplus process referring to the same time (in the paper, these are the surplus values just before and after the ruin), then the resulting renewal equation is much similar to the equation for the simple ruin probability, and owing to this fact, it can be solved, or, at least, investigated.

This observation allows them to obtain their profound results, which have numerous consequences and give rise to further research. Among such consequences, I would like to emphasize:

(i) Formula (3.3) for the joint density of the aforementioned two values of the surplus at the time of ruin given initial surplus $U(0)=0$

(ii) A generalization of Dickson's formula given in Section 6.

Now, let me list several remarks concerning the paper. First, for the experienced mathematician, the paper contains too many details, and it can be shortened without loss of clarity. For example, the arguments yielding the renewal equation (2.34) can be replaced by the arguments used in the last part of Section 3, yielding the same equation more directly. But I realize that the audience of this paper includes not only mathematicians.

Second, there are some minor incorrectnesses that do not affect the final results. For example, some regularity conditions for the claim amount distribution are needed to guarantee the existence of R.

Third, the authors use the Laplace transform to state many of their results. It should be kept in mind that the inversion problem is difficult in many real cases.

Fourth, it is well-known that there exists a similarity between risk and queuing theories. In queuing, similar problems were also considered and various methods for their solution (including the Wiener-Hopf factorization, not mentioned in the paper) were elaborated. In particular, this can allow the investigation of the Andersen risk model in which the occurrence times form a renewal process. Let me refer to Borovkov (1976) (especially Chapter 4), Cohen (1969) (especially Section II.5.3), Takács (1967) (see Chapter 7), and Gnedenko and Konig (1983). These books contain many relevant topics and further references.

In conclusion, I hope that the results presented in this paper will be implemented to find new characteristics of risk processes and will result in new research in actuarial science.

REFERENCES

BOROVKOV, A. 1976. *Stochastic Processes in Queuing Theory.* New York: Springer.

COHEN, J. 1969. *The Single Server Queue.* Amsterdam: North Holland.

TAKÁCS, L. 1967. *Combinatorial Methods in the Theory of Stochastic Processes.* New York: J. Wiley.

GNEDENKO, B. AND KÖNIG, D. eds. 1983. *Handbuch der Bedienungtheorie.* Berlin: Akademie-Verlag.

GÉRARD PAFUMI*

The mathematics of this paper are interesting by themselves. At the same time, they open the door to several financial applications, such as the pricing of perpetual put options, see the paper by Gerber and Shiu (1997), and the optimal choice of dividend strategies, as described in Section 7 of the paper. The purpose of this discussion is to show how certain results of this paper can be used to calculate the net single premium for a *perpetual default* (or *insolvency*) *reinsurance.*

As in Section 2 we assume that a company starts with an initial surplus $u \geq 0$, receives premiums continuously at a rate $c>0$, and has to pay claims, which constitute a compound Poisson process $\{S(t)\}$. We consider the following contract: Whenever the surplus is negative, the reinsurer makes the necessary payment to bring the surplus back to zero. A typical sample path of the surplus process is depicted in Figure 1. Thus the reinsurer will pay the amounts Y_1, Y_2, Y_3, ... The net single premium of the contract is the expectation of the sum of their discounted values and is denoted by the symbol $A(u)$.

For $u=0$, there is a surprisingly simple and explicit result:

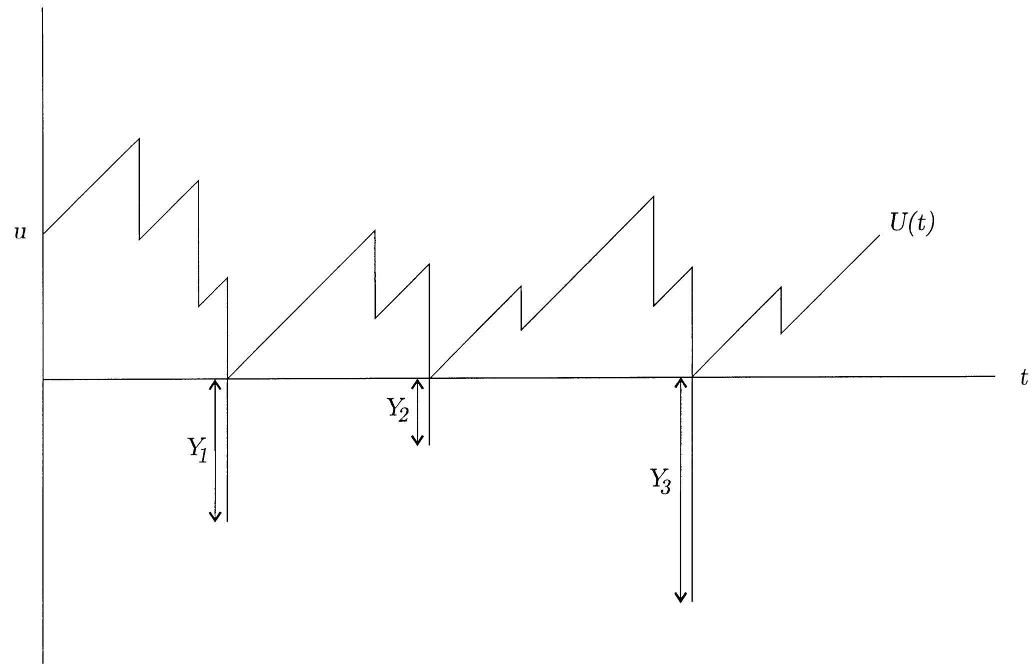
$$A(0) = \frac{1}{\rho} - \frac{c - \lambda p_1}{\delta}. \tag{1}$$

We note that $A(0)$ can also be interpreted as the unloaded reinsurance premium for the guarantee that the surplus will never drop below the initial level.

Formula (1) can be obtained as follows. When the surplus is negative for the first time, the reinsurer has to make an immediate payment of Y_1 and reserve the amount $A(0)$ for the future payments Y_2, Y_3, ...

*Gérard Pafumi is a doctoral student at the Ecole des HEC (Business School), University of Lausanne, CH-1015 Lausanne, Switzerland.

FIGURE 1
A TYPICAL SAMPLE PATH OF THE SURPLUS PROCESS UNDER DEFAULT INSURANCE



Hence

$$A(0) = \int_0^\infty [y + A(0)]g(y)dy, \tag{2}$$

with

$$g(y) = \frac{\lambda}{c} e^{\rho y} \int_y^\infty e^{-\rho z}\, p(z)dz \tag{3}$$

as defined by Formula (2.30) of the paper. Formula (2) is a linear equation for $A(0)$. Its solution is

$$A(0) = \frac{\int_0^\infty yg(y)dy}{1 - \int_0^\infty g(y)dy}. \tag{4}$$

According to Formula (3.9) of the paper, the denominator equals $\delta/(c\rho)$. To evaluate the numerator, we use (3) and change the order of integration to obtain

$$\frac{\lambda}{c} \int_0^\infty e^{-\rho z}\, p(z) \left(\int_0^z ye^{\rho y}\, dy\right) dz.$$

Since

$$\int_0^z ye^{\rho y}\, dy = \frac{1}{\rho} ze^{\rho z} - \frac{1}{\rho^2} e^{\rho z} + \frac{1}{\rho^2},$$

the numerator is

$$\frac{\lambda p_1}{c\rho} - \frac{\lambda}{c\rho^2} + \frac{\lambda}{c\rho^2} \hat{p}(\rho).$$

Because ρ is a solution of Lundberg's equation (2.22), the numerator can further be simplified to

$$\frac{\lambda p_1}{c\rho} - \frac{\lambda}{c\rho^2} + \frac{1}{c\rho^2} \ell(\rho) = \frac{\lambda p_1}{c\rho} + \frac{\delta}{c\rho^2} - \frac{1}{\rho}.$$

Substituting this in the numerator and $\delta/(c\rho)$ in the denominator of (4), we obtain (1).

If $u>0$, we have

$$A(u) = \phi(u),$$

corresponding to the penalty function

$$w(x, y) = y + A(0), \quad y > 0.$$

An explicit expression can be obtained for exponential claim amount distributions. Then it follows from (5.43) that

$$\begin{aligned} A(u) &= A(0)e^{-Ru} \\ &= \left(\frac{1}{\rho} - \frac{c - \lambda/\beta}{\delta}\right) e^{-Ru}. \end{aligned}$$

Here $-R$ and ρ are the roots of (3.11).

REFERENCE

GERBER, H.U., AND SHIU, E.S.W. 1997. "From Ruin Theory to Option Pricing," *Joint Day Proceedings Volume of the XXVIIIth International ASTIN Colloquium/7th International AFIR Colloquium*, Cairns, Australia, pp. 157–76.

AUTHORS' REPLY

HANS U. GERBER AND ELIAS S.W. SHIU

We are grateful for receiving four discussions that add much breadth and depth to the paper. We thank the discussants for their thoughtful contributions.

Drs. De Vylder and Goovaerts have provided a most interesting alternative proof of Formula (3.3) by considering the claim number of the individual claim causing ruin and by applying Lagrange's expansion formula. Their formula (7), together with (12), is similar to, but not the same as, the first formula in Gerber (1988, Theorem 1(a)). Following a suggestion by Willmot, Shiu (1989, p. 248) has shown that Gerber's (1988) first formula can be derived by Lagrange's formula. There is a second version of Lagrange's formula. Shiu (1989, p. 248) has pointed out that Gerber's second formula (Gerber 1988, Theorem 1(b)) follows from the second version of Lagrange's formula. Panjer and Willmot (1992, Corollary 11.7.1) have used Lagrange's formula to give an explicit expression for $\rho(\delta)$.

The two proofs of (3.3) are given in the order that they were discovered. Because (3.3) looked uncomplicated, we kept asking ourselves whether there would be a simpler proof. Then we came up with the martingale proof in Section 5.

Dr. Dickson has provided an efficient way to derive the Laplace transform of ϕ, starting with (2.16). His derivation of (2.26) from (2.16) is indeed insightful. Drs. Dickson and Kalashnikov seem to have different opinions with respect to the difficulty in inverting Laplace transforms.

We agree with Dr. Kalashnikov that, for an experienced mathematical audience, the paper can be shortened. Dr. Kalashnikov points out that some technical conditions are needed to guarantee the existence of R. Although we did say in Section 2 that "if the individual claim amount density function, p, is sufficiently regular, Equation (2.22) has one more root," we should have been more precise when discussing R at the beginning of Section 4. For an illustration of this point, we now refer the reader to Example 13.4.3 on page 412 of *Actuarial Mathematics*. We appreciate the references to queuing theory. To the list of books, we would add the papers by Seal (1972) and Taylor (1976).

We take this opportunity to point out that Formula (3.3) holds even if Inequality (2.5) is reversed, that is, even if the loading is negative. The only modification needed is for the limiting case $\delta=0$. If

$$c < \lambda p_1,$$

we have

$$\lim_{\delta\to 0} \rho(\delta) = \rho(0) > 0$$

and

$$\lim_{\delta\to 0} R(\delta) = R(0) = 0.$$

Consequently, if $\delta = 0$, we have

$$f(x, y|0) = \frac{\lambda}{c} e^{-\rho(0)x} p(x + y), \; x > 0, \; y > 0, \quad \text{(R.1)}$$

but the symmetry (3.7) does not hold any more.

Mr. Pafumi has presented a very clever idea, which can be applied to price the so-called reset guarantees (Gerber and Shiu 1998). His arguments can also be used to price a contract that provides protection against the first n deficits only. Let $A_n(u)$ denote the net single premium for the payments $Y_1, \ldots, Y_n$. [Mr. Pafumi's $A(u)$ is $A_\infty(u)$ here.] Thus

$$\begin{aligned} A_1(0) &= \int_0^\infty y\, g(y)\, dy \\ &= \frac{\lambda p_1}{c\rho} + \frac{\delta}{c\rho^2} - \frac{1}{\rho}, \end{aligned} \quad \text{(R.2)}$$

as derived by Mr. Pafumi. For $n = 2, 3, \ldots,$ we have the recursive formula

$$\begin{aligned} A_n(0) &= \int_0^\infty [y + A_{n-1}(0)]\, g(y)\, dy \\ &= A_1(0) + \left(1 - \frac{\delta}{c\rho}\right) A_{n-1}(0). \end{aligned}$$

It follows that, for $n = 1, 2, 3, \ldots,$

$$\begin{aligned} A_n(0) &= \frac{c\rho}{\delta}\left[1 - \left(1 - \frac{\delta}{c\rho}\right)^n\right] A_1(0) \\ &= \left[1 - \left(1 - \frac{\delta}{c\rho}\right)^n\right] A_\infty(0). \end{aligned} \quad \text{(R.3)}$$

Furthermore,

$$A_n(u) = \phi(u)$$

with

$$w(x, y) = y + A_{n-1}(0).$$

Again, an explicit formula can be obtained for exponential claim amount distributions; it follows from (5.43) that, for $n = 1, 2, \ldots,$

$$A_n(u) = A_n(0)\, e^{-Ru}, \; u > 0. \quad \text{(R.4)}$$

Let us briefly look at the case in which the *income process* is a Wiener process, with constant parameters μ and σ^2. Then Lundberg's fundamental equation is

$$\frac{1}{2}\sigma^2\xi^2 + \mu\xi - \delta = 0, \tag{R.5}$$

yielding

$$\rho = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2\delta}}{\sigma^2} \tag{R.6}$$

and

$$R = \frac{\mu + \sqrt{\mu^2 + 2\sigma^2\delta}}{\sigma^2}. \tag{R.7}$$

We gather from Formula (1) in Mr. Pafumi's discussion that

$$A_\infty(0) = \frac{1}{\rho} - \frac{\mu}{\delta}.$$

From this, together with (R.6) and (R.7), we obtain $A_\infty(0) = 1/R$; hence, for $u \geq 0$,

$$A_\infty(u) = \frac{1}{R}e^{-Ru}. \tag{R.8}$$

In particular, it follows that

$$A'_\infty(0) = -1, \tag{R.9}$$

which can be compared to (7.4).

Finally, we would like to explain how the results of our paper can be generalized to the case in which there are n types of claims with frequencies $\lambda_1, \lambda_2, \ldots, \lambda_n$, so that

$$p(x) = \sum_{i=1}^{n}\frac{\lambda_i}{\lambda}p_i(x),\ x > 0. \tag{R.10}$$

(See Theorem 12.4.1 of *Actuarial Mathematics*.) We suppose that the penalty at ruin depends on the type of claim that causes ruin and the n penalty functions are $w_1(x, y), \ldots, w_n(x, y)$. Hence the expected discounted penalty is

$$\phi(u) = \sum_{i=1}^{n}\int_0^\infty\int_0^\infty w_i(x, y)\, f_i(x, y|u)\,dx\,dy, \tag{R.11}$$

with

$$f_i(x, y|u) = \int_0^\infty e^{-\delta t} f_i(x, y, t|u)\,dt. \tag{R.12}$$

(Do not confuse the notation here with that of De Vylder and Goovaerts.) We proceed as in Section 2, except that now we have

$$\omega(x) = \sum_{i=1}^{n}\int_u^\infty w_i(u, x-u)\frac{\lambda_i}{\lambda}p_i(x)\,dx. \tag{R.13}$$

Formulas (2.34) and (2.32) are still valid, in particular

$$\phi(0) = h(0) = \frac{\lambda}{c}\int_0^\infty e^{-\rho z}\omega(z)\,dz. \tag{R.14}$$

Substituting (R.13) in the right-hand side of the above, and comparing the resulting formula with (R.11) (with $u = 0$), we found that

$$f_i(x, y|0) = \frac{\lambda_i}{\lambda}e^{-\rho x}p_i(x+y),\ x > 0,\ y > 0, \tag{R.15}$$

$i = 1, 2, \ldots, n$. If we integrate this formula over x, over y, or both, we generalize (3.4), (3.5) and (3.9). For an alternative proof of (R.15), see Gerber and Shiu (1998).

We thank all five discussants for their thought-provoking comments.

REFERENCES

BOWERS, N.L., JR., GERBER, H.U., HICKMAN, J.C., JONES, D.A., AND NESBITT, C.J. 1997. *Actuarial Mathematics*, 2nd ed. Schaumburg, Ill.: Society of Actuaries.

GERBER, H.U. 1988. "Mathematical Fun with Ruin Theory," *Insurance: Mathematics and Economics* 7:15–23.

GERBER, H.U., AND SHIU, E.S.W. 1998. "From Ruin Theory to Pricing Reset Guarantees and Perpetual Put Options," *Insurance: Mathematics and Economics*, in press.

PANJER, H.H., AND WILLMOT, G.E. 1992. *Insurance Risk Models*. Schaumburg, Ill.: Society of Actuaries.

SEAL, H.L. 1972. "Risk Theory and the Single-server Queue," *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker* 72:171–78.

SHIU, E.S.W. 1989. "Ruin Probability by Operational Calculus," *Insurance: Mathematics and Economics* 8:243–49.

TAYLOR, G.C. 1976. "Use of Differential and Integral Inequalities to Bound Ruin and Queuing Problems," *Scandinavian Actuarial Journal*, 197–208.

Additional discussions on this paper can be submitted until July 1, 1998. The authors reserve the right to reply to any discussion. See the Submission Guidelines for Authors for detailed instructions on the submission of discussions.