STABILITY OF NUMERICAL METHODS FOR COMPUTING PADÉ APPROXIMANTS

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1. NOTATIONS AND DEFINITIONS.

Let $f(x) = \sum_{i=o}^{\infty} c_i.x^i$ be a given power series, with $c_o \neq o$. Let R_n^m be the class of (ordinary) rational functions $r = \frac{p}{q}$ where p and q are polynomials of degree at most m and n respectively such that $\frac{p}{q}$ is irreducible. The Padé approximation problem for f of order (m,n) is to find an element $r = \frac{p}{q}$ in R_n^m satisfying $f.q-p = O(x^{m+n+1+j})$, where j is an integer which is as high as possible. This problem has a unique solution for all values of m and n, denoted by $r_{m,n}$. The elements $r_{m,n}$ can be ordered in a 2-dimensional array, called the Padé table, as follows

$$\begin{array}{cccccc} r_{00} & r_{01} & r_{02} & r_{03} & \cdots \\ r_{10} & r_{11} & r_{12} & r_{13} & \cdots \\ r_{20} & r_{21} & r_{22} & r_{23} & \cdots \\ r_{30} & r_{31} & r_{32} & r_{33} & \cdots \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \end{array}$$

This table has a "block structure" which means that equal elements appear in square blocks. An element is called normal if it appears only once in the table. Further in this paper we will consider only the case of a normal Padé table which means that every element in the table is assumed to

1 Supported in part by the FKFO under grant number 2.0021.75

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ISBN: 0-12-171050-5

be normal.

If $r_{m,n} = \frac{p_{m,n}}{q_{m,n}}$, with $p_{m,n} = \sum_{i=o}^{m} a_i^{(m,n)}.x^i$ and $q_{m,n} = \sum_{i=o}^{n} b_i^{(m,n)}.x^i$ is normal then $a_o^{(m,n)} \neq o$, $a_m^{(m,n)} \neq o$, $b_o^{(m,n)} \neq o$, $b_n^{(m,n)} \neq o$ and

$$f(x).q_{m,n}(x) - p_{m,n}(x) = \sum_{i=o}^{\infty} d_i^{(m,n)}.x^{m+n+1+i} \quad \text{with } d_o^{(m,n)} \neq o. \qquad (1)$$

This implies that $a_i^{(m,n)}$, $b_i^{(m,n)}$ satisfy

$$\sum_{i=o}^{n} c_{k-i}.b_i^{(m,n)} = a_k^{(m,n)} \quad \text{for } k = 0,1,2,\ldots,m \qquad (2.a)$$

$$\sum_{i=o}^{n} c_{m+k-i}.b_i^{(m,n)} = o \quad \text{for } k = 1,2,3,\ldots,n \qquad (2.b)$$

where $c_i = o$ if $i < o$. Remark that $a_i^{(m,n)}$ can be computed easily from (2.a) as soon as a solution of (2.b) is known. Before describing the basis of some recursive methods for solving (2.b) we consider the conditioning of the Padé approximation problem.

2. THE CONDITIONING OF THE PADE APPROXIMATION PROBLEM

Let m and n be fixed and $r = \frac{p}{q}$ be the Padé approximant of order (m,n) for f. What is the effect of "small" changes on the coefficients c_i of f on the coefficients of p and q. Since r only depends on $c_o, c_1, \ldots, c_{n+m}$ we will only consider changes in these coefficients. Let $c = [c_o, c_1, \ldots, c_{n+m}]^T$ and $\|c\| = \sup_{o \leq i \leq n+m} |c_i|$. A neighbourhood U_δ of f, with $\delta > o$, be defined as the set of power series $f' = \sum_{i=o}^{\infty} c_i'.x^i$ such that $\|c-c'\| < \delta$. Using a characterization theorem for normal Padé approximants ([14], p.34) it is not difficult to prove the following result.

THEOREM. Let f have a normal Padé approximant of order (m,n), then there exists a neighbourhood U_δ of f such that every f' in U_δ has a normal Padé approximant of order (m,n).

This result guarantees that the system (2.a), (2.b) has a unique solution if the coefficients c_i are changed "slightly". This system will therefore be well-conditioned if the condition number of its matrix of coefficients is not too large ([10], p.38). There are however examples ([9], p. 242) for which this condition number is very large. Therefore the problem of finding a Padé approximant of a given order for f can be ill-conditioned. This fact is illustrated numerically in [12].

Let T be the operator which associates $r_{m,n}$ to f. We conjecture that, for certain classes of functions f, the operator T satisfies a local Lipschitz condition, or $\|Tf-Tf'\| \leq K.\|f-f'\|$ for all f' in a neighbourhood" of f and where the norm must be chosen suitably.

3. FORMULATION OF RECURSIVE ALGORITHMS FOR COMPUTING PADÉ APPROXIMANTS

It has been remarked in ([8], [16]) that several algorithms for computing a Padé approximant can be interpreted as algorithms for factorizing a Toeplitz or Hankel matrix. A more extensive study of this fact is given in [3]. Here we will only give some of the basic ideas of these algorithms and their relations with other algorithms in the field, of which the exact references can be found in [15].

3A. Algorithms for constructing a row or column in the Padé table.

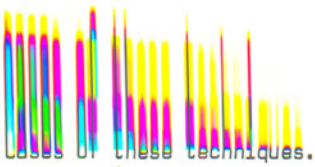
Let m be fixed, then (1) and (2b) for different values of n can be combined to give : T.U = L where T is a Toeplitz matrix with $t_{ij} = c_{m+1+i-j}$, U is upper triangular with $u_{ij} = b_{i-1}^{(m,j-1)}$, and L is lower triangular with $\ell_{ij} = d_{i-j}^{(m,j-1)}$. Using (2.a) and (2.b) we get T'.L' = U' where $t'_{ij} = c_{m+i-j}$, L' is lower triangular with $\ell'_{ij} = u_{i-j+1,n-j+2}$, and U' is upper triangular with $u'_{ij} = a_{m-j+i}^{(m,n-j+1)}$.

The equations T.U. = L and T'.L'=U' can be combined to give a three-term recurrence relation between the denominators of the row $\{r_{m,j}\}_{j\geq 0}$ in the Padé table. It is also possible to derive relations between the elements on neighbouring rows and columns in the Padé table. Similar relations (of Frobenius-type) can be obtained for the elements of the column $\{r_{i,n}\}_{i\geq 0}$.

The algorithms of W.F. Trench (1964), J. Rissanen (1974), O. Bussonais (1978), A. Bultheel (1978) are special cases of these techniques, which are in fact algorithms to factorize a Toeplitz matrix or its inverse.

3B. Algorithms for constructing a diagonal or staircase in the Padé table.

The equations (1), (2.b), corresponding to the elements on a diagonal in the Padé table, can be written in the form H.V = L where H is a Hankel matrix with $h_{i,j} = c_{m-n+i+j-1}$, V is upper triangular with $v_{ij} = b_{j-i}^{(m+j-1,j-1)}$ and L is lower triangular with $\ell_{ij} = d_{i-j}^{(m+j-1,j-1)}$. Similar relations as above can now be derived between the elements on a diagonal (or anti-diagonal) in the Padé table. The algorithms of Massey J.L. (1969), Rissanen J. (1971), Mills W.H. (1975), C. Brezinski (1976) are special



cases of these techniques.

By combining the relations which correspond to neighbouring diagonals (or antidiagonals) we get algorithms for computing the elements on a descending (or ascending) staircase in the Padé table. The algorithms of Thacher H.C. (1960), Watson P.J. (1973), Murphy J.A. and O'Donohoe M.R. (1977) for computing the elements on a descending staircase and of Baker G.A. (1970), Claessens G. (1975) and McEliece R.J. and Shearer J.B. (1978) for computing the elements on an ascending staircase are special cases of these techniques. They are in fact based on factorizing a Hankel matrix or its inverse. Also Bussonais O. (1978) has this Hankel-type recursions. Remark that it is also possible to use a classical factorization algorithm (Gauss, Crout,...) to solve (2.b). These algorithms use however $O(n^3)$ operations, instead of $O(n^2)$ in case the special structure of a Hankel or Toeplitz matrix is exploited. The disadvantage (with respect to stability) of exploiting this structure is however that no pivoting is possible.

4. STABILITY OF RECURSIVE ALGORITHMS

A detailed analysis of the stability of the algorithms mentioned in the preceding section would require a detailed description of these algorithms. Since this is not possible here we only mention some of the results obtained in this field and refer to the literature for further details.
In [6] it is shown that the algorithms of Rissanen and Massey-Berlekamp (and of a variant of it) are unstable. A stable variant of the algorithm of Rissanen is proposed, but it uses $O(n^3)$ operations.
In [5] the algorithm of Durbin for factorizing a real symmetric positive definite Toeplitz matrix is considered. This algorithm uses $O(n^2)$ operations and shows similar stability properties as the Cholesky algorithm, which uses $O(n^3)$ operations.
A more general result was obtained in [4], for several recursive techniques to factorize Toeplitz matrices or their inverses. It is proved that these algorithms are unstable when applied to the whole class of Toeplitz matrices. However if these algorithms are restricted to the class of symmetric positive definite Toeplitz matrices, bounded away from singularity, then they are stable. The Levinson algorithm or its continued fraction form is mentioned as a special case from this class of stable algorithms.
Remark that the system (2.a), (2.b) is based on a representation of the numerator and denominator of a rational function using powers of x as base functions. It is an open problem whether another choice of base functions

would improve the general situation.

5. EVALUATION OF THE VALUE OF A PADE APPROXIMANT

Most of the algorithms mentioned in section 3 can also be interpreted as algorithms for computing the coefficients in continued fractions whose convergents form a sequence of neighbouring elements in the Padé table. For example the algorithms of Watson and Brezinski can be used to find the coefficients e_i in the continued fraction $e_o + \underset{i=1}{\overset{\infty}{\Sigma}} \left| \frac{e_i \cdot x}{1} \right.$, whose convergents are the elements $r_{o,o}, r_{1,o}, r_{1,1}, r_{2,1}, r_{2,2}, \ldots$ on a staircase in the Padé table. If the value of a Padé approximant, from this sequence, must be computed for several values of x then the continued fraction representation of this element can be used.

The problem of finding the value of a finite part of this continued fraction is directly connected with the problem of evaluating three-term recurrence relations. Two basic techniques can be used to do this (see [7]): forward and backward recursion. It is known that forward recursion can be unstable. The use of backward recursion seems to be more appropriate [11]. Certain variants of these basic techniques do exist [13] and can be seen as algorithms for solving a special tridiagonal system of equations.

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