DIVISION ALGORITHMS FOR CONTINUED FRACTIONS AND THE PADE TABLE

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Abstract

A long known algorithm attributed to Viskovatov gives a straightforward way for obtaining corresponding continued fractions. Its convergents are the elements of a descending staircase in the Padé table. Thus it is identical to the Thacher algorithm. The idea of dividing power series can however, without much difficulty, be generalized to non normal Padé tables. This gives the Berlekamp-Massey algorithm. A similar division algorithm mentioned by Hart gives the associated continued fraction and it coincides with Brezinski's algorithm. This can be generalized in a similar way. Other paths of the Padé table like a sawtooth, a row or a column require "constrained" division algorithms. The divisions are interrupted to introduce terms for the monitoring of the degree of the numerator and/or of the denominator of the convergents.

1. Introduction

Given a formal power series $f(z)$, we denote by $[L/M]$ a padé approximant i.e. the couple of polynomials $P^{[L/M]}$, $Q^{[L/M]}$ that is a solution of

$$\begin{cases} d^\circ\, P^{[L/M]} \leq L \\ d^\circ\, Q^{[L/M]} \leq M \\ P^{[L/M]} \text{ and } Q^{[L/M]} \text{ are relative prime} \\ Z^{[L/M]} = Q^{[L/M]} f - P^{[L/M]} = 0(z^N) \;;\; N \geq L+M+1 \\ \text{normalization} \end{cases}$$

If the solution exists then it is unique.

The normalization fixes the constant factor.

$Z^{[L/M]}$ is called the residual and N the approximation order

The padé table is a matrix with $[L/M]$; $L,M = 0,1,2,\ldots$ as elements.

If all $[L/M]$ exist the we call the padé table normal.

If L and M are the lowest possible values (if they exist) such that $f(z) = [L/M]$ then $[L/M]$ is the left top element of an infinite block of padé approximants that are all equal, but in a normal padé table all other elements are different. The infinite block described, if it exists, will be called the function block. If it does not exist then we will use "function block" for a block that can never be reached. The structure of a non normal padé table is well known or can be looked up ([8] e.g.). It is divided into square blocks that are called singular if their size is greater than 1x1. One exception to this may be a horizontal band where the padé approximants do not exist . This happens if the first coefficients in the power series $f(z)$ are zero. The left top element of a block is called its indicator.

We will consider the following paths in a padé table:

$T_k = \{[\,k/0\,],[\,k+1/0\,],[\,k+1/1\,],\ldots,[\,k+\ell/\ell\,],[\,k+\ell+1/\ell\,],\ldots\}$ $k \geq +0$

$= \{[\,0/-k\,],[\,0/1-k\,],[\,1/1-k\,],\ldots,[\,\ell/\ell-k\,],[\,\ell/\ell+1-k\,],\ldots\}$ $k \leq -0$

(remark that $T_{+0} \neq T_{-0}$)

T_k is called a descending staircase.

The following are descending diagonals:

$D_k = \{[\,k/0\,],[\,k+1/1\,],\ldots,[\,k+\ell/\ell\,],\ldots\}$ $k \geq 0$

$= \{[\,0/-k\,],[\,1/1-k\,],\ldots,[\,\ell/\ell-k\,],\ldots\}$ $k < 0$

An ascending staircase: (only for $k \geq 0$)

$U_k = \{[\,k/0\,],[\,k-1/0\,],[\,k-1/1\,],\ldots,[\,k-\ell/\ell\,],[\,k-\ell-1/\ell\,],\ldots,[\,0/k\,]\}$

An upward diagonal: (only for $k \geq 0$)

$E_k = \{[\,k/0\,],[\,k-1/1\,],\ldots,[\,k-\ell/\ell\,],\ldots,[\,0/k\,]\}$

Sawteeth:

$S_k^+ = \{[\,k/0\,],[\,k+1/0\,],[\,k/1\,],[\,k+1/1\,],\ldots,[\,k/\ell\,],[\,k+1/\ell\,],\ldots\}$ $k \geq +0$

$= \{[\,0/-k\,],[\,0/1-k\,],[\,1/-k\,],\ldots,[\,\ell/-k\,],[\,\ell/1-k\,],\ldots\}$ $k \leq -0$

$S_k^- = \{[\,k+1/0\,],[\,k/0\,],[\,k+1/1\,],[\,k/1\,],\ldots,[\,k+1/\ell\,],[\,k/\ell\,],\ldots\}$ $k \geq +0$

$= \{[\,0/1-k\,],[\,0/-k\,],[\,1/1-k\,],[\,1/-k\,],\ldots,[\,\ell/1-k\,],[\,\ell/-k\,],\ldots\}$ $k \leq -0$

(again $S_{+0}^{\mp} \neq S_{-0}^{\mp}$)

The rows and columns are defined by:

$L_k = \{[\,k/0\,],[\,k/1\,],\ldots,[\,k/\ell\,],\ldots\}$ $k \geq +0$

$= \{[\,0/-k\,],[\,1/-k\,],\ldots,[\,\ell/-k\,],\ldots\}$ $k \leq -0$

We call a path P normal if all its elements exist and are different.[+] A normal path can be a part of a non normal table. If the path P is is not normal then we will indicate by $\hat{P}$ the reduced path consisting of all the relevant elements of P, i.e. only those elements are retained that are equal to indicators and are not duplicate elements.

[+] except eventually for the elements of the funtionblock.

For a formal power series f(z) we define the following continued fractions (CF):

An <u>equivalent</u> CF: Its k-th convergent is the k-th partial sum of f(z). It is of the form

$$a_0 + \mathop{\K}_{i=1}^{\infty} \frac{a_i z}{b_i + c_i z}$$

or one that is equivalent with it.

A <u>corresponding</u> CF: Its k-th convergent is an order k+1 approximation to f(z). It is of the form

$$a_0 + \mathop{\K}_{i=1}^{\infty} \frac{a_i z}{1}$$

or one that is equivalent with it.

An <u>associated</u> CF: Its k-th convergent is an order 2k+1 approximation of f(z). It is of the form

$$a_0 + \mathop{\K}_{i=1}^{\infty} \frac{a_i z^2}{b_i + c_i z}$$

or of an equivalent one.

2. A first division algorithm

Suppose we want to find a CF corresponding to

$$f(z) = \frac{\alpha_{10}+\alpha_{11}z+\ldots+\alpha_{1m}z^m}{\alpha_{00}+\alpha_{01}z+\ldots+\alpha_{0n}z^n} = \frac{S_1(z)}{S_0(z)}$$

m,n may or may not be infinite.

This can be done by successively dividing power series like this:

$$f(z) = \frac{S_1(z)}{S_0(z)} = \frac{\alpha_{10}}{\frac{S_0(z)}{\hat{S}_1(z)}} \quad ; \quad \hat{S}_1(z) = S_1/\alpha_{10}$$

and

$$\frac{S_0(z)}{\hat{S}_1(z)} = \alpha_{00} + z\,\frac{\alpha_{20}+\alpha_{21}z+\ldots+\alpha_{2k}z^k}{\alpha_{10}+\alpha_{11}z+\ldots+\alpha_{1m}z^m} = \alpha_{00} + z\,R_1$$

with

$$k = \max(n,m) - 1$$

$$\alpha_{2i} = \alpha_{10}\;\alpha_{0,i+1} - \alpha_{00}\;\alpha_{1,i+1} \qquad i=0,1,2,\ldots$$

Letting $\alpha_{0i} = 0$ for $i > n$ and $\alpha_{1i} = 0$ for $i > m$ and

$$S_i(z) = \sum_{j=0}^{\infty} \alpha_{ij}\, z^j \qquad i = 0,1,2$$

then

$$S_2 = \frac{1}{z}\,(\alpha_{10}\,S_0 - \alpha_{00}\,S_1) = \frac{1}{z}\,(S_1(0)S_0 - S_0(0)S_1)$$

and $R_1 = S_2/S_1$

R_1 is then treated similar to f(z) etc...

We resume in the following

Theorem 1

Given $f(z) = S_1(z)/S_0(z)$; $d^\circ S_1 \leq m$; $d^\circ S_0 \leq n$

(m and/or n can be infinite)

Consider the algorithm RTOV1:

RTOV1: initialization:

$$R_0 = f(z) = S_1(z)/S_0(z)$$

$$v_0 = S_0(0) \quad ; \quad v_1 = S_1(0)$$

$$d_0 = n \quad ; \quad d_1 = m$$

for $i = 1,2,3,\ldots$

$$d_{i+1} = \max(d_{i-1}, d_i) - 1$$

$$S_{i+1}(z) = \frac{1}{z}(S_i(0)S_{i-1}(z) - S_{i-1}(0)S_i(z)) = \frac{1}{z}(v_i S_{i-1} - v_{i-1} S_i) \quad (1)$$

$$v_{i+1} = S_{i+1}(0)$$

If we set $R_i = S_{i+1}/S_i$

then:

1) $d^\circ S_{i+1} \leq d_{i+1}$

2) $$f(z) = \cfrac{v_1|}{|v_0} + \sum_{i=1}^{\ell} \cfrac{v_{i+1} z|}{|v_i} + \cfrac{zR_{\ell+1}|}{|1} \qquad \ell = 0,1,2,\ldots \qquad (2)$$

3) if m and n are finite then $S_\ell = 0$ for $\ell \geq \max(2m+1, 2n+2)$

4) for $\ell = 1,2,\ldots$ we have that if $v_i \neq 0$, $i = 0,1,\ldots,\ell$ then the ℓ-th convergent

$$C^{(\ell)} = \frac{P^{(\ell)}}{Q^{(\ell)}} = \cfrac{v_1|}{|v_0} + \sum_{i=1}^{\infty} \cfrac{v_{i+1} z|}{|v_i} \qquad (3)$$

is an order ℓ approximation to $f(z)$ and $z^\ell S_{\ell+1}$ is proportional to the residual

$$Q^{(\ell)} f - P^{(\ell)} = (-1)^{\ell+1} z^\ell S_{\ell+1}/S_0 \qquad (4)$$

Proof

1) and 2) follow directly from the construction of the algorithm.

3) Suppose $m \leq n < \infty$ then we have $d_2 = n-1$, $d_3 = n-1$, $d_4 = n-2$, $d_5 = n-2$ etc... and in general $d_{2k} = d_{2k+1} = n-k$. The degree of S_i is decreasing "staircasewise". $S_\ell = 0$ if $d_\ell \leq -1$. This is for $\ell \geq 2n+2$

For $n < m < \infty$ we have $d_2=m-1, d_3=m-2, d_4=m-2, \ldots$ and $d_{2k-1}=d_{2k}=m-k$ for $k \geq 2$. Thus $S_\ell = 0$ for $\ell \geq 2m+1$

4) This proof is by induction.

For $\ell = 1$ clearly

$$v_0 \frac{S_1(z)}{S_0(z)} - v_1 = z \frac{S_2(z)}{S_0(z)}$$

The inductionstep follows from the train of equalities below:

$$\begin{aligned} z^{\ell+1} S_{\ell+2} &= z^{\ell}(v_{\ell+1} S_\ell - v_\ell S_{\ell+1}) \\ &= (-1)^{\ell+1} z^{\ell} [v_{\ell+1} (Q^{(\ell-1)} f - P^{(\ell-1)}) S_0 z^{-(\ell-1)} \\ &\qquad + v_\ell (Q^{(\ell)} f - P^{(\ell)}) S_0 z^{-\ell}] \\ &= (-1)^{\ell+1} S_0 [(v_{\ell+1} z Q^{(\ell-1)} + v_\ell Q^{(\ell)}) f - \\ &\qquad (v_{\ell+1} z P^{(\ell-1)} + v_\ell P^{(\ell)})] \\ &= (-1)^{\ell+1} S_0 (Q^{(\ell+1)} f - P^{(\ell+1)}) \end{aligned}$$

□

This algorithm is long known [15,11] and is sometimes refered to as Viskovatov's algorithm [7] or Routh's algorithm [10].

Although it is a "division algorithm" there is nowhere a division of constants to be made, so there is at the first sight no objection against some of the v_i being zero on condition that the RHS of (3) is evaluated with the forward recursion formula. This however is of no use as is seen from the following

Proposition 1

If in the algorithm RTOV1 we have that $v_{\ell+1} = 0$ then

1) there exist polynomials $p_k(z)$ of degree at most $[\frac{k}{2}]$ such that

$$P^{(\ell+k)} = p_k(z)\, P^{(\ell)} \quad \text{and} \quad Q^{(\ell+k)} = p_k(z)\, Q^{(\ell)} \qquad k = 1,2,\ldots$$

2) $C^{(\ell)} = f(z)$ or $L \geq \ell$ exists such that $P^{(L)} = Q^{(L)} = 0$.

Proof

1) $P^{(\ell+1)} = v_\ell\, P^{(\ell)}$

$P^{(\ell+2)} = v_{\ell+2}\, z\, P^{(\ell)}$

$P^{(\ell+3)} = (v_{\ell+3}\, v_\ell + v_{\ell+2}^2)\, z\, P^{(\ell)}$

...

$P^{(\ell+k)} = v_{\ell+k-1}\, P^{(\ell+k-1)} + v_{\ell+k}\, z\, P^{(\ell+k-2)}$

$= (v_{\ell+k-1}\, p_{k-1}(z) + v_{\ell+k}\, z\, p_{k-2}(z))\, P^{(\ell)}$

$= p_k(z)\, P^{(\ell)}$

and $d^\circ p_k(z) \leq \max([\frac{k-1}{2}], [\frac{k-2}{2}]+1) = [\frac{k}{2}]$

2) Suppose $C^{(\ell)} \neq f(z)$ thus $C^{(\ell)}$ has a finite order of approximation then the algorithm must give zero in the end for P and Q because the order of approximation is increasing in every step as was proved in 4) of theorem 1 , but the degree of $P^{(\ell+k)}$ is only increasing every two steps, so there must be an L such that equation (4) can only be satisfied for $Q^{(L)} = P^{(L)} = 0$

If $C^{(\ell)} = f(z)$ then all $S^{(\ell+k)} = 0$ for $k=1,2,\ldots$

□

If we say in the following that an algorithm collapses we mean that something like the phenomenon described in property 1 happens.

Property 1 tells us that the only case of interest is $v_\ell = 0$ iff $S_\ell = 0$. We thus have a situation like this: $v_i \neq 0, i = 0,1,\ldots,\ell$ and $f(z) = C^{(\ell)}$ with $v_{\ell+k} = C^{(\ell+k)} = 0$, $k = 1,2,\ldots$

Let us therefore go over the properties of

$$v_\ell(z) = \cfrac{v_1}{|v_0} + \sum_{i=1}^{\ell-1} \cfrac{v_{i+1}z}{|v_i} \tag{5}$$

if $v_i \neq 0$, $i = 0,1,\ldots, \ell$

This we will call a V1-type CF (V for Viskovatov).

Proposition 1

If $v_\ell(z)$ is a V1 type CF then

1) it can be brought into the forms

$$t_\ell(z) = \cfrac{1}{|t_0} + \sum_{i=1}^{\ell-1} \cfrac{z}{|t_i} \qquad \text{(T1-type (T for Thiele))} \tag{6}$$

or

$$c_\ell(z) = \cfrac{c_0}{|1} + \sum_{i=1}^{\ell-1} \cfrac{c_i z}{|1} \qquad \text{(C1-type (C for corresponding))} \tag{7}$$

by equivalence transformations.

2) If $C^{(k)} = P^{(k)}/Q^{(k)}$ is the k-th convergent of $v_\ell(z)$ $(1 \leq k \leq \ell)$ then

a) $\begin{cases} d°P^{(2k+1)} \text{ is exactly } k \\ d°Q^{(2k)} \text{ is exactly } k \end{cases}$ and $\begin{cases} d°Q^{(2k+1)} \leq k \\ d°P^{(2k)} \leq k-1 \end{cases}$

b) $P^{(k)}(0) = v_1.c \neq 0$ and $Q^{(k)}(0) = v_0.c \neq 0$

c) $P^{(k)}/Q^{(k)}$ is irreducible.

Proof

1) is a well known result.

2)a) It is easily seen that the highest degree coefficient of $P^{(2k+1)}$ is $v_1v_3\ldots v_{2k+1} \neq 0$ and the highest degree coefficient of $Q^{(2k+1)}$ is $v_2v_4\ldots v_{2k} \neq 0$.

b) The constant terms are:

$$P^{(k)}(0) = v_1v_1v_2v_3\ldots v_{k-1}$$

$$Q^{(k)}(0) = v_0v_1v_2v_3\ldots v_{k-1}$$

c) We have to prove that $P^{(k)}(a) = 0 \Rightarrow Q^{(k)}(a) \neq 0$ and that $Q^{(k)}(a) = 0 \Rightarrow P^{(k)}(a) \neq 0$. a=0 can not be a zero of $P^{(k)}$ or $Q^{(k)}$ because of b). Now it can be proved that

$$P^{(k)}Q^{(k-1)} - P^{(k-1)}Q^{(k)} = (-1)^{k-1} v_1 \ldots v_k \, z^{k-1} \qquad (8)$$

Suppose that $P^{(k)}(a) = 0$ then we can not have $Q^{(k)}(a) = 0$ otherwise (8) would imply that a = 0. The other implication is likely verified.

□

It is a trivial matter to build the equivalence transformations mentioned in proposition 1 directly into the algorithm RTOV1. Let us call the so adapted algorithms RTOT1 and RTOC1. To obtain them replace in RTOV1 the recursions by

RTOT1 : $S_{i+1}(z) = \frac{1}{z}(S_{i-1}(z) - t_{i-1}S_i(z))$; $t_i = (R_i(0))^{-1}$

RTOC1 : $S_{i+1}(z) = \frac{1}{z}(c_{i-1}S_{i-1}(z) - S_i(z))$; $c_i = R_i(0)$

Hart [9] gives an algol version of the algorithm RTOT1 (which is called RTOC there)

The whole thing can be repeated for a CF of V1 type but with a constant in front. However this is not essentially different as is shown in the following.

Theorem 2

Given $f(z) = S_0(z)/S_1(z)$; $d^\circ S_0 = m$; $d^\circ S_1 = n$

Apply the algorithm

RTOV2: initialization:

$$1/R_0 = f(z) = S_0(z)/S_1(z)$$

$$v_0 = S_0(0) \quad ; \quad v_1 = S_1(0)$$

$$d_0 = m \quad ; \quad d_1 = n$$

for $i = 1,2,\ldots$

$$d_{i+1} = \max(d_i, d_{i-1}) - 1$$

$$S_{i+1} = \frac{1}{z}\left(S_i(0)S_{i-1} - S_{i-1}(0)S_i \right) = \frac{1}{z}\left(v_i S_{i-1} - v_{i-1} S_i \right) \qquad (9)$$

$$v_{i+1} = S_{i+1}(0)$$

Set $R_i = S_{i+1}/S_i$, $i = 0,1,\ldots$

then

1) $d^\circ S_{i+1} \leqslant d_{i+1}$

2) $$f(z) = \frac{1}{v_1}\left(v_0 + \sum_{i=1}^{\ell} \left| \frac{v_{i+1} z}{v_i} \right| + \left| \frac{z R_{\ell+1}}{1} \right|\right) \quad \ell = 0,1,\ldots \qquad (10)$$

3) if m and n are finite then $S_\ell = 0$ for $\ell \geqslant \max(2m+1, 2n+2)$

4) the ℓ-th convergent (supposed $v_i \neq 0; i = 1,2,\ldots,\ell$)

$$C^{(\ell)} = P^{(\ell)}/Q^{(\ell)} = \frac{1}{v_1}\left(v_0 + \sum_{i=1}^{\ell} \left| \frac{v_{i+1} z}{v_i} \right| \right) \qquad (11)$$

is an order $\ell+1$ approximant to $f(z)$ and $z^{\ell+1} S_{\ell+2}$ is proportional to the residual:

$$Q^{(\ell)} f - P^{(\ell)} = (-1)^\ell z^{\ell+1} S_{\ell+2}/S_1 \qquad \ell = 0,1,\ldots \qquad (12)$$

Proof

The algorithm RTOV1 and RTOV2 do the same computations actually.

E.g. the relation (10) can be found like this:

Apply RTOV1 to $1/f(z) = S_1(z)/S_0(z)$ to give

$$\frac{S_1}{S_0} = \cfrac{v_1}{v_0 + \sum_{i=1}^{\ell} \left| \cfrac{v_{i+1} z}{v_i} \right| + \left| \cfrac{z R_{\ell+1}}{1} \right|} \qquad (13)$$

so that

$$f(z) = S_0(z)/S_1(z) = \frac{1}{S_1/S_0} = \frac{1}{v_1}\left(v_0 + \sum_{i=1}^{\ell} \left| \frac{v_{i+1} z}{v_i} \right| + \left| \frac{z R_{\ell+1}}{1} \right|\right) \qquad (14)$$

The relation (12) can be found from (4) because the k-th convergent

of (14) is the (k+1)-th convergent of (13) inverted. Thus (4) implies

$$P^{(k-1)} S_1/S_0 - Q^{(k-1)} = (-1)^{k-1} z^k S_{k+1}/S_0$$

or

$$P^{(k-1)} - Q^{(k-1)} S_0/S_1 = (-1)^{k-1} z^k S_{k+1}/S_1$$

giving (12) □

Remarks

1) We can now allow $f(0) = 0$ if $f'(0) \neq 0$ (i.e. $v_0 = 0$) without a breakdown of the algorithm but this is the only exception on the rule, so that, like for RTOV1, $v_i \neq 0 \ \forall\, i \geq 1$.

2) The previous remark implies that not only RTOV1 can be used to produce (10) but conversely also RTOV2 can be used to produce (2) Therefore give it as input $g(z) = zf(z)$. The result will be

$$g(z) = \frac{1}{v_1}\left(0 + \sum_{i=1}^{\ell} \frac{v_{i+1} z \,|}{|\, v_i} + \frac{z\, R_{\ell+1} \,|}{|\, 1} \right)$$

and thus

$$f(z) = \frac{v_2/v_1 \,|}{|\, v_1} + \sum_{i=2}^{\ell} \frac{v_{i+1}\, z \,|}{|\, v_i} + \frac{z\, R_{\ell+1} \,|}{|\, 1}$$

A CF of the form (11) with $v_i \neq 0$, $i \geq 1$ will be called a V2-type fraction.

If $v_0 = 0$ then proposition 1 is directly applicable to

$$\frac{1}{z} v_\ell(z) = \frac{v_2/v_1 \,|}{|\, v_1} + \sum_{i=2}^{\ell} \frac{v_{i+1}\, z \,|}{|\, v_i}$$

We can thus state the following without proof:

Proposition 2

If $v_\ell(z)$ is a V2-type fraction then

1) It can be brought into the forms

$$\text{T2:} \qquad t_0 + \sum_{i=1}^{\ell} \frac{z|}{|t_i} \tag{15}$$

or

$$\text{C2:} \qquad c_0^{-1} + \sum_{i=1}^{\ell} \frac{c_i z|}{|1} \tag{16}$$

by equivalencetransformations.

2) If $C^{(k)} = P^{(k)}/Q^{(k)}$ is the k-th convergent of $v_\ell(z)$ $(0 \leqslant k \leqslant \ell)$

then

a) $\begin{cases} d^\circ P_{2k+1} \text{ is exactly } k+1 \\ d^\circ Q_{2k} \text{ is exactly } k \end{cases}$ and $\begin{cases} d^\circ P_{2k} \leqslant k \\ d^\circ Q_{2k+1} \leqslant k \end{cases}$

b) $P^{(k)}(0) = v_0 . c$ and $Q^{(k)}(0) = v_1 . c$

c) $P^{(k)}/Q^{(k)}$ is irreducible.

□

The equivalence transformations for (15) and (16) built into RTOV2 gives:

RTOT2: $S_{i+1} = \frac{1}{z}(S_{i-1} - t_{i-1} S_i)$; $t_i = (R_i(0))^{-1}$

RTOC2: $S_{i+1} = \frac{1}{z}(c_{i-1} S_{i-1} - S_i)$; $c_i = R_i(0)$

3. The first division algorithm and the padé table

The relation between a corresponding CF and the padé table is well known in the normal case. We mention

Proposition 3

If the staircase $\top_{-0}$ ($\top_{+0}$) is a normal staircase then its successive elements are the convergents of the CF of V1-(V2-)type constructed by the algorithm RTOV1 (RTOV2) up to some normalization.+

Proof

The normality implies that $v_i \neq 0 \ \forall\, i$ and then e.g. proposition 1 says that $d°P^{(\ell)} \leq [\frac{\ell-1}{2}]$ and $d°Q^{(\ell)} \leq [\frac{\ell}{2}]$ and that $C^{(\ell)} = P^{(\ell)}/Q^{(\ell)}$ is irreducible. Theorem 1 says that $C^{(\ell)}$ is an order $\ell+1$ approximation. Therefore $C^{(\ell)}$ must be a padé approximant up to some normalization.

□

The normalization of T (C) (1 or 2) is such that $P^{(2k+1)}$ and $Q^{(2k)}$ are monic ($Q^{(k)}(0) = 1$) except for T2 with $t_0 \neq 0$ then $t_0^{-1} Q^{(2k)}$ is monic.

The other staircases of the padé table can be constructed if we first perform a division up to the k-th term before starting a RTO.. algorithm. After this division we have something of the form

$$\frac{P^{[k/0]}}{Q^{[k/0]}} + z^{k+1} \frac{S_1}{S_0} \qquad \text{or} \qquad \frac{P^{[0/-k]}}{Q^{[0/-k]}} + \frac{z^{-k+1} S_0}{S_1} \tag{17}$$

We suppose in the following that this division has been done and that we only have to compute staircases like $\top_{+0}$ or $\top_{-0}$.

Of course there must be a connection with other algorithms computing such staircases or corresponding CF's.

The Thacher algorithm as described by Claessens [5] computes $\top_{+0}$ as the convergents of

+We will omit the clause "up to some normalization" in the following propositions because there is alwayse "some" normalization.

$$c_0 + \frac{c_1 z}{1} + \sum_{i=2}^{\ell} \frac{z}{t_i}$$

This is almost of T2 type. The coefficients t_i are computed as the ratio of two leading terms in successive residuals. This is exactly what is done in RTOT2. So, we have proved that, except for some minor change in the first iteration, the Thacher algorithm and the RTOT algorithms are essentially the same.

The Watson algorithm [16] results in the computation of rational forms that are the convergents $C^{(k)}$ of a CF of C-type (normalization $Q^{(k)}(0) = 1$). Here however the denominators (and eventually the numerators) of the convergents are explicitly computed and the c_i are expressed directly as a function of the denominator coefficients, but as the algorithms indicate (see also Bussonnais [4]) it is possible to express them as ratio's of residual coefficients. This would be the Thacher algorithm with a $Q^{(k)}(0) = 1$ normalization.

Property 2

The staircase T_{-0} (T_{+0}) is normal iff the RTOV1 (RTOV2) algorithm constructs a CF of type V1 (V2) i.e. having all $v_i \neq 0$ for $i \geq 0$ ($i \geq 1$) until the function block is reached, then the approximation is exact.

Proof

If v_{k+1} were zero then by property 1 we have $C^{(k+1)} = \frac{v_k}{v_k} C^{(k)}$, thus having two identical padé approximants in the staircase. This is only possible if the staircase is non normal.

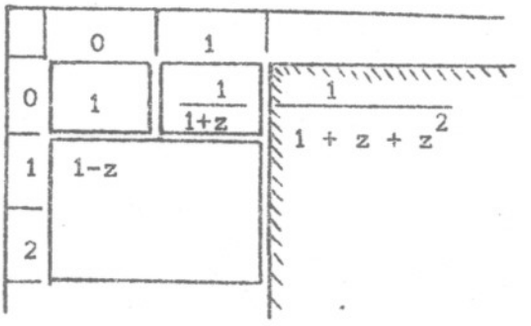
The converse is true because the degree statements of propositions 1 and 2 do not allow two convergents to be equal. □

Example

The RTOV1 algorithm gives

$$\frac{1}{1 + z + z^2} = \frac{1}{|1} + \frac{z}{|1} - \frac{z}{|1} + \frac{z}{|1} \qquad (18)$$

having convergents in $\top_{-0}$. [1/1] is in a singular block. The third convergent of (18) is $\frac{1 - z}{1}$



Remarks

1) There are of course relations with other algorithms too. E.g. the Thiele algorithm of reciprocal differences (see e.g. [11]) also constructs f(z) as in (10) but in that algorithm the quantity $R_i = S_{i+1}/S_i$ is treated as a whole instead of the S_i separately. This makes the Thiele algorithm so difficult to implement for an automatic computer program.

2) If we were given $f(x^k)$ with a non normal padé table then we can make a normal table by substituting $z = x^k$. E.g. k = 2 makes it possible to treat even power series or k = -1 gives asymptotic power series. The so called P-algorithm of Patry and Gupta [14] derives CF's of the form $\sum_{i=1}^{\infty} \frac{B_i}{|x}$ to correspond to the power series

$$A(x) = \sum_{i=1}^{\infty} A_i \, x^{-2i+1}$$

The RTOC1 algorithm gives for

$$\frac{A_1}{x} + \frac{A_2}{x^3} + \ \ldots \ = \frac{1}{x}\left(A_1 + \frac{A_2}{x^2} + \ldots \right)$$

with $z = x^{-2}$:

$$A(z) = \frac{1}{x}\left(\frac{c_0}{1} + \sum_{i=1}^{\infty} \frac{c_i z}{1}\right) = \sum_{i=1}^{\infty} \frac{c_i}{x}$$

The P-algorithm is thus a special case of RTOC1

So far for the normal staircases T_k. In the following section we will investigate how the first division algorithm can be generalized to non normal staircases.

4. Generalization of the first division algorithm to non normal staircases.

For non normal staircases we know that the foregoing algorithms break down but the division of the power series can go on. E.g. we could obtain CF's of the form $\sum_i \left| \frac{z^{\mu_i}}{F_i} \right.$ or of an equivalent form, eventually with a constant in front. This however is not a good strategy because the convergents we would obtain are of a growing order all right, but the degrees of numerator and denominator are growing as well and it is not guaranteed that we will find a padé approximant [K/L] i.e. which is at least of order K+L+1. The reason is that finding a convergent in function of the last two convergents by a combination of the form

$$P^{(k+1)} = F_k P^{(k)} + z^{\mu_k} P^{(k-1)} \tag{19}$$

is too simple a strategy as is described in [3] . We can however generalize (19) to a relation of the form

$$P^{(k+1)} = p_k P^{(k)} + z^{\mu_k} P^{(k-1)} \tag{20}$$

where p_k is some polynomial of appropriate degree. How this can be done is explained in the following

Theorem 3

Given $f(z) = S_1(z)/S_0(z)$; $d°S_1 \leq m$; $d°S_0 \leq n$

and suppose $S_1(0) \neq 0$ and $S_0(0) \neq 0$. Apply the algorithm:

RTOGV1: initialization

$$v_0 = S_0(0) \quad ; \quad v_1 = S_1(0)$$

$$\mu_0 = 0 \quad ; \quad p_0 = 1 \quad ; \quad \nu_0 = 0$$

$$d_0 = n \quad ; \quad d_1 = m$$

for i = 1,2,3,... do the following

+ Find μ_i and S_{i+1} such that $0 \neq S_{i+1}(0) < \infty$ and

$$z^{\mu_i} S_{i+1} = v_i S_{i-1} - p_{i-1} v_{i-1} S_i \qquad (21)$$

+ Set $v_{i+1} = S_{i+1}(0) \qquad (\neq 0)$

+ $d_{i+1} = \max(d_{i-1}, d_i + \nu_{i-1}) - \mu_i$; $\nu_i = \mu_i - \nu_{i-1} - 1$

+ Find p_i as the polynomial of degree $\leq \nu_i$, consisting of the first ν_i+1 terms of the series

$$\frac{v_{i+1} S_i(z)}{v_i S_{i+1}(z)}$$

Setting $R_i = S_{i+1}/S_i \qquad \forall\, i$ then we have:

1) $d^\circ S_{i+1} \leq d_{i+1}$; $\mu_{i+1} \geq \nu_i + 1$ and $\nu_i \geq 0$

2) $$f(z) = \cfrac{v_1}{|v_0} + \sum_{i=1}^{\ell} \cfrac{v_{i+1} z^{\mu_i}}{|v_i p_i} + \cfrac{z^{\mu_{\ell+1}} R_{\ell+1}}{|1} \qquad \ell = 0,1,\ldots \qquad (22)$$

3) If m and n are finite then the algorithm will terminate i.e. there exists a finite L such that $S_\ell = 0 \quad \forall\, \ell \geq L$.

4) For $\ell = 1,2,\ldots$ we have that the ℓ-th convergent

$$C^{(\ell)} = \frac{P^{(\ell)}}{Q^{(\ell)}} = \cfrac{v_1}{|v_0} + \sum_{i=1}^{\ell-1} \cfrac{v_{i+1} z^{\mu_i}}{|p_i(z)} \qquad (23)$$

is an order $N = \mu_1+\mu_2+\ldots+\mu_\ell$ approximation to f(z) and $z^N S_{\ell+1}$ is proportional to the residual:

$$Z^{(\ell)} = Q^{(\ell)} f - P^{(\ell)} = (-1)^\ell z^N S_{\ell+1} / S_0 \qquad (24)$$

Proof

1) and 2) again follow from the construction.

3) Since $d_{i+1} = \max(d_{i-1}, d_i+\nu_{i-1}) - \mu_i$ and $\nu_{i-1} \geq 0$ and $\mu_i \geq \nu_{i-1}+1$ we have in the worst case that $d_{i+1} = \max(d_{i-1}, d_i) - 1$ corresponding to the normal case. The algorithm will then terminate at least at the L-th step where $L \leq \max(2m+1, 2n+2)$. Whenever

there is some $\mu_i > 1$ then the algorithm will terminate earlier.

4) For $\ell = 1$

$$v_0 f(z) - v_1 = \frac{1}{S_0} (v_0 S_1 - v_1 S_0) = - \frac{1}{S_0} z^{\mu_1} S_2$$

Inductionstep:

$$z^{\mu_\ell} S_{\ell+1} = v_\ell S_{\ell-1} - p_{\ell-1} v_{\ell-1} S_\ell$$

so that

$$(-1)^{(\mu_1+\mu_2+\ldots+\mu_\ell)} S_{\ell+1}/S_0 = v_\ell z^{\mu_{\ell-1}} z^{(\ell-1)} + p_{\ell-1} v_{\ell-1} z^{(\ell)}$$

and the forward recursion formula for the CF gives the conjecture.

□

Comment

1) The restriction that $S_1(0) \neq 0$ and $S_0(0) \neq 0$ is not a severe one because if one or both were zero then by setting some suitable power of z in front we can start the algorithm anyway.

2) It is also seen that if all $\mu_i = 1$ then the previous algorithm is again the RTOV1 algorithm.

We will call a CF of the form (23) a generalized V1 or GV1 fraction if $v_i \neq 0$, $p_i(0) = 1$, $d^\circ p_i \leq \nu_i = \mu_i - \nu_{i-1}$ and $\mu_i > \nu_{i-1}$.
Again a property similar to property 1 would indicate that $v_i \neq 0$ $\forall$ $i \leq \ell$ is the only case of interest.
To see if the convergents of a GV1 fraction are padé approximants we have to state the analog of proposition 1, but before we give a technical lemma that is directly verified.

Lemma 1

If $\nu_0 = c$; $\mu_0 = 0$ and for $i = 1,2,\ldots: \nu_i = \mu_i - \nu_{i-1} - 1$

then

$$\nu_{2k} = \sum_{i=1}^{k} (\mu_{2i} - \mu_{2i-1}) + c$$

$$\nu_{2k+1} = \sum_{i=0}^{k} (\mu_{2i+1} - \mu_{2i}) - (c+1)$$

□

Proposition 4

If

$$v_\ell(z) = \frac{v_1}{|v_0} + \sum_{i=1}^{\ell-1} \frac{v_{i+1} z^{\mu_i}}{|v_i p_i(z)}$$

is of GV1 type then

1) It can be brought into the form

$$t_\ell(z) = \frac{1}{|t_0} + \sum_{i=1}^{\ell-1} \frac{z^{\mu_i}}{|t_i p_i(z)} \quad \text{(GT1 type)} \tag{25}$$

or

$$c_\ell(z) = \frac{c_0}{|1} + \sum_{i=1}^{\ell-1} \frac{c_i z^{\mu_i}}{|p_i(z)} \quad \text{(GC1 type)} \tag{26}$$

by equivalence transformations.

2) If $C^{(k)} = P^{(k)}/Q^{(k)}$ is the k-th convergent of $v_\ell(z)$ $(1 \leq k \leq \ell)$ then

a) $d^\circ P^{(2k-1)}$ is exactly $\mu_0+\mu_2+\ldots+\mu_{2k-2}$ and its coefficient of highest degree is $v_1 v_3 \cdots v_{2k-1}$.

$d^\circ Q^{(2k)}$ is exactly $\mu_1+\mu_3+\ldots+\mu_{2k-1}$ and the coefficient of highest degree is $v_2 v_4 \cdots v_{2k}$.

$d^\circ P^{(2k)} \leq d^\circ Q^{(2k)} - 1$

$d^\circ Q^{(2k-1)} \leq d^\circ P^{(2k-1)}$

b) $P^{(k)}(0) = v_1 . c \neq 0$ and $Q^{(k)}(0) = v_0 . c \neq 0$

c) $C^{(k)} = P^{(k)}/Q^{(k)}$ is irreducible.

Proof

We only prove 2)

a) This is true for k=1. Now

$$P^{(2k-1)} = v_{2k-2} p_{2k-2} P^{(2k-2)} + v_{2k-1} z^{\mu_{2k-2}} P^{(2k-3)}$$

$$d^\circ (p_{2k-2} P^{(2k-2)}) \leq \mu_{2k-2} + \sum_{i=0}^{k-2} \mu_{2i+1} - 1$$

and

$$d^\circ (z^{\mu_{2k-2}} P^{(2k-3)}) = \sum_{i=0}^{k-1} \mu_{2i}$$

the lemma implies that the degree of the second term is strictly higher than the degree of the first and thus the highest degree coefficient of $P^{(2k-1)}$ is v_{2k-1} times the highest degree coefficient of $P^{(2k-3)}$.

The other conjectures are similarly proved.

b) is verified by checking that

$$P^{(k)}(0) = v_1 v_1 v_2 \cdots v_{k-1}$$

$$Q^{(k)}(0) = v_0 v_1 v_2 \cdots v_{k-1}$$

c) The proof is like in proposition 1. Now we have

$$P^{(k)} Q^{(k-1)} - P^{(k-1)} Q^{(k)} = (-1)^{k-1} v_1 \cdots v_k z^{(\mu_1+\mu_2+\ldots+\mu_k)} \quad (27)$$

□

It is left as an exercise to formulate the RTOGT1 and RTOGC1 algorithms as well as the generalizations of theorem 2 and proposition 2 concerning the RTOGV2 algorithmand its companion versions.

We directly give the generalization of proposition 3:

Proposition 5

The RTOGV1(2) algorithm constructs a GV1(2)-type CF whose successive convergents are the elements of the reduced staircase $\hat{T}_{-0}(\hat{T}_{+0})$

□

Proposition 5 thus states that the algorithm RTOGV1-2 are essentially Thacher-like versions of the Berlekamp-Massey algorithm as generalized in [3] . Thacher-like because just like in the Thacher algorithm we compute the partial numerators and denominators from the residual coefficients $(S_i(z))$.

Example

$$f(z) = \frac{1 + z^4}{1 - z^5}$$

$\hat{T}_{-0}$ is generated from

$$\frac{1}{|1} + \frac{-z^4}{|1-z+z^2-z^3} + \frac{-2z^4}{|-1} + \frac{2z}{|2}$$

and $\hat{T}_{+0}$ from

$$1 + \cfrac{z^4}{1-z+z^2-z^3} + \cfrac{z^4}{1} + \cfrac{2z}{1} + \cfrac{-z}{1}$$

If we want to construct an other staircase e.g. $\hat{T}_{-3}$ then set in RTOGV1 $p_0 = Q^{[0/3]}$ and $\nu_0 = 3$ as an initialization. This gives:

$$\hat{T}_{-3}: \quad \cfrac{1}{1} - \cfrac{z^4}{1} + \cfrac{z}{1} + \cfrac{z}{1} + \cfrac{z^2}{1-z} + \cfrac{2z^2}{1} + \cfrac{z}{1}$$

5. A second division algorithm

By using a contraction formula for a V-type fraction we would obtain an associated CF whose convergents would fit the given power series twice as far. A general term being of the form indicated in the introduction. This fraction can also be obtained directly by again successively dividing power series. We summarize this in the following

Theorem 4

Given $f(z) = S_0(z)/S_1(z)$; $d^\circ S_0 \leq m$; $d^\circ S_1 \leq n$

Consider the algorithm

RTOA: initialization:

$$1/R_0 = f(z) = S_0/S_1$$

$$d_0 = m \; ; \; d_1 = n$$

preliminar division:

$$S_2(z) = \frac{1}{z}\,(\,S_1(0)\,S_0 - S_0(0)\,S_1) \; ; \; d_2 = \max(d_0,d_1) - 1$$

for $i = 1,2,\ldots$

$$\begin{aligned} d_{i+2} &= \max(d_i - 1, d_{i+1}) - 1 \\ S^{\star}_{i+2} &= \frac{1}{z}\,(\,S_{i+1}(0)\,S_i - S_i(0)\,S_{i+1}) \\ S_{i+2} &= \frac{1}{z}\,(\,S_{i+1}(0)\,S^{\star}_{i+2} - S^{\star}_{i+2}(0)\,S_{i+1}) \end{aligned} \tag{28}$$

Set

$$\begin{aligned} u_i &= (S_{i+1}(0))^2 \\ v_i &= S_i(0)\,S_{i+1}(0) \\ w_i &= S^{\star}_{i+2}(0) \\ R_i &= S_{i+1}/S_i \end{aligned}$$

then

1) The degree of S_{i+2} is at most d_{i+2}

2) $$f(z) = \frac{1}{S_1(0)}\left(S_0(0) + \frac{1}{z}\sum_{i=1}^{\ell}\cfrac{u_i z^2}{v_i + w_i z} + \cfrac{z^2 R_{\ell+1}}{1}\right) \tag{29}$$

3) If m and n are finite then $S_\ell = 0$ for $\ell \geq \max(m,n) + 2$ (29)

4) The ℓ-th convergent

$$C^{(\ell)} = \frac{P^{(\ell)}}{Q^{(\ell)}} = \frac{1}{S_1(0)} \left(S_0(0) + \frac{1}{z} \sum_{i=1}^{\ell} \left| \frac{u_i z^2}{v_i + w_i z} \right| \right) \tag{30}$$

is a order $2\ell+1$ approximation to $f(z)$ and $z^{2\ell+1} S_{\ell+2}$ is proportional to the residual:

$$Q^{(\ell)} f - P^{(\ell)} = (-1)^\ell z^{2\ell+1} S_{\ell+2} / S_1 \tag{31}$$

or we have that $Q^{(\ell)} = P^{(\ell)} = 0$

Proof

1) and 2) follow directly from the construction.

3) If $m > n$ then $d_2 = m-1$, $d_3 = m-2, \ldots, d_k = m-k+1$

If $m \leq n$ then $d_2 = n-1$, $d_3 = n-2, \ldots, d_k = n-k+1$

So $S_\ell = 0$ for $\ell \geq \max(m,n) + 2$

4) $\ell = 0,1$ is directly verified and the inductionstep is like this:

$$z S_{\ell+2} = S_{\ell+1}(0) S^{\star}_{\ell+2} - S^{\star}_{\ell+2}(0) S_{\ell+1}$$

$$z^2 S_{\ell+2} = u_\ell S_\ell - v_\ell S_{\ell+1} - w_\ell z S_{\ell+1}$$

So

$$\frac{z^2 S_{\ell+2}}{S_1} = u_\ell (-1)^\ell z^{2\ell-3} Z^{(\ell-2)} - (v_\ell + w_\ell z)(-1)^{\ell+1} z^{2\ell-1} Z^{(\ell-1)}$$

Using the forward recursion formula for the CF (30) we get the desired result.

□

Remark

We could also give a formulation without the starred series.

Denoting prime for derivative we have:

$$S_{i+2} = \frac{1}{z^2} \left(u_i S_i - (v_i + z w_i) S_{i+1} \right) \tag{32}$$

with $u_i = (S^2_{i+1})_{z=0}$; $v_i = (S_{i+1} S_i)_{z=0}$; $w_i = (S_{i+1} S_i)'_{z=0}$

Again we suppose that u_i and v_i are non zero for $i \leq \ell$ and $S_{\ell+2} = 0$ because it is seen from the definition that u_i and v_i are non zero for $i \leq \ell$ and $u_{\ell+1} = 0$ implies $v_{\ell+1} = 0$ too. Similar to property 1 we can prove that from then on $C^{(\ell+k)}$ will be reducible to $C^{(\ell)}$.

We call thus

$$a_n(z) = \frac{1}{v_0}\left(u_0 + \frac{1}{z}\sum_{i=1}^{n}\cfrac{u_i z^2}{v_i + w_i z}\right) = \frac{1}{v_0}\left(u_0 + \sum_{i=1}^{n}\cfrac{u_i}{\frac{v_i}{z} + w_i}\right) \qquad (33)$$

an A-type fraction (A for associated) if u_i and $v_i \neq 0$ for $i \leq n$.

By equivalence transformations we can also obtain the forms

$$i_n(z) = u_0 + \frac{1}{z}\sum_{i=1}^{n}\cfrac{u_{i+1} z^2}{u_i + w_i z} = u_0 + \sum_{i=1}^{n}\cfrac{u_{i+1}}{u_i/z + w_i} \quad \text{(I-type)} \qquad (34)$$

$$j_n(z) = u_0 + \frac{1}{z}\sum_{i=1}^{n}\cfrac{u_i z^2}{1 + w_i z} = u_0 + \sum_{i=1}^{n}\cfrac{u_i}{1/z + w_i} \quad \text{(J-type)} \qquad (35)$$

$$k_n(z) = u_0 + \frac{1}{z}\sum_{i=1}^{n}\cfrac{z^2}{v_i + w_i z} = u_0 + \sum_{i=1}^{n}\cfrac{1}{v_i/z + w_i} \quad \text{(K-type)} \qquad (36)$$

and if all w_i are non zero then

$$l_n(z) = u_0 + \frac{1}{z}\sum_{i=1}^{n}\cfrac{u_i z^2}{v_i + z} = u_0 + \sum_{i=1}^{n}\cfrac{u_i}{v_i/z + 1} \quad \text{(L-type)} \qquad (37)$$

J stands for Jacobi and the J-type is known in the literature. A J-CF is called CF of special form by Hart [9]. The other names are chosen because they are alfabetically adjacent.

Proposition 6

If $a_n(z)$ is a A-type fraction with convergents $C^{(k)} = P^{(k)}/Q^{(k)}$ $(0 \leq k \leq n)$ then

1) $d^\circ P^{(k)} \leq k$ and $d^\circ Q^{(k)} \leq k$ and at least one of them is precisely equal to k and strict inequality $d^\circ P^{(k)} < k$ ($d^\circ Q^{(k)} < k$) can not be true for two consecutive values of k.

2) $P^{(k)}(0) = u_0.c \neq 0$ and $Q^{(k)}(0) = v_0.c \neq 0$.

3) $P^{(k)}/Q^{(k)}$ is irreducible.

Proof

1) This point is verified for k = 0,1,2.

Let us denote by $H_P^{(k)}$ ($H_Q^{(k)}$) the coefficient of z^k in $P^{(k)}$ ($Q^{(k)}$) and $H_{P/Q}^{(k)}$ is one of $H_P^{(k)}$ or $H_Q^{(k)}$. We have that

$$H_{P/Q}^{(k)} = w_k H_{P/Q}^{(k-1)} + u_k H_{P/Q}^{(k-2)}$$

We will investigate several possibilities:

a) $H_{P/Q}^{(k-1)} = 0$ then $H_{P/Q}^{(k-2)} \neq 0$ and thus $H_{P/Q}^{(k)} = u_k H_{P/Q}^{(k-2)} \neq 0$

b) $H_{P/Q}^{(k-2)} = 0$ then $H_{Q/P}^{(k-2)} \neq 0$ and so $H_{Q/P}^{(k)} = u_k H_{Q/P}^{(k-1)} \neq 0$

c) $H_{P/Q}^{(k-1)} \neq 0$ and $H_{P/Q}^{(k-2)} \neq 0$ then we can not have that $H_P^{(k)}$ and $H_Q^{(k)}$ are both zero because then we would have that

$$w_k H_P^{(k-1)} + u_k H_P^{(k-2)} = w_k H_Q^{(k-1)} + u_k H_Q^{(k-2)} = 0$$

or

$$w_k = - u_k H_P^{(k-2)}/H_P^{(k-1)} = u_k H_Q^{(k-2)}/H_Q^{(k-1)}$$

Now it can be proved by induction that

$$H_P^{(k-1)}/H_Q^{(k-1)} = H_P^{(k-2)}/H_Q^{(k-2)}$$

can never be true on condition c). The inductionstep herefore is

$$\frac{H_P^{(k)}}{H_Q^{(k)}} = \frac{H_P^{(k-1)}}{H_Q^{(k-1)}} \text{ would imply } \frac{w_k H_P^{(k-1)} + u_k H_P^{(k-2)}}{w_k H_Q^{(k-1)} + u_k H_Q^{(k-2)}} = \frac{H_P^{(k-1)}}{H_Q^{(k-1)}} \quad (38)$$

If $w_k = 0$ then this is not true by induction hypothesis.

If $w_k \neq 0$ then (38) would imply

$$H_P^{(k-2)}/H_Q^{(k-2)} = H_P^{(k-1)}/H_Q^{(k-1)}$$

This is not true by induction hypothesis again.

This ends the rather technical proof of 1).

2) and 3) are proved like in proposition 1 the underlying identities being:

$$P^{(k)}(0) = u_0 v_1 v_2 \cdots v_k$$

$$Q^{(k)}(0) = v_0 v_1 v_2 \cdots v_k$$

for 2) and

$$P^{(k)} Q^{(k-1)} - P^{(k-1)} Q^{(k)} = (-1)^{k-1} v_0 u_1 u_2 \cdots u_k z^{2k-1}$$

for 3) □

We can again derive companion versions of the RTOA algorithm, directly implementing the equivalence transformations. The J-type being the most economic with respect to its evaluation for a certain value of z, we redefine the recursions (28) for RTOJ.

RTOJ:

$$u_0 = S_0(0)/S_1(0)$$

$$S_2 = \frac{1}{z}(S_0 - u_0 S_1)$$

for $i = 1,2,\ldots$

$$S^{\star}_{i+2} = \frac{1}{z}\left(\frac{S_{i+1}(0)}{S_i(0)} S_i - S_{i+1}\right)$$

$$S_{i+2} = \frac{1}{z}\left(S^{\star}_{i+2} - \frac{S^{\star}_{i+2}(0)}{S_{i+1}(0)} S_{i+1}\right) \qquad (39)$$

$$u_i = \frac{S_{i+1}(0)}{S_i(0)}$$

$$w_i = \frac{S^{\star}_{i+2}(0)}{S_{i+1}(0)}$$

Hart [9] gives an algol program for this algorithm in the case of a finite m and n = m+1. The others are left as an exercise.

6. The second division algorithm and the padé table

With theorem 4 and proposition 6 it is easy to prove

Proposition 7

The diagonal $\mathbb{D}_0$ is normal iff its successive elements are the convergents of the CF of A-type constructed by the RTOA algorithm modulo some normalization.

□

The most common normalization $Q^{(k)}(0) = 1$ is obtained by the J-form. Because there is no difference between $\mathbb{D}_{+0}$ and $\mathbb{D}_{-0}$ we do not need a A1- and A2- version of the algorithm.

The other diagonals of the padé table can be found too by expanding the preliminar division in algorithm RTOA up to the required degree.

Proposition informs us that the RTOJ algorithm is a Thacher-like variant of the algorithm of Brezinski [2]. Here we compute the partial numerators and denominators from coefficients of the residual Z like in the Thacher algorithm while Brezinski computes the denominators explicitly and finds the required coefficients from the denominator representation like in Watson's algorithm.

This is in harmony with a result of Bussonnais [4].

Example

$\mathbb{D}_0$ for $\dfrac{1}{1 + z + z^2}$ is obtained from

$$1 - \cfrac{z}{|1+0z} + \cfrac{z^2}{|1+z}$$

and for $\mathbb{D}_2$ we have:

$$1 - z + \cfrac{z^3}{|1+z} + \cfrac{z^2}{|1+0z}$$

7. Generalization of the second division algorithm to non normal diagonals

The generalization of RTOA for non normal diagonals is now an obvious task.

Theorem 5

Given $f(z) = S_0(z)/S_1(z)$; $d°S_0 \leq m$; $d°S_1 \leq n$

with $S_0(0)\, S_1(0) \neq 0$

Consider the algorithm

RTOGA: initialization:

$$u_0 = S_0(0) \quad ; \quad v_0 = S_1(0)$$

$$d_0 = m \; ; \; d_1 = n \; ; \; \nu_0 = 0 \; ; \; \mu_0 = 0$$

preliminar division .

+ Find μ_1 and S_2 such that $0 \neq S_2(0) < \infty$ and

$$z^{\mu_1} S_2 = S_1(0)\, S_0 - S_0(0)\, S_1$$

+ $d_2 = \max(d_0, d_1) - \mu_1$ (40)

+ $u_1 = (S_2(0))^2 \quad ; \quad v_1 = S_1(0)\, S_2(0)$

for i = 1,2,...

+ $d_{i+2} = \max(d_i, d_{i+1} + \nu_i) - \mu_{i+1}$

+ $\nu_i = \mu_i - \nu_{i-1}$

+ Find p_i as the polynomial of degree ν_i, consisting of the first $\nu_i + 1$ terms of the series

$$\frac{u_i\, S_i}{v_i\, S_{i+1}}$$

+ Find μ_{i+1} and S_{i+2} such that $0 \neq S_{i+2}(0) < \infty$ and

$$z^{\mu_{i+1}} S_{i+2} = u_i\, S_i - v_i\, p_i\, S_{i+1}$$

(41)

+ $u_{i+1} = (S_{i+2}(0))^2 \quad ; \quad v_{i+1} = S_{i+1}(0) S_{i+2}(0)$

Setting $R_i = S_{i+1}/S_i$ then

1) $d^\circ S_i \leqslant d_i$; $p_i(0) = 1$; $\mu_{i+1} \geqslant \nu_i + 1$ and $\nu_i \geqslant 1$

2) $$f(z) = \frac{1}{v_0}\left(u_0 + \frac{1}{z}\sum_{i=1}^{\ell}\frac{u_i z^{\mu_i}}{|v_i\ p_i(z)} + \frac{z^{\mu_{\ell+1}} R_{\ell+1}}{|1}\right) \qquad (42)$$

3) If m and n are finite then the algorithm will terminate i.e. there exists a finite L such that $S_\ell = 0 \quad \forall\ \ell \geqslant L$

4) For $\ell = 1,2,\ldots$ we have that the ℓ-th convergent

$$C^{(\ell)} = P^{(\ell)}/Q^{(\ell)} = \frac{1}{v_0}\left(u_0 + \frac{1}{z}\sum_{i=1}^{\ell}\frac{u_i z^{\mu_i}}{|v_i\ p_i(z)}\right) \qquad (43)$$

is an order $N = \mu_1+\mu_2+\ldots+\mu_{\ell+1}$ approximation to f(z) and

$$Q^{(\ell)} f - P^{(\ell)} = (-1)^\ell z^N S_{\ell+2}/S_1 \qquad (44)$$

The proof is completely in parallel with the previous. □

We will call (43) a GA-type fraction which is equivalent with GI-, GJ- and GK-type fractions, defined in an obvious way.

The properties of a GA-type fraction are summarized in

Proposition 8

If $C^{(k)} = P^{(k)}/Q^{(k)}$ is the k-th convergent of a GA-type fraction (i.e. $u_i \neq 0$, $v_i \neq 0$, $p_i(0) = 1$, $d^\circ p_i \leqslant \nu_i$, $\nu_i = \mu_i - \nu_{i-1} - 1$) then

1) For $N = \mu_1+\mu_2+\ldots+\mu_k$: $d^\circ P^{(k)} \leqslant N$ and $d^\circ Q^{(k)} \leqslant N$

At least one of these is an equality and we can not have inequalities for $P^{(k)}$ ($Q^{(k)}$) for two successive values of k.

2) $P^{(k)}(0) = u_0.c \neq 0$ and $Q^{(k)}(0) = v_0.c \neq 0$

3) $C^{(k)}$ is irreducible.

The proof uses lemma 1 and is like in proposition 4 □

Thus we also have proved

Proposition 9

The RTOGA algorithm constructs a GA-type fraction whose convergents are the elements of the reduced diagonal $\hat{\mathbb{D}}_0$. □

We now have a Thacher-like version of the Berlekamp-Massey algorithm for diagonals. A Watson-like derivation was given in [3].

Example

For $\frac{1 + z^4}{1 - z^5}$ $\mathbb{D}_0$ is generated from

$$1 + \cfrac{z^4}{1 - z + z^2 - z^3 + z^4} - \cfrac{2z^5}{1 + z}$$

and e.g. $\mathbb{D}_2$ from

$$1 + \cfrac{z^4}{1 - z} + \cfrac{z^2}{1 + z} + \cfrac{z^3}{1 - z + z^2} - \cfrac{2z^3}{1}$$

8. A third division algorithm

The RTOT1 algorithm goes in the normal case along the following steps

$$\frac{S_1}{S_0} = t_0 + \cfrac{z\,S_2}{S_1} \qquad ; \; zS_2 = S_0 - t_0 S_1 \; ; \; t_0 = \frac{S_0(0)}{S_1(0)}$$

$$= t_0 + \cfrac{z}{t_1} + \cfrac{zS_3}{S_2} \qquad ; \; zS_3 = S_1 - t_1 S_2 \; ; \; t_1 = \frac{S_1(0)}{S_2(0)} \qquad (45)$$

$$= t_0 + \cfrac{z}{t_1} + \cfrac{z}{t_2} + \cfrac{zS_4}{S_3} \qquad ; \; zS_4 = S_2 - t_2 S_3 \; ; \; t_2 = \frac{S_2(0)}{S_3(0)}$$

$$= \ldots$$

The degree of the numerator will generally increase like 0,1,1,2,2,... Now if we want the degree of the numerator to oscillate like 0,1,0, 1,... then we have to interrupt the successive divisions of S_i/S_{i+1} for the introduction of partial denominators t_i that make the numerator of the convergents again of degree zero. So after the second step in (45) we introduce $t_2^{\star}$ to obtain a constant numerator.

$$\frac{S_0}{S_1} = t_0 + \cfrac{z}{t_1} + \cfrac{z}{t_2^{\star}} + \cfrac{S_4}{S_3} \qquad ; \; S_4 = S_2 - t_2^{\star} S_3 ; \; t_2^{\star} = -t_0$$

$$= t_0 + \cfrac{z}{t_1} + \cfrac{z}{t_2^{\star}} + \cfrac{z}{t_3} + \cfrac{zS_5}{S_4} \qquad ; \; zS_5 = S_3 - t_3 S_4 ; \; t_3 = \frac{S_3(0)}{S_4(0)}$$

etc...

We call this a constrained division algorithm. A more general description is given in the following

Theorem 6

Given $f(z) = S_0(z)/S_1(z)$ and suppose $S_0(z)S_1(z) \not\equiv 0$.

Apply the algorithm

RTOV3: initialization:

$$1/R_0 = f(z) = S_0(z)/S_1(z)$$

$$v_0 = S_0(0) \; ; \; v_1 = S_1(0)$$

$$S_2 = \frac{1}{z}(v_1 S_0 - v_0 S_1)$$

$$v_2 = S_2(0) \; ; \; v_2^{\star} = -v_0$$

$$S_3 = \frac{1}{z}(v_2 S_1 - v_1 S_2)$$

$$v_3 = S_3(0) \quad ; \quad \overset{\star}{v}_3 = v_2$$

$$F_2 = v_0 v_1 \quad ; \quad G_2 = v_2$$

for $i = 2,3,\ldots$

$$S_{2i} = \overset{\star}{v}_{2i-1} S_{2i-2} - \overset{\star}{v}_{2i-2} S_{2i-1}$$

$$v_{2i} = S_{2i}(0) \quad ; \quad \overset{\star}{v}_{2i} = - \overset{\star}{v}_{2i-2} F_i$$

$$F_{i+1} = F_i(\overset{\star}{v}_{2i-2} v_{2i-1} + v_{2i}) \qquad (46)$$

$$S_{2i+1} = \frac{1}{z}(v_{2i} S_{2i-1} - v_{2i-1} S_{2i})$$

$$v_{2i+1} = S_{2i+1}(0) \quad ; \quad \overset{\star}{v}_{2i+1} = G_i$$

$$G_{i+1} = G_i v_{2i}$$

Set $R_i = S_{i+1}/S_i$ then

1)
$$f(z) = \frac{1}{v_1}\left(v_0 + \frac{v_2 z}{\Big| v_1} + \sum_{i=1}^{\ell} \left(\frac{\overset{\star}{v}_{2i+1} z}{\Big| \overset{\star}{v}_{2i}} + \frac{v_{2i+2}}{\Big| v_{2i+1}} \right) + \frac{z\, R_{2\ell+2}}{\Big| 1} \right)$$

$$= \frac{1}{v_1}\left(v_0 + \frac{v_2 z}{\Big| v_1} + \frac{\overset{\star}{v}_3 z}{\Big| \overset{\star}{v}_2} + \sum_{i=1}^{\ell} \left(\frac{v_{2i+2}}{\Big| v_{2i+1}} + \frac{\overset{\star}{v}_{2i+3} z}{\Big| \overset{\star}{v}_{2i+2}} \right) + \frac{R_{2\ell+3}}{\Big| 1} \right) \qquad (47)$$

2) If $C^{(k)} = P^{(k)}/Q^{(k)}$ is the k-th convergent of the CF (47) and if the algorithm has not collapsed (cfr. property 1 and below) then $C^{(k)}$ is an approximation to $f(z)$ of order at least $[\frac{k}{2}] + 1$ and the residual is given by

$$Z^{(k)} = Q^{(k)} f - P^{(k)} = (-1)^k z^{[\frac{k}{2}]+1} \frac{S_{k+2}}{S_1} \qquad (48)$$

3) $d^\circ P^{(k)} \leqslant 1$ and

$$p^{(2k)} = F_k \overset{\star}{v}_{2k} \quad ; \quad p^{(2k+1)} = F_{k+1} + G_{k+2} z$$

$$d^\circ Q^{(k)} \leqslant [\tfrac{k}{2}]$$

and the highest degree coefficients are:

for $Q^{(2k)}$: $v_1 \overset{\star}{v}_3 \overset{\star}{v}_5 \cdots \overset{\star}{v}_{2k+1}$

for $Q^{(2k+1)}$: $v_1 \overset{\star}{v}_3 \overset{\star}{v}_5 \cdots \overset{\star}{v}_{2k+1} v_{2k+1}$

The constant terms are

$$Q^{(2k)}(0) = v_1^2(v_2^{\star}v_3+v_4) \ \ldots \ (v_{2k-2}^{\star}v_{2k-1}+v_{2k}) \ v_{2k}^{\star} = \frac{v_1}{v_0} F_{k+1} \ v_{2k}^{\star}$$

$$Q^{(2k+1)}(0) = v_1^2(v_2^{\star}v_3+v_4) \ \ldots (v_{2k}^{\star}v_{2k+1}+v_{2k+2}) = \frac{v_1}{v_0} F_{k+2}$$

The proof is like in theorem 1 □

Comment

There is no finiteness statement for the algorithm because the degree of the numerator of the convergents is ≤ 1 and thus we can not find f(z) exactly with a finite CF in general.

The even convergents are of course irreducible and for the odd convergents we have like for (8) of proposition 1 that

$$P^{(2k+1)}Q^{(2k)} - P^{(2k)}Q^{(2k+1)} = z^k v_{2k+2}v_{2k+1}^{\star}\cdots v_4^{\star}v_3^{\star}v_2^{\star}v_1 \qquad (49)$$

Thus the odd convergents are irreducible if $Q^{(2k+1)}(0)P^{(2k+1)}(0) \neq 0$

i.e. if $F_k \neq 0 \ \forall k$ or $v_{2i}^{\star}v_{2i+1}+v_{2i+2} \neq 0 \ \forall i$.

It can also be seen that if we do not want the algorithm to collapse then we must require $v_{2i+1}^{\star} \neq 0$ and $v_{2i} \neq 0 \ \forall i$. The last one implies the other and is also sufficient for the irreducibility as the following property proves.

Property 3

If in algorithm RTOV3 we have that $v_{2i} \neq 0 \ \forall i$ then F_i and $v_{2i+1}^{\star} = G_i$ are non-vanishing $\forall i$

Proof

By construction $v_{2i+2} = v_{2i+1}^{\star} v_{2i} - v_{2i}^{\star} v_{2i+1}$ so that

$$v_{2i}^{\star}v_{2i+1} + v_{2i+2} \neq 0 \quad \Leftrightarrow \quad v_{2i+1}^{\star} v_{2i} \neq 0$$

□

(48) then implies that for $v_{2i} \neq 0 \ \forall i$ we have that the even convergents have an order that is exactly $\frac{k}{2}+1$. The order of the odd convergents can still be higher than $[\frac{k}{2}]+1$.

We call a CF like (47) of V3-type if $v_1 \prod_{i=0}^{\ell} v_{2i} \neq 0$.

Equivalent forms are

$$t_0 + \left|\frac{z}{t_1}\right| + \sum_{i=1}^{\ell} \left(\left|\frac{z}{t^{\star}_{2i}}\right| + \left|\frac{1}{t_{2i+1}}\right| \right) \quad \text{(T3-type)} \tag{50}$$

and

$$c_0^{-1} + \left|\frac{c_1 z}{1}\right| + \sum_{i=1}^{\ell} \left(\left|\frac{c^{\star}_{2i} z}{1}\right| + \left|\frac{c_{2i+1}}{1}\right| \right) \quad \text{(C3-type)} \tag{51}$$

the last one exists only when all partial denominators are non zero.

Proposition 10

If in the padé table of f(z) $\lfloor_{+0}$ is a normal row then the successive elements of the sawtooth S^{+}_{+0} are the convergents of the CF constructed by the algorithm RTOV3

Proof

It follows from theorem 6 that they are padé approximants and if $\lfloor_{+0}$ is normal then [0/k] is of exact order k+1 and this means that $v_{2k} \neq 0$. Property 3 states that this is sufficient for the algorithm not to collapse. □

Example

RTOT3 would generate for $\frac{1}{1 + z + z^2}$: $1 - \left|\frac{z}{1}\right| + \left|\frac{z}{1}\right| + \left|\frac{1}{0}\right| + \left|\frac{z}{1}\right|$

Here the algorithm ends because the function block is reached. The other sawteeth can also be treated in a similar way on condition that the first row of the sawtooth is normal. E.g. S^{+}_{k} $(k \geq +0)$ first compute [k/0]. The role of v_0 is taken over by the highest degree coefficient of the numerator of [k/0], which is non zero because of the normality condition. Then RTOV3 runs as before. Before tackling the generalization of this algorithm we will first look at the contracted form of it. (I.e. the algorithm computing a row of the padé table.)

9. A forth division algorithm

The proof of the following theorem is as in theorem 1 and therefore omitted.

Theorem 7

Given $f(z) = S_1(z)/S_0(z)$ and suppose $S_0(0)S_1(0) \neq 0$

If the algorithm RTOE is defined as follows

RTOE: initialization:

$$R_0 = f(z) = S_1(z)/S_0(z) \quad ; \quad v_0 = S_0(0) \quad ; \quad v_1 = S_1(0)$$

$$S_2 = \frac{1}{z}(v_1 S_0 - v_0 S_1) \quad ; \quad v_2 = S_2(0)$$

$$S_3 = \frac{1}{z}(v_2 S_1 - v_1 S_2) \quad ; \quad v_3 = S_3(0)$$

$$u_3 = -v_1 v_3 \quad ; \quad w_3 = +v_1 v_2$$

for $i = 3,4,\ldots$

$$S^{\star}_{i+1} = \frac{1}{z}(v_i S_{i-1} - v_{i-1} S_i)$$

$$S_{i+1} = w_{i-1} S^{\star}_{i+1} - v_i S_i \quad ; \quad v_{i+1} = S_{i+1}(0) \qquad (52)$$

$$u_{i+1} = -v_{i+1} w_i \quad ; \quad w_{i+1} = -v_i w_i$$

and setting $R_i = S_{i+1}/S_i$ then we have:

1)
$$f(z) = \left|\frac{v_1}{v_0}\right| + \left|\frac{v_2 z}{v_1}\right| + \sum_{i=3}^{\ell} \left|\frac{u_i z}{w_i + v_i z}\right| + \left|\frac{z R_\ell}{1}\right| \qquad (53)$$

2) If $C^{(k)} = P^{(k)}/Q^{(k)}$ is the k-th convergent of the CF (53) and if $v_i \neq 0 \ \forall\, i$ then $C^{(k)}$ is an approximant of order k and

$$Z^{(k)} = Q^{(k)} f - P^{(k)} = (-1)^k z^k S_{k+1}/S_0 \qquad (54)$$

3) $P^{(k)} = v_1 v_1 w_3 \ldots w_k$

$d^{\circ} Q^{(k)} = k-1$ (if all $v_i \neq 0$)

$Q^{(k)}(0) = v_0 v_1 w_3 \ldots w_k$

The highest degree coefficient of $Q^{(k)}$ is $v_2 v_3 \ldots v_k$ $(k \geq 2)$

□

The algorithm will not collapse if all $v_i \neq 0$ and then equivalent forms for

$$\frac{v_1}{v_0}\Big|+\frac{v_2 z}{v_1}\Big|+\sum_{i=3}^{k}\frac{u_i z}{w_i+v_i z}\Big| \qquad \text{(E-type (E for Equivalent))} \qquad (55)$$

are

$$\frac{f_0}{1}\Big|+\frac{f_1 z}{1}\Big|+\sum_{i=3}^{k}\frac{u_i z}{w_i+z}\Big| \qquad \text{(F-type)} \qquad (56)$$

$$\frac{1}{g_0}\Big|+\frac{z}{g_1}\Big|+\sum_{i=3}^{k}\frac{z}{w_i+v_i z}\Big| \qquad \text{(G-type)} \qquad (57)$$

$$\frac{h_0}{1}\Big|+\frac{h_1 z}{1}\Big|+\sum_{i=3}^{k}\frac{u_i z}{1+v_i z}\Big| \qquad \text{(H-type)} \qquad (58)$$

Again we can prove that the convergents are irreducible if $v_i \neq 0$ $\forall i$ and thus:

Proposition 11

If in the padé table of f(z) $\lfloor_{+0}$ is a normal row then RTOE computes a CF whose successive convergents are the elements of the row $\lfloor_{+0}$ □

The interpretation of the algorithm is best seen from the G-type version. w_i is the first term in the division of S_{i-1}/S_i and $v_i z$ is a term that is added to make the numerator of the convergent of zero degree again.

Example

$\lfloor_{+0}$ for $\dfrac{1}{1+z+z^2}$ are the convergents of $\dfrac{1}{1}\Big|+\dfrac{z}{1}\Big|-\dfrac{z}{1+z}\Big|$

We could of course set a constant in front of the CF constructed by RTOE. We do not give this in detail. The CF would be of the form

$$\frac{1}{v_1}\left(v_0+\sum_{i=1}^{k}\frac{u_i z}{w_i+v_i z}\Big|\right) \qquad (59)$$

For the example this gives $\lfloor_{+0}$ as $1-\dfrac{z}{1+z}\Big|-\dfrac{z}{1+z}\Big|$

The generalization of RTOE to the non normal case is not so difficult if the interpretation as a constrained division algorithm is kept in mind and if we only compute the zeroth row, but there may rise a problem if we want to compute other rows of the padé table and similar difficulties arise for RTOV3. If we want to compute say $\lfloor_k$ $(k \geqslant +0)$ and if we set

$$f(z) = [\,k/0\,] + z^{k+1}\, g(z)$$

then if we substitute for g(z) the CF representing its zeroth row then we do not obtain the k-th row of the padé table of f(z) if this is not a normal row. Let us look at an isolated step first. Suppose two successive elements in the row $\lfloor_k$ $(k \geqslant +0)$ have numerators

$$P'' = \sum_{i=0}^{d''} a''_i\, z^i \qquad \text{and} \qquad P' = \sum_{i=0}^{d'} a'_i\, z^i \qquad d', d'' \leqslant k$$

and that we are left with a residual

$$R = z^{\mu}\, \frac{S'(z)}{S''(z)} \qquad \mu \geqslant 1$$

Then compute p as the polynomial of degree k-d' being the first terms of the series (division of) S"/S' and let $S^{\star}$ be the remainder

$$S'' = pS' + S^{\star} \quad \text{(terms in S" and S' ordered from low to high degree)} \tag{62}$$

We then have to take away the degree augmenting effect of z^{μ} (it would make the next numerator of degree $\mu+d''$) Therefore find a polynomial q to "peel off" the terms of $P''\, z^{\mu}$ that exceed degree k. This can be done as follows:

Set $N = d'' - d' + \mu$. If $N \geqslant k - d' + 1$ then make the division

$$\frac{z^{\mu} P''}{P'} = \frac{a''_{d''}\, z^{d''+\mu} + a''_{d''-1}\, z^{d''+\mu-1} + \ldots + a''_0\, z^{\mu}}{a'_{d'}\, z^{d'} + a'_{d'-1}\, z^{d'-1} + \ldots + a'_0} =$$

$$v_N\, z^N + \ldots + v_{k-d'+1}\, z^{k-d'+1} + \frac{r_k\, z^k + \ldots + r_0}{P'} \tag{63}$$

q is then defined as $v_N z^N + \ldots + v_{k-d'+1} z^{k-d'+1}$ and r as the remainder

If $N \leq k-d'$ then set $q = 0$ and $r = P''$

We then have

$$R = z^{\mu} \frac{S'}{S''} = \frac{z^{\mu}}{p + \frac{S^{\star}}{S'}} = \left| \frac{z^{\mu}}{p - q} \right| + \left| \frac{z^{\hat{\mu}} S}{S'} \right|$$

with $S^{\star} = S'' - p S'$ and $z^{\hat{\mu}} S = S^{\star} + q S'$.

The next convergent has then as numerator $P'(p - q) + z^{\mu} P''$

and this is constructed to be of a degree not exceeding k.

We will indicate the operation

(62) as $\left[\frac{S''}{S'} \right]^{+}_{k-d'} = (p, S^{\star})$ and (63) as $\left[\frac{z^{\mu} P''}{P'} \right]^{-}_{k} = (q, r)$

The difference is the order in which the terms are set to make the division. The first uses an increasing degree order, the second a decreasing degree order.

From the foregoing it seems inevitable to compute the numerators of the successive convergents when treating an arbitrary row of a non normal padé table.

We are now ready to formulate the RTOGE algorithm for an arbitrary row.

Theorem 8

Given $f(z) = S_{-1}(z)/S_0(z)$ with $S_0(0) \neq 0$

Compute the k-th row by RTOGE as follows ($k \geq 1$)

RTOGE: initialization :

$P_{-1} = 1$; $d_{-1} = 0$

$P_0 = P^{[k-1/0]}$ find its strict degree $d_0 \leq k-1$

$z^{\mu_1} S_1 = S_{-1} - [k-1/0]\, S_0$; $(\mu_1 \geq k)$; $S_1(0) \neq 0$

for i = 1,2,...

Find

$$\left[\frac{S_{i-1}}{S_i} \right]^{+}_{k-d_{i-1}} = (p_i , S^{\star}_{i+1})$$

and

$$\left[\frac{z^{\mu_i} P_{i-2}}{P_{i-1}} \right]_k^- = (q_i , \bar{S}_{i+1}) \tag{64}$$

$$z^{\mu_{i+1}} S_{i+1} = S^{\star}_{i+1} + q_i S_i \qquad (S_{i+1}(0) \neq 0)$$

$P_i = P_{i-1} p_i + \bar{S}_{i+1}$ and determine its strict degree

$d_i \leqslant k.$

Then:

1)
$$f(z) = [\, k-1/0 \,] + \sum_{i=1}^{\ell} \left| \frac{z^{\mu_i}}{p_i - q_i} \right| + \left| \frac{z^{\mu_{\ell+1}} S_{\ell+1}}{S_\ell} \right|$$

2) The ℓ-th convergent $C^{(\ell)}$ has as numerator P_ℓ and if its denominator is Q_ℓ then

$$Q_\ell f - P_\ell = (-1)^\ell z^N S_{\ell+1}/S_0 \quad ; (\ell \geqslant 1) \; ; \; N = \sum_{i=1}^{\ell+1} \mu_i$$

3) $\bar{S}_i(0) = 0 \; ; \; \mu_{i+1} > k-d_{i-1} \; (i > 1) \; ; \; p_i(0) = S_{i-1}(0)/S_i(0) \neq 0$

$P_i(0) = P_0(0)\, p_1(0) \ldots p_i(0) = P_0(0)\, S_0(0)/S_i(0) \neq 0$

$Q_i(0) = Q_0(0)\, S_0(0)/S_i(0) \neq 0$

$d^\circ Q_i < \mu_1+\mu_2+\ldots+\mu_{i+1}$

4) The convergents are irreducible

5) The convergents form the elements of the reduced row $\hat{L}_k$

(except perhaps for the zeroth convergent) □

The exception for the zeroth convergent can be avoided if we start the CF with $[\, k/0 \,]$ instead of with $[\, k-1/0 \,]$. This would be the equivalent of starting RTOE with a constant in front. The zeroth row is then not a special case. To get the zeroth row in the formulation above we need $S_{-1}(0) \neq 0$ and this is the straightforward generalization of the RTOE algorithm. One can solve this as an exercise.

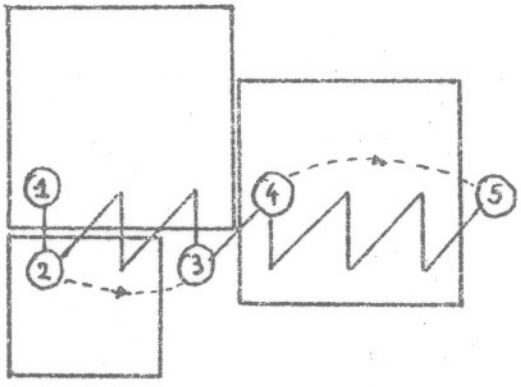
Example

For $\frac{1+z^4}{1-z^5}$ we get :

$$\hat{L}_{+0} : \frac{1}{|1} + \frac{z^4}{|1} + \frac{z}{|1+z} + \frac{z^3}{|1-z^3} + \ldots$$

$$\hat{L}_2 : 1 + \frac{z^4}{|1-z+z^2-z^4} + \frac{z^3}{|1+z} + \frac{z}{|1-z^3} + \ldots$$

The sawtooth algorithm RTOV3 can also be generalized. If the sawtooth succession is interrupted by a singular block then we go along a row (the first or the second of the sawtooth) by the RTOGE algorithm as is depicted in the example below.



Computationally however it is not a simple matter at the situation (2) e.g. to predict if the next element is on the first or on the second row. Therefore we have to know the size of the singular blocks of (1) and (2) . This can be found if we know the order of approximation, the strict degree of the numerator and the column position. In a situation like (4) we have that the approximation order is greater than expected in the normal case and then we know that the next element (5) is on the first row of the sawtooth. Because of the more complicated character we will not persue this algorithm in further detail here.

10. Other algorithms

Some other algorithms can be develloped. E.g. a sawtooth S^{-}_{+0} has a corresponding CF of the form

$$\frac{1}{v_1}\left(v_0 + v_0' z + \sum_{i=1}^{\ell} \left(\frac{v^{\star}_{2i} z}{v^{\star}_{2i-1}} + \frac{v_{2i+1}}{v_{2i} z} \right) \right)$$

where the v_i originate from the division and the $v^{\star}_i$ are introduced for degree reducing purposes.

We can also find column algorithms as duals for the row algorithms presented here.

The algorithms for ascenting staircases and diagonals like the Baker algorithm [1,5] and the Cordellier algorithms [6] fall somewhat out of this context. It is seen that we have to monitor the degrees of numerator and of denominator with techniques in the style of the constrained division algorithms. The caretaking of this will "prevent" to insert a term from the division of two successive S_i and therefore we clearly do not have an increase of approximation order. This is also predicted by relations of Bussonnais [4] indicating that the recursion coefficients are not expressed as a function of residual coefficients. For completeness we will state a theorem similar to the previous:

Theorem 9

If $f(z) = S_{-1}(z)/S_0(z)$; $S_0(0) \neq 0$; $L = N$ or $N+1$ is given and apply the following algorithm:

initialization

$$P_{-1} = z^{L}$$

$$P_0 = P^{[N/0]} \text{ and find its strict degree } d_0 \leq N$$

$$S_1 = z^{-N}(S_{-1} - P_0 S_0)$$

for $i = 1,2,\ldots$ while $d^\circ P_i \geq 0$

$$\left[\frac{P_{i-2}}{P_{i-1}}\right]^{-}_{d_{i-1}-1} = (q_i , P_i) \quad ; \text{ find the strict degree } d_i \text{ of } P_i$$

$$S_{i+1} = S_{i-1} + q_i S_i$$

then

$$f(z) = P_0 + z^L \sum_{i=1}^{\ell} \left|\frac{1}{-q_i}\right| + \left|\frac{S_{\ell+1}}{S_\ell}\right| \qquad (65)$$

and the residual of the convergent is $z^N S_{\ell+1}/S_0$

Proof

see Cordellier [6] □

The convergents can possibly be reducible. I.e. they can be of the form $\frac{z^k}{z^k}\frac{P}{Q}$ with $\frac{P}{Q}$ a padé approximant.

Apart from this and some normalization, the convergents of (65) are the elements of the reduced ascending staircase $\hat{U}_N$ for $L = N$ and of the reduced upward diagonal $\hat{E}_N$ for $L = N+1$.

Example

$$\frac{1 + z^4}{1 - z^5} = 1 + z^4 + \left|\frac{z^5}{-z}\right| + \left|\frac{1}{z^3}\right| + \ldots \text{giving } \hat{U}_4$$

$$= 1 + z^4 + \left|\frac{z^6}{-z^2}\right| + \left|\frac{1}{z^2}\right| + \ldots \text{ giving } \hat{E}_4$$

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