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Eggs, Bubbles, and Wormholes in semi-classical Quantum Gravity

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Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Science (PhD): Physics

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To the ones I love

Preface

These are the very first words of a little booklet I would like to call 'my thesis'. Allow me to kick off by directly turning myself to my dearest reader wishing him/her a warm welcome. This PhD has been quite a journey and it is my pleasure to take you on a journey through it, at least through the physics part of it. This thesis consists of a (massaged) reprint of existing papers [2–4], as well as a preprint of a future paper; since the copyright is retained by the authors in the respective journals, the reproduction of the published papers is permitted.

Thesis plan

Quantum gravity is one of the hottest topics in contemporary theoretical physics. Although an accepted theory is lacking, string theory seems to be the predominant approach. Despite its beauty and elegance, string theory has a setback: it is difficult, and this in many aspects. Many smart people are racking their brains about its structure, formulation, top–down constructions, etc. In this thesis, I will not attempt to compete with these brainboxes. Instead, the main interest here is semi–classical gravity with 'stringy' ingredients: inter alia, are there any imprints of quantum gravity already at the semi–classical level, once we start cooking with the key properties of string theory (extra dimensions, branes, fluxes, etc.)?

The introductory chapter contains a 'build–up' towards string theory and discusses its ingredients at a very conceptual level. It is written in an informal style that is meant to be –largely– understandable for the layman. Its raison d'être is essentially to serve as a 'prequel' to the chapters 3 and 4, for which the presented conceptual

storyline is sufficient. The reader familiar with the concepts of string theory (in particular flux compactifications, the landscape, and the swampland) can of course skip this entire chapter without being afraid of missing essential information.

In chapter 2, we provide the prerequisite background material with which the rest of this thesis frequently makes contact. It is reviewed how our universe can be modelled as a Friedmann–Lemaître–Robertson–Walker universe, that has an inherent cosmological constant problem. Next, the first steps toward quantum gravity in the context of cosmology are made, leading to the (semi–classical) framework of quantum cosmology.

The cosmological constant problem serves as the main motivation for chapter 3, which proposes a dynamical mechanism to relax it. Using the ingredients of string theory, in particular flux compactifications, we cook 'cosmic eggs': a toy model of cosmological initial conditions and a proposal for a Hartle–Hawking(–like) quantum state (an explicit calculation is presented in appendix B).

The motivation for chapter 4 is the difficulty of constructing de Sitter vacua in string theory; it contains an alternative attempt in the context of braneworlds. This chapter is accompanied by appendix C, which outlines the basic formalism to treat hypersurfaces in General Relativity. While drawing motivation from the swampland, the 'Dark Bubble' scenario is introduced. This string–inspired model realizes a Friedmann–Lemaître–Robertson–Walker universe on top of an expanding bubble of a 'less false' vacuum that has nucleated in the background of an unstable anti–de Sitter vacuum. We extend the literature in two ways. Firstly we make contact with quantum cosmology, leading to a Vilenkin–type weighting of the (bubble) nucleation probability. Secondly we construct the gravitational fluctuations in the bulk that represent (cosmological) gravitational waves on the expanding bubble.

In chapter 5, we examine the saddle point expansion in Euclidean quantum gravity: are gravitational instantons with a non–trivial topology to be included or not? Wormholes provide a particularly interesting playground for this purpose, and it is a long–standing debate whether or not they contribute to the saddle point expansion. This has been puzzled out fairly recently for axionic (i.e. Giddings–Strominger)

wormholes. The existing literature hosts several incorrect results on this matter. We scrutinize one approach and point out its mistake(s). By connecting the scalar field formulation of the axion to the form field formulation, we try to alleviate the debate even more.

In appendix A, we present the conventions followed in this thesis, as well as some useful formulae. The reader may wish to consult this prior to chapter 2. I apologize in advance to the reader for sourcing confusion by introducing 'almost identical' symbols (in the sense that they are difficult to distinguish by handwriting or pronunciation) that mean vastly different things. I blame this on our alphabet, even supplemented by the Greek one, having an insufficient amount of letters.

Thank you

During the four years I spent at the ITF as a PhD student, I was lucky to have the support from many different people around me, and from many different –sometimes unexpected– corners. Expressing my gratitude towards them by 'just words' is by far insufficient and, possibly, impossible. Let me nevertheless try to fulfill this daunting task.

The first person I would like to thank, is my original supervisor, Bert Vercnocke. Thank you, Bert, for hiring me as one of your PhD students. I am grateful I could spend four –more– wonderful years in Leuven, which ultimately led to meeting my girlfriend (this probably constitutes a nice example of the butterfly effect). Although we did not collaborate very much, I really enjoyed the discussions we had together.

The biggest 'thank you' goes out to my current supervisor, Thomas Van Riet. Thomas, I am extremely grateful that you adopted me as one of your PhD students; my whole PhD would definitely not have succeeded without you. Thank you for always having my back; it felt safe 'under your wings'![1] To everyone reading this: please reread the last sentence and acknowledge its value. You are –in many aspects– one of the greatest persons I have met in my life. I particularly enjoyed every minute of our physics (and non–physics) discussions: thank you for making

[1] I ~~sincerely hope~~ am certainly convinced that you know what I mean with this short sentence.

time for those, especially when I needed it the most. Thomas, thank you for raising me from a mediocre student to a scientist/researcher. It is my pleasure to acknowledge that it is because of you that I learned what (physics) research *really* entails. You are such a nice person and I really feel honored to have worked withyou. Thomas, thanks for everything!

I also wish to express my thanks to the amazing people I met during my time as a PhD student. It has been a pleasure and a privilege to work with my collaborators Bert, Kwinten, Gideon, Thomas (×2), Simon, Ulf, and Daniel, with whom I had plenty of enjoyable 'physics talk'. They allowed me to expand my knowledge by soaking up theirs. (Also thanks for not making fun of me when I came up with silly questions). In the second place, I would like to thank the friends I made at the ITF, especially Jonas and Vincent, proving that physics and I have some social side. I really liked the way we could talk –and argue– about practically anything imaginable, not necessarily related to physics at all. Lastly, thanks to Anneleen and Filip (who all of ITF will miss!) for taking care of all administrative and ICT–related stuff, which I am practically incapable of doing.

The last people I would like to thank for their direct support are Pierre Savaria, Thomas Van Riet, and Lise Verstreken for carefully reading (parts of) the draft version of this thesis and commenting on all the horrible spelling and grammar mistakes I made. This is often an underappreciated job, but it was of crucial importance to me. Thank you very much!

Besides the support from the academic world, I also owe much to people very close to me. In the first place: my parents. I owe everything to my parents and I could write an entire, new thesis about this. I am so grateful that they gave me the opportunity to study at a university, and that they supported me for every step I took. Many thanks go out to my dad for all the cozy car rides from Pelt to Leuven, some of which were earlier in the morning than others. My parent's greatest achievement is raising me with a critical and scientific mindset (which probably started with my mother's famous words 'Jezuske straft niet' ☺). Mom and dad, thank you for raising me from a little boy to the person I am now. Een heel

warm woord van dank gaat zeker uit naar mijn Oma en Opa† Fina, twee personen aan wie ik enorm veel gehad heb in mijn leven. Zij zullen altijd een speciaal plaatsje in mijn hart hebben. In fact, I have to acknowledge my entire family, but especially my Peter and Tante Renild for all the good times we had –and certainly will have– together. To finish, I want to thank my girlfriend for her support and capability to listen to me nagging around. I hope we can in the foreseeable future –taking into account that time is relative– start a journey of our own.

I would also like to thank the members of the jury: Nathal Severijns, Thomas Hertog, Joseph Indekeu, Ruth Gregory and Gideon Koekoek. I genuinely appreciate that you found the time to read and assess my thesis.

Finally, I should also acknowledge (or recognize) the Doctoral Committee for teaching me the true meaning of 'box ticking' and how superfluous administration significantly contributes to a better understanding of the physics that governs this world. Rather ironically, these people will most likely not read this acknowledgment.

Goodbye

These are the very last words of the very first words of this little booklet. It is with pain in my heart that I leave the field of physics. Thanks to everyone who made it possible that I could follow my passion, thanks to universe for existing giving me something to study, and to physics I must say 'goodbye'!

†My grandfather left me at the start of my PhD. I still miss him. I hope he would have been very happy if he could have been here...

List of publications

[1] K. Fransen, G. Koekoek, R. Tielemans and B. Vercnocke, *Modeling and detecting resonant tides of exotic compact objects*, *Phys. Rev. D* **104** (2021) 044044 [2005.12286]

[2] T. Hertog, R. Tielemans and T. Van Riet, *Cosmic eggs to relax the cosmological constant*, JCAP **08** (2021) 064 [2101.01437].

[3] U. Danielsson, D. Panizo, R. Tielemans and T. Van Riet, *Higher-dimensional view on quantum cosmology*, *Phys. Rev. D* **104** (2021) 086015 [2105.03253].

[4] U. Danielsson, D. Panizo and R. Tielemans, *Gravitational waves in dark bubble cosmology*, *Phys. Rev. D* **106** (2022) 024002 [2202.00545].

Abstract

Current observations suggest that our universe is asymptoting towards a de Sitter space with a small amount of vacuum energy. From a quantum field theory perspective, this smallness presents a perplexing puzzle –the cosmological constant problem– and it is the hope that a theory of quantum gravity, such as string theory, provides an opportunity to address it. However, the status of our universe in string theory is not yet clear and it appears fruitful to construct string–inspired models to understand some semi–classical aspects of gravity in a cosmological setting.

In chapter 3, we argue that in theories with extra dimensions, such as string theory, the cosmological hierarchy problem can be thought of as the unnaturally large radius of the observable universe in Kaluza–Klein units. An early universe scenario is proposed in which three large spatial dimensions arise through tunneling from a 'cosmic egg', an effectively one–dimensional configuration with all spatial dimensions compact and of comparable, small size. A quantum cosmological treatment of a toy model egg predicts that, in a variant of the Hartle–Hawking state, cosmic eggs 'break' to form higher–dimensional universes with a small, but positive cosmological constant. Cosmic eggs serve as natural initial conditions for cosmology, whose breaking dynamically relaxes the cosmological constant problem.

In chapter 4, motivated by the difficulty in finding explicit de Sitter solutions within string theory, we argue in favor of the recently proposed Dark Bubble model, where a de Sitter cosmology is induced on the surface of a Brown–Teitelboim bubble in a five–dimensional anti–de Sitter space. From a quantum cosmological point of view, the boundary conditions are unambiguously fixed by demanding consistency

with the known physics of bubble nucleation; this selects the Vilenkin weighting for the amplitude. In order to generalize the connection between Dark Bubbles and quantum cosmology to include gravitational perturbations, we construct the five–dimensional uplift of the four–dimensional gravitational waves in a de Sitter cosmology. We also use this uplift to explain the interpretation of the apparently negative energy contributions in the effective four–dimensional Einstein equations, which distinguish the Dark Bubble scenario from Randall–Sundrum.

In chapter 5, we turn to Euclidean axion wormholes and their status as saddle points in the path integral. We confirm the recent result that Giddings–Strominger wormholes are perturbatively stable saddle points in both the three–form and scalar field formulation of the axion. Our analysis also provides a clean interpretation of the boundary conditions, which seem to have been applied erroneously in the literature.

Beknopte samenvatting

Kosmologische observaties suggereren dat ons universum evolueert richting een de Sitter ruimtetijd met een zeer kleine hoeveelheid aan vacuümenergie. Dit is vanuit het perspectief van de kwantumveldentheorie een vervelende kwestie –het kosmologische constante probleem– en de hoop is dat een theorie van kwantumgravitatie, zoals snaartheorie, hierop licht kan werpen. De status van ons heelal in snaartheorie is echter niet volledig begrepen; het is daarom nuttig om modellen geïnspireerd op snaartheorie te construeren om zo semi–klassieke aspecten van gravitatie in een kosmologisch kader te begrijpen.

In hoofdstuk 3 beargumenteren we dat in theorieën met extra dimensies, zoals snaartheorie, men het kosmologische constante probleem kan zien als de onnatuurlijke grootte van het zichtbare heelal in Kaluza–Klein eenheden. Er wordt een scenario voor het vroege universum voorgesteld waarin drie grote, ruimtelijke dimensies natuurlijkerwijs verschijnen door een tunneleffect uit een 'cosmic egg'. Dit is een effectieve, ééndimensionale configuratie waarin alle ruimtelijke dimensies compact en van vergelijkbare grootte zijn. Een kwantumkosmologische analyse van dit model voorspelt dat, in een variant van de Hartle–Hawking toestand, deze cosmic eggs 'breken' en zo hoger dimensionale universa vormen met een kleine, maar positieve kosmologische constante. Cosmic eggs vormen dus mogelijks natuurlijke initiële condities voor kosmologie en het breken ervan relaxeert het kosmologische constante probleem op een dynamische manier.

In hoofdstuk 4 pleiten we, geleid door de moeilijkheid om expliciete de Sitter oplossingen in snaartheorie te vinden, in het voordeel van het recent voorgestelde 'Dark Bubble' model. Hierin wordt een de Sitter kosmologie geïnduceerd op de rand van een Brown–Teitelboim bubbel in een vijfdimensionale anti–de Sitter ruimtetijd. Vanuit het perspectief van de kwantumkosmologie worden de randvoorwaarden in dit geval ondubbelzinnig bepaald wegens consistentie met de gekende fysica van bubbelnucleatie; dit selecteert de Vilenkin amplitude. Om de connectie tussen de Dark Bubbles en de kwantumkosmologie uit te breiden met gravitationele fluctuaties construeren we vijfdimensionale gravitatiegolven die reduceren naar de gekende vierdimensionale golven in een de Sitter kosmologie. Bovendien kunnen we deze hoger dimensionale gravitatiegolven gebruiken om de schijnbare negatieve energie bijdragen in de effectieve vierdimensionale Einstein vergelijking uit te leggen.

In hoofdstuk 5 kijken we naar Euclidische, axionische wormgaten en hun status als zadelpunten in de padintegraal. We bevestigen het recente resultaat dat Giddings–Strominger wormgaten perturbatief stabiele zadelpunten zijn in zowel de drie–vorm als de scalaire formulering voor het axionveld. Onze analyse voorziet ook een duidelijke interpretatie van de randvoorwaarden die in het verleden in de literatuur niet altijd correct werden opgelegd.

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Chapter 1

Introduction

1.1 Physics is curiosity

Curiosity touches everyone. It is something that is deeply rooted in our human existence, something that makes us stand right here, something that has even brought us to the Moon and back. The flavors of curiosity range from our basic instinct to survive to pure wonder about why things are what they are. We have simply evolved to be curious. It is the seed, the foundation even, of science and philosophy. It goes hand in hand with mystery, and with the failure to understand. At some point in our lives, we start asking ourselves basic questions about the things we perceive, like 'why does the Moon rise and set every day?', but also about the things that are beyond our perception, like 'what is everything around us ultimately made of?'. We are always craving for explanations.

In the early days, many phenomena were attributed to the divine. To give one mythical example, Selene, the Greek goddess of the Moon, drove her shiny silver chariot every day across the night sky on her way to her lover, thereby illuminating the Earth. Such personifications are age–old attempts to explain the observed phenomena in nature. Lacking any argumentative grounds, these man–made stories were questioned by the pre–Socratic philosophers. These proto–scientists

were critical thinkers and distanced themselves from any mythical considerations. Instead, they believed in some minimal set of principles –a unifying theory one could say– that would govern our world. These principles can either stem from our everyday experience 'à la Aristotle', or our intellectual and mathematical mind 'à la Plato'. This is the foundation of contemporary science: our curiosity drives us to describe certain mysteries in terms of mathematical principles or models and these can –or in fact should– be verified experimentally.

'**It's human nature to stretch, to go, to see, to understand. Exploration is not a choice, really; it's an imperative.**'

— *Michael Collins*

Our curiosity is tremendously difficult –if not impossible– to saturate; if we can see the horizon, we probably also want to know what lies beyond. Democritus' philosophical search for the fundamental building blocks of nature neatly illustrates this insatiability. The Moon consists of rocks and sand, things that we can grasp and understand from our everyday experience. These building blocks themselves are composed of molecules and atoms, things beyond our sensory perception. Yet, our curiosity drove us to invent instruments, like electron microscopes, that allow us to see them. These atoms consist of even more elementary particles, quarks and electrons, for which we even need state–of–the–art particle accelerators to probe them. Can we go on? Unfortunately, we do not have the technology available to probe what quarks and electrons are (ultimately?) made of. However, our curiosity and our imagination do not stop at this 'experimental horizon'. Instead, if we believe string theory, all these elementary particles are basically tiny vibrating strings. Most likely, we will never be able to directly explore these tiny hypothetical objects, but why would this mean that we should stop wondering?

'**Big, bigger, smallest...**'

This question about the elementary constituents of matter is many centuries old. Yet, it has propagated into modern physics. Another age–old mystery that is still present in contemporary physics relates to the stars that we see at night. Looking at the twinkling stars, we realize we are just small humans on a big planet in a large solar system in a huge galaxy in a humongous, unimaginably huge universe. Yet we are intrigued by those big cosmological questions about our existence: 'Why are we here? How did we end up here? Where will we be going?' This curiosity has led us to develop a wealth of theories to describe the universe we live in.

Looking at those same stars, in trying to find answers, we noticed something very peculiar around the turn of the millennium: those stars, the other galaxies, seem to be moving away from us. The universe expands; it is not some static and eternal entity, as was a common (divine–inspired?) belief prior to the second millennium. Instead, the universe must have had a beginning and, ever since then, it seems to be growing bigger and bigger! As curious as this might sound, this is not the end of the story. Remarkably, those very same stars revealed that the rate at which the universe grows is increasing: our universe is expanding at an accelerating, exponential rate! Although the discovery of the expansion of the universe seems to be of first importance to which the mere fact of acceleration seems to add a simple detail, the converse truly holds. The discovery of acceleration is possibly one of the most important[1] ones in contemporary high–energy physics, and it triggered a major revolution in the field of cosmology and in the field of fundamental physics. Our curiosity now strikes once again: 'If our universe is not eternal, what was there before it began? What causes the *accelerated* expansion we currently observe?'

At this point in time, there is no convincing answer to the first question about some 'embryonic phase' of our universe. This has a very simple reason: we do not have an accepted theory that is able to describe the moment the universe was born, let alone what happened before that (whatever 'before' or 'born' would mean in this context). Neither is there a true widespread consensus about the second question. There

[1] As demonstrated by Nobel Prize in physics 2011 awarded to Perlmutter, Schmidt, and Riess [5].

is a common concordance, however, that seems to fit the current observational data: the accelerated expansion is caused by a very special form of energy, called 'dark energy', which –at least as our observations suggest– corresponds to 'vacuum energy'. Dark energy is the energy that resides in completely empty space. This sounds disturbing: why would there be energy stored in essentially nothing? The reason is that quantum mechanics has taught us that we cannot fully determine what happens on very small scales: empty space is not truly empty. For a brief moment of time, a particle–anti particle pair can easily pop into existence and disappear again in yoctoseconds. The continuous occurrence of this is one of the causes that the vacuum energy is non–zero. What is disturbing, in fact, is that this vacuum energy is extremely tiny; to get the energy equivalent of a small 9V battery, one would require a box of about 35.000 cubic kilometers filled with nothing but empty space. The stars thus seem to tell us a mysterious David and Goliath tale: the tiny amount of vacuum energy, determined by what happens on the tiniest 'quantum' scales, determines how the universe behaves on the largest 'cosmological' scales! Is there something about our universe that we can learn from this David and Goliath story?

The 'ultimate curiosity in physics' would be some physical theory that could provide answers to the aforementioned questions (and possibly more). This quest is far from finished, but there are indications that string theory could be on the right track (although much work remains to be done!). The curiosity in this thesis, however, is not directly about string theory, but rather about some of its possible 'imprints' that might touch our cosmology. We need very little of the full formalism of string theory; rather, we start 'cooking' with its ingredients and construct some string–inspired toy models. Of course, one has to be even more curious than this and ask whether these toy models can ultimately be embedded in string theory; the relevance of these toy models crucially depends on the (non–)existence of such an embedding. However, we do not explicitly discuss this. This introductory chapter provides –at the risk of oversimplifying– a very conceptual and informal outline of these ingredients. The (experienced) reader might skip this chapter without losing continuity of the story. As a disclaimer: we do not provide all the relevant references in this chapter but postpone some of them to later chapters where it is more natural to put them.

1.2 Unity in diversity

The idea of *unification* has always been a major way forward in physics and considerable effort has been put into pushing this idea. Unification between two (or more) phenomena means that it is –in fact– possible to view these phenomena as different manifestations of the same entity. Put differently, unifying means something like 'accommodating known physical laws into some more general, encompassing super–law'. This is progress at best: explaining why different phenomena share a common explanatory structure. Typically, unification points toward an understanding of nature on smaller and smaller length scales. Figure 1.1, which will be referred to manifold, gives an overview of the relevant scales in physics, from the smallest to the largest imaginable. The unification arrow typically points leftward. Unification is not necessarily to be viewed as the essential aim of physics; rather, it is an epiphenomenon of physical research.

'**Everything works out in the end.**
If it hasn't worked out yet, then it's not the end.'

— *Tracy McMillan*

Historically, the first major breakthrough regarding a unifactory practice happened in 1865 when Maxwell accommodated the phenomena of electricity and magnetism to one single consistent theory, electrodynamics. In the many decades that followed, new frameworks for physical theories 'beyond Newton' arose. At the beginning of the twentieth century, the emergence of quantum mechanics and special relativity drastically changed the view on matter, energy, space, time, etc. Quantum theory and Relativity marked the birth of modern physics, but yet were mutually inconsistent. The need for an encompassing framework was evident and the first foundations of a unification between Special Relativity and quantum mechanics were laid by Dirac in 1927. Further developments in that direction led to the coherent framework of quantum field theory, providing the basis of our current understanding of particle physics. Well–established examples of quantum field theories include quantum electrodynamics, unifying Maxwell's electrodynamics with quantum mechanics, and the descriptions of the weak and

strong interactions. In the 1960s, Glashow, Salam and Weinberg were able to provide a unified description of the electromagnetic and weak interaction as an 'electroweak' interaction. During this century, unification has thus led to an understanding of physics from human–like scales all the way down to subnuclear scales (about 10^{-18} m).

The electroweak and strong interactions make up the Standard Model of particle physics. This is currently the best description of fundamental particles that we have, in the sense that it can be experimentally verified. Up to TeV scales (near the dash–dot line in figure 1.1), its predictions are well–tested by numerous experiments to an astonishing accuracy [6]. The most remarkable example is probably the magnetic moment of the electron for which the agreement between theory [7] and experiment [8] is no less than 13 significant digits!

While the Standard Model is very appealing, it still does not offer a complete and fundamental description of elementary particles for several reasons. Most obviously, it does not include everything we know –or that we think we know– about elementary particles, such as neutrino masses and, more speculatively, dark matter. Besides this, the theory of not even 'self–contained' in the sense that it still requires the experimental input of its 19 free 'fundamental' parameters to make any prediction at all. The Standard Model is thus by no means a unique theory. Within the theory itself, there is also no obvious reason for why these parameters take the values they take (nor why they should not take these values). Of all these parameters, the Higgs mass is possibly the most provocative and the most difficult one to understand from theoretical considerations. Besides these completeness issues, the Standard Model can also not be truly regarded as a unifying theory. The sector of strong interactions is seemingly pasted 'ad hoc' to the electroweak sector; both forces now merely coexist within the same theory. Any further unification between these two interactions into some Grand Unifying Theory is still an unsettled issue. Admittedly, such Grand Unification is a rather speculative expectation (or desire even) of many physicists. In fact, statements about physics to the left of the dash–dot line inherently remain somewhat speculative since the highest energy scales that are experimentally available reach TeV scales. Lastly, the recently found discrepancies between Standard Model predictions and experimental

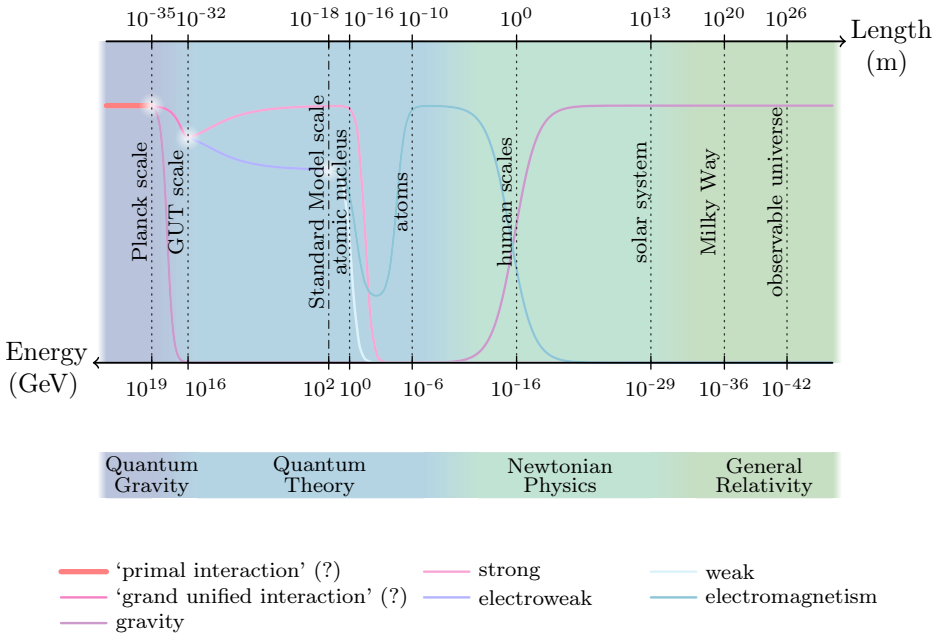


Figure 1.1: Overview of all the length scales and corresponding energy scales known in physics. The dash–dot line corresponds roughly to the experimental limit, put by the Large Hadron Collider (LHC). The color shading relates to the appropriate theoretical framework as indicated by the legend; blue colors refer to an inherently quantum framework, while green colors refer to a classical framework. The relative importance of the fundamental interactions is represented by the height of the background lines. Their intersections, marked by the bright spots, indicate a 'unification'.

measurements of the mass of the W boson [9] and the magnetic moment of the muon [10] could also provide indications that the Standard Model cannot even be the end of the particle physics story; awaiting any independent confirmation of these results, it might be possible that new physics is lurking at energy scales explorable by current particle accelerators.

Besides the interactions in the Standard Model, our current understanding of nature includes one more interaction that remained silent in the previous paragraphs: gravity. While Newton first described it hundreds of years ago, and while it is the first fundamental interaction known to mankind, it is still the one that is the least understood. In a relativistic framework, gravity is well described

by Einstein's General Relativity. In contrast to the Standard Model, General Relativity is a classical theory. Its predictions have been thoroughly tested by various experiments, see e.g. [11]. The most prominent prediction is possibly the existence of gravitational waves, first detected in 2015 [12]. They provide a window into the strong–field regime of gravity and allow for parameterized tests of the theory [13]. So far, no significant deviations from General Relativity have been found; it is our best, testable description of gravity from millimeter scales [14] onwards; it even accurately describes our universe on the largest, cosmological sales of about 10^{26} m! Since it is a classical theory, it can not offer a complete description of gravity the short–distance regime; it is expected to break down at very high energy, such as near a black hole's singularity or in the very early universe, when quantum effects become important.

'**Oh gravity, thou art a heartless bitch.**'

— *Sheldon Cooper* (in the Big Bang Theory)

The two paragraphs above seem to discuss a complementarity. On the one hand, we observe, to excellent approximation, that the universe seems to be governed by the laws of quantum mechanics and quantum field theory: all dynamical fields have quantum properties and live on a non–dynamical, Poincaré–invariant spacetime. On the other hand, the long–distance properties of gravity are governed by General Relativity: gravity is described by a classical, dynamical field that is sourced by all forms of energy, it is a theory of spacetime itself. In some regards, both theories hold each other's kryptonite. The Standard Model accurately governs the physics on small scales ranging from 10^{-3} m down to 10^{-18} m, while General Relativity governs the larger–scale physics from 10^{-3} m up to cosmological distances. The current cosmological model, which suggests that about a quarter of our universe consists of dark matter, is not at all realized in the Standard Model for the simple reason that the latter does not offer any candidate dark matter particle! If the ultimate dream is to have a unified or complete theory, an implementation of gravity into a quantum framework is absolutely necessary. However, this is easier said than done, since these two highly successful theories seem to be mutually inconsistent; gravity resisted many attempts that try reconciling it with quantum

theory. The conflict that arises is apparent: the dynamics of spacetime must be quantum mechanical in nature, or equivalently, the non–dynamical spacetime on which an ordinary quantum field theory is built does not exist. Clearly, some concepts in this view on quantum theory and gravity are to be changed and both theories should compromise somehow. It is to say, gravity really keeps us grounded.

'**Look here, not there!**'

The belief that there might be some greater, possibly unifying, theory overarching the well–established theories of today does not imply that, for instance, quantum field theory and General Relativity are suddenly rendered useless. On the contrary, one should think of these as being 'effective field theories': low–energy, coarse–grained approximations of some more fundamental theory. They provide a relatively simple description of certain phenomena at hand by putting the focus on that specific energy corner at which the phenomena occur. The decoupling principle lies at the heart of their simplicity: what happens macroscopically is a direct consequence of what happens microscopically, nevertheless, the microscopic description is often superfluous and contains unnecessary degrees of freedom. Put differently, the decoupling principle is a statement that physics seems to be organised in scales: physics on one scale does not depend sensitively on the details of the physics on another scale.

To exemplify using figure 1.1: when studying the planetary orbits in the solar system, the dominant interaction is gravity and the fundamental objects are planets. The effective theory that describes this fairly well is classical Newtonian gravity. There is thus no need to consider, for instance, the electromagnetic interaction or to analyze the planets at atomic level. The details of the physics at smaller scales is irrelevant for the problem at hand. This principle of decoupling is really strong, important, and not to be underestimated: it is the main reason that progress in science is possible, since it shows we can understand phenomena at a certain energy scale without having to understand everything else. The price to be paid when using such an effective theory approach, however, is that at smaller length scales effective theories typically break down since –by their very definition–

they do not capture the correct physics away from the energy corner zoomed into. Nevertheless, when we ask sufficiently simple questions, effective theories provide sufficiently accurate answers.

All of our well–established theories today (the Standard Model, General Relativity, etc.) are thus in principle to be interpreted as effective theories. An estimate for the energy scale at which all of them surely break down is near *the Planck scale*, where gravity 'strikes again'. Conceptually, at energies near the Planck scale, quantum theory is –definitely– plaguing General Relativity; it can be understood as the energy scale at which the Schwarzschild radius $r_{\rm s} \sim 2Gm/c^2$ of a particle of mass m, characterizing strong–gravity effects, becomes comparable to the particle's own Compton wavelength $\lambda_{\rm c} \sim h/mc$, characterizing quantum effects:

$$E_{\rm P} \sim \sqrt{\frac{\hbar c^5}{G}} \sim 1.22 \times 10^{19}\ \text{GeV}. \tag{1.1}$$

This corresponds to the extremely tiny length scale $L_{\rm P} \sim 1.62 \times 10^{-35}\,\text{m}$, commonly referred to as the Planck length. This is a rather conservative estimate and it may very well be possible that quantum effects of gravity already kick in at lower energies. After all, there is in principle no reason to keep trusting the Standard Model above a TeV scale where its validity has not been experimentally verified. Whatever physics is hiding near the Planck scale, is neither captured by the Standard Model nor by General Relativity: we lack a UV[2] completion of both theories! Only a fully consistent theory of quantum gravity will be able to unravel the mysteries lurking at the Planck scale, but it seems, unfortunately, far from straightforward to construct such a theory. Multiple, often mutually orthogonal, attempts exist. Given that spacetime is dynamical according to General Relativity, one attempt is to quantize spacetime itself. Other approaches, such as string theory, start from a completely different quantum theory. Whatever the path chosen by nature is, it is the hope that quantum gravity requires a unification between gravity and the other interactions.

[2]After ultraviolet, meaning 'in the high–energy regime'. Conversely, the IR (after infrared) denotes the low–energy regime.

‘**Then God said, ‘Let there be strings’.**’

— *Genesis 1:3* (almost)

The most promising candidate theory that realizes this hope is possibly string theory. It unifies all forces and particles in a deep and consistent way: their fundamental description is no longer as a quantum field or as a point particle, but rather as a quantum string of a typical size, set by the string scale. Particles appear as vibrational modes of a string: at low energies, there is no sufficient resolution to see the spatial extent of a string. String theory thus naturally accommodates a vast variety of particles given the many ways a string can vibrate. There is evidence that this large tower of particles could constitute gauge groups large enough to accommodate the Standard Model (although this has not been accomplished yet). Furthermore, this tower must also include a graviton for consistency; gravity is inevitable and takes place beside the other fundamental interactions in a unified way. The fields, that one is used to in quantum field theory or General Relativity, are emergent in string theory. Since by construction it is a quantum theory, string theory was proposed as a candidate theory of quantum gravity by Scherk and Schwarz in 1974. In contrast to other attempts to quantum gravity, string theory is not a ‘quantization of General Relativity’; rather, it should be viewed as a successor to it. In the same spirit, the most correct way to look at string theory is probably that it also forms an extension of the Standard Model, including gravity, to a higher energy scale; it tries to make the left region of figure 1.1 accessible (at least theoretically).

While being very promising, string theory by itself is not a unique unifying theory (but different in the sense that the Standard Model is not unique). In the mid–1980s, no less than five different consistent string theories were discovered. This seems odd with the idea that string theory realizes the hope of being a unifying theory. In the years that followed, however, it was found that these theories were not at all unrelated to one another; instead, they are interconnected through a web of dualities. In 1995, Witten claimed that there is a ‘mother’ theory of all the string theories at hand; they were just specific corners of a single, even more unified, theory dubbed M–theory. While it all sounds very promising that M–theory could be a true, unique, unifying theory, it is still in its infancy and remains quite

mysterious up until this day; mysterious in regard to its mathematical structure, mysterious in regard to its formulation, and –completely– mysterious in regard to its predictions and its experimental verification.

What do we hope to achieve?

From a purely theoretical point of view, string theory is very appealing, as it is a consistent description of quantum gravity that furthermore comes close to fulfilling the ultimate dream of unification. Upon closer inspection, string theory appears to be 'difficult'. Historically, string theory was only understood perturbatively, but with the advent of AdS/CFT and the string dualities we get a window into its non–perturbative nature as well. Much of the work done in string theory is performed in the so–called point–particle approximation, where all length scales of interest are taken to be large compared to the string length. Just as Maxwell's electrodynamics arises as a low–energy effective field theory from quantum electrodynamics, so does string theory at low energies (i.e. in the point–particle limit) reduce to supergravity. This is a classical field theory that forms a supersymmetric[3] extension of Einstein's gravity. Many string theorists are thus actually focusing on supergravity. This opens opportunities –at least those understood corners– to address some of the fundamental questions in General Relativity and quantum theory. In this thesis, we are concerned about two questions in particular that we briefly outline below.

The occurrence of spacetime singularities is rather generic in General Relativity. A prime example is the initial singularity of the universe, the Big Bang. At very early times, all matter in the universe was crushed into a particularly small region with an extremely large density, thereby reaching energies near the Planck scale. What is the nature of these singularities? How should one interpret them? From the classical perspective, this is completely unclear. Since their formation requires that matter is being crushed to a region smaller than the Compton wavelength, it is the hope that a fully consistent theory of quantum gravity might resolve these pathologies. If quantum gravity effects indeed took place during these early stages, their data might be scattered all around us.

[3]The premise of supersymmetry is that every fermion has a boson counterpart and vice versa.

The equivalence principle applied to a quantum field theory implies that all fields must interact gravitationally. This has a particular consequence: vacuum energy 'feels' all complicated small–scale physics inherent to particle physics, but it must also gravitate at the same time. What is disturbing, in fact, is that the gravitational interaction on cosmological scales seems to suggest the presence of an amount of vacuum energy that is far below what one would expect[4] from quantum field theory considerations. This is *again* the David and Goliath story! In string theory, particle physics seems to be inherently tied to cosmology, and it is the hope that the theory might resolve the pathological mixing between small–scale and large–scale physics.

We have to mention that in this thesis we do not approach these problems from a 'top–down' perspective, which might have appeared to be the case from the above discussion. Rather, we approach quantum gravity from the other side: by combining insights of quantum mechanics with General Relativity, we attempt to understand some of the implications once specific string theory ingredients are added. In light of the correspondence principle, this could provide some insight into the aforementioned questions.

1.3 Stringredients

This section introduces some of the key properties of string theory that are relevant in the context of this thesis. String theory is a vast subject and we do not aim to go into its details; instead, we choose to give a simple and simplistic discussion and to remain conceptual. This proves to be sufficient for later chapters.

String theory is a unifying theory, an explicit quantum model of gravity and Yang–Mills forces. A remarkable aspect of the theory is its self–containedness and completeness; many things just come 'for free'. Firstly, in contrast to particle theories, string theory is free of any UV divergences, a fact that can be traced back to the finite extent of string. Furthermore, the possible interactions amongst the particles in a low–energy limit are dictated, making it therefore much more robust than any field–theoretical description. A second remarkable feature is that the

[4] I really do mean 'expect': a physicist's gut feeling.

string scale is the only dimensionful parameter in theory and that everything is in principle expressable in terms of this one single parameter: string theory has the potential to explain low–energy parameters that were otherwise left unexplained in quantum field theory (such as particle masses). Lastly, while the fundamental entity is one–dimensional, Polchinski claimed in 1995 that string theory must democratically also involve fundamental, higher–dimensional membranes (coined D–branes) that come with their own dynamics. Sting theory cannot only be about strings, it has to be about membranes too! Conceptually this makes sense: open strings can end on membranes and their rigidity is ruled out in a gravitational theory. Below we discuss how we can start cooking with these 'free ingredients'.

'Gratis bestaat niet.[5]'

— *Jan Tielemans*

String theory requires, besides supersymmetry, ten spacetime dimensions. This is fixed by the mathematical consistency of the theory and should be appreciated since no theory has been capable of making such prediction. However, can such a theory at all account for the reality we see in our everyday experience? In particular, supersymmetry seems to be absent (broken) in any situation of phenomenological interest. It should be recognized, though, that supersymmetry merely serves as a computational tool; when broken, we quickly lose calculational control. In principle –if one has infinite computational power– supersymmetry should thus not form an obstruction to construct our world from string theory. That being said, the excess of spatial dimensions could be much more bothersome. Firstly because it seems to be in direct contradiction to our everyday experience: where (and how and why) are the six residual dimensions of string theory hidden? Secondly, since gravity can permeate throughout all spacetime, including the extra dimensions, why do we still see the typical Newtonian gravitational force with its characteristic inverse–square law?

[5] A wisdom of my father. I think he took inspiration from the English variant 'There ain't no such thing as a free lunch'.

Some of the 'free things' in string theory thus come at a cost: how does the theory make contact with observations? Any theory that involves more than four spacetime dimensions, such as string theory, should allow for configurations that represent our world, i.e. that solve the two aforementioned issues: at the energy scales we have access to (LHC, table–top experiments, astronomical observations), we should perceive an effective four–dimensional world without supersymmetry. We will, however, not comment on supersymmetry breaking, but focus entirely on the issue of hiding extra dimensions. As a matter of fact, there are two main ideas that make such a hide–and–seek scenario of extra dimensions plausible and consistent with what we see, or rather, what we do not see.

'**Living large**'

A first proposal of how extra dimensions could hide dates back to Kaluza and Klein [15,16] from the 1920s. It is based on the conceptual idea that extra dimensions are simply 'too small to be observed'. In this so–called *compactification* scenario, one looks for solutions to the higher–dimensional equations of motion in which the extra dimensions span a compact manifold of a tiny size. The typical size of this so–called internal manifold is commonly referred to as the Kaluza–Klein or compactification scale. This scenario is conceptually illustrated in figure 1.2a for a compactification of one dimension to a circle. The smallness of the extra dimensions guarantees that only high–energy experiments provide the necessary resolution to probe the extra dimensions: spacetime thus effectively appears four–dimensional for observers that perform experiments at sufficiently low energies, i.e. at distances exceeding the compactification scale. Given the operational energy of the LHC, and the fact that no signature of any extra compact dimensions is found, their size is constrained to be smaller than at least 10^{-18} m.

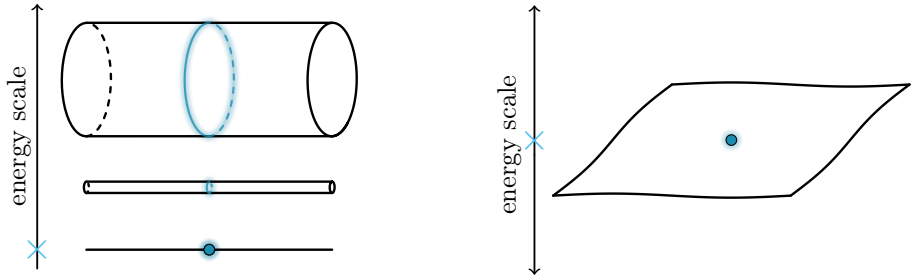
Particles, as we perceive them at low energies, correspond to field excitations that are smeared out over the extra dimensions. Gravity, being the dynamics of spacetime, manifests itself as ordinary gravity in the lower–dimensional theory *and* as the dynamics of the extra dimensions (in figure 1.2a this would correspond to the cylinder not having the same circumference everywhere). For phenomenological

purposes, it is absolutely required that the compactification is stable: the compact manifold can not indefinitely grow to a large size (or decompactify), nor can it entirely collapse. It is highly non–trivial to find such stable configurations in a theory with extra dimensions. This is the main issue for these scenarios: finding a stable compactification.

'**Living on the edge**'

At the turn of the millennium, Randall and Sundrum [17, 18] proposed a different scenario as an alternative to compactification. Their idea is that our four–dimensional world could simply be a membrane that is embedded in some higher–dimensional bulk spacetime, hence the name '*braneworld*'. This is illustrated in figure 1.2b. In these scenarios the extra dimensions are not necessarily small, nor compact. On the contrary, they can be even infinite in extent. The wave function of a particle is strongly peaked around the membrane, effectively localizing it onto the brane. Similarly to Kaluza–Klein theories, the extra dimensions can only be detected by high–energy experiments: high–energy collision of confined particles can produce unconfined particles that disappear from the brane. This scenario is made manifest by string theory. Branes naturally exist and localization of (Standard Model–like) fields arises spontaneously: open strings end on D–branes and their end points are by definition localized states on the brane.

Insofar as these matter fields and their mutual interactions are concerned, the extra dimensions are invisible at low energies. The main issue for these scenarios is gravity. Gravity, in contrast to the other interactions or the other matter fields, is sensitive to the entire spacetime; it must persist everywhere to have a consistent theory! This is a crucial restriction for many constructions: at low energies, or equivalently at sufficiently large distances, gravity should effectively be described by four–dimensional gravity as we know it. In contrast to Kaluza–Klein scenarios, this is highly non–trivial to achieve. One option, as pursued by Randall and Sundrum, is that gravity can be 'trapped' on the brane. Put differently, there exists a graviton wave function that is strongly peaked around the brane, which would mediate the effective four–dimensional gravitational force on the brane.



(a) Compactification. One dimension is curled up into a small circle, which becomes 'invisible' at low energies. This effectively removes this one dimension. Particles (blue) are being 'smeared' over the extra dimension.

(b) Braneworld. The orthogonal direction becomes 'invisible' at low energies. Particles (blue) are strongly localized on the brane.

Figure 1.2: Two common approaches to hiding extra dimensions. The cross '×' marks the energy scale we can do experiments at.

'**E pluribus unum (?)**'

A compactification or a braneworld? It is not yet clear what the path chosen by nature is, and it might even be a combination of both scenarios or something completely different (this option is explored later in this thesis, chapter 4). At first sight, braneworlds look very appealing from the perspective of string theory which naturally presents the necessary ingredients. At the same time, it turns out to be difficult to concretely realize them in a top–down construction. Most attempts in constructing lower–dimensional models from string theory focus on compactifications rather than on braneworld scenarios. For the remainder of this chapter, we also focus on the former: is there a compactification down to four dimensions that describes our universe?

Since string theory is a unique theory, it could have been the hope that it singles out some unique and some very specific four–dimensional configuration that could serve as a realization of our universe. While this might in fact be true, there is little to no evidence for this claim. Instead –at least within our current understanding of the theory– early work suggested that a shedload of consistent and inequivalent four–dimensional theories can exist within string theory. Support for

this claim comes from the vast amount of allowed six–dimensional compactification spaces to compactify on; the extracted low–energy physics (particle content, interactions among them, etc.) crucially depends on the choice of compact space. Compactifications thus naturally entail an arbitrariness and, at this point, there is no principle that would suggest some preferred choices. Early estimates yielded about 10^{500} possible effective four–dimensional theories. This extremely large amount of possibilities is called the *string landscape* [19]: the collection of all possible four–dimensional physics one can get from compactifications of string theory. This (in)famous number almost became a joke in string theory and it has to be said that not everyone believes in the existence of a landscape [20].

While the existence of such a huge landscape might seem undesirable, the situation in ordinary quantum field theory is not much of a difference. In fact, there are infinitely many fully consistent quantum field theories and the Standard Model is just one of them. The advantage of the landscape, to be argued below, is the fact that it becomes calculable –at least in principle– what the low–energy quantum field theory actually is. Most of these do not have any chance to describe our universe, yet on the other hand with 10^{500} options (or more) how hard can it be to find but one that approximately describes our world?

'**Yes, how hard can it possibly be?**'

— *Me* (almost every time trying to do a calculation)

Ironically, it turns out to be *really* difficult to embed our universe (at least in our current understanding of it) in string theory and it is seriously being debated whether this is possible at all. Current research is not only about finding that needle in the haystack, but also about that needle's sheer existence. If a compactification is to describe our universe, it should (at least) meet the following requirements:

- the compact manifold should be stable, or at least metastable on cosmological time scales[6], and the compactification should exhibit a clean scale separation with three large spatial dimensions and six small ones;

[6]This means the compactification should be able to exist for about 13.6 billion years, roughly the current estimated lifetime of our universe, before collapsing or decompactifying.

- on the (small–scale) particle physics side, the physics should be described by the Standard Model and supersymmetry should be broken;
- on the (large–scale) gravitational side, the four–dimensional effective theory should fit the present cosmological data.

Is it obvious that string theory can do this?

Once a compact manifold has been chosen, the higher–dimensional gravitational dynamics dictates that its size and shape can vary throughout the lower–dimensional space. The size and shape of the internal space are described by so–called *moduli*. An example of a modulus could be the circumference of the cylinder in figure 1.2a as a function of the horizontal position. Moduli are not *a priori* fixed parameters, but, instead, descend naturally to a set of scalar fields, here denoted ϕ, in the lower–dimensional theory with a dynamics of their own.

Given the possible complexity of six–dimensional compact spaces, there are numerous moduli fields –typically thousands of them– in a generic compactification. This proliferation is rather problematic. For starters, moduli endanger the recovery of the desired small–scale physics, since they could give rise to a 'fifth interaction' which is clearly absent in the Standard Model. Secondly, the requirement that the internal space should be stable with a clear separation of scales, implies that the moduli fields should be stabilized themselves and are not allowed to fluctuate very fiercely. What this thus advocates for, is the need for a potential energy $V(\phi)$ that admits local minima that are sufficiently 'narrow' in which the moduli can safely 'rest'. Figure 1.3 provides a conceptual illustration of the possible shapes a moduli potential for a single modulus can take (in reality with a multitude of moduli, the potential is much more complicated, but its behaviour is qualitatively analogous as in the figure). Stable (or metastable) compactifications correspond to the (local) minima, indicated in blue and green. These are called *vacua* and are able to stabilize the moduli fields. It is also possible for the potential to feature a runaway behaviour which signals a decompactification or a collapse of the compactification manifold. To reconcile the two aforementioned issues, the potential should be able to generate a sufficiently high mass for the moduli, at least above TeV scale (but probably much higher). In the figure, this corresponds to vacua that are sufficiently

'narrow', i.e. of type *b,c,e*. Configurations in which the moduli are fixed[7] inside a vacuum state provide solutions to string theory, also called *vacua of string theory*. These are low–energy effective theories whose parameters are determined by the vacuum expectation[8] value of the moduli. A single vacuum of string theory is thus characterized by the choice of compact space and the position in the moduli potential. As alluded to before the belief is that there exists a vast landscape of these vacua. A specific vacuum, i.e. particular compactification determines the properties of the low–energy effective theory (particle masses, coupling constants, etc.). So while string theory a priori has no free parameters, the moduli fields provide some freedom and arbitrariness in the effective theory. This would mean that the constants of nature, as in the Standard Model, are not just 'God given', nor 'fundamental'. Rather, they are rather dynamical quantities that may take different values, but –in principle– these are computable once a certain compactification is found; they have a certain value for a good reason.

The problem of moduli stabilization relates to the question of what might generate such a potential $V(\phi)$. It is certainly not for free and some 'extra effects' are required to generate a potential. Among these are quantum corrections to the potential which can generate masses for the scalars. Unless in very specific circumstances (such as when the vacuum has unbroken supersymmetry or when the scalar masses are protected by a symmetry of some sort), massless scalars do not appear in practice. While this might seem an evident way out, it certainly is not. Our capabilities to compute quantum corrections are limited to the so–called weak coupling regime, where supergravity is an accurate approximation for string theory. When quantum corrections are sufficiently large to generate a local minimum of the potential, one typically leaves the weak coupling regime and loses computational control. This tautology is known as the Dine–Seiberg problem [21]. Luckily, string theory itself provides other 'extra effects' that could serve as a way around this issue. Firstly, branes are inherent to the theory. These objects can wrap around the extra dimensions and serve as 'elastic rubber bands' that make it

[7] Minima of the potential that correspond to supersymmetric configurations (of the type *a* in figure 1.3) do not have their moduli stabilized in the sense argued for in the text. Nevertheless, these are still called supersymmetric *vacua* of string theory.

[8] One should consider the expectation value since the moduli can quantum mechanically fluctuate.

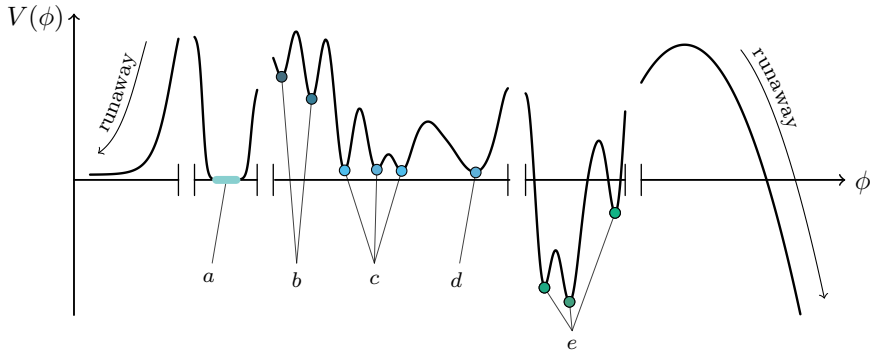


Figure 1.3: Artistic representation of various shapes of a modulus potential $V(\phi)$. Runaway behaviour typically signals an instability where the moduli cannot be stabilized. Moduli stabilizaton is the quest for local minima. There exist different kinds of vacua and phenomenology amounts to finding a minimum that could describe our universe:

- supersymmetric vacua (a) forming a flat plain with massless moduli
- de Sitter vacua (b–d) with a positive vacuum energy that could potentially describe our universe, but
  - (b) has a too large vacuum energy
  - (c) has a sufficiently low vacuum energy, but there could be many distinct vacua with a radically different particle physics content
  - (d) has the moduli not sufficiently stabilised
- anti–de Sitter vacua (e) with a negative vacuum energy

more difficult (or easier, depending on the type of brane) for the dimensions they wrap to expand. Besides these objects, the string spectrum contains many fields in the low–energy regime. In particular, it contains 'generalised' electromagnetic fields. Like Maxwell's electromagnetic field couples to particles (particles can be electrically charged and produce electric fields), these generalised gauge fields couple to the fundamental objects of string theory. Strings, or more generally branes, can thus carry some generalised charge. These fields can acquire a non–zero vacuum expectation value, even in the absence of the charged brane sources; it thus becomes possible to consider vacuum configurations in which some fluxes thread the compact manifold. Generically fluxes tend to make it more difficult for the extra dimensions to collapse. This picture is intuitively clear: it costs energy to push field lines together. The advantage of these two 'extra effects' is that they generate a potential for the moduli already at the classical level.

The existence of such a moduli potential with sufficiently sharp local minima would thus provide a tool to meet the first requirement, but it far from guarantees that the second requirement is also met. The absence of a long–range fifth force, that would be mediated by massless moduli, does not imply that the matter content of the low–energy theory matches (or extends) the content of the Standard Model. On the contrary, it generically will not. In figure 1.3 this would correspond to the different vacua of type c. It is still unknown how many vacua –if any at all– exist that approximately describe the Standard Model. Since the interest in this thesis is in cosmology, we do not further discuss this particle physics side.

The real struggle is the third requirement for a vacuum to describe our universe: does it match with the cosmological data we currently have? In light of the decoupling principle, this is really surprising, puzzling even: cosmology is about the largest scales, while string theory informs us about the smallest scales. How could cosmology then possibly depend on the details of string theory? In fact, there are many reasons to believe that in string theory the small–scale particle physics side seems to be inseparable from large–scale cosmology side [22]. One of the reasons is that string theory allows for a precise calculation of the vacuum energy (influencing the large–scale physics) of a certain compactification using the ingredients of string theory (small–scale physics) such as branes, fluxes, compact manifolds, etc. The vacuum energy of a certain compactification corresponds to the value of the moduli potential evaluated at that vacuum. The interplay between the UV and IR stems directly from the moduli potential. For instance, 'adding particles' by letting branes wrap the compact manifold generically affects the moduli potential and *hence* the vacuum energy. String theory thus forms an exception to the decoupling principle.

The vacuum energy forms the first and most important test for a string vacuum: a compactification should describe a universe with a tiny positive value for the vacuum energy in order to have any chance to describe our world. A universe with positive (constant) vacuum energy is called a *de Sitter* universe, indicated in blue in figure 1.3. If the vacuum energy is negative it is called an *anti–de Sitter* universe, indicated in green. Cosmological data suggests we inhabit a de Sitter universe. Vacua of type e are then unacceptable for phenomenological purposes.

Vacua of the de Sitter type are not always admissible either: their vacuum energy could be too large (as in *b*), or, if sufficiently low as in *c* or *d*, it is not evident at all that the second requirement is met.

'**As we know, there are known knowns...**
We also know there are known unknowns...
But there are also unknown unknowns...'

— *Donald Rumsfeld*

An interesting phenomenological question is whether string theory allows for de Sitter vacua. Preliminary calculations in flux compactifications, such as the model of Bousso and Polchinski [23], suggested a huge number of possible flux choices which all give rise to de Sitter vacua with densely spaced values for the vacuum energy (type *c*). This fed the belief that string theory allows for a whole landscape of de Sitter vacua. Although these immature calculations seemed promising, it is far from established that string theory actually admits such a landscape of de Sitter vacua. On the contrary, string theory seems to be rather resistant to de Sitter vacua; not even a single one has been rigorously constructed, let alone a whole landscape [24]. One of the difficulties in constructing de Sitter vacua is that these are never supersymmetric (supersymmetric vacua always have vanishing or negative vacuum energy) and computational control is thus easily lost. To complicate matters even more, there are several no–go theorems asserting that certain specific constructions can never yield a de Sitter vacuum. Although these are not all–encompassing statements and although there are ways to avoid them, the theorems at least give a flavor of the idea that string theory might be conspiring against de Sitter vacua. Almost provocatively, anti–de Sitter vacua seem to be rather abundant and very well understood.

Historically, much work in string theory has been done from such a top–down approach, but, at the same time, it also appeared rather cumbersome. It seems to be a needle in a haystack: one needs to 'pick' the right compact manifold and right fluxes, branes etc. to find one candidate vacuum we could be living in. Furthermore, we also lack the machinery to find that needle: it is computationally extremely difficult to reliably carry out the compactification prescription. On the other hand,

there is also no known top–down mechanism which suggests a privileged choice of a compactification. At least within our current understanding, string theory does *not* offer any satisfactory reason for why our universe exhibits four large extended dimensions and six small ones. Indeed, such configurations seem highly contrived and far from natural. Yet, these are always used as an essential input in phenomenological compactifications. This poses an interesting question (see chapter 3): what is the reason behind scale separation?

'What are you doing in my swamp?'

— *Shrek*

Almost inspired by the many things we do not fully understand about top–down constructions, the converse question appeared in 2005, causing a major shift in paradigm [25]: is there a bottom–up way of understanding the effective low–energy theories that can arise from string theory? Given the proclaimed huge number of vacua and the appearance that anything seems plausible in this vast landscape, should we really worry about top–down constructions? Phrased differently, can we 'order' physics from string theory? This, however, does not seem to be the case: string theory is certainly constrained in what low–energy theories might emerge. Despite the richness of the landscape, not everything is possible and a bottom–up approach *most likely* does not work; it is a false hope that any effective field theory (such as the Standard Model), when coupled to gravity, has a sensible UV completion. Hence in contrast to the landscape, one defines *the swampland*: the collection of apparently[9] consistent low–energy theories that cannot be completed into a consistent theory of quantum gravity in the UV [26]. This is illustrated in figure 1.4: the landscape appears to be surrounded by an even bigger swampland.

The swampland program is about the delineation of the border between theories that fall in the swampland and ones that fall in the landscape. The goal is to formulate a set of criteria that a low–energy theory must satisfy in order to be extendable to a UV complete theory of quantum gravity. These criteria are intimately linked to gravitational considerations and capture the UV imprint of

[9]This means the theory is not faulty by itself at low energies but becomes inconsistent at high energies when coupled to gravity.

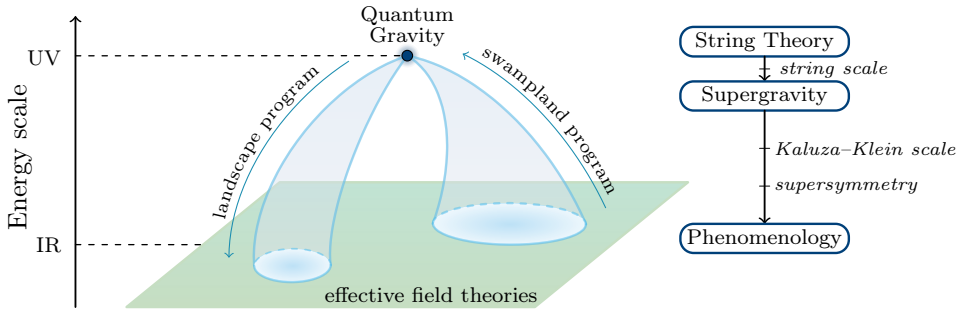


Figure 1.4: An artistic representation of the set of all effective field theories. Only a subset, called the landscape (blue), can arise from string theory. All other theories are said to be in the swampland (green).

quantum gravity on the IR. This again demonstrates that string theory breaks with the decoupling principle. Ultimately, the criteria should be rigorously derived from a top–down perspective, but this turns out to be extremely difficult, if not impossible. For this reason, one often relies on patterns that emerge in reliable string vacua, or on lessons from semi–classical gravity. In that regard, making strong claims is seldomly possible and one rather speaks about swampland *conjectures*, unproven statements for which there exists –some degree of– evidence. The program is still in its infancy and remains somewhat heuristic, and some conjectures are more controversial than others.

One example of a well–established swampland conjecture is the severe restriction on conserved quantities that might exist in a low–energy theory. The conjecture claims that such conserved charges can*not* exist unless there are interactions, directly associated with the charge itself, amongst charged objects.[10] For instance, electric charge may exist in a consistent theory of quantum gravity: the amount of electric charge of an isolated system is conserved and the electrically charged objects interact through the electromagnetic interaction. An example of a conserved quantity that may not exist in a quantum gravity theory is the total lepton

[10]This is the conjecture that there do not exist global symmetries in a consistent theory of quantum gravity [27]. Gauge symmetries, on the other hand, may exist.

number[11]. This is a perfectly conserved charge in the Standard Model, but there is no interaction that couples directly to the lepton number (leptons can interact electromagnetically, but this force couples to the electric charge of the lepton, not directly to the lepton number).

As discussed above, de Sitter vacua are notoriously difficult to construct in a top–down way. This could find its origin in an obstruction that is either purely technical, or either in an obstruction that is deeply rooted within string theory itself. Recently, this has been cast into a –controversial– swampland conjecture [28, 29]: the moduli potential $V(\phi)$ does not allow for local minima with positive vacuum energy. This conjecture is far from being proven and not everyone agrees; it remains an open question whether de Sitter spaces are in the landscape or in the swampland. We remain agnostic about this, but we think it is important to stress this possibility. If de Sitter vacua turn out to be in the swampland, how would this be reconcilable with our cosmological data suggesting the current expansion of the universe is driven by a constant vacuum energy? Would this mean string theory is just wrong? Before jumping to such a drastic conclusion, one could examine at least two options if it turns out that de Sitter vacua are not realizable in string theory.

While our cosmological data is consistent with the idea that the current expansion of our universe is driven by vacuum energy that is constant in space and time, this idea is by no means sacrosanct. Experimental data should never tell a theory what to predict or not! Physics is about the converse: can we build a model that accurately enough describes the world we observe? If de Sitter is swampy after all, string theory would inform us that there does not exist any universe in which the expansion can be driven by a *constant* vacuum energy, but it in principle has no obstruction to a *dynamical* vacuum energy. In these scenarios, called quintessence scenarios, not all the moduli fields are fixed in a local minimum of the potential but (part of them) are rather rolling slowly over the potential (this could for instance happen in the runaway parts of figure 1.3) making the vacuum energy time–dependent. It should be noted that this weakens the requirement of moduli stabilization discussed before. Furthermore, unstabilized moduli are not

[11] This counts the number of leptons (electrons, muons, taus, and their respective neutrinos) minus the number of anti–leptons.

necessarily in contradiction with experiments: if the rolling moduli do not couple to the visible matter fields like the ones in the Standard Model, but rather to some invisible 'dark[12]' sector, there is no contradiction. Although much work has to be done, this could be a viable alternative to pure de Sitter.

Another option is to indeed accept that current cosmological data points to a constant vacuum energy and find a way to reconcile this in string theory while at the same time accepting that string theory does not allow for de Sitter vacua that come from compactifications. An example of an alternative scenario is braneworlds. The power of braneworlds is that de Sitter space can be realised on top of the brane: observers that are confined to the brane would measure a constant vacuum energy. Although no concrete braneworld in string theory has been found and much work has to be done, this equally sounds like a viable alternative to pure de Sitter.

Much more can be said about string theory, the landscape, and the swampland. However, at this point, we have established the aim of this section: conceptually outlining the basics that appear in the following chapters of this thesis. We therefore stop the discussion here, hoping to have contented the reader.

[12]This might seem speculative, but, in fact, even the Standard Model contains a dark sector: right–handed neutrinos are completely sterile and do not interact with the Standard Model forces, but only via gravity. It is thus not unimaginable that a dark sector arises from string theory (in fact, we better have one to account for dark matter).

Chapter 2

Classical and Quantum Cosmology

At the largest scales, our universe seems to resemble a de Sitter space. This is a model in which the universe is dominated by a constant form of vacuum energy: a cosmological constant. Cosmology and vacuum energy play a prominent role in this thesis and this chapter provides the necessary background material. A minimalistic introduction to cosmology is provided in section 2.1; for a proper introduction to the topic, the reader is referred to the literature (see e.g. [30–33]). Section 2.2 discusses how vacuum energy –in particular its origin– poses one of the big questions[1] in cosmology: it seems to be 'unnaturally small' from a quantum field theory point of view. This is the (in)famous cosmological constant problem [34] and serves as the main motivation for chapter 3. In section 2.3 it is outlined how quantum mechanics and cosmology combine into the framework of quantum cosmology. This mainly motivated by the quest of cosmological initial conditions, but we aim to provide an introduction to the topic that is largely restricted to well–debated aspect of how to compute the quantum state of the universe (for more details, see e.g. [35–37]). The presented treatment is sufficient to make contact with the quantum cosmological aspects of chapters 3 and 4. Another (puzzling)

[1] The question itself might also be questioned...

quantum aspect of cosmology is that it appears to be possible for our universe to quantum mechanically change topology and 'give birth' to a baby universe through so–called Euclidean wormholes [38]. This is the topic of chapter 5.

2.1 Classical cosmology

Soon after Einstein published his theory of General Relativity in 1915, the question arose whether this new gravitational theory could describe our universe as a whole. In the decades that followed, pioneering work by de Sitter, Einstein, Friedmann, Hubble and Lemaître provided the foundations for the so–called *Standard Model of Cosmology* or *the Hot Big Bang model* that we know today. It accurately describes our universe all the way back to early times. Our aim is to give a brief overview of its basic concepts, for further details the reader may consult the aforementioned references.

The three main pillars on which the model is built are

- the Copernican principle, asserting that we do not occupy some special or privileged position in the universe;
- the assumption that the gravitational dynamics of the universe is correctly described by the Einstein equation, and the discovery of Hubble in 1929 that the universe expands;
- the hydrodynamical, perfect–fluid model of matter and energy.

At very large scales (about 10^8 ly) and on a very coarse–grained level, the Copernican principle holds. It implies that at these scales the universe is approximately spatially homogeneous and isotropic[2] [39]. This high degree of symmetry restricts the form of the metric to be of the Friedmann–Lemaître–Robertson–Walker (FLRW) type

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -N^2(t)\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\Sigma_3^2, \tag{2.1}$$

[2]Homogeneity means that there no preferred points and isotropy is the statement that there are no preferred directions.

where $N(t)$ is a lapse function, $a(t)$ is the scale factor describing the expansion or contraction of the spatial slices whose metric $\mathrm{d}\Sigma_3^2$ is, by virtue of the first pillar, a maximally symmetric Euclidean metric. There are three possibilities: a negatively curved space (open universe) where $\mathrm{d}\Sigma_3^2$ is the metric on the hyperboloid $\mathbb{H}^3$, a space with vanishing curvature (flat universe) where $\mathrm{d}\Sigma_3^2$ is the metric on flat space $\mathbb{R}^3$, and a positively curved space (closed universe) where $\mathrm{d}\Sigma_3^2$ is the metric on the sphere S^3. Two common coordinate systems are

$$\mathrm{d}\Sigma_3^2 \equiv \gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = \frac{\mathrm{d}\rho^2}{1-k\rho^2} + \rho^2\,\mathrm{d}\Omega_2^2 = \mathrm{d}\chi^2 + \rho_k^2(\chi)\,\mathrm{d}\Omega_2^2, \tag{2.2a}$$

where $k \in \{-1, 0, +1\}$ labels the curvature of the spatial slices (respectively an open, flat, or closed universe), $\mathrm{d}\Omega_2^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ is the metric on S^2, and $\rho_k(\chi)$ is the function given by

$$\rho_k(\chi) = \begin{cases} \sinh\chi & \text{if } k=-1 \\ \chi & \text{if } k=0 \\ \sin\chi & \text{if } k=+1 \end{cases}, \tag{2.2b}$$

Time reparametrization invariance is made manifest in (2.1) by the presence of an arbitrary lapse function $N(t)$. There are two convenient gauge choices to work with. In the proper–time gauge, where $N = 1$, t corresponds to *cosmic time* or *proper time* that is measured by comoving observers. In the conformal–time gauge, where $N = a$, t describes *conformal time* and the metric becomes conformal to Minkowski space. The time coordinate t is often relabelled by τ (proper time) or η (conformal time) respectively.

Two comments about the FLRW metric are in order. First, the lapse function and scale factor appear squared in the metric. This means that the physics is insensitive to the absolute sign of N or a. Typically, one identifies a with $-a$ and N with $-N$. Secondly, the spatial metric (2.2a) does not provide any global information. For instance, in the $k = 0$ case it describes the metric on the infinite $\mathbb{R}^3$ space as well as on the compact torus $T^3 \cong \mathbb{R}^3/\mathbb{Z}^3$.

The FLRW metric (2.1) describes a universe in which the observed distance between two galaxies may vary with time (ignoring their relative peculiar motion). In particular, the physical distance r between two spatial points separated by a coordinate distance χ is given by $r = a\chi$. Their relative velocity v therefore increases as the objects are farther separated:

$$v = Hr, \tag{2.3a}$$

where H is the *Hubble rate* defined by

$$H \equiv \frac{\dot{a}}{aN} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\tau}. \tag{2.3b}$$

The velocity–distance relation (2.3) is known as the *Hubble–Lemaître law*. For a static universe ($H = 0$) any red– or blueshift in the emission spectra of distant objects finds its origin entirely in the Doppler effect due to peculiar motion. This, however, is not what Hubble discovered in the data [40]. It appeared that there was a velocity–distance relation of the form (2.3), suggesting that our universe is not static, but instead is expanding[3] ($H > 0$). The observed redshift is of a *cosmic* origin: almost all galaxies seem to recede away from the Earth as a consequence of the expanding space between them. A major consequence of a non–zero Hubble rate is that it is seemingly possible for galaxies to recede faster than the speed of light[4] if they are sufficiently separated. The boundary set by $v = 1$ occurs at a (proper) distance known as the Hubble radius

$$r_{\mathrm{H}} = \frac{1}{H}. \tag{2.4}$$

This defines the *the Hubble volume*. Galaxies outside this region recede superluminally and are consequently left unobservable for an observer at the center.

How the universe expands is captured by the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa_4^2 T_{\mu\nu}, \tag{2.5}$$

[3]The notion of expanding (or contracting) in principle only makes sense for three–spaces of finite volume. Nevertheless, we can define this notion locally as the divergence (or convergence) of comoving observers, as outlined in the text.

[4]This does not contradict the laws of special relativity: locally everything is at rest with respect to the Hubble flow (that is, everything is comoving with the expansion).

where $\mathcal{R} = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar (trace of the Ricci tensor)[5], Λ is the cosmological constant and $T_{\mu\nu}$ is the energy–momentum tensor containing all forms energy in the universe. Under the assumption that the energy–momentum can be described by a perfect fluid that respects the FLRW symmetries and that is following the Hubble flow, the energy–momentum tensor is of the form

$$T_{\mu\nu} = (p + \rho)\, u_\mu u_\nu + p g_{\mu\nu} \tag{2.6}$$

where $u^\mu = (1/N, 0, 0, 0)$ is the fluid's four–velocity, and ρ and p are, respectively, its energy density and pressure. Any viscosity, vorticity or other dissipative properties are absent by the assumption that the cosmic fluid is of an ideal nature. For a single component of the fluid, both quantities are related through an equation of state. In a cosmological context, the simple relation[6]

$$p = w\rho, \tag{2.7}$$

for some constant w is often assumed. The null dominant energy condition[7] restricts its value to $|w| \leqslant 1$. Covariant conservation of the energy–momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, implies that the energy density scales as

$$\rho \propto a^{-3(w+1)}. \tag{2.8}$$

In the Standard Model of Cosmology, the dynamics of our Universe is determined by the three forms of energy that are summarized in table 2.1:

- radiation or relativistic matter. Photons and neutrinos are the prime examples hereof. Since these constituents dilute the most quickly as the universe expands, they form the smallest fraction of the total energy content of the universe at present, approximately 0.01%.

[5]For the conventions regarding the curvature symbols, see appendix A.

[6]This linear relation remarkably captures all the essential physics in the Standard Model of Cosmology, contained in table 2.1. Nevertheless, cosmological models involving more general equations of state also exist, such as, for instance, scalar field driven cosmologies [41–48], or the Chaplygin gas model [49–51].

[7]The null dominant energy condition states that for any null vector m^μ, the energy–momentum tensor satisfies (1) $T_{\mu\nu}m^\mu m^\nu \geqslant 0$ and (2) $T^\mu{}_\nu\, m^\nu$ is a non–spacelike vector. Physically, this implies that the vacuum is stable.

Table 2.1: Cosmological parameters as measured in our observable patch, data taken from [52].

Cosmological parameter	
Hubble rate	$H_0 = 1.44 \pm 0.01 \times 10^{-42}$ GeV

Components in the Friedmann equation				
		Equation of state	**Scaling**	**Present value**
Energy content	Radiation (r)	$w = \frac{1}{3}$	$\rho_\mathrm{r} \sim \rho_{\mathrm{r}0} a^{-4}$	$\rho_{\mathrm{r}0} = 3.35 \pm 0.04 \times 10^{-15}$ eV^4
	Matter (m)	$w = 0$	$\rho_\mathrm{m} \sim \rho_{\mathrm{m}0} a^{-3}$	$\rho_{\mathrm{m}0} = 1.15 \pm 0.03 \times 10^{-11}$ eV^4
	Dark energy (Λ)	$w = -1$	$\rho_\Lambda \sim \rho_{\Lambda 0}$	$\rho_{\Lambda 0} = 2.56 \pm 0.04 \times 10^{-11}$ eV^4
	Curvature	$w = -\frac{1}{3}$	$\rho_\mathrm{c} \sim a^{-2}$	$\rho_{\mathrm{c}0} = -2.60 \pm 7.05 \times 10^{-14}$ eV^4

- ordinary, pressureless matter. This is for instance ordinary baryonic matter, the main constituents of planets and stars, but this also includes contributions from cold dark matter particles. All together, these are responsible for approximately 31% of the total energy content.

- dark energy. It represents a constant form of energy that does not dilute as the universe expands; rather, it is an intrinsic property of space. The associated energy–momentum tensor (2.6) is $T^{(\Lambda)}_{\mu\nu} = -\rho_\Lambda g_{\mu\nu}$. It is thus equivalent to the cosmological constant in the Einstein equation (2.5) upon the identification $\Lambda \equiv \kappa_4^2 \rho_\Lambda$. The dark energy density is comparable[8] to the matter density today, and it constitutes about 69% of the energy content. It is currently the most abundant form of energy in the universe. Aspects of the cosmological constant are further elaborated in section 2.2.

The total amount of energy in the Universe can thus be summarized as

$$\rho = \rho_\mathrm{r} + \rho_\mathrm{m} + \rho_\Lambda = \rho_{\mathrm{r}0} \left(\frac{a}{a_0}\right)^{-4} + \rho_{\mathrm{m}0} \left(\frac{a}{a_0}\right)^{-3} + \rho_{\Lambda 0}, \tag{2.9}$$

[8] This is kown as 'cosmic coincidence' [53].

where the subscript $_0$ denotes the value of a quantity *today*. Each of the above components dilutes differently. This suggests the existence of different eras in which each one of the components became the dominant source of expansion. As should be clear from table 2.1, the universe is currently in a stage dominated by dark energy, whose negative pressure drives the observed accelerated expansion. As time goes on, every component dilutes except for dark energy; our universe is asymptoting towards a *de Sitter*[9] universe.

The Einstein equation (2.5) determines how the universe responds to the presence of forms of energy. When the metric (2.1) and energy–momentum tensor (2.6) are inserted, the Friedmann equations emerge:

$$H^2 = \frac{1}{N^2}\left(\frac{\dot{a}}{a}\right)^2 = -\frac{k}{a^2} + \frac{\kappa_4^2}{3}\rho, \tag{2.10a}$$

$$\frac{1}{N}\dot{H} = \frac{k}{a^2} - \frac{\kappa_4^2}{2}(\rho + p) \tag{2.10b}$$

where ρ represents the *total* energy density, as in (2.9). The curvature contribution can be interpreted as a fictitious form of energy ($\rho_\mathrm{c} \equiv -3k/\kappa_4^2 a^2$) with its own equation of state as listed in table 2.1.

The first Friedmann equation is a constraint equation and the second one, consequently, is redundant. The first equation can, in the proper time gauge, be written as a classical energy conservation equation with an effective potential

$$\frac{1}{2}\left(\frac{\mathrm{d}a}{\mathrm{d}\tau}\right)^2 + V(a) = 0, \qquad V(a) = -\frac{\kappa_4^2}{6}\rho a^2. \tag{2.11}$$

In the Standard Cosmological Model, this potential is given by (figure 2.1)

$$V(a) = -\frac{\kappa_4^2}{6}a_0^2\left(\frac{\rho_{\mathrm{r}0}}{(a/a_0)^2} + \frac{\rho_{\mathrm{m}0}}{a/a_0} + \rho_{\mathrm{c}0} + \rho_{\Lambda 0}(a/a_0)^2\right). \tag{2.12}$$

Components of the cosmic fluid that satisfy the strong energy condition (such as matter and radiation) cause the effective potential to diverge near $a \to 0$; this opens up the possibility that the scale factor must have vanished at some finite

[9] A spacetime solving the Einstein equation, or equivalently the Friedmann equation (2.10a), with a cosmological constant Λ only, is called a de Sitter space if $\Lambda > 0$. If $\Lambda < 0$, it is called an anti–de Sitter space.

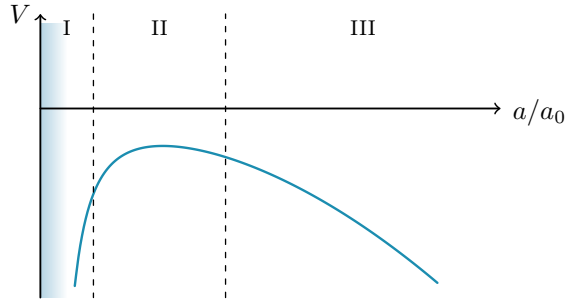


Figure 2.1: Qualitative plot of the effective potential. The three different epochs of radiation–domination (I), matter–domination (II) and dark energy–domination (III) are indicated. The shaded region illustrates conceptually where General Relativity is expected to break down.

time in the past. At this pathological point, the energy density (2.8) is typically divergent. The FLRW geometry (2.1) at this point is singular, as can be seen from the Kretschmann scalar

$$K = \frac{12k}{a^4}\left(k + 2H^2a^2\right) + 12H^4 + 12\left(H^2 + \frac{1}{N}\dot{H}\right)^2 = \frac{\kappa_4^4}{3}\left((\rho + 3p)^2 + 4\rho^2\right) \quad (2.13)$$

which diverges if the scale factor vanishes. This is simply a manifestation of the Hawking–Penrose singularity theorems [54]. This singularity was also encountered by Lemaître [55] in 1933 who tried to give it a physical interpretation. Lemaître originally called this singularity '*a primeval atom*' [56, 57]: a metastable, quantum state in which the notion of spacetime ceases to exist and whose decay event represents the beginning of the universe. In the context of Standard Cosmology, the singularity is conventionally taken to be the moment $t = 0$ and it is called the *the Big Bang*. It is –conceptually– identified with 'the beginning'; it symbolizes the creation of the universe from some initial singular state (that we do not understand), where all matter is crushed down to a single point of infinite density. This means that, early on, the universe must have been in a very small, very dense and very hot state (hence the name *hot* Big Bang model) that gradually cooled down upon expansion.

The idea that the universe could have had a beginning has a profound consequence: any causal patch must be of finite extent. In particular, no signal could have traveled further than a coordinate distance

$$\chi = \int_0^{t_0} \mathrm{d}t \, \frac{N(t)}{a(t)} \,. \tag{2.14}$$

where t_0 represents the value of the time coordinate today. Clearly, this distance χ coincides with the conformal time coordinate η introduced before. It defines the distance to the cosmological horizon beyond which one is never able to look; points that are further separated were *never* in causal contact. This is not to be confused with the Hubble volume that measures how far one is able to see *now*. The size L of the *observable patch* of the universe is restricted by both scales, estimated to be

$$L \sim 10^{10} \,\mathrm{ly}. \tag{2.15}$$

Despite its successes, the Standard Model of Cosmology cannot be the end of the story [58–60]. As any theory, it relies on a set of assumptions, based on our current understanding of Nature. It is worth to revisit them; all of them have a common weakness, namely their failure to capture the correct physics near the Big Bang. First of all, Standard Cosmology suffers from a singularity, the Big Bang. This is insurmountable within the framework of General Relativity and it is the result of extrapolating this classical theory into a quantum regime (shaded area in figure 2.1). This singularity should not be treated as a physical reality; rather, it is are mathematical pathology that signals the breakdown of General Relativity (as well as the hydrodynamical description of matter). Instead, under the severe conditions in the early universe, effects of quantum gravity presumably become important and the Standard Model of Cosmology can simply not account for this; it only provides a reliable, effective description when the universe has evolved sufficiently to a state where the Hubble radius is much larger than the Planck length, $\kappa_4 H \ll 1$. A second issue is that the assumption of homogeneity and isotropy on the largest scales is left unexplained within the Standard Model of Cosmology. Generic cosmological solutions to the Einstein equation are neither isotropic, nor homogeneous; the FLRW universe is a very particular model and there is a priori no reason why Nature would have picked this highly uniform solution above any other. Still, the

universe cannot be pure FLRW on smaller scales since some initial[10] anisotropies are definitely required for structures (galaxies, clusters, etc.) to form at late times. Even very small fluctuations are sufficient because of the attractive nature of gravity, but the question remains what caused these primeval deviations from a smooth FLRW universe. Besides these issues related to the pillars of the theory, there are signals in the observational data that seem to be in some tension (see e.g. [60]).

Could quantum mechanics come to the rescue? Our theory of cosmology crucially relies on it through, for instance, the calculation of fusion rates that lie at the heart of Big Bang nucleosynthesis. However, when trying to reconcile quantum mechanics with the theory of cosmology, one also runs into some trouble. We will devote the rest of this chapter to this.

2.2 The cosmological constant problem

The cosmological constant has a complicated history. Over the course of the decades, it has recurrently hopped in and out the Einstein equation and it has introduced numerous problems for the physics community, already right from the start (see e.g. the historical reviews [61–63]). The most puzzling one that has stood the test of time is the cosmological constant problem [34]: our (apparent?) failure to understand the large–scale behavior of the universe based on its well–known behavior at small scales. It is a really subtle problem that seems to be rather difficult to formulate precisely; different formulations exist and each one captures –in some regards– some aspect of the problem. The purpose of this section is to briefly highlight the essentials. In the first part we argue that in the framework of Standard Cosmology, a cosmological constant represents vacuum energy. Put simplistically, the puzzle with the observed vacuum energy density is that is unnaturally small by particle physics standards. This is substantiated in the second part.

[10] The singularity at $t = 0$ prevents a well–posed initial value problem. An 'initial conditions' in this context means a condition at some point at which General Relativity can be trusted. This could be, for instance, when $\kappa_4 H \approx 1$.

2.2.1 What is a cosmological constant?

The cosmological constant was originally introduced by Einstein in 1917 as an attempt to construct the first cosmological model of a static and eternal universe [64]. It served in a way as 'repulsive gravity': a way to counterbalance the gravitational attraction between ordinary matter and radiation, that is required to render the universe static. Einstein's universe, however, turned out to be unstable [65]. Thanks to the work of Lemaître [66], Friedmann [67, 68] and Hubble [40], the then belief of staticity ended in 1931, two years *after* Hubble observed the expansion of the universe. This refuted Einstein's motivation for the cosmological constant [69]; it was needed for a static universe, but not necessarily for an expanding one (unless it accelerates). Cosmological observations at the turn of the millennium [5, 70], however, seem to indicate that our universe is currently undergoing a phase of exponentially accelerated expansion. In the Standard Model of Cosmology this acceleration is driven by the negative pressure associated with a tiny (in Planck units), but definitely non–zero cosmological constant

$$\Lambda_{\text{obs}} \approx 4.38 \times 10^{-84}\,\text{GeV}^2. \tag{2.16}$$

We added the subscript to emphasize that this is the cosmological constant that we observe. In our Cosmological Model, it thus turns out that a cosmological constant is needed after all. Yet, its purpose is radically different from its original role. Despite the successes of the model, we have some theoretical issues with the cosmological constant.

The doubting Thomas[11] could raise some concerns. In physics, it is really strong to claim that a certain parameter in a theory is absolutely constant. Are we really sure that Λ should be there? Maybe it is a vanishing quantity –due to some yet unknown mechanism– and one must assign the exponential expansion to a different source of energy (see e.g. the review [71]). A possible approach in this spirit is to modify the matter content. Examples include scalar–field based models realizing dynamical (rather than constant) dark energy, such as quintessence [41–45] or k–

[11] After the biblical figure, of course. I do not intent to refer to my supervisor (although it could be appropriate as well...).

essence [46–48], or models involving an exotic equation of state for the cosmic fluid such as the Chaplygin gas [49–51]. Another possibility to construct dynamical dark energy is through modifications of the underlying gravitational theory and including higher–curvature corrections to the Einstein–Hilbert action [72, 73]. The difficulty with these alternative approaches to dark energy is that current observations either serve as severe constraints, or are inconclusive to prefer them above a standard cosmological constant [74, 75].

Although some of the aforementioned alternatives to a cosmological constant might sound appealing, setting Λ to zero is in fact highly unjustified by the lore of effective field theories: it is allowed by general covariance. On general grounds, we thus expect[12] gravity, coupled to some matter Lagrangian $\mathscr{L}_{\mathrm{m}}$, to be described by

$$S = \int \mathrm{d}^4x\sqrt{|g|}\left[\frac{1}{2\kappa_4^2}\left(\mathcal{R} - 2\Lambda\right) + \mathscr{L}_{\mathrm{m}}\right]. \tag{2.17}$$

Since there is no symmetry known that enforces the vanishing of Λ, it would be unjustified not to include it in the Einstein equation. This is the point of view that we take throughout this thesis: from a bottom–up perspective we must include the cosmological constant and we can only ponder its value, from a top–down perspective the UV theory will inform us about its value. In the context of cosmology, an effective field theory with semi–classical gravity, we are restricted to the former scenario.

At the heart of the cosmological constant problem lies the interpretation of Λ [76, 77]. The essence is that it has mass dimension two from a quantum field theory perspective. From the point of view of pure General Relativity, there is no natural energy scale that might set this; the cosmological constant is another constant of nature just as the gravitational constant. The meaning of Λ should thus be clarified from a different angle. In the above action (2.17), the cosmological constant could be absorbed into the matter Lagrangian density through a constant shift:

$$\mathscr{L}'_{\mathrm{m}} = \mathscr{L}_{\mathrm{m}} - \frac{1}{\kappa_4^2}\Lambda. \tag{2.18}$$

[12] We restrict to the simplest, non–trivial, covariant action for the metric that only contains second–order derivatives of the metric.

This shows that the cosmological constant represents a constant[13] background energy density, called *dark*[14] *energy*, that permeates all space. In ordinary quantum field theory, only energy differences are physical and Λ is of no concern. However, the gravitational field equation is sensitive to the value of Λ, as dictated by the equivalence principle: gravity couples to all forms of energy, including a constant background energy. What this energy density truly represents can be seen from its energy–momentum tensor

$$T^{(\Lambda)}_{\mu\nu} = -\kappa_4^{-2}\Lambda g_{\mu\nu}, \tag{2.19}$$

which coincides with the energy–momentum tensor of an ideal fluid with equation of state parameter $w = -1$ and energy density $\rho_\Lambda = \kappa_4^{-2}\Lambda$. Since the metric is the only tensor that respects the symmetries of the vacuum (locally Lorentz invariant), the tensor (2.19) must describe the energy–momentum that resides in the vacuum. The cosmological constant thus represents *vacuum energy*. In the Standard Model of Cosmology, vacuum energy, a cosmological constant, and dark energy are *synonymous*[15] and will be used interchangeably.

The energy density of the vacuum is one of the smallest scales that is observed in nature (cf. table 2.1):

$$\rho_{\text{obs}} \sim 10^{-47}\ \text{GeV}^4. \tag{2.20}$$

Since this is non–zero, the vacuum gravitates. This is a highly non–trivial and remarkable statement. It is non–trivial in the sense that the vacuum (often being thought of as 'nothing' or 'empty space') actually contains energy. It is remarkable in the sense that vacuum energy dictates the evolution of our universe at late times when the Hubble radius is particularly large. This seems to be a loophole in the standard paradigm of effective field theories asserting a decoupling of scales [78]; one needs to understand the smallest scales (the vacuum) to understand the largest scales (its influence on cosmology). This unusual feature lies at the heart of the cosmological constant problem.

[13] Constant in time and space, except for a finite amount of phase transitions the universe possibly has gone through that changed its value.

[14] The reason for coining it 'dark' is that only gravity is used to infer its existence, namely through the exponential expansion of the universe. Besides that, we do not 'see' it. In principle, dark energy can be any –unknown– form of energy with negative pressure that is capable of doing to work that causes this expansion (see e.g. [71]); we, however, will always identify it with a cosmological constant.

[15] This ceases to be true in e.g. braneworld scenarios, see chapter 4.

2.2.2 What is the cosmological constant?

Vacuum energy, or a cosmological constant, is undoubtedly present in every quantum field theory. However, it is usually swept under the rug since nothing is actually coupling to it. This ceases to be true if the theory is semi–classically coupled to gravity; we then have a truly meaningful cosmological constant in the theory. In this semi–classical approach, gravity is an external, classical field that is merely meant to detect vacuum energy, while all other fields are quantized. It is important to emphasize that this is merely an effective field theory approach. Given this framework, the following question naturally arises: can we compute –or at least understand– the cosmological constant from principles of quantum field theory?

We therefore briefly discuss vacuum energy in the simplest quantum field theory without gravity[16]: a scalar field[17] subjected to some field potential $V(\phi)$ in flat space with the action given by

$$S = \int \mathrm{d}^4x \left[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \right]. \tag{2.21}$$

The associated energy–momentum tensor is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + V(\phi) \right). \tag{2.22}$$

Upon quantization, this becomes an operator[18] and, in the vacuum state, its trace is directly proportional to the vacuum energy density:

$$\langle\rho\rangle = -\frac{1}{4}\langle\eta^{\mu\nu}T_{\mu\nu}\rangle = \frac{1}{4}\langle\partial_\mu\phi\partial^\mu\phi\rangle + \langle V(\phi)\rangle. \tag{2.23}$$

[16]Turning off gravity can be interpreted as a choice of a local frame in which the vacuum energy is computed while leaving out all gravitational effects (in the language of equation (2.24), diagrams with gravitons running in loops are ignored). It should be noted that the ability to turn off gravity does not imply that corrections of this kind will necessarily be subleading. In fact, they are definitely not! Since the contributions from high–energy gravitons cannot be reliably calculated without a theory of quantum gravity, we leave them out of the analysis. This does not make much of a difference; as we argue, quantum contributions from matter fields (which are better understood) are already problematic.

[17]This simple model contains all the essential physics to discuss vacuum energy; the situation for other fields is analogous, see e.g. [79].

[18]The notation $\langle\mathcal{O}\rangle \equiv \langle 0|\mathcal{O}|0\rangle$ denotes the vacuum expectation value of the operator $\mathcal{O}$.

Obtaining an analytic result, however, is impossible for generic potentials and one often relies on perturbative approaches. Pictorially and schematically,

$$\langle\rho\rangle \quad \approx \quad \bullet \quad + \quad \frac{m^2}{4} \quad + \quad \ldots \tag{2.24}$$



The notation is as follows (we discuss their implications below). The first term, represented by the bullet, denotes a classical contribution to the vacuum energy; formally, this is what survives in the limit $\hbar \to 0$. All the other terms represent quantum corrections and necessarily involve closed–loop Feynman diagrams. The first quantum correction (second term) is always present and contains the mass parameter m of the field. The ellipses denote higher–order loop corrections that typically originate from interaction terms in $V(\phi)$. Including these makes the analysis much more cumbersome, so we do not explicitly discuss higher–loop diagrams. The physical interpretation of these quantum contributions is describing field fluctuations about the vacuum configuration; intuitively, this can be understood as virtual particles popping in and out of existence[19]. Whereas classical contributions are strongly model–dependent, quantum contributions are more generic and always present. This is a direct consequence of the Heisenberg uncertainty principle and similar to the appearance of zero–point energies in quantum mechanics. In quantum field theory, however, a quantized field corresponds to an infinite collection of harmonic oscillators, each of which has a non–zero zero–point energy. This means that, naively, the energy density of the vacuum would be infinite. To reconcile this, the normal–ordering procedure is often invoked, which formally amounts to subtracting this infinite contribution. We can, however, ask the question of whether this ad hoc procedure is consistent when the theory is coupled to gravity since we throw away a source of the gravitational field itself. In the remainder of this section, we discuss –based on some of the classic reviews [79–81]– how to give meaning to (2.24).

The first term is the easiest one to handle and, as we argue, is not much of a problem. Classically, a scalar field in the vacuum state no longer has kinetic energy and settles down in a minimum ϕ_0 of the potential. The background energy $V(\phi_0)$

[19] Although this picture can be a bit misleading. In general, the number operator does not commute with the Hamiltonian, and one cannot assign a definite particle number to a state that is not an eigenstate of the number operator (such as the vacuum state).

acts as vacuum energy. Generally, classical contributions to the vacuum energy are remnants of phase transitions the early universe has gone through, similar to the release of latent heat when, for instance, water freezes.

Within the context of the Standard Model, the Brout–Englert–Higgs field and the associated electroweak symmetry breaking generates such a classical contribution to the cosmological constant. The field potential in the Glashow–Weinberg–Salam model is given by

$$V(\phi) = m^2|\phi|^2 + g|\phi|^4. \tag{2.25}$$

The electroweak phase transition in the early universe is triggered by the decrease in temperature as the universe expands. It is thus more natural to go to the framework of finite temperature field theory to analyze this (see e.g. [82]). Without going into the details, it is a standard result that, already at one–loop, the potential becomes dependent on the temperature T, as illustrated in figure 2.2. In particular, the mass parameter $m^2 = m^2(T)$ increases with temperature. The location of the minimum also becomes temperature dependent:

$$|\phi_0| = \begin{cases} 0 & \text{if } m^2(T) \geqslant 0 \\ \sqrt{-\dfrac{m^2(T)}{2g}} & \text{if } m^2(T) < 0 \end{cases}. \tag{2.26}$$

There exists a critical temperature T_{cr} above which the gauge symmetry is manifest. At lower temperatures, the Higgs field can acquire a non–zero vacuum expectation value, which breaks the gauge symmetry spontaneously. This electroweak phase transition shifts the value of the background vacuum energy to a lower value. Using the current data [83], this contribution to the vacuum energy can be estimated to be approximately

$$V(\phi_0) = -\frac{m^4(T)}{4g} \approx -1.18 \times 10^8 \text{ GeV}^4, \tag{2.27}$$

which is 55 orders of magnitude off the observed value of the vacuum energy.

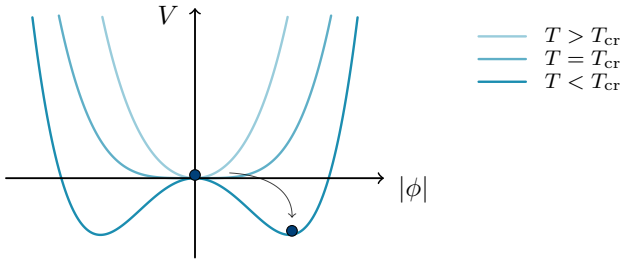


Figure 2.2: The electroweak phase transition. The potential (2.25) is plotted for different temperatures. At temperatures above (below) $T_{\rm cr}$, the gauge symmetry is (un)broken.

The electroweak transition is just one prototypical example of a phase transition the early universe has gone through. Many more transitions are believed to have occurred, for instance,

$$\ldots \quad \to \quad \text{SUSY breaking} \quad \to \quad \text{GUT breaking} \quad \to \quad \ldots$$

$$\to \quad \text{Electroweak breaking} \quad \to \quad \text{Confinement of quarks} \quad \to \quad \ldots$$

Each phase transition contributes to the vacuum energy. Clearly, the vacuum energy can acquire huge contributions already at the classical level. It is highly unlikely that all these contributions will be (almost) exactly balancing. However, this is not really problematic: after all, one could simply add a 'bare' cosmological constant to the action, whose sole purpose is to absorb these large numbers. In fact, this is how we interpret Λ in (2.17): it is the *bare* cosmological constant. The observed vacuum energy $\rho_{\rm obs}$ is then the combination between the bare vacuum energy and the computed one:

$$\rho_{\rm obs} = \kappa_4^{-2}\Lambda + \langle\rho\rangle \sim 10^{-47}\ \text{GeV}^4. \tag{2.28}$$

Considering only the classical contributions to $\langle\rho\rangle$, such as (2.27), the need to fine–tune the bare cosmological constant to 55 digits already feels unnatural. As we argue below, this is not really an issue; it becomes an issue, however, when the quantum contributions to $\langle\rho\rangle$ are taken into account.

The quantum contributions are indeed more problematic and lie at the heart of the cosmological constant problem. This is *not* because they typically diverge in the UV. In fact, this should not be disturbing at all; such 'infinities' are omnipresent in quantum field theory and we know how to deal with them through renormalization. From a quantum field theory point of view, it thus seems natural to renormalize the vacuum energy just as any other constant appearing in the action. We note that Weinberg [34] originally only regularized the divergent integrals by a cut–off procedure. We present a slight improvement of this by renormalizing the vacuum energy. This has been considered in various reviews, e.g. see [79–81].

The main idea of renormalization is that, after following a suitable[20] regularization scheme, counter–terms are introduced to cancel off any divergent pieces. For the vacuum energy, this means that any divergence coming from the loop diagram(s) in (2.24) is absorbed into the bare cosmological constant. This reveals its true role: the bare cosmological constant is a counter–term. The fine–tuning that felt unnatural (see above) is thus without justification: the counter–term is by definition fine–tuned to cancel infinite orders, so why would one bother about cancelling 'only' 55 orders more?

To leading order in perturbation theory, the renormalized result is

$$\frac{m^2}{4}$$  $$= \frac{m^4}{64\pi^2}\log\left(\frac{m^2}{4\pi\mu^2}\right), \tag{2.29}$$

where μ is an arbitrary renormalization–group scale, whose meaning is still unclear in this context [84]. Since this issue is not particularly relevant for the purpose of this thesis, it will not be further discussed. The above result reveals two important features of the vacuum energy. First of all, it is –as expected– a renormalized parameter. This means one can*not* calculate it from first principles of quantum field theory; instead, one *has to measure* it, just as, for instance, the electron mass. This resonates with the previous paragraph: one can get any value for the cosmological constant by suitably choosing the counter–term. Secondly, it demonstrates that each particle species with mass m contributes $\sim m^4$ to the vacuum energy. Since we renormalized the result, this represents a true physical effect.

[20] We adopt the Lorentz invariant dimensional regularization scheme.

The first insight implies that fine–tuning the bare cosmological constant is *not* a problem by itself; one should set aside the attitude that the vacuum energy is computable by first principles of quantum field theory. As argued by [81,85,86], the problem is that the procedure of tuning the counter–term is radiatively unstable: at each successive order in perturbation theory (the higher–loop diagrams, collected in the ellipses of equation (2.24)) one has to retune the bare cosmological constant with roughly the same precision. This discloses the true nature of the cosmological constant problem: it is a *high–energy* problem! This resonates with the second insight: the dominant contribution to the vacuum energy comes from the heaviest particles in the theory with masses near the cut–off scale. Hence any fine–tuning of the bare cosmological constant to cancel contributions from particles in loops depends *sensitively* on the UV.

This UV sensitivity entails a highly unnatural feeling: how can a parameter in a low–energy effective theory depend crucially on the *details* of the high–energy completion? This awkward feeling is known as a *naturalness problem*, a breakdown of the decoupling principle. Below we briefly outline how the concept of naturalness arises in effective field theories (see e.g. [87–89]) and how the tiny value of the vacuum energy constitutes a naturalness problem. This establishes the formulation of the cosmological constant problem that is relevant for the context of this thesis. Before we proceed, we must mention that naturalness is *not* a top–down, rigorously proven theorem or a 'dogma' that must, by all means, be imposed on effective field theories; rather, it serves as a guideline. At least, that is our perception and even not everyone might agree (see e.g. [90–95]).

Remaining ignorant about what the UV completion should look like, an effective field theory valid up to a cut–off[21] scale M can be constructed in a bottom–up way by including all local operators $\mathcal{O}_i$ of mass dimension Δ_i into the effective action that are consistent with the (presumed) symmetries:

$$S[\phi] = S_0[\phi] + \sum_i \int \mathrm{d}^4x \, g_i \frac{\mathcal{O}_i[\phi]}{M^{\Delta_i - 4}}, \tag{2.30}$$

[21] We assume that M coincides with the characteristic scale of the UV theory.

where ϕ represents a collection of fields, S_0 the associated free action, and where the dimensionless coupling constants g_i are called the Wilson coefficients. From a top–down perspective, this action would result from integrating out high–energy modes with energies exceeding M; the Wilson coefficients are thus –in principle– calculable from the UV theory. This is how the low–energy theory 'knows' about high–energy degrees of freedom: through the values of the Wilson coefficients. A coupling constant g_i is said to be *natural* if $g_i \sim \mathcal{O}(1)$ [87]. This means that there is no sensitive mixing between the UV and the IR that might cause an uncomfortable feeling [88].

Although (2.30) in principle constitutes an infinite sum, only the relevant[22] operators have to be taken into account at energies well–below the cut–off [96]. This is a blessing and a curse since relevant operators can be sensitive to physics in UV. If the coupling constant for a relevant operator is not of order unity, then there are two possible takes on this:

- There is a conspiracy between the UV and the IR: the bare coupling constant is fine–tuned in such a way to absorb the large quantum contributions from the UV [88]. In this case, the coupling is called *unnatural*. Light scalars provide an example [87] and the electroweak hierarchy problem (put simplistically, the Higgs mass is a relevant coupling and UV sensitive, yet unnaturally small compared to a reasonable Standard Model cut–off) is a particular incarnation of this.

- The limit where the coupling constant vanishes enhances the symmetry of the action. In this case, the coupling constant is said to be *technically natural* [97]. An example of this are the mass terms associated with light fermions.

[22] For processes occurring at an energy scale $E < M$, dimensional analysis suggests that the i–th term in the sum (2.30) contributes $\sim (E/M)^{\Delta_i - 4}$. Based on this behavior, the operators $\mathcal{O}_i$ can be classified into three types based on their mass dimension Δ_i: (1) irrelevant operators have $\Delta_i > 4$ and become irrelevant in the IR, (2) relevant operators have $\Delta_i < 4$ and are important at low energy, (3) marginal operators with $\Delta_i = 4$. Note that quantum effects are left out in the previous discussion. In most cases, there only exists a finite amount of relevant and marginal operators.

Naturalness is a *guideline* in constructing effective theories that asserts that all relevant couplings should take their natural $\mathcal{O}(1)$ values or, if they do not, that they should at least be technically natural. Naturalness thus means that there should be no conspiracy between UV and IR physics, which would break with the decoupling principle [88]. If neither of the aforementioned conditions are met, the effective theory is unnatural; this may give a physicist the awkward feeling that the low–energy effective theory is UV sensitive. We reemphasize that naturalness is not absolute, but rather a guideline that seemed to have worked in the past in the sense that 'unnaturalness' typically pointed towards some new physics at higher energies (for examples see [87]).

For the case of the cosmological constant, the vacuum energy is a relevant operator with mass dimension zero. As argued above, it is very sensitive to the UV[23]: it receives quantum corrections that are *quartic* in the cut–off. Yet, the observed value seems to be many orders of magnitude below any reasonable[24] cut–off scale M_{NP} of the effective field theory under consideration, say the Standard Model,

$$\rho_{\mathrm{obs}} \ll M_{\mathrm{NP}}^4. \tag{2.31a}$$

We added the subscript 'NP', emphasizing it represents a 'new physics' scale. Furthermore, a would–be vanishing of the vacuum energy does not seem to enhance symmetry; it can, therefore, not be a technically natural parameter. The only possible conclusion is that the cosmological constant is an unnaturally small parameter, compared to the cut–off, and that (2.31a) constitutes a naturalness problem. Conservatively, one often takes M_{NP} to be Planck scale; the disagreement between the natural and observed value is then about 120 orders of magnitude (compare with [34]). We wish to reemphasize that the effective field theory under

[23] If we play God and add a heavy particle of mass m_{UV} in the UV, which would leave the low–energy effective field theory otherwise unaffected, the vacuum energy would receive a contribution proportional to $\sim m_{\mathrm{UV}}^4$. This means the bare cosmological constant should conspire and be retuned by the same amount. Ardent opponents (of naturalness), however, could argue that, if we play God anyway, retuning the bare cosmological constant is peanuts compared to inventing a new particle, and that Gods might as well do this retuning. Other opponents (of God?) could argue that one may not assume an unconstrained UV theory, such that one can in principle not say anything about the hierarchy.

[24] The vacuum energy would be natural in an effective theory with a meV cut–off scale. This is well within reach of particle accelerators and no surprising new physics seems to be lurking here. A 'bottom–up' requirement of naturalness does not seem to work here.

consideration does *not* predict the value of the vacuum energy; the statement (2.31a) is *not* of a mathematical nature. An effective field theory only seems to suggest that its observed value is unnatural from a quantum field theory point of view. I will leave it to the reader how severe this is. Formulated in terms of the energy scales $M_\Lambda \sim 1\,\text{meV}$, setting the scale of the vacuum energy, and M_{NP}, the cosmological constant problem amounts to the observation that

$$M_\Lambda \ll M_{\text{NP}}. \tag{2.31b}$$

In words, the dark energy density is always unnaturally small in units of a possible new–physics scale M_{NP}. This 'unnaturalness' points toward a UV/IR mixing which goes against the spirit of decoupling that underlies effective field theories. Cosmology, in particular the cosmological constant, thus seems to provide a loophole [22, 78].

What is the cosmological constant problem trying to tell us? The UV sensitivity seems to suggest it is problem of the highest–energy scales. However, the problem would not be there if gravity was absent; it crucially involves gravity as well. We view the cosmological constant problem as a puzzle in which gravity at high energies seems to affect our low–energy late–time cosmology, an awkward UV/IR mixing from a quantum field theory point of view. This is further discussed in section 3.3 of chapter 3.

2.3 Quantum cosmology

Quantum mechanics is an unavoidable part of our current understanding of nature, and in particular of the early universe. Indeed, it underlies calculations of, for instance, fusion rates that lie at the basis of Big Bang nucleosynthesis. As far as we know, quantum mechanics is much more fundamental than any classical theory; classical properties emerge as some sort of approximation to a quantum theory for certain macroscopic systems. In this spirit, could the conventional, classical theory of cosmology also be an approximation to a more fundamental quantum theory of cosmology?

As a matter of fact, conventional wisdom of quantum mechanics argues that the entire universe is itself a closed quantum system without any (classical) external observer; it should thus be described by quantum mechanics. Admittedly, it is unclear whether quantum mechanics as understood today applies to the entire universe (however, there is no evidence that it would break down on cosmological scales [98]). The quickest option to analyze some quantum aspects of cosmology which looks consistent with our present knowledge, is to construct such a theory of quantum cosmology by naively combining quantum mechanics and cosmology. In particular, it is the hope that such a theory can shed light on some of the pressing issues of classical Big Bang cosmology; it is certain that our universe cannot have had a classical beginning. The aim of this section is not to provide a complete review of the current status of quantum cosmology, but rather to present the essentials of the framework that are needed in the context of this thesis. The interested reader is referred to some of the literature of the subject, mentioned at the beginning of this chapter.

2.3.1 General formalism

Since gravity is the dominant interaction on cosmological scales, it is inevitable that quantum cosmology is intimately linked with a theory of quantum gravity. However, quantum cosmology is not that ambitious and does not pretend to be a theory of quantum gravity; it is, in fact, merely a canonical quantization of General Relativity which –as we know– has it limitations at high energy. The premise is that in its semi–classical limit it must agree with the semi–classical limit of the full quantum gravity theory (whatever that may be). For this reason one seldom works beyond the semi–classical approximation in quantum cosmology.

Canonical quantization of General Relativity proceeds in a similar fashion as quantum mechanics is canonically 'built' from classical mechanics with the aid of a Hamiltonian. This follows Dirac's prescription and is a three–step procedure. The canonical variables of the classical theory are defined first. Secondly, one introduces the notion of a quantum state Ψ represented by a vector in a Hilbert space. The canonical variables are promoted to operators with canonical commutation

relations that act linearly on the Hilbert space of states. The last step consists of imposing the classical constraints by turning them into quantum constraints, i.e. operators that annihilate the physical state vectors. As a result of the procedure, one obtains a quantum theory in which the evolution of a state vector is governed by a Schrödinger–like equation. The aim of Quantum Cosmology is to compute the wave function of the universe Ψ, which in principle encodes all the information about it, and extract predictions out of it. The program is clearly ambitious and several aspects of its basic principles remain of a source of debate, as argued below.

Applying the quantization procedure as outlined above to General Relativity, however, involves four subtleties:

- How to identify possible canonical variables?
- Which canonical variables are to be chosen?
- How to solve that 'Schrödinger' equation?
- Which interpretation is given to Ψ?

The first two of these concerns are related to a theory of the dynamics while the third and fourth are more inherent to a quantum framework and related to a theory of the quantum state. It is very likely that both should be derivable from a complete theory of quantum gravity, such as string theory [98]. Lacking one, the theory of the dynamics is usually taken to be General Relativity as a low–energy approximation and the theory of the quantum state is seemingly imposed by hand as an additional law of physics.

The first subtlety arises because of the different nature of time in the conventional quantum theory and General Relativity. In quantum theory, time is absolute and in order to define some canonical variables a distinct, global time coordinate is required. Yet this seems to be contrary to the whole spirit of General Relativity where time is dynamical and a preferred time coordinate simply does not exist. The best one can do is to select an arbitrary one. Such a choice manifestly breaks

general covariance. Lorentzian manifolds $\mathcal{M}$ that admit a global time coordinate are always globally hyperbolic and time–orientable; these admit an ADM or $3+1$ decomposition [99] separating space from time (see also [100]).

Assuming $\mathcal{M}$ is globally hyperbolic, some particular timelike coordinate t can be chosen. The spacelike hypersurfaces of constant time Σ_t provide a foliation of $\mathcal{M}$ whose leaves are all diffeomorphic through the flow of time. The manifold $\mathcal{M}$ is thus diffeomorphic to $\mathbb{R} \times \Sigma$, with Σ some three–dimensional manifold with fixed topology[25]. The leaves of the foliation are the images of a time–dependent embedding $\Sigma \hookrightarrow \Sigma_t$. The interpretation of the ADM decomposition is thus a reformulation of Einstein's equation into a Cauchy problem: the time evolution of a manifold Σ, given some initial data. For simplicity, we will also restrict to situations where Σ is compact.

The 3+1 decomposition amounts to constructing a local coordinate system on $\mathcal{M}$ using the privileged time coordinate t. In addition to t, this requires some coordinates on Σ. As before, a preferred system does not exist and one needs to select an arbitrary one. Taking such a coordinate system x^i on Σ, one can naturally construct local coordinates on $\mathcal{M}$ that are adapted to the foliation:

$$x^\mu = \left(t, x^i\right). \tag{2.32}$$

The most general expression for the metric on $\mathcal{M}$ in these coordinates is[26]

$$\mathrm{d}s^2 = -\left(N^2 - N_i N^i\right)\mathrm{d}t^2 + 2N_i\,\mathrm{d}x^i\,\mathrm{d}t + h_{ij}(t,x)\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{2.33}$$

where N and N^i are arbitrary functions called, respectively, the lapse and shift function, and where h_{ij} is the induced metric on Σ_t. The arbitrariness in the lapse and shift functions reflect that they are pure gauge and can be absorbed by a redefinition of t and x^i. General covariance can thus be recovered through the arbitrariness of the chosen foliation.

[25] Topology–changing processes are discussed in chapter 5.

[26] In the proper time ($N = 1$) and comoving ($N^i = 0$) gauge, one recovers the Gaussian normal coordinate system (see appendix C).

Once a time coordinate has been selected, the road is clear to identify –or define– the canonical variables. This requires an understanding of the classical dynamics which, in this case, is determined by the Einstein–Hilbert action, supplemented by the Gibbons–Hawking–York boundary term, coupled to matter:

$$S = \frac{1}{2\kappa_4^2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{|g|}\left({}^{(4)}\mathcal{R} - 2\Lambda\right) + \frac{\xi}{\kappa_4^2}\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{|\gamma|}\,\mathcal{K} + S_{\mathrm{m}}[\phi], \tag{2.34}$$

where γ is the metric on a boundary $\partial\mathcal{M}$, $\xi = -(+)1$ if the boundary is spacelike (timelike), $\mathcal{K}$ is the trace of the extrinsic curvature[27] and ϕ represents the matter fields. The gravitational part of the action can be written in terms of the Ricci and extrinsic curvature of Σ with the aid of the Gauss–Codazzi equations

$$S = \frac{1}{\kappa_4^2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{h}N\left[K_{ij}K^{ij} - \mathcal{K}^2 + {}^{(3)}\mathcal{R} - 2\Lambda\right] + S_{\mathrm{m}}[\phi], \tag{2.35a}$$

where the extrinsic curvature K_{ij} is given by

$$K_{ij} = \frac{1}{2N}\left(-\dot{h}_{ij} + 2D_{(i}N_{j)}\right), \tag{2.35b}$$

with D_i the covariant derivative on Σ and $\dot{} \equiv \partial_t$. This shows that the only dynamical variable is the three–metric h_{ij}; the Cauchy problem thus translates to the time evolution of the metric h_{ij} on Σ. The lapse and shift function are non–dynamical and act as Lagrange multipliers enforcing constraints on the dynamics of h_{ij}.

The configuration space on which this dynamics takes place is called *superspace* $\mathcal{S}(\Sigma)$: the infinite dimensional space of physically distinct three–metrics

$$\mathcal{S}(\Sigma) \equiv \left\{h_{ij}(x), \phi(x) \,\middle|\, x \in \Sigma\right\} \Big/ \mathrm{Diff}_0(\Sigma), \tag{2.36}$$

where $\mathrm{Diff}_0(\Sigma)$ represents the set of three–dimensional diffeomorphisms connected to the identity. The constraints enforced by the lapse and shift are most clearly displayed when the action is written in its Hamiltonian form

$$S = \int \mathrm{d}^4x\left[\pi^{ij}\dot{h}_{ij} + \pi_\phi\dot{\phi} - N\mathcal{H} - N^i\mathcal{H}_i\right], \tag{2.37a}$$

[27] See appendix C for conventions and definitions.

where

$$\mathcal{H} = 2\kappa_4^2 G_{ijk\ell}\pi^{ij}\pi^{k\ell} - \frac{1}{2\kappa_4^2}\sqrt{h}\left({}^{(3)}\mathcal{R} - 2\Lambda\right) + \mathcal{H}_{\rm m} \tag{2.37b}$$

$$\mathcal{H}_i = -2D^j\pi_{ij} + \mathcal{H}_{{\rm m},i}. \tag{2.37c}$$

The metric $G_{ijk\ell}$ is called the DeWitt metric and, when supplemented with a suitable metric on the configuration space of the matter fields, it constitutes a metric on superspace. It is given by

$$G_{ijk\ell} = \frac{1}{2\sqrt{h}}\left(h_{ik}h_{j\ell} + h_{i\ell}h_{jk} - h_{ij}h_{k\ell}\right). \tag{2.38}$$

The contributions $\mathcal{H}_{\rm m}$ and $\mathcal{H}_{{\rm m},i}$ arise from the matter fields and are related to components of the matter energy–momentum tensor. Conventionally, one refers to $\mathcal{H}$ as *the* Hamiltonian. The constraints that are enforced by the shift and lapse functions are, respectively, the diffeomorphism or momentum constraint and the Hamiltonian constraint:

$$\mathcal{H}_i = 0, \qquad\qquad \mathcal{H} = 0. \tag{2.39}$$

These are the equivalent of the tt– and the ti–components of the Einstein equation.

Since all the dynamics is encoded in the three–metric, it suggests that h_{ij} could be an appropriate configuration variable to adopt as one of the canonical variables. This is the second sublety in quantizing General Relativity: the choice of canonical variables is not unique and different choices will typically give different quantum theories upon quantization. For the choice (h_{ij}, π^{ij}) as canonical variables, with π^{ij} the momentum conjugate to h_{ij}, given by

$$\pi^{ij} = \frac{\sqrt{h}}{2\kappa_4^2}\left(K^{ij} - \mathcal{K}h^{ij}\right), \tag{2.40}$$

the resulting quantum theory is called *quantum cosmology*. Using (h_{ij}, π^{ij}) as canonical variables, Dirac's procedure of quantization can be followed. One introduces a wave function(al) $\Psi[h, \phi; \Sigma]$, defined on superspace, that describes the quantum state of the universe. It specifies an amplitude that the universe contains a hypersurface Σ with three–metric h_{ij} and matter field configuration ϕ.

The canonical variables are replaced by the operators

$$\begin{aligned} h_{ij} &\to h_{ij}, & p^{ij} &\to -i\frac{\delta}{\delta h_{ij}}, \\ \phi &\to \phi, & \pi_\phi &\to -i\frac{\delta}{\delta \phi}, \end{aligned} \tag{2.41}$$

that satisfy canonical commutation relations. Physical states are annihilated by the operator counterparts of the classical constraints (2.39):

$$\mathcal{H}_i \Psi = 0, \qquad \mathcal{H}\Psi = 0. \tag{2.42}$$

The first constraint guarantees that the wave function is invariant under three–dimensional diffeomorphisms, as required. The second constraint enforces that Ψ is 'timeless', and it is called the *Wheeler–DeWitt equation* [101]:

$$\left[-2\kappa_4^2 G_{ijk\ell}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{k\ell}} - \frac{1}{2\kappa_4^2}\sqrt{h}\left({}^{(3)}\mathcal{R} - 2\Lambda\right) + \mathcal{H}_{\rm m}\right]\Psi = 0. \tag{2.43}$$

The Wheeler–DeWitt equation is the analog of the Schrödinger equation and it is the central equation in the study of quantum cosmology. As written above, it is very formal. We also ignored any issue of operator ordering; this is indeed not relevant in the semi–classical limit.

There is a strong similarity to the situation in ordinary quantum mechanics where wave functions are defined as solutions to the Schrödinger equation. In quantum cosmology, the wave function is defined as a solution to the Wheeler–DeWitt equation playing the role of the stationary, 'zero energy' Schrödinger equation. A tool to solve the Schrödinger equation in quantum mechanics is via Feynman's path integral. It computes the probability amplitude for a particle to travel within a given time interval between two points by summing over all possible paths that interpolate between the boundary points. This path integral approach has been uplifted to quantum field theory and the transition amplitude associated with the field evolution from $\phi_{\rm i}$ to $\phi_{\rm f}$ within a specified timespan $\triangle t$ is

$$Z = \langle \phi_{\rm f}|e^{-iH\triangle t}|\phi_{\rm i}\rangle = \int_{\phi(t_{\rm i})=\phi_{\rm i}}^{\phi(t_{\rm f})=\phi_{\rm f}} \mathcal{D}\phi\, e^{iS_{\rm m}[\phi]}, \tag{2.44a}$$

with S_{m} the action that describes the matter fields, H the Hamiltonian of the system, and where one integrates over all possible field configurations that connect the boundary conditions ϕ_{i} with ϕ_{f}. This heavily oscillating integrand is, in contrast to the quantum–mechanical path integral, mathematically ill–defined and suffers from convergence issues. A way to circumvent this is to perform a Wick rotation to Euclidean signature $r = it$, where r is called Euclidean time. The integrand becomes exponentially damped (improving convergence):

$$Z = \int \mathcal{D}\phi \, e^{-S_{\mathrm{E,m}}[\phi]}, \tag{2.44b}$$

where (here and in the following) the boundary conditions in the path integral are suppressed for notational simplicity.

Generalizing this path integral further to gravitational systems, however, entails various conceptual and calculational issues. Formally, one can define the amplitude to evolve from a state defined on a three–surface Σ_{i}, characterized by a time t_{i}, with three–metric $h_{ij}^{(\mathrm{i})}$ and matter field configuration ϕ_{i} to a state defined on a three–surface Σ_{f}, characterized by a time t_{f} with three–metric $h_{ij}^{(\mathrm{f})}$ and matter field ϕ_{f} by a Lorentzian path integral

$$Z = \langle h_{ij}^{(\mathrm{f})}, \phi_{\mathrm{f}}; \Sigma_{\mathrm{f}} | h_{ij}^{(\mathrm{i})}, \phi_{\mathrm{i}}; \Sigma_{\mathrm{i}} \rangle = \sum_{m} \int \mathcal{D}g\mathcal{D}\phi \, e^{iS[g,\phi]}, \tag{2.45}$$

where S is the action (2.34) and where the sum over m a priori runs over the different topologies of the four–manifold and integration over all metrics $g_{\mu\nu}$ that these can be equipped with. Inspired by the Euclidean path integral in quantum field theory, one also formally defines a Euclidean path integral for gravity

$$Z = \sum_{m} \int \mathcal{D}g\mathcal{D}\phi \, e^{-S_{\mathrm{E}}[g,\phi]}, \tag{2.46a}$$

where S_{E} is the Euclidean counterpart of (2.34):

$$S_{\mathrm{E}} = -\frac{1}{2\kappa_4^2} \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{g} \left[{}^{(4)}\mathcal{R} - 2\Lambda \right] - \frac{1}{\kappa_4^2} \int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{\gamma} \, \mathcal{K} + S_{\mathrm{E,m}}[\phi]. \tag{2.46b}$$

Despite the similarity, the Euclidean path integral for gravity is in a worse shape than in quantum field theory. The first reason is the domain of four–manifolds that enter in the path integral. The sum over topologies is a contentious

issue and cannot even be performed since four–manifolds are not classifiable [37]. Furthermore, because of general covariance one must impose some gauge condition in the path integral [102]. A last issue with this gravitational (Euclidean) path integral concerns its convergence. As in quantum field theory, divergences are encountered if the path integral is expanded in a perturbative series. The problem here, however, is that it is impossible to tame these by a finite amount of counter–terms: gravity is non–renormalizable. This means that (2.46) does not define a full theory of quantum gravity. Another convergence issue is of a non–perturbative nature: the gravitational part of the action (2.46b) is unbounded from below. This is the (in)famous conformal factor problem [103, 104]. To achieve convergence, the inclusion of complex metrics into the path integral is necessary and the path integral should consequentially be regarded as an abstract contour integral in the space of complex metrics. Conceptually, this contour can regarded as the class of metrics that are summed over in the path integral. On the positive side, convergent contours can be found, but, on the negative side, there does not exist a unique or preferred contour and the outcome of the integration crucially depends on which one is chosen [105].

Despite all these difficulties, the path integral of gravity remains of great heuristic value and one of its advantages is that the semi–classical approximation, which is often the only limit in which quantum cosmology is studied, is more apparent. The semi–classical approximation amounts to the stationary–phase approximation (in Lorentzian signature) or saddle–point approximation (in Euclidean signature). The path integral is then approximated as a sum over the saddles (solutions to the equations of motion, which extremize the action)

$$Z \approx \sum_{\text{saddles}} \mathcal{A}\, e^{iS_0} \qquad \text{(Lorentzian)}, \tag{2.47a}$$

$$Z \approx \sum_{\text{saddles}} \mathcal{A}\, e^{-S_{\text{E},0}} \qquad \text{(Euclidean)}, \tag{2.47b}$$

where S_0 ($S_{\mathrm{E},0}$) is the (Euclidean) action evaluated on the saddle point and $\mathcal{A}$ represents a suitable coefficient that represents the weight of the particular saddle. In the Euclidean approach one also calls the saddles '*instantons*' and (2.47b) is referred to as the instanton expansion[28].

Using the path integral, either in Lorentzian or Euclidean signature, the wave function of the universe can also be defined as

$$\Psi[h,\phi;\Sigma] = \int \mathcal{D}g\mathcal{D}\phi\, e^{iS[g,\phi]} \qquad \text{(Lorentzian)}, \tag{2.48a}$$

$$\Psi[h,\phi;\Sigma] = \int \mathcal{D}g\mathcal{D}\phi\, e^{-S_{\mathrm{E}}[g,\phi]} \qquad \text{(Euclidean)}, \tag{2.48b}$$

where the initial state (the ket state in (2.45) from which the state $|h_{ij},\phi;\Sigma\rangle$ evolved) is left unspecified. This wave function satisfies both the momentum constraint and the Wheeler–DeWitt equation (2.42) and should thus be considered as a valid quantum state of the universe for any choice of initial conditions. This is the third subtlety encountered in quantum cosmology: to select one single wave function of the universe amongst the many allowed by the dynamics, boundary conditions are absolutely necessary to single out one unique solution to the Wheeler–DeWitt equation (2.43) or, equivalently, to compute the path integral (2.48). In the context of contour integration, specifying boundary conditions amounts to specifying a choice of contour to uniquely define the state. Usually in quantum mechanics, boundary conditions are to be imposed by the classical observer who is external to the quantum system. In the framework of quantum cosmology, however, the system is not coupled to a classical environment and it is therefore less clear which boundary conditions are to be imposed. In fact, it is possibly one of the most debated aspects of quantum cosmology. It could be mathematical consistency as DeWitt [101] originally had hoped (although not realised), but we will adopt the attitude that it is the physics that ultimately decides this (see e.g. the discussion in [106]). This is highly non–trivial to implement, since we lack a UV complete Hamiltonian of gravity. Throughout the literature there are two common proposals which are briefly outlined below.

[28] A more systematic account of instanton physics is postponed until section 5.1. It is possible –though not strictly needed– to consult this before proceeding without losing continuity of the storyline.

The first proposal, due to Hartle and Hawking [107, 108], is that the universe is in its 'ground state' and completely 'self–contained'. The idea of a ground state in cosmology is realized –in analogy with quantum mechanics– through a Euclidean path integral (2.48b), which is assumed to be fundamental in the approach to quantum gravity. The proposal is inspired by the observation that the early universe seems to be simpler (more homogeneous, isotropic, etc.) than the late–time universe; simplicity usually points towards a ground state. The self–containedness refers to the four–manifolds summed over in the path integral. These should be compact with Σ as their *only* boundary and the matter fields summed over should be everywhere regular (semi–classically at least, see also [109]). There is thus no other boundary in the past. For this reason, the Hartle–Hawking boundary condition is also known as the *no–boundary* proposal. The interpretation of the no–boundary wave function is as being the amplitude that the given three–geometry arises –through some quantum mechanical process– from 'nothing', a zero three–geometry. In the semi–classical approximation, this would lead to imaginary–time initial conditions on $h_{ij}(x)$ and $\phi(x)$, and conditions on their derivatives. Admittedly, the no–boundary proposal does not entirely fulfill the self–containedness promise: it does not uniquely specify the integration contour [110] (although Hartle and Hawking make a concrete suggestion for a particular toy model, see section 2.3.2).

Inspired by the idea that it is possible for a universe to quantum–mechanically tunnel from 'nothing', Vilenkin proposed a quantum state which draws a strong analogy to quantum tunneling, as e.g. occuring in α–decay (hence the name *tunneling* proposal [111–113]). A tunneling boundary condition in quantum mechanics is a statement about outgoing modes at infinity. An uplift to quantum cosmology is far from straightforward since the notion of outgoing versus ingoing strongly depends on the existence of a timelike Killing vector which in superspace is generically absent. Nevertheless, this is better (though not always) defined in the semi–classical approximation and in many toy examples. The proposal is that the wave function of the universe should be everywhere bounded and at

the singular boundaries[29] of superspace solely consists of outgoing modes. This proposal directly restricts the possible solutions to the Wheeler–DeWitt equation. In the path integral formulation the Vilenkin boundary condition starts from the Lorentzian path integral (2.48a) and restricts the class of geometries summed over to those having a vanishing three–geometry in the past.[30] These are compact, Lorentzian four–manifolds; in that sense these geometries also close off in the past, akin to the no–boundary proposal.

The last subtlety in quantum cosmology is how to give an interpretation to (i.e. how to extract prediction from) the wave function. In ordinary quantum mechanics, the Copenhagen interpretation is the most common approach. It presumes an open quantum system that is able to interact with an external, classical observer in the laboratory. This is clearly inapplicable to quantum cosmology since the universe breaks with these two premises: it is a closed quantum system without an external observer. In quantum cosmology, the wave function is interpreted using the so–called 'decoherent histories' approach. Since prediction making is not relevant in this thesis, we do not discuss this in detail, but refer the reader to the appropriate literature [35, 36, 114–118]. Instead, we state the –rather simplistic– way how the wave function is interpreted in this thesis. We are always concerned with the saddle point approximation (2.47) where the instantons describe the spontaneous nucleation of a four–dimensional cosmology; we interpret the exponential as giving the nucleation probability. Although a more rigorous analysis can be given, this has never been persued by the author.

2.3.2 A tractable model in minisuperspace

The principles above were outlined to some degree of generality. However, working in the full, infinite–dimensional superspace is practically impossible. The common attitude in quantum cosmology is to consider particular toy models that enjoy a certain degree of symmetry. Formally, this corresponds to freezing an

[29] This refers to configurations on Σ that are in some sense singular. If these singular three–configurations correspond to four–manifolds that are *also* singular, then this is called a singular boundary of superspace.

[30] The equivalence between these two definitions has not been shown in general.

infinite number of degrees of freedom and truncating the infinite–dimensional system down to a finite–dimensional one. The resulting finite–dimensional configuration space is called *minisuperspace*. The minisuperspace approximation makes quantum cosmology 'tractable' and we also follow this attitude in the rest of this thesis. Because of the Heisenberg uncertainty principle, this minisuperspace approximation is sometimes argued to be without motivation, since it fixes certain degrees of freedom together with their corresponding momenta.

This section provides the simplest toy model (following [35, 36]) in quantum cosmology to illustrate the aforementioned principles. The model turns out to be relevant in the context of this thesis; we treat it rather superficially, but to an extent that is sufficient to convey the messages of the rest of this thesis (see chapter 3, and section 4.4 of chapter 4). We are aware of some recent developments (see e.g. [119]) and some, associated, recent claims [120]. We, however, remain agnostic about these technicalities and simply follow the 'old school' approach led by Hartle, Hawking, and Vilenkin [107, 112].

The toy model considers the quantization of an empty, spatially compact FLRW universe (2.1) with cosmological constant. Inhomogeneous fluctuations of the metric are frozen, effectively yielding a model on one–dimensional minisuperpace with the scale factor as the only dynamical variable. The classical dynamics is determined by the action (2.34) and, in the variables[31] of the model, given by

$$S = \frac{\Omega_3}{\kappa_4^2} \int \mathrm{d}t\, N \left[-\frac{3}{N^2} a\dot{a}^2 + 3a - \Lambda a^3 \right], \tag{2.49}$$

where $\Omega_3 = 2\pi^2$ is the volume of the unit S^3. The canonical variables are (a, π_a), with π_a the momentum conjugate to a

$$\pi_a = -\frac{\Omega_3}{\kappa_4^2} \frac{6}{N} a\dot{a}. \tag{2.50}$$

The Hamiltonian for this system is

$$\mathcal{H} = -\frac{\kappa_4^2}{\Omega_3} \frac{\pi_a^2}{12a} + \frac{\Omega_3}{\kappa_4^2} a \left(\Lambda a^2 - 3 \right). \tag{2.51}$$

[31] Recall that one identifies a with $-a$ and N with $-N$.

Notice how the classical singularity at $a = 0$ is still present in the Hamiltonian. Upon quantization, the momentum is replaced by the operator $\pi_a \to -i\partial/\partial a$, and the Hamiltonian constraint $\mathcal{H} = 0$ is replaced by the Wheeler–DeWitt equation $\mathcal{H}\Psi = 0$. In this specific minisuperspace model, the Wheeler–DeWitt equation resembles the stationary Schrödinger equation in an effective potential:

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial a^2} + \frac{\Omega_3^2}{\kappa_4^4}V(a)\right]\Psi(a) = 0, \tag{2.52a}$$

with the potential given by

$$V(a) = 6a^2\left(3 - \Lambda a^2\right). \tag{2.52b}$$

The central question here is whether –and, if so, by what probability– an empty FLRW universe can nucleate out of nothing (that is, from a vanishing scale factor). This notion of quantum tunneling is related to the existence of a potential barrier, $V(a) > 0$, that separates $a = 0$ from a classical region with $V(a) < 0$. From (2.52b), it is clear that this requires a positive cosmological constant; only a spatially compact de Sitter universe can nucleate. When $\Lambda > 0$ two different regions can be identified, as qualitatively shown in figure 2.3a: the quantum or Euclidean region (Region I), and a classical or Lorentzian region (Region II). Both are separated by a turning point at $a_* \equiv \sqrt{3/\Lambda}$ which is, not coincidentally, the minimal radius of a positively curved de Sitter universe.

The wave function is generically non–vanishing and exponential in region I; this demonstrates the possibility for quantum tunneling from one turning point at $a = 0$ to the other turning point a_*. The question of nucleation probability can only be answered if the wave function itself is known. The Wheeler–DeWitt equation is in the semi–classical (WKB) approximation solved by

$$\Psi(a) = \frac{1}{|V(a)|^{1/4}}\begin{cases}\left(\mathcal{A}e^{S(a,0)} + \mathcal{B}e^{-S(a,0)}\right) & \text{if } a < a_* \\ \left(\mathcal{C}e^{iS(a,a_*)} + \mathcal{D}e^{-iS(a,a_*)}\right) & \text{if } a > a_*\end{cases}, \tag{2.53a}$$

where $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}, \mathcal{D}$ are pairs of complex numbers, related to one another through the WKB connection formulae, and where $S(a, a_i)$ is defined by

$$S(a, a_i) \equiv \frac{\Omega_3}{\kappa_4^2} \int_{a_i}^{a} \mathrm{d}a' \, \sqrt{2|V(a')|}. \tag{2.53b}$$

The arbitrariness in $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ reflects the well–known fact that there are infinitely many possible solutions to the Wheeler–DeWitt equation with vastly different behavior. The wave function is only uniquely fixed with a choice for $(\mathcal{A}, \mathcal{B})$ or, equivalently, a choice for $(\mathcal{C}, \mathcal{D})$. This corresponds to fixing the boundary conditions that the Wheeler–DeWitt equation should be supplemented with. A priori there is no good reason to favor either the Harle–Hawking or the Vilenkin boundary condition; we discuss the implementation of both below.

The implementation of the Hartle–Hawking boundary condition proceeds via a Euclidean path integral. In the semi–classical approximation, the most dominant contributions the the path integral are the instantons. These extremize the Euclidean action and thus solve the Euclidean Friedmann equations

$$\frac{1}{a}\frac{\mathrm{d}^2 a}{\mathrm{d}r^2} = -\frac{\Lambda}{3}N^2 - \frac{1}{aN^3}\frac{\mathrm{d}a}{\mathrm{d}r}\frac{\mathrm{d}N}{\mathrm{d}r}, \qquad \frac{1}{N^2}\left(\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}r}\right)^2 = \frac{1}{a^2} - \frac{\Lambda}{3}. \tag{2.54}$$

We adopt a gauge in which $\dot{N} = 0$ and $r \in [0, 1]$. The boundary conditions are $a(1) = a_0$, which corresponds to the only boundary, and $a(0) = 0$ which ensures that the geometry smoothly caps off. The solution that satisfies the boundary conditions is given by

$$a(r) = \sqrt{\frac{3}{\Lambda}} \sin\left(\sqrt{\frac{\Lambda}{3}} N r\right), \tag{2.55a}$$

with the lapse function being determined by the boundary conditions:

$$N^{(-)} = \sqrt{\frac{3}{\Lambda}} \sin^{-1}\left(\sqrt{\frac{\Lambda}{3}} a_0\right), \tag{2.55b}$$

$$N^{(+)} = \sqrt{\frac{3}{\Lambda}} \left[\pi - \sin^{-1}\left(\sqrt{\frac{\Lambda}{3}} a_0\right)\right]. \tag{2.55c}$$

This determines the two[32] saddle points. If $a_0 > a_*$, these saddle points correspond to complex geometries, in which case the path integral would yield an oscillatory wave function in the classical region. In the Euclidean signature ($a_0 < a_*$) these saddle points are real and correspond geometrically to a less–than–half S^4 $(-)$ or a more–than–half S^4 $(+)$ with radius a_*. In the latter case the path integral computes the wave function inside the potential barrier. We will be interested in this case and use the WKB connection formulae to construct the wave function in region II. The action for any these instantons is

$$S_{\mathrm{E}} = -\frac{6\Omega_3}{\kappa_4^2\Lambda}\left(1 \mp \left(1 - \frac{\Lambda}{3}a_0^2\right)^{3/2}\right) \tag{2.56}$$

The Hartle–Hawking prescription does not specify any contour to integrate over; that is, it does not prescribe which saddle point(s) are to be included in the semi–classical approximation (2.47). Nevertheless, Hartle and Hawking originally argued that the dominant saddle point is the *less–than–half* saddle point [107]. We will take this to be the *definition* of the Hartle–Hawking wave function in this particular minisuperspace model.

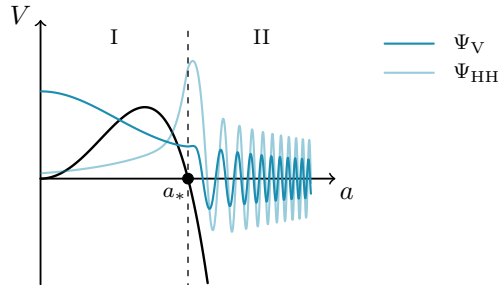
With this additional information, the no–boundary wave function is uniquely determined. In region I, it is thus of the form $\Psi \sim e^{-S_{\mathrm{E}}}$, with S_{E} the action of the less–than–half saddle point. The wave function is of exactly the WKB form (2.53) with $(\mathcal{A}, \mathcal{B}) \sim (1, 0)$: the Hartle–Hawking boundary condition amounts to considering the increasing exponential only. Through the usual WKB connection formulae, the wave function can be constructed in region II. In this specific minisuperspace model, the no–boundary wave function is explicitly given by

$$\Psi_{\mathrm{HH}}(a) = \frac{1}{|V(a)|^{1/4}} \begin{cases} \exp\left[\frac{\Omega_3}{\kappa_4^2}\frac{6}{\Lambda}\left(1 - \left(1 - \frac{\Lambda}{3}a^2\right)^{3/2}\right)\right] & \text{if } a < a_* \\ 2e^{S_*}\cos\left(S(a, a_*) - \frac{\pi}{4}\right) & \text{if } a > a_* \end{cases}, \tag{2.57}$$

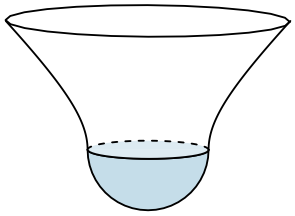
where S_* is defined by

$$S_* \equiv S(a_*, 0) = \frac{6\Omega_3}{\kappa_4^2\Lambda}. \tag{2.58}$$

[32] In fact, the lapse functions are only defined modulo a multiple of π; consequently, there are in principle infinitely many saddle points, that geometrically correspond to a chain of contiguous four–spheres of radii $\sqrt{3/\Lambda}$. However, we do not consider these saddles (for details, see [35, 105]).



(a) Plot of the minisuperspace potential $V(a)$ in black, and the Vilenkin (real part) and Hartle–Hawking wave function in blue shades.



(b) The de Sitter instanton (blue) glued to a Lorentzian de Sitter space, representing the nucleation of a de Sitter universe.

Figure 2.3: Wave function and instanton for the minisuperspace FLRW toy model.

Note that S_* coincides with *minus* the action associated with the de Sitter instanton.

The implementation of the Vilenkin boundary condition proceeds more naturally through the WKB solution to the Wheeler–DeWitt equation (2.53). In particular, it argues to only consider outgoing waves in the classical region, i.e. $(\mathcal{C}, \mathcal{D}) \sim (0, 1)$. Using the connection formulae, the wave function in region I can be constructed:

$$\Psi_{\mathrm{V}}(a) = \frac{1}{|V(a)|^{1/4}} \begin{cases} e^{S_*} \exp\left[-\frac{\Omega_3}{\kappa_4^2}\frac{6}{\Lambda}\left(1-\left(1-\frac{\Lambda}{3}a^2\right)^{3/2}\right)\right] & \text{if } a < a_* \\ e^{-iS(a,a_*)-i\frac{\pi}{4}} & \text{if } a > a_* \end{cases}. \tag{2.59}$$

A graphical comparison between the Hartle–Hawking and Vilenkin wave function can be found in figure 2.3.

The knowledge of the wave function(s), allows one to naively extract the probability that a universe with cosmological constant Λ nucleates from nothing. The nucleation probabilities associated with each of these wave functions are given by

$$P_{\mathrm{HH}} \propto \exp\left(+\frac{24\pi^2}{\kappa_4^2\Lambda}\right), \qquad P_{\mathrm{V}} \propto \exp\left(-\frac{24\pi^2}{\kappa_4^2\Lambda}\right). \tag{2.60}$$

We take this as the definition for a *Hartle–Hawking* (HH) or a *Vilekin* (V) *weighting*, and we remain agnostic about the claims in [120]. The former peaks at a small positive cosmological constant, while the opposite is true for the tunneling wave function. This would suggest that the most naive interpretations of the wave function might then be at odds with the tunneling wave function if one wants to address the cosmological constant problem. But since we have not provided a full integration measure on the space of cosmological constants, such claims have no meaning.

Chapter 3

Cosmic Eggs

In theories involving extra dimensions, such as string theory, one is forced to provide a physical basis for their apparent absence in the perceived physical world. One of the possibilities is to consider compactification scenarios à la Kaluza and Klein [15, 16], where a hierarchy of scales is imposed *by hand*. Its purpose is to enforce that extra dimensions are unobservably small in a low–energy theory. Although other possibilities exist (such as warped compactifications or braneworlds, discussed in chapter 4), this chapter deals exclusively with Kaluza–Klein compactifications. These scenarios generically miss one crucial aspect, namely *why* our world, if it results from a compactification, appears four–dimensional. In string theory, it is sometimes believed that this could have a dynamical explanation, such as a spontaneous (de–)compactification. Yet, a concrete realization of such a mechanism has not been found (see however [121–124]).

Compactification scenarios typically entail a vast amount of possible lower–dimensional effective theories. While it is far from established that four–dimensional de Sitter vacua can be realized in string theory, the consensus in the community seems to be that there are either none of them [24] or a vast landscape of them [23]. This chapter explores the consequences of a multitude of de Sitter

vacua[1]. This naturally poses the question *which* vacuum, given the many possible choices, realizes the universe we inhabit. This is neither well–established, and it is sometimes devoted to an anthropic mechanism [19, 125, 126].

In this chapter we argue how the cosmological constant problem might provide some insight into the two aforementioned questions. In section 3.1, it is first briefly demonstrated how Kaluza–Klein compactifications 'work' with the aid of an example, and why scale separation is an essential ingredient. The remainder of this chapter parallels the discussion presented in [2]. Section 3.2 conceptually discusses how the cosmological constant problem manifests itself in compactifications of string theory, and what its relation to scale separation is. In section 3.3 we propose, based in the acquired insights, a dynamical mechanism that might relax the cosmological constant problem from a string–inspired perspective.

3.1 Hiding extra dimensions: compactifications

This section provides a toy model of a Kaluza–Klein (KK) compactification (see e.g. [127, 128]). To give a flavor of these scenarios, it is illustrative to consider a compactification of six dimensions down to four, where the two extra dimensions span a two–sphere. While the focus is put on these specific dimensionalities and on this specific compact manifold, generalizations to other situations are straightforward and the procedure applies equally well –though it is computationally more involved. Leaving aside the details, we comment on how this procedure is implemented in string theory.

Schematically, given a six–dimensional theory, the quest is open to find solutions which are of the phenomenological form

$$\mathcal{M}_6 = \mathcal{M}_4 \times S^2, \tag{3.1}$$

[1]The content of this chapter likely carries over to quintessence scenarios.

where the two–sphere is parametrically smaller than the four–dimensional space $\mathcal{M}_4$. It is, however, not always obvious that the higher–dimensional theory admits solutions to the field equations that exhibit this property. For future reference, the coordinates on $\mathcal{M}_6$ are denoted $X^M = (x^\mu, \theta^m)$ with x^μ (θ^m) the coordinates on $\mathcal{M}_4$ (S^2).

To illustrate how extra dimensions can hide, it is instructive to first consider a real scalar field ϕ with mass parameter M propagating on a *fixed* six–dimensional background $\mathcal{M}_6$ whose metric is denoted

$$\mathrm{d}s_6^2 = \mathrm{d}s_4^2 + R^2\,\mathrm{d}s_2^2, \tag{3.2}$$

where $\mathrm{d}s_4^2$ is the line element on $\mathcal{M}_4$ and $\mathrm{d}s_2^2$ on the unit S^2 with R representing its *fixed* radius. The fixation of the background amounts to ignoring the dynamics of gravity. This is common in any quantum field theory approach, but, as argued below, this cannot always be justified. Nevertheless, it is for the moment illustrative to proceed in this manner. The action for the scalar in six dimensions is given by

$$S = \int \mathrm{d}^6X \sqrt{|g_6|}\left[-\frac{1}{2}\partial_M\phi\partial^M\phi - \frac{1}{2}M^2\phi^2\right]. \tag{3.3}$$

The compactness of S^2 allows to separate variables and expand the scalar field ϕ into spherical harmonics[2]

$$\phi(x,\theta) = \frac{1}{R}\sum_{\ell\in\mathbb{N}}\sum_{|m|\leqslant\ell}\phi_{\ell,m}(x)Y_\ell^m(\theta). \tag{3.4}$$

The reality condition on the field implies $\phi_{\ell,m} = (-)^m\phi^\dagger_{\ell,-m}$. The complex coefficients $\phi_{\ell,m}$ are called the Kaluza–Klein modes. This expansion can be inserted in the action and any angular dependence can be integrated out, yielding an effective four–dimensional action

$$S = \int \mathrm{d}^4x\sqrt{|g_4|}\sum_{\substack{\ell\in\mathbb{N}\\|m|\leqslant\ell}}\left[-\frac{1}{2}\partial_\mu\phi_{\ell,m}\partial^\mu\phi^\dagger_{\ell,m} - \frac{1}{2}\left(M^2 + \frac{\ell(\ell+1)}{R^2}\right)|\phi_{\ell,m}|^2\right] \tag{3.5}$$

[2]We adopt the convention that in standard spherical coordinates $\theta^m = (\theta^1, \theta^2)$ the harmonics are given by $Y_\ell^m(\theta^1,\theta^2) = (-)^m\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}e^{im\theta^2}P_\ell^m\left(\cos(\theta^1)\right)$ where P_ℓ^m are the associated Legendre polynomials.

From the perspective of $\mathcal{M}_4$, there appears to be an infinite number of scalar fields $\phi_{\ell,m}$ with respective masses

$$M^2_{\ell,m} = M^2 + \frac{\ell(\ell+1)}{R^2}. \tag{3.6}$$

This illustrates a generic feature of compactifications: all fields living on the higher–dimensional space descend naturally to an infinite tower of fields with ever increasing masses in the lower–dimensional spacetime. This is indeed the only way a lower–dimensional theory can know about the existence of extra dimensions: through the KK modes. It is important to stress that (3.5) is completely equivalent to (3.3); it simply offers a different interpretation.

This rewriting of the action illustrates how extra dimensions can remain out of sight. The length scale $L_{\rm KK} \sim R$, characterizing the size of the extra dimensions, determines an energy scale $M_{\rm KK} \sim 1/R$. Both are commonly referred to as the Kaluza–Klein scale or the compactification scale. It sets the scale of the internal manifold and the scale at which a particle in the tower –except for the lightest one– can be created on–shell. Exciting KK modes is, in light of (3.4), equivalent to introducing a dependence on the extra dimensions. Consequently, extra dimensions can remain unobservable for low–energy observers if the observers' characteristic energy scale M_4 is 'remote' from the KK scale in order that the inhomogeneous KK modes, and thus the extra dimensions, escape detection. The condition of 'remoteness' is commonly referred to as *scale–separation*:

$$M_4 \ll M_{\rm KK}, \tag{3.7a}$$

which can also be formulated in terms of the length scales $L_{\rm KK}$ and $L_4 \sim 1/M_4$:

$$L_{\rm KK} \ll L_4. \tag{3.7b}$$

This is the physical translation of the intuition that the extra dimensions are too small to be observed. As long as experiments are performed at energies below the KK scale, all the KK modes –except for the zero mode– become irrelevant

and can be discarded[3] from the effective theory that describes the low energy physics. At these low energies, therefore, the background (3.2) genuinely appears four–dimensional for the observers, and particles correspond to field excitations that are homogeneously smeared over the extra dimensions. This illustrates a generic approach to compactifications: one is interested in constructing a low–energy effective theory and to that extent one makes field Ansätze in which the fields do not depend on the extra dimensions. This is called a Kaluza–Klein Ansatz and relies on the fact that KK modes –except for the zero mode– are too massive to be excited and can be consistently truncated.

The background (3.2) with the imposed scale separation (3.7) is highly artificial and, as anticipated, it should be questioned whether this exists at all from the higher–dimensional perspective. In particular, the gravitational dynamics dictates that the geometry (3.2) is in fact dynamical and can a priori not be fixed as above. The action that determines the dynamics of six–dimensional gravity is taken to be pure Einstein–Hilbert without a cosmological constant:

$$S = \frac{1}{2\kappa_6^2} \int \mathrm{d}^6 X \sqrt{|g_6|}\, {}^{(6)}\mathcal{R}. \tag{3.8}$$

The S^2 should be regarded as being dynamical, meaning its radius in principle carries a position dependence $R = R(x, \theta)$. This might result in the awkward realization that (3.2) with fixed R does not exist or, even worse, that it develops a singularity if the dynamics allow $R \to 0$. To examine how the six–dimensional geometry behaves when compactifying to four dimensions, one adopts, as argued above, a KK Ansatz for the metric:

$$\mathrm{d}s_6^2 = e^{-\sqrt{\kappa_6/\Omega_2}\,\varphi(x)} g_{\mu\nu}(x)\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu + \kappa_6 e^{\sqrt{\kappa_6/\Omega_2}\,\varphi(x)}\, \mathrm{d}s_2^2, \tag{3.9}$$

where $\mathrm{d}s_2$ is the metric on the unit S^2 with volume $\Omega_2 = 4\pi$. The metric is chosen in the Einstein frame which ensures that the effective four–dimensional action contains the Einstein–Hilbert term of four–dimensional gravity (equivalently: $g_{\mu\nu}$ describes lower–dimensional gravity). The scalar field φ sets the radius of S^2 that now

[3] In principle, it should always be verified that it is consistent to discard fields from the theory. In the presented case the Klein–Gordon equation demonstrates that all KK modes decouple from one another, justifying the truncation.

becomes dynamical:

$$R(x) = \kappa_6^{1/2} \exp\left(\frac{1}{2}\sqrt{\frac{\kappa_6}{\Omega_2}}\,\varphi(x)\right). \tag{3.10}$$

Parameters that determine properties (size, shape, etc.) of the internal manifold are called *moduli*. In this simple toy model the field φ thus represents a volume modulus. Moduli correspond to fields in the lower–dimensional effective theory; what the internal geometry thus looks like can vary from point to point in $\mathcal{M}_4$.

When the Ansatz (3.9) is inserted into (3.8), the integral over the compact space becomes trivial and the effective four–dimensional action takes the form

$$S = \int \mathrm{d}^4x\sqrt{|g|}\left[\frac{\Omega_2}{2\kappa_6}\,{}^{(4)}\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \frac{\Omega_2}{\kappa_6^2}e^{-2\sqrt{\kappa_6/\Omega_2}\,\varphi}\right], \tag{3.11}$$

where the last term is generated by curvature of the compact manifold. This neatly illustrates three generic outcomes of a compactification of gravity. Firstly, the gravitational coupling constant in the lower–dimensional theory is determined by the coupling in the higher–dimensional theory and by the volume of the extra dimensions, in this example[4]

$$\kappa_4^2 \equiv \frac{\kappa_6}{\Omega_2}. \tag{3.12}$$

Secondly, the size (and under more general circumstances also the shape) of the internal manifold are not a priori fixed parameters, but are represented by moduli fields with a dynamics of their own. Their kinetic terms are generated by the Einstein–Hilbert action, i.e. by the higher–dimensional gravitational theory itself. Lastly, the curvature of the internal manifold is able to generate a potential for the moduli. In this case, the volume modulus is subjected to the potential

$$V(\varphi) = -\frac{e^{-2\kappa_4\varphi}}{\kappa_4^4\Omega_2}, \tag{3.13}$$

which exhibits runaway behavior as $\phi \to -\infty$, or equivalently as $R \to 0$. The sphere thus inevitably collapses into a singularity. In this regime, the field theory approximation is obviously no longer to be trusted. The proclaimed stable configuration (3.2) is thus non–existent, let alone a scale–separated one!

[4] In fact, this should be interpreted in a vacuum where it is reasonable to speak about four–dimensional gravity.

This demonstrates a universal feature of compactifications of pure Einstein gravity: it is impossible to classically stabilize the internal manifold without some 'extra ingredients'.

In theories with extra dimensions, one is facing a problem of stabilizing the moduli fields. In string theory fluxes, for instance, could provide a stabilization mechanism (see e.g. [129–131]). The low–energy effective action of string theory always contains some p–form fields (with p depending on the type of string theory under consideration). This makes it possible to consider configurations in which a non–zero flux of the field strength permeates the compactification manifold. These are commonly known under the name of Freund–Rubin compactifications [132]. To illustrate how fluxes could come to the rescue, the Einstein–Hilbert action (3.8) can be supplemented by a Maxwell field

$$S = \int \mathrm{d}^6 X \sqrt{|g_6|} \left[\frac{1}{2\kappa_6^2} {}^{(6)}\mathcal{R} - \frac{1}{4} F^{MN} F_{MN} \right]. \tag{3.14}$$

We will not be interested in compactifications of the Maxwell field down to four dimensions, nor will we examine their fluctuations as for the scalar field theory above; rather, we examine how its background vacuum expectation value affects the compactification. The KK Ansatz for gravity is again (3.9), and for the field strength the Ansatz is such that it only has legs S^2:

$$F_{MN} = \begin{cases} Q\varepsilon_{mn} & \text{if } M,N = m,n \\ 0 & \text{else} \end{cases}, \tag{3.15}$$

where ε is the canonical volume form on the compactification manifold and Q some arbitrary[5] constant. The magnetic flux that threads the compact space is

$$\int_{S^2} F = \Omega_2 Q. \tag{3.16}$$

[5]In fact, after charge quantization, Q is quantized. This subtlety is not particularly relevant in the context of this thesis.

By following the same procedure, the effective four–dimensional action is given by

$$S = \int \mathrm{d}^4x\sqrt{|g|}\Big[\frac{1}{2\kappa_4^2}\,{}^{(4)}\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \\ + \frac{1}{\kappa_4^4\Omega_2}\left(e^{-2\kappa_4\varphi} - \kappa_4^2Q^2e^{-3\kappa_4\varphi}\right)\Big], \tag{3.17}$$

where the second line represents (minus) the potential for the volume modulus:

$$V(\varphi) = \frac{1}{\kappa_4^2\Omega_2}\left(Q^2e^{-3\kappa_4\varphi} - e^{-2\kappa_4\varphi}\right). \tag{3.18}$$

This shows a generic feature of flux compactifications: fluxes contribute positively to the potential giving the extra dimensions a tendency to expand. In the presented toy model, the competition between the tendency to collapse (caused by the positive curvature of compact space) and the tendency to expand (caused by the flux) makes it possible to stabilize the volume modulus at the minimum of the potential where the sphere has a fixed radius

$$R = \sqrt{6\pi}\,\kappa_4^2Q. \tag{3.19}$$

Indeed, this geometry is a solution to the six–dimensional field equations and realizes the aforementioned promise. The last feature this toy model neatly illustrates, is that compactifications allow for a calculation of the cosmological constant in the lower–dimensional spacetime: if the modulus settles in the minimum of the potential, the effective action (3.17) becomes

$$S = \frac{1}{2\kappa_4^2}\int \mathrm{d}^4x\sqrt{|g|}\left({}^{(4)}\mathcal{R} - 2\Lambda\right), \tag{3.20a}$$

where the four–dimensional cosmological constant is given by

$$\Lambda = -\frac{1}{27}\frac{1}{\kappa_4^6Q^4} = -\frac{4\pi}{3}\frac{\kappa_4^2}{R^4}. \tag{3.20b}$$

This also indicates a ‘preference’ of flux compactifications: these constructions easily yield a negative cosmological constant, a positive one is much more difficult to obtain. The vacuum configuration found in this example is thus $\mathrm{AdS}_4 \times S^2$, which realizes (3.1).

An important comment about this toy model is in order. The $\mathrm{AdS}_4 \times S^2$ vacuum is *not* genuinely four–dimensional and the KK Ansatz is thus *not* justified from this perspective. This follows from the fact that the scale associated with AdS_4 is of the same order of magnitude as the KK scale set by the radius of the sphere. The S^2 does not fulfill the promise of 'small' extra dimensions. This issue also appears when fixing the volume modulus in the minimum of the potential, thereby discarding it as a field from the four–dimensional theory. It can only be successfully integrated out if it is sufficiently massive compared to the energy scale of AdS_4. The mass parameter, however, is given by

$$m^2 = \frac{8\pi\kappa_4^2}{R^4} = 6|\Lambda|, \tag{3.21}$$

which is clearly comparable to the AdS scale. The promised scale separation (3.7) is thus *not* realized. Nevertheless this toy model turned out to be useful to illustrate the general machinery behind Kaluza–Klein compactifications.

3.2 String compactifications and the cosmological constant problem

As discussed at length in section 2.2, the cosmological constant problem arises in any effective field with semi–classical gravity. The problem stems directly from the fact that the vacuum energy is a relevant operator, meaning it is highly sensitive to the exact details of the UV physics. Indeed, the vacuum energy receives contributions from every particle species running in loops and the most dominant contribution comes from UV particles with masses near the cut–off of the theory. A straightforward formulation of the cosmological constant problem is:

> *The observed dark energy density is extremely –unnaturally– small in units of the cut–off of the theory.*

This is simply the content of equation (2.31). In terms of the length scales $L_\Lambda = \sqrt{3/\Lambda}$ and $L_{\rm NP} = M_{\rm NP}^{-1}$, where 'NP' denotes the cut–off scale at which new physics appears, the cosmological constant problem amounts to the observation that the observed Hubble scale L_Λ is unexpectedly many orders of magnitude larger than its natural size $L_{\rm NP}$:

$$L_{\rm NP} \ll L_\Lambda. \tag{3.22}$$

The supreme difficulty of the cosmological constant problem stems precisely from the fact that, at least within our current theoretical framework, it intertwines the largest and the smallest scales in physics. This strong interdependence –the naturalness problem– is puzzling in light of the decoupling principle of effective field theories, and it might even suggestive of the possibility that cosmological constant problem might not be resolvable within such a framework. Indeed, it is very difficult to modify the low–energy framework (such as the Standard Model) in such a way as to reconcile the cosmological constant problem without at the same time spoiling its successes.

The UV sensitivity of the cosmological constant strongly indicates that it is a *high*–energy quantum gravity problem, although the energy scale involved is the lowest one known in nature! Indeed, inherent to the formulation of the problem are quantum aspects (particles running in loops) as well as gravitational aspects (the equivalence principle). A proper analysis –and hopefully a resolution– should therefore be formulated in the context of a UV complete description of gravity such as string theory [133]. It is a striking, yet underappreciated, property of string theory that it enables a precise calculation of the vacuum energy[6] in certain controlled corners, at weak coupling and low energies. The puzzling UV/IR mixing mentioned above also manifests itself in the context of string theory, but it gets a precise interpretation. String theory is an exception to the decoupling principle since particle physics (UV) is not inseparable from cosmology (IR) in the framework: the calculation of vacuum energy is completely *dependent* on the details of the compactification [22, 24]. This is exactly what the cosmological constant problem is telling us!

[6]For instance $\mathrm{AdS}_5 \times S^5$ is a well–trusted background in type IIB string theory with a negative vacuum energy that does not receive significant quantum corrections.

It thus seems very natural to analyze the cosmological constant problem from a string theory perspective. String theory brings to the table genuinely novel ingredients which bear directly on this problem. Here we concentrate on two such elements:

- the possibility to (de–)compactify dimensions;
- the sheer abundance of string theory vacua.

Since vacuum energy is indeed computable within string theory, would the theory or (any compactification) then still be plagued by a cosmological constant problem? To answer this question, it is vital to reemphasize two crucial aspects of the cosmological constant problem. First of all, the cosmological constant problem is *not* a calculational issue; rather, it is a qualitative statement about the value of a certain parameter in the theory which goes against one's expectation or intuition. Secondly, the cosmological constant problem as formulated so far is inherent to the four–dimensional framework of quantum field theory coupled to cosmology. If it occurs in string theory, it must do so in solutions where four–dimensional quantum field theory arguments apply, i.e. in genuinely four–dimensional vacua.

In string phenomenology one attempts to find solutions by constructing a metric Ansatz in which six dimensions are compactified and three spatial dimensions remain non–compact, schematically[7]

$$\mathrm{d}s_{10}^2 = \mathrm{d}s_4^2 + \mathrm{d}s_6^2. \tag{3.23}$$

A genuinely four–dimensional compactification like this requires the scale separation

$$L_{\mathrm{KK}} \ll L_\Lambda, \tag{3.24}$$

where L_{KK} is the Kaluza–Klein (KK) length scale associated with the compact dimensions and L_Λ is the possibly time–dependent Hubble scale on $\mathrm{d}s_4$. This means the extra dimensions should be stabilized at unobservable length scales.

[7] In this chapter we focus on Kaluza–Klein compactifications 'pur sang' where the metric is factorizable, as in (3.23). The possibility of warped extra dimensions and how it impacts the cosmological constant problem is not considered in this chapter (see e.g. [134, 135]).

Under the assumption that a huge landscape of possible de Sitter vacua exists, it is not at all obvious which one corresponds to our universe. Frequently, anthropic arguments are invoked since string theory appears to provide the necessary ingredients[8] for anthropic reasoning. The central object in this kind of reasoning is the observer: 'cogito ergo sum'. It is simple probability theory in the landscape, conditioned on the fact that observers exist. This idea loses any of its power if all vacua enable life to emerge. This, however, does not seem to be the case, and the vacuum we inhabit is clearly fine–tuned to our own existence. This in particular bears on the cosmological constant: if too large, galaxies would not have formed, and if too small, the universe would have collapsed. This naturally opens a window for the vacuum energy in which it is possible –although not guaranteed– for the observers to emerge [139]:

$$-10^{-45}\,\mathrm{GeV}^4 \lesssim \rho_\Lambda \lesssim 10^{-45}\,\mathrm{GeV}^4. \tag{3.25}$$

While this still differs by several orders of magnitude from the observed value, it is not unthinkable that progress in this direction –by for instance including other considerations– could sharpen this bound. While anthropics is fed by the string landscape and suggests an easy solution (or alleviation) to the cosmological constant problem, we argue below that it could be in tension with some of the ideas in string theory.

From a cosmological viewpoint, an Ansatz (3.23) satisfying (3.24) amounts to a highly contrived starting point. After all, how did the universe get there? How did it achieve the required separation of scales (3.24) to appear four–dimensional? In fact, a construction of this kind, i.e. scale separation, already *implies* a cosmological constant problem because L_{KK} is definitely a new physics scale. Consequently, the requirement for scale separation (3.24) is identical to the cosmological constant problem (3.22) [140]: four–dimensional compactifications of string theory that obey (3.24) have a cosmological constant problem built in right from the start! This is how the cosmological constant problem manifests itself in string theory:

[8]Being (1) a substantial amount of de Sitter vacua with densely spaced values for the cosmological constant and (2) a mechanism in hopping between these vacua. The former is realized by the landscape paradigm [23] and the latter possibly by bubble nucleation [136–138], for instance.

> *In theories with extra dimensions, the cosmological hierarchy problem can be thought of as the unnaturally large radius of the observable universe in Kaluza–Klein units.*

In other words, the proclaimed landscape is full of de Sitter universes with an inherent cosmological constant problem[9]. This is consistent with our view that the cosmological constant problem cannot be resolved within four–dimensional effective field theory. It still haunts string theory, but in a different manner: the failure for a solution to appear genuinely four–dimensional is equivalent to not having a cosmological hierarchy. The hierarchy problem is reflected in the difficulties to construct (meta)stable vacua exhibiting scale separation.

String theory might shed some new light onto the cosmological constant problem. It is sometimes believed that string theory can provide a dynamical explanation for the perceived dimensionality of the universe (such as string gas cosmology [121]). On the other hand, string theory could accommodate the anthropic principle, suggesting an anthropic solution to the cosmological problem. These two viewpoints are clearly inconsistent given how the cosmological constant problem appears in theories involving extra dimensions [140] and the former approach is –in our opinion– the most exciting. Secondly, string theory argues that quantum field theoretical considerations are insufficient to address it: equation (3.24) means that standard four–dimensional arguments apply only when the vacuum *already* has an unnaturally small cosmological constant. Both considerations point towards the view that, to properly address the problem, one must go beyond four–dimensional quantum field theory and probe the origin of four–dimensional scale separated configurations for otherwise one might run a risk of being caught in a dangerous circular reasoning.

[9]De Sitter universes with a huge (natural) cosmological constant do not appear in this landscape since, by construction, they would break with the premise of scale separation (3.24).

3.3 Cosmic eggs

Since string theory suggests that a deeper understanding –or maybe a resolution– of the cosmological constant problem might be hidden in the origin of scale separated four–dimensional vacua, we formulate a different 'line of attack' to the cosmological constant problem. We consider a scenario in which all spatial dimensions are initially compactified and of comparable, small size. Such configurations can be thought of as effectively one–dimensional vacua or, as we wish to name them, 'cosmic eggs'. The absence of any scale separation means that cosmic eggs do not suffer from a cosmological constant problem. In this regard they are 'democratic' configurations.

In order for cosmic eggs to serve as possible initial conditions for cosmology, these configurations should be metastable: they should have the possibility to decay to a configuration with large expanding dimensions. This implies they probably should break supersymmetry. However, there are no one–dimensional vacua in a strict sense once supersymmetry is broken. One should therefore not expect cosmic eggs to be perfectly static configurations; rather, they exhibit a 'breathing' volume modulus [141].

In our opinion configurations that allow a certain number of dimensions to grow large (through dynamical decompactification), form a natural starting point for cosmological considerations. Our proposed 'cosmic egg cosmology' is in that regard a two–step emergent universe scenario. These scenarios typically rely on a spontaneous decompactification of three spatial dimensions. Another example of an emerging universe scenario is the Brandenberger–Vafa mechanism [121,142,143] in which at most three spatial dimensions of a T^9 can spontaneously decompactify.

It should be noted that the sheer abundance of string vacua also means that there is not a unique cosmic egg. On the contrary, even the most ardent advocates of the swampland program would agree that string theory encompasses a 'landscape' of possible low–energy laws, albeit one with a set of interesting theoretical patterns that constrain its phenomenology. On general grounds one expects that vacua with a lower number of large dimensions are statistically hugely favored since there are

more compact manifolds, more cycles to wrap branes and fluxes, etc. The landscape of higher–dimensional vacua may well be a set of measure zero in the landscape of all vacua.

3.3.1 Construction

To implement our scenario we are thus led to consider compactifications of string theory down to one dimension, time. The landscape of one–dimensional string theory vacua is relatively unexplored and we will not attempt to find new compactifications to one dimension (but see e.g. [144–146]). Instead we work with a toy model landscape that admits configurations that enable us to implement a cosmic egg scenario.

An essential ingredient of the scenario is the possibility for the cosmic eggs to evolve to four–dimensional universes[10]. A convenient way to proceed is to dimensionally reduce four–dimensional effective theories. To this end, we introduce the following action

$$S = \int \mathrm{d}^4x\sqrt{|g|}\left[\frac{1}{2\kappa_4^2}\left(\mathcal{R} - 2\Lambda\right) - \frac{1}{12}H^{\mu\nu\rho}H_{\mu\nu\rho}\right], \tag{3.26}$$

where H is an axion three–form field strength and Λ is a four–dimensional cosmological constant, not necessarily small. These are the simplest ingredients needed to convey the cosmic egg idea. We note that the possibility to add more axions, even two–form fluxes, or to consider a rolling scalar potential instead of a cosmological constant, does not significantly change the qualitative discussion we wish to pursue here.

[10] In principle one could also imagine a cosmic egg cosmology with a different number of large dimensions and the mechanism we suggest below would definitely allow for the birth of higher–dimensional universes as well. So, unless there is an obstruction in the string landscape, there is no reason to exclude these. Yet to date no proposal in string theory exists to construct flux vacua with moduli stabilisation that feature scale separation in dimensions larger than four.

A dimensional reduction of the system (3.26) over the three spatial dimensions amounts effectively to adopting a FLRW Ansatz[11] for the metric:

$$\mathrm{d}s^2 = -\frac{N^2(t)}{x(t)}\,\mathrm{d}t^2 + x(t)\,\mathrm{d}\Omega_3^2, \tag{3.27a}$$

where N is a lapse function, $x(t)$ the scale factor (or in the language of compactifications: the volume modulus), and $\mathrm{d}\Omega_3^2$ the line element on the compact space which, for simplicity and for cosmological purposes, we take to be Einstein. The spatial curvature can be normalised as usual, in FLRW language to $k = -1,\ 0,\ +1$, with all three values consistent with compactness. The Ansatz for the axion field strength is

$$H = Q\mathcal{E}, \tag{3.27b}$$

where $\mathcal{E}$ is the volume form on the three–dimensional spatial manifold and Q is called the axion flux. The effective action for the scale factor is of the form

$$S = \frac{\Omega_3}{\kappa_4^2}\int \mathrm{d}t\; N\left[-\frac{3}{4N^2}\dot{x}^2 + U(x)\right], \tag{3.28a}$$

where, in the present toy model, the effective potential is given by

$$U(x) = 3k - \frac{\kappa_4^2 Q^2}{2x^2} - \Lambda x, \tag{3.28b}$$

and displayed as the dashed line in figure 3.1a.

Gravity in one dimension is non–dynamical and governed by a Hamiltonian constraint (the first Friedmann equation) that is enforced by the lapse equation of motion. Hence, a one–dimensional, time–independent, (meta)stable vacuum only exists when U has a minimum where it *exactly* vanishes. This seems to be impossible in the present model since $U''(x) < 0$ everywhere. The closest one can get to a configuration in which the scale factor is 'immobilized' is when the spatial curvature is positive and the axion flux and cosmological constant are mutually tuned to be

$$Q^2 = \frac{8k}{\kappa_4^2\Lambda^2}. \tag{3.29}$$

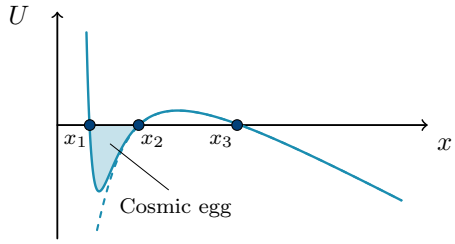
[11] This is (2.1) with a redefinition of the scale factor $x(t) = a^2(t)$, playing the role of a volume modulus, and with a redefinition of time $\mathrm{d}t \to \mathrm{d}t/x(t)$.

Since the potential then has a maximum at zero energy, one can hardly call this a vacuum. This maximum corresponds to the unstable Einstein static Universe and is thus *not* a candidate cosmic egg. Furthermore, since Q is quantized it is not guaranteed that the required tuning can be achieved for any Λ. The eggs, if they exist, are to be found at yet smaller values of the volume modulus.

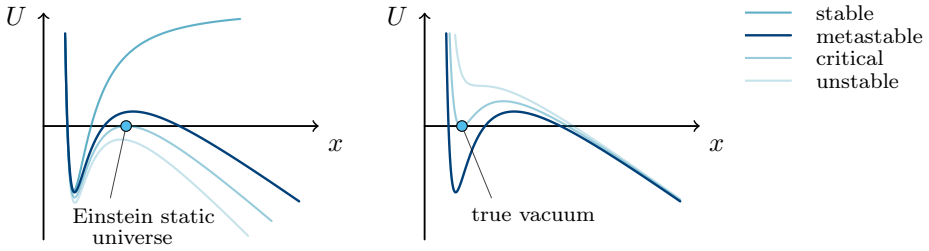
Going beyond the present model potential (3.28b), it seems hard –if not impossible– to achieve a vacuum in the defined strict sense with broken supersymmetry, since quantum corrections are unlikely to sum exactly to zero. Since we do not want to trade one fine–tuning problem for another, the notion of a vacuum could be relaxed: for our purposes, it luckily suffices for the effective potential merely to have a 'pocket' region which can trap the scale factor, as illustrated by the shaded region in figure 3.1a. This would (in a weak sense) stabilize the volume modulus: it can be trapped in the pocket and oscillate back and forth, creating the possibility of an effective one–dimensional cosmological phase. The existence of such a pocket region lies at the heart of a cosmic egg cosmological scenario.

The potential (3.28b), in fact, does not admit a pocket region. However, the existence of these regions is a statement about the behavior of the potential at small x. Typically, the four–dimensional effective field theory is not to be trusted at small scale factor (cf. the shaded area in figure 2.1) and one expects quantum corrections to modify the effective potential at small x. Typically, these corrections add terms involving higher inverse powers of x. For example, an F^4 higher–derivative correction in the Maxwell field, for instance, would introduce a term proportional to $P^4 x^{5/2}$, with P the magnetic flux. To compute whether the effective potential will ultimately be repulsive or attractive at very small volume requires full control over the quantum gravity theory, which is not within reach.

We therefore proceed heuristically and add terms to the potential (3.28b) that we speculate to be originating from quantum corrections. These should only modify (3.28b) at small x. The simplest possibility that creates the desired pocket region is adding a single repulsive term as $x \to 0$. Interestingly, in fact, models of loop quantum gravity suggest such a repulsive behavior [147]. Admittedly, there is no reason to expect repulsion away from small x and the potential might just as well



(a) The effective potential $U(x)$ without quantum corrections (dashed), equation (3.28b), and with corrections (solid), equation (3.30).



(b) Effective, quantum corrected potential for different values of Λ. Lower graphs correspond to higher Λ values

(c) Effective, quantum corrected potential for different values of Q. Lower graphs correspond to higher Q values.

Figure 3.1: Cosmic eggs: figure (a) demonstrates the construction of cosmic eggs by adding putative quantum corrections to the effective potential; figures (b,c) illustrate that their existence crucially depends on the value of the cosmological constant and axion flux.

remain attractive here. Classically, this might be unappealing since it allows for a complete vanishing of the scale factor. We therefore propose a model with repulsion having an effective, quantum–corrected potential

$$U(x) = \frac{A^2}{x^\alpha} + 3k - \frac{\kappa_4^2 Q^2}{2x^2} - \Lambda x. \tag{3.30}$$

The first term establishes the repulsive barrier as $x \to 0$ provided $\alpha > 2$. We leave open the exact form of this repulsive term so that our cosmig egg cosmology is at least to some extent model–agnostic. There is no way around the fact that the existence of cosmic eggs –at least in within this model– will

intimately be tied to quantum gravity. Since this is not fully understood, our arguments remain qualitative and we avoid making statements about the putative quantum corrections.

Figure 3.1 qualitatively illustrates the behavior of the effective potential U with $k = +1$. Whether in our model an appropriate potential well, i.e. 'cosmic egg', exists depends on three properties:

- The spatial curvature k should be positive.
- The cosmological constant must be positive in order for the cosmic egg to be metastable: negative cosmological constants typically yield stable configurations. Furthermore, Λ should be bounded from above by the value that roughly corresponds to the Einstein static universe[12]

$$\Lambda \lesssim \Lambda_{\mathrm{cr}} \equiv \frac{\sqrt{8}}{\kappa_4 Q}. \tag{3.31}$$

  If the cosmological constant exceeds this critical value, the configuration is an unstable runaway. This is illustrated in figure 3.1b.
- There should be a sufficient amount of axion flux which allows the potential to 'dip' below zero at small x. This is illustrated in figure 3.1c. Alternatively, this requirement is equivalent to the quantum corrections being not 'too big' so that they affect the potential already near[13] $x = \kappa_4 Q/\sqrt{6}$. Since this is ultimately a statement of the UV completion, we do not wish to elaborate on this. It merely illustrates the need for the assumption that quantum corrections only modify the potential at truly small values of x. In the following we will always assume the quantum corrections to be sufficiently small and impute the existence of cosmic eggs solely to the cosmological constant.

Hence, metastable cosmic eggs in this toy model are always spatially closed with a cosmological constant in the range $0 < \Lambda \lesssim \Lambda_{\mathrm{cr}}$ and with sufficiently small quantum corrections or, equivalently, a sufficient amount of axion flux $Q \gtrsim Q_{\mathrm{cr}}$. We always

[12] Since only Q^2 has a physical meaning, we write Q rather than $|Q|$.
[13] This value corresponds to approximately the turning point x_2 in figure 3.1a.

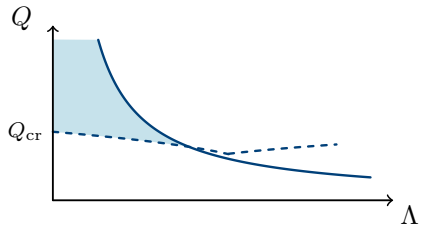


Figure 3.2: Conceptual illustration of the toy model landscape of cosmic eggs with fixed quantum correction parameters A and α. The solid line corresponds to critical configurations exhibiting an Einstein static universe as in figure 3.1b. Below the dashed line, quantum corrections are too large, so that the amount of axion flux is not able to create a cosmic egg. As a boundary of the landscape, the dashed line corresponds to the true vacua as in figure 3.1c.

assume the latter condition to be valid. As anticipated, even in this simple model, there exists a large number of these cosmic eggs characterized by the parameters (A, α, Q, Λ). This 'landscape' is conceptually illustrated in figure 3.2.

Cosmic egg cosmology, we claim, is the idea that any of these eggs can serve as a possible initial condition for the universe in the following sense. First, the scale factor is trapped inside 'the cosmic egg'. These configurations are neither de Sitter nor anti–de Sitter but instead are breathing, with an oscillating volume modulus. As a historical comment, we note that the cosmic egg configurations we consider differ from Lemaître's conception of the primeval atom which he thought of as purely quantum, i.e. exhibiting no (classical) notion of space and time. Lemaître's primeval atom seems more akin to the Euclidean region of the conventional no–boundary saddle points. Secondly, the scale factor can quantum mechanically tunnel through the potential barrier, thereby 'breaking' the egg. This causes the three spatial dimensions to decompactify and grow exponentially: the era of four–dimensional cosmology 'starts' here.

3.3.2 Breaking cosmic eggs

The proliferation of cosmic eggs means that a cosmic egg cosmology is neither complete nor predictive without specifying a state on its configuration space that defines a notion of typicality. We therefore embed cosmic egg cosmology into the framework of quantum cosmology. This provides a unified treatment of cosmic egg cosmology, from the quantum formation of the classical oscillatory egg phase, to its breaking and the subsequent expansion of the universe; the wave function of the universe provides a relative weighting of different egg–born cosmological histories from which a measure for observations can be obtained.

The compactified egg phase is ideally suited to be treated in a minisuperspace approximation. The Kaluza–Klein Ansatz guarantees that the overall volume of the three compact spatial dimensions is the only remaining light degree of freedom. Of course new degrees of freedom, the inhomogeneous KK modes, will become relevant when the expansion gets underway. However our toy model does not include these, and hence does not describe a realistic universe. We are primarily interested here in modeling the earliest stages of evolution, the egg and tunneling phases, and to determine the relative weighting of expanding universes emerging from it. For these purposes it is reasonable to assume that the minisuperspace approximation should apply.

The Wheeler–DeWitt equation for this minisuperspace model is

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{3\Omega_3^2}{2\kappa_4^4}U(x)\right]\Psi(x)=0, \tag{3.32}$$

where $U(x)$ is the potential (3.28b) *plus* some quantum corrections modifying its behavior at small x. It now remains to specify the minisuperspace wave function, i.e. the boundary conditions to be imposed.

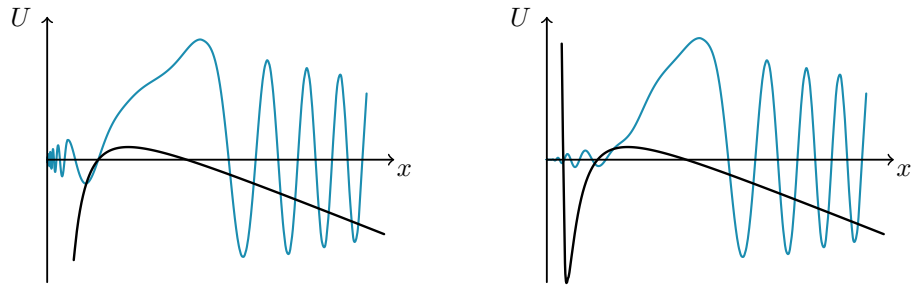
In compactifications where $U(x)$ is repulsive as $x \to 0$, as proposed in (3.30), the wave function must definitely decay and eventually tend to zero in the small volume limit. This is because, for a repulsive potential, the problem resembles that of a particle subjected to an infinite potential barrier in quantum mechanics. This

choice of boundary condition is akin to the original Hartle–Hawking proposal. It embodies its motivation that the big bang is a genuine beginning in a physical sense.

Alternatively, in compactifications where $U(x)$ is attractive as $x \to 0$, it would seem that one has a choice of boundary conditions on the wave function at $x = 0$. One can either require the wave function to vanish at the origin, or one can impose boundary conditions for which the wave function diverges at the origin. The former case is again akin to the original Hartle–Hawking proposal. We will see below that the no–boundary condition on the wave function acts as an effective quantum barrier at small x, giving rise to an effective pocket in which the wave function oscillates and describes a classical breathing egg just as in the repulsive case above. In the latter case the wave function would diverge as $x \to 0$ and it is doubtful that it describes a physically meaningful, let alone approximately classical, egg phase.

For both the attractive and repulsive model, we therefore require the wave function to decay to zero as $x \to 0$. Figure 3.3 shows the wave function Ψ with the 'no–boundary' condition $\Psi(0) = 0$, first in a model with an attractive small x potential such as (3.28b) potential and then with a repulsive potential of the form (3.30). In both cases one can clearly identify the classical oscillatory egg state, the exponential behavior of the wave function under the barrier, and finally the classical expansion at large volume.

The steepness of the potential in the small volume limit causes the WKB approximation to break down there. Indeed, semi–classical gravity should not apply in a regime of full quantum gravity. This is rather different from the standard applications of Hartle–Hawking initial conditions in the context of inflation involving gentle and finite, positive potentials in the small volume regime, in which the minisuperspace wave function can be expressed in terms of regular semi–classical saddle points. In appendix B we present an approximate solution of the small x wave function with Hartle–Hawking boundary conditions using a different approximation scheme. On the other hand the WKB approximation becomes accurate for larger x. The general behavior of the WKB solution in the



(a) Attractive potential. **(b)** Repulsive potential.

Figure 3.3: Illustration of the behavior of the wave function with Hartle–Hawking boundary condition $\Psi(0) = 0$ imposed. The wave function is everywhere real and shown in blue, the potential is represented in black. Both models clearly exhibit a classical oscillatory egg state connected through quantum tunneling to an expanding four–dimensional de Sitter–like universe.

large volume $x \gg x_3$ regime is

$$\Psi(x) \approx \frac{1}{|2U(x)|^{1/4}} \left(\mathcal{C} e^{iS(x,x_3)} + \mathcal{D} e^{-iS(x,x_3)} \right), \tag{3.33a}$$

where x_3 is the large x endpoint of the potential barrier (figure 3.1a) and $S(x, x_i)$ is defined as

$$S(x, x_i) \equiv \sqrt{\frac{3}{2}} \frac{\Omega_3}{\kappa_4^2} \int_{x_i}^{x} \mathrm{d}x' \sqrt{2U(x')}. \tag{3.33b}$$

The amplitudes $\mathcal{C}$ and $\mathcal{D}$ are determined by the behavior of the wave function in the small x region. In general, the WKB connection formulae suggest the form

$$\mathcal{C} = e^{S_{23} - i\frac{\pi}{4}} \mathcal{H} + e^{-S_{23} + i\frac{\pi}{4}} \mathcal{V}, \tag{3.34a}$$

$$\mathcal{D} = e^{S_{23} + i\frac{\pi}{4}} \mathcal{H} + e^{-S_{23} - i\frac{\pi}{4}} \mathcal{V}, \tag{3.34b}$$

where $S_{23} \equiv S(x_3, x_2)$ and where the coefficients $\mathcal{H}$ and $\mathcal{V}$ are complicated functions of the parameters in the problem, $\mathcal{H} = \mathcal{H}(A, \alpha, Q, \Lambda)$ and $\mathcal{V} = \mathcal{V}(A, \alpha, Q, \Lambda)$, that are fixed by the small x behavior of the wave function. These generically depend on the choice of boundary condition on Ψ and are obtained by matching the small x wave function (beyond WKB) to the WKB form at large x. Since we do not have full control over a quantum theory of gravity in the small x regime, we use

a parametrized approach and leave open their exact forms (and refer to appendix B for the details). We have numerically verified and analytically substantiated in Appendix B that both $\mathcal{H}$ and $\mathcal{V}$ are generically non–zero for Hartle–Hawking boundary conditions at $x = 0$.

The large x wave function is given by

$$\Psi(x) \approx \frac{2}{|2U(x)|^{1/4}} \Big[\mathcal{H} e^{S_{23}} \cos\Big(S(x, x_3) - \frac{\pi}{4}\Big) + \mathcal{V} e^{-S_{23}} \cos\Big(S(x, x_3) + \frac{\pi}{4}\Big) \Big] \tag{3.35}$$

The leading dependence on Q and Λ is encoded in S_{23}. To understand this, note from (3.33b) that S_{23} is given by the surface area of the function $\sqrt{2U}$ between the two turning points x_2 and x_3. With h the maximum height of this function and $\triangle x = x_3 - x_2$, the distance between the turning points, the area lies between the bounds

$$\frac{1}{2} h \triangle x < S_{23} < h \triangle x, \tag{3.36}$$

since U is concave between x_2 and x_3. Under the reasonable assumption that quantum corrections (e.g. the term $\sim A^2/x^\alpha$) are sufficiently damped out for $x > x_2$, one finds

$$h \approx \sqrt{6 - 3\left(\kappa_4 Q \Lambda\right)^{2/3}} \qquad\qquad \triangle x \approx \frac{3}{\Lambda} - \frac{\kappa_4 Q}{\sqrt{6}}. \tag{3.37}$$

This yields the following estimate

$$S_{23} \sim \zeta \sqrt{\frac{3}{2}} \frac{\Omega_3}{\kappa_4^2} \left(\frac{3}{\Lambda} - \frac{\kappa_4 Q}{\sqrt{6}}\right) \sqrt{6 - 3\left(\kappa_4 Q \Lambda\right)^{2/3}}, \tag{3.38}$$

with $1/2 < \zeta < 1$. This is sufficient to demonstrate the scaling of S_{23}. In the limit where $\Lambda \ll \Lambda_{\rm cr}$, this becomes

$$S_{23} \sim \zeta \frac{9\Omega_3}{\kappa_4^2 \Lambda} \left[1 - \frac{1}{2}\left(\frac{\Lambda}{\Lambda_{\rm cr}}\right)^{2/3} + \mathcal{O}\left(\frac{\Lambda}{\Lambda_{\rm cr}}\right)\right] \tag{3.39}$$

This might be compared with (2.58) and (2.60). Hence the intermediate tunneling dynamics in cosmic egg cosmology naturally produces a wave function in the large volume limit (3.35) that involves a superposition of the typical Hartle–Hawking behavior $\sim e^{S_{23}}$ with an exponentially suppressed Vilenkin tail $\sim e^{-S_{23}}$. The

mixing of both wave functions arises because the quantum dynamics of the egg cosmology involves a combination of Hartle–Hawking initial conditions to create the egg, and quantum tunneling from the egg to a large expanding universe. The characteristic Hartle–Hawking weighting favors the nucleation of large universes with a low value of the four–dimensional cosmological constant. This sheds some new light on the cosmological constant problem:

> *A seemingly unnaturally small cosmological constant is seen to arise from natural initial conditions.*

Needless to say, the cosmic egg cosmologies we have considered are not realistic. The sole purpose of the toy models we have proposed, is to exhibit a concrete mechanism in relaxing the cosmological constant problem. It will require much further work to embed this scenario in somewhat realistic models of the early universe.

Chapter 4

Dark Bubbles

The dominant approach to hiding the extra dimensions of string theory is by a compactification. The previous chapter explored –or exploited even– parts of the associated landscape program. Nevertheless, we have to admit that, despite all the effort that has been put into trying to construct de Sitter vacua in this manner, this program has not yielded any satisfactory outcome because of the extreme subtleties and difficulties involved [24]. The reason for this prominent difficulty is still an unsettled issue; it could either be due to technical difficulties, associated with supersymmetry breaking and moduli stabilization, or it could either be that there is a deep obstruction within string theory itself. This has led to the formulation of the conjecture that de Sitter belongs to the swampland [28,29,148,149]. This is not proven and is still under debate. We remain agnostic about this, but we believe it presents a living question: does de Sitter space belong to the swampland?

This chapter is motivated by the difficulty of finding de Sitter solutions in string theory and considers an alternative route: braneworlds. We first outline the basic ideas underlying these scenarios in section 4.1 and we review the key aspects of the 'classic' braneworlds à la Randall–Sundrum [17, 18] in section 4.2. From section 4.3 onward, we focus on a recently proposed braneworld scenario: the Dark Bubble [150]. We take a string–inspired (and swamp–inspired), bottom–up point of view and do not worry about a concrete string realization, although a concrete

embedding is crucial for the viability of this model (for progress, see [151, 152]). The model realizes a de Sitter cosmology on top of an expanding bubble of a 'less false' vacuum that has nucleated in an unstable five–dimensional anti–de Sitter background. In section 4.4 we explain how the nucleation event makes contact with quantum cosmology and how this affects the choice of boundary conditions. In section 4.5 we construct five–dimensional gravitational waves that reduce to the known four–dimensional waves on top of the bubble. These are used to explain the apparent 'wrong sign' of the energy in the four–dimensional Einstein equation in section 4.6.

In this chapter, we are interested in four–dimensional branes embedded in a five–dimensional bulk. The bulk is always taken to be anti–de Sitter and, for notational convenience, we often switch notation between the five–dimensional cosmological constant Λ_5 and the associated anti–de Sitter scale k, defined by

$$\Lambda_5 \equiv -6k^2. \tag{4.1}$$

We also define the scale K set by the four–dimensional cosmological constant on the brane Λ_4 by

$$\Lambda_4 \equiv \pm 3K^2, \tag{4.2}$$

with $-(+)$ for an (anti–)de Sitter brane; a Minkowski brane corresponds to $K = 0$.

4.1 Hiding extra dimensions: braneworlds

The idea that the perceived world could be a membrane embedded ('braneworld') in some higher–dimensional space goes –at least– back to Rubakov and Shaposhnikov [153]. They put forward a field–theoretic toy model (neglecting gravitational effects) with a single, infinitely large extra dimension in which ordinary, visible particles are confined to a domain wall. Such localization of matter[1] is a direct consequence of the interactions between the fields and the domain wall itself; the higher–dimensional equations of motion admit solutions that are sufficiently peaked around the wall. Analogous to KK theories, there exist modes –typically also very

[1] Localizing gauge fields in an analogous manner is more subtle, see e.g. [154].

massive, but not necessarily discrete– that represent particles with momentum in the extra dimension. The difference with KK scenarios from the particle physics side is how the extra dimension 'opens up' at high energies. In KK theories this corresponds to the existence of a tower of very massive particles that are still –in principle– perceivable in the lower–dimensional world, while in localization scenarios this corresponds to particles being able to completely disappear into the bulk. In the former case, particle physics experiments directly constrain the size of the extra dimensions, while in the latter case it only constrains the 'thickness' of the membrane. It is of course highly non–trivial to establish the localization of all Standard Model fields. Nevertheless, the Rubakov–Shaposhnikov model makes it plausible that confinement to a domain wall or, more generally, to a membrane, can be achieved. Later, string theory made this manifest with the advent of D–branes that localize open–string states [155].

Rather crucially, the Rubakov–Shaposhnikov model neglects all gravitational effects. On the positive side, this means that –insofar as all interactions except gravity are concerned– the low–energy physics effectively appears four–dimensional because of confinement. On the other side, a consistent theory of gravity requires a graviton to propagate in the full spacetime; since gravity is fundamentally higher–dimensional, some mechanism is required to recover four–dimensional gravity, with its characteristic Newtonian $\sim 1/r^2$ scaling *on the membrane* (at least at sufficiently low energies).

A first attempt to include gravity in a braneworld scenario is provided by the toy model of Arkani–Hamed, Dimopoulos, and Dvali (ADD) [156]. It constitutes a 'hybrid' between a KK compactification and confinement: it presumes that the extra dimensions are flat and compact and –as in the aforementioned sense– that all visible matter can be localized on a three–dimensional membrane. As argued before, this means that, as far as particle physics is concerned, the world appears four–dimensional. Gravity, on the contrary, is able to propagate throughout all spacetime; from the standard lore of KK theories, the gravitational interaction at long distances is mediated by the KK zero mode. This would imply that Newtonian gravity is recovered at energies well–below the compactification scale. The short–distance behavior of gravity is determined by the inhomogeneous KK modes, which

modify Newtonian gravity. Since gravity is the only interaction that is sensitive to the bulk, the constraints on the size of the compact manifold become weaker since gravity is not tested at LHC scales. Current bounds on the ADD model require more than four extra dimensions with a size smaller than 10^{-12} m [157].

Although the ADD model involves gravity, it does not incorporate it in an entirely satisfying manner: the model neglects the backreaction of the brane onto the geometry. A gravitating brane, in fact, has a non–trivial effect: the geometry becomes 'warped'. A warped geometry generalizes a factorizable (KK) metric. For instance, a warped compactification of ten dimensions down to four, for concreteness denoting the coordinates by (x^μ, θ^m), proceeds through the Ansatz

$$\mathrm{d}s_{10}^2 = A^2(\theta)\,\mathrm{d}s_4^2 + \mathrm{d}s_6^2, \tag{4.3}$$

where A is called the *warp factor*. It describes how the four–dimensional geometry $\mathrm{d}s_4$ is deformed by moving through the extra dimensions. This geometry preserves the desired four–dimensional symmetries, and, as such, generalizes the KK Ansatz, see e.g. (3.23). Although warped compactifications have been considered before (see e.g. [134]), the first scenario proposed by Randall and Sundrum [18], also called the RS I model, exhibits novel features due to the presence of a gravitating brane. Like ADD, the RS I model constitutes a hybrid between a (warped!) compactification and localization. Crucially, this model still involves a –not necessarily small– compact extra dimension and two branes, one of which has a negative tension. Since the RS I model is not particularly relevant in the context of this thesis, it will not be further discussed.

It is worth taking a minor detour to discuss one of the motivations for studying braneworld scenarios: as realized in [18, 156], it provides an opportunity to relax the electroweak hierarchy problem[2]. With a n compact extra dimensions, the four–dimensional Planck scale M_{Planck} can be expressed in terms of the fundamental, higher–dimensional Planck scale $M \equiv \kappa_{4+n}^{-2/(n+2)}$, with κ_D the D–dimensional

[2]Put simplistically, the electroweak hierarchy problem poses that it is *unnatural* (in the sense of section 2.2) to have a large hierarchy between the electroweak scale $M_{\mathrm{EW}} \sim 1$ TeV and the Planck scale $M_{\mathrm{Planck}} \sim 10^{19}$ GeV.

gravitational constant, as

$$M^2_{\text{Planck}} = M^{n+2} \int \mathrm{d}^n\theta \sqrt{|\gamma|} A^2(\theta), \tag{4.4}$$

with γ the metric on the compact n–dimensional space. This suggests a 'quick fix': the fundamental scale M can be pushed to order TeV scale if the integration over the warp factor is sufficiently large, thereby removing the hierarchy between the fundamental scales (higher–dimensional gravity and electroweak scale). This is the approach followed by ADD. The RS I scenario proposes to do the opposite and to raise the fundamental electroweak scale, set by the Higgs mass, to a Planckian value. This is achievable if our conventional, visible matter is located on a membrane that is sufficiently warped; if it is located in the extra dimensions at some coordinates $\theta = \theta_0$, then the 'fundamental' Higgs mass m relates to the physical Higgs mass m_0 by $m = A(\theta_0)m_0$. Clearly, m can be Planck scale while m_0 is TeV scale if the geometry is sufficiently warped. This detour illustrated –very roughly– how extra dimensions might help to alleviate hierarchy problems; this is in line with the idea that cosmic eggs (chapter 3) potentially relax the cosmological hierarchy problem.

In a subsequent article, Randall and Sundrum [17] demonstrated that the negative tension brane can be consistently removed from the set–up. This model, commonly known as the RS II model, involves only one brane which warps the geometry and an *infinitely* large extra dimension. In that sense, it provides 'an alternative to compactification'. Prior to RS II, it was standard lore that the extra dimensions are required to be sufficiently small: even if localization of particles can be achieved, gravity still permeates into the extra dimensions, thereby possibly spoiling Newtonian gravity even at low energies on the brane. The revolutionary aspect of the RS II model, which avoids contradiction with low–energy gravitational experiments, is the proof of principle that higher–dimensional gravity has excitations that can be localized onto the brane. This is a direct consequence of the coupling between the graviton and the energy of the membrane (akin to confinement in the Rubakov–Shaposhnikov picture, and crucially absent in the ADD model). This localized mode of the graviton gives rise to effectively four–dimensional gravity on the brane.

Randall–Sundrum models (or suitable generalizations thereof) provide an interesting phenomenological playground that goes against the spirit of compactifying extra dimensions. From a top–down perspective of string theory, no clear phenomenologically relevant construction has been obtained so far; the situation one is facing seems to be akin to constructing de Sitter vacua from a compactification [158]. Motivated by both difficulties, this chapter introduces an alternative braneworld scenario that comes close to being UV–completable within string theory [150]. It can be instructive to compare some of its aspects with the 'classic' braneworld scenarios of Randall–Sundrum (and its generalization, known as Karch–Randall). The main features of these scenarios are therefore briefly discussed first.

4.2 Randall–Sundrum and Karch–Randall braneworlds

In this section, we briefly review the set–up of braneworlds as originally envisioned by Randall and Sundrum [17,18], later extended by Karch and Randall [159]. Much of the covered material can be found in the original papers, as well as in many excellent reviews on this matter (see e.g. [160–163]).

The key ingredient of these scenarios is a four–dimensional *gravitating* brane in a five–dimensional bulk. At scales larger than the brane thickness (typically set by particle physics scales), the brane can be approximated by a delta–function source of the gravitational field. Geometrically, it then spans a hypersurface in the bulk, as was conceptually illustrated in figure 1.2b. This is the approach that we will follow in this chapter: we leave aside any microscopic description of a brane and treat it as a separate entity with properties of its own. The general formalism for describing hypersurfaces in General Relativity, conventions and definitions on this matter are outlined in appendix C to which the reader is referred at this point.

For convenience, the five–dimensional bulk coordinates are written as $x^\mu = (y^m, z)$ and the brane is put fixed at $z = 0$. The action determining the dynamics of this system is given by

$$S = \frac{1}{2\kappa_5^2}\int \mathrm{d}^5x\sqrt{|g|}\left({}^{(5)}\mathcal{R} - 2\Lambda_5\right) + \frac{1}{\kappa_5^2}\int \mathrm{d}^4x\sqrt{|\zeta|}\,\mathcal{K} - \sigma\int \mathrm{d}^5x\sqrt{|h|}\delta(z). \quad (4.5)$$

The first term is the conventional Einstein–Hilbert term describing the gravitational dynamics in five dimensions; the subscript on Λ_5 emphasizes that it represents the cosmological constant in the bulk. The second term is the Gibbons–Hawking–York boundary term, with ζ being the induced metric on the boundary and $\mathcal{K}$ the trace of its extrinsic curvature. In principle, there can be multiple boundaries, but in the present model only the brane constitutes one as $z \to 0^{\pm}$ (in which case ζ coincides with the induced metric on the brane). This demonstrates how we view the set–up: two spacetimes $\mathcal{M}_{\pm}$ are glued across a thin-shell Σ which acts as the boundary of both spacetimes, i.e. $\Sigma = \partial\mathcal{M}_{\pm}$. The last term in the action describes the presence of this brane. For simplicity, it is taken to be a simple membrane that is characterized by just one parameter, its tension σ. The induced metric on the brane is denoted by h.

The equations of motion are obtained by varying the action with respect to the five–dimensional metric. This yields the five–dimensional Einstein equation away from the brane

$$ {}^{(5)}R_{\mu\nu} - \frac{1}{2}\,{}^{(5)}\mathcal{R}g_{\mu\nu} + \Lambda_5 g_{\mu\nu} = 0, \qquad (z \neq 0) \tag{4.6a} $$

and, as outlined in appendix C, the Israel equation which describes how the geometry reacts to the presence of brane

$$ \triangle K_{mn} - \triangle\mathcal{K}h_{mn} = \kappa_5^2 S_{mn}, \tag{4.6b} $$

where $\triangle$ denotes the jump of a certain quantity across the brane and where $S_{mn} = -\sigma h_{mn}$ is the brane energy–momentum tensor. The bulk Einstein equation (4.6a) ensures that the spacetime away from the brane is maximally symmetric: Minkowski (MkW), de Sitter (dS) or anti–de Sitter (AdS). This symmetry is not realized globally but broken by the non–tensionless brane.

In braneworld scenarios, these equations of motion are often solved under the simplifying assumption that the brane itself is also maximally symmetric. The most general Ansatz that preserves the desired four–dimensional symmetries (Poincaré, de Sitter, or anti–de Sitter invariance) is the warped geometry

$$ \mathrm{d}s^2 = A^2(z)\gamma_{mn}(y)\,\mathrm{d}y^m\,\mathrm{d}y^n + \mathrm{d}z^2, \tag{4.7} $$

where γ_{mn} is metric of MkW_4, dS_4, or AdS_4 in some y^m coordinate system. The cosmological constant associated with the maximally symmetric brane will be denoted Λ_4, the subscript emphasizing it is the one perceived by observers confined to the brane. This Ansatz provides an alternative view of the set–up, namely choosing coordinates in the bulk by foliating it with a specific set of leaves whose geometry is fixed by γ. The warp factor $A(z)$ then describes how the leaves are distorted when moving transversely. Whether such a foliation is possible, depends on whether a solution to the Einstein equation in the bulk (4.6a) exists. The physics enters the stage through the junction condition (4.6b): two halves of two such foliations are glued together across a three–brane. A brane of non–zero tension typically sources a jump discontinuity in $A'(z)$.

The induced metric on the brane is given by

$$h_{mn} = A^2(0)\gamma_{mn}. \tag{4.8}$$

With the above Ansatz, the Einstein equation away from the brane (i.e. $z \neq 0$) becomes

$$\left(\frac{A'}{A}\right)^2 = -\frac{\Lambda_5}{6} + \frac{\Lambda_4}{3A^2}, \tag{4.9a}$$

$$\frac{A''}{A} = -\frac{\Lambda_5}{6}, \tag{4.9b}$$

and the Israel equation is given by

$$\triangle A'(0) = A'(0^+) - A'(0^-) = -\frac{1}{3}\kappa_5^2\sigma A(0). \tag{4.10}$$

The bulk Einstein equation (4.9a) does not always admit a viable solution for the warp factor; its existence crucially depends on the signs of the bulk and brane cosmological constant. This is summarized in table 4.1. The Randall–Sundrum and Karch–Randall scenarios focus on an AdS_5 bulk. While two other options are equally viable (see e.g. [164]), we follow their lines of thought and restrict our attention to a negative cosmological constant in the bulk[3]. By combining the

[3]Although not explicitly discussed here, it is possible to obtain the warp factor for a dS_4 brane in a MkW_5 bulk by formally taking the limit $k \to 0$ of (4.14), as well as the warp factor for a dS_4 brane in a dS_5 bulk by analytically continuing $k \to ik$ in (4.14) [165].

Brane \ Bulk	AdS_5	MkW_5	dS_5
AdS_4	✓	×	×
MkW_4	✓	×	×
dS_4	✓	✓	✓

Table 4.1: Allowed brane configurations. In principle MkW_5 can be foliated with MkW_4 slices, but since this is trivial (and would correspond to a tensionless brane) we indicated MkW_4/MkW_5 with a cross.

Israel equation (4.10) and the first Einstein equation (4.9a), it can be seen that the multi–valuedness of (4.9a) must be exploited to have a non–zero brane tension. In particular, this requires a $\mathbb{Z}_2$ symmetry across the brane; the brane thus sits at a fixed point. The reason for this is the identical cosmological constant Λ_5 on both sides of the brane. This demonstrates that in the Randall–Sundrum and Karch–Randall set–ups the brane glues together two *identical* spacetimes (in the notation used before $\mathcal{M}_+ = \mathcal{M}_-$).

Restricting attention to positive brane tensions only, one finds that the *effective* four–dimensional cosmological constant is given by

$$\Lambda_4 = \frac{\kappa_5^4 \sigma^2}{12} - 3k^2. \tag{4.11}$$

Its dependence on the brane tension is plotted in figure 4.1. The explicit usage of 'effective' emphasizes that Λ_4 is the cosmological constant perceived by observers confined to the brane; it does *not* correspond to the vacuum energy on the brane, which is controlled by the brane tension. This is peculiar to warped geometries [134] and manifestly different from non–braneworld scenarios where 'vacuum energy' and 'cosmological constant' are synonymous. In braneworld scenarios, it is a combination of the bulk and brane vacuum energy that determines the four–dimensional cosmological constant. The cosmological constant problem (i.e. the unnaturally small Λ_4 in Planck units (2.31)) thus manifests itself as the unnatural

fine–tuning between the bulk cosmological constant and the brane tension. In other words, it poses the question why our brane –if we would happen to live on one– is nearly flat [135, 166].

As clear from (4.11), a flat brane is achieved when the brane tension is fine–tuned to the *critical* value $\sigma_{\rm cr} \equiv 6k/\kappa_5^2$, which makes Λ_4 vanish. This critical brane tension is a particular value that separates an AdS_4 brane and a dS_4 brane. To sum up, the three possible brane configurations in AdS_5 are [165]:

- Subcritical branes with $\sigma < \sigma_{\rm cr}$. These have an induced anti–de Sitter metric and the warp factor and brane tension are given by

$$A(z) = \frac{K}{k}\cosh\left[\cosh^{-1}\left(\frac{k}{K}\right) - k|z|\right], \quad \kappa_5^2\sigma = 6K\sqrt{\left(\frac{k}{K}\right)^2 - 1}. \tag{4.12}$$

  Clearly, there is a maximal value for the AdS_4 scale K, reached when the brane tension is zero. In this case, the AdS_4 radius matches the AdS_5 radius.

- Critical branes with $\sigma = \sigma_{\rm cr}$. These branes have a Minkowski geometry and correspond to the four–dimensional scale $K \to 0$. This was the original focus of Randall and Sundrum. The warp factor and tension are given by

$$A(z) = e^{-k|z|}, \qquad \kappa_5^2\sigma_{\rm cr} = 6k. \tag{4.13}$$

  The brane tension is fine–tuned such that it exactly cancels the contribution from the bulk cosmological constant to the induced curvature.

- Supercritical branes with $\sigma > \sigma_{\rm cr}$. These have an induced de Sitter metric and the warp factor and brane tension are given by

$$A(z) = \frac{K}{k}\sinh\left[\sinh^{-1}\left(\frac{k}{K}\right) - k|z|\right], \quad \kappa_5^2\sigma = 6K\sqrt{\left(\frac{k}{K}\right)^2 + 1}. \tag{4.14}$$

  This coordinate system exhibits a singularity at $k|z| = \sinh^{-1}(k/K)$. Since the bulk is pure AdS_5, it is merely a coordinate singularity which represents a Rindler–like horizon. Consequentially, the range of the coordinate z is restricted to $k|z| < \sinh^{-1}(k/K)$.

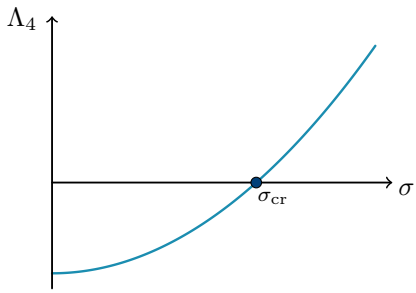


Figure 4.1: The dependence of the effecive four–dimensional cosmological constant Λ_4 on the brane tension for the $\mathbb{Z}_2$–symmetric braneworlds of Randall–Sundrum and Karch–Randall. The Randall–Sundrum model with a flat brane corresponds to tuning the brane tension to the critical one.

By virtue of the $\mathbb{Z}_2$ symmetry, and the non–negative brane tension, the positive and negative z regions both represent the same 'inside'[4] region of an AdS_5 space. For this reason, these constructions are often referred to as *inside–inside* constructions.

One of the crucial aspects that makes these braneworld models viable for phenomenology is that, although the extra dimension is infinite in extent, gravity still appears four–dimensional for low–energy observers on the brane. A first clue of this can be found in the weak–field limit of linearized gravity, where perturbative sources can be added to the brane to examine whether they experience four–dimensional gravity. To this extent, we consider gravitational fluctuations around the brane background (4.7), corresponding to

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}. \tag{4.15}$$

Indices are raised and lowered with the background metric $g_{\mu\nu}$. Following [159], we impose a Gaussian normal gauge

$$\delta g_{z\mu} = 0, \tag{4.16}$$

[4]This is definitely true for the MkW_4 and dS_4 brane. For the AdS_4 brane this is more subtle since the brane reaches the boundary of AdS_5. We will, nevertheless, call this inside–inside constructions.

and consider[5] only fluctuations that are transverse and tracefree (TT)

$$\nabla^\mu \delta g_{\mu\nu} = 0, \qquad\qquad g^{\mu\nu}\delta g_{\mu\nu} = 0. \tag{4.17}$$

The linearized Einstein equation for these fluctuations becomes

$$\frac{1}{2}\Box_5 \delta g_{mn} + {}^{(5)}R_{ambn}\delta g^{ab} = 0, \tag{4.18}$$

where $\Box_5 \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the five–dimensional covariant d'Alembertian. After some manipulations, this reduces to [167]

$$\frac{1}{A^2}\left(\Box_4 \delta g_{mn} - \frac{2\Lambda_4}{3}\delta g_{mn}\right) + \delta g''_{mn} - \frac{2A'^2 + 2AA''}{A^2}\delta g_{mn} = 0, \tag{4.19}$$

where $\Box_4 \equiv \gamma^{mn}D_m D_n$ with D_m the brane covariant derivative and where a prime denotes a derivative with respect to z. The gravitational wave equation in four dimensions is recognized in the first term in brackets; it suggests performing a separation of variables and solving for modes. This is akin to KK reductions where a higher–dimensional theory induces KK states at low energies.

We proceed with the Ansatz

$$\delta g_{mn}(y,z) = A^{1/2}(z)\Psi(z)H_{mn}(y), \tag{4.20}$$

where H_{mn} describes massive four–dimensional gravitons whose mass m, to be determined below, is characterized by a particular separation constant:

$$\left(\Box_4 - \frac{2}{3}\Lambda_4\right)H_{mn} = m^2 H_{mn}. \tag{4.21}$$

The z–dependent part of (4.19) can be simplified by introducing the conformal coordinate ξ, defined by

$$\xi \equiv \int_0^z \frac{\mathrm{d}z'}{A(z')}. \tag{4.22}$$

In terms of ξ, equation (4.19) reduces to a Schrödinger–like equation

$$-\frac{\mathrm{d}^2\Psi}{\mathrm{d}\xi^2} + V(\xi)\Psi = m^2\Psi, \tag{4.23a}$$

[5] A treatment about non–TT modes can be found in the original Karch–Randall paper [159], or in the review [162].

with potential

$$V(\xi) = \frac{3}{4}\left(\frac{A'}{A}\right)^2 + \frac{3}{2}\frac{A''}{A}, \tag{4.23b}$$

where a prime now denotes a derivative with respect to ξ. The appearance of the second derivative in (4.23b) is crucial: since the warp factor in the present set–up generically involves an absolute value, see (4.12)–(4.14), the effective potential $V(\xi)$ typically involves a delta–function. This is illustrated in figure 4.2 for three types of braneworlds. Close to the brane, the potential is 'peaked' with a negative delta–function at the location of the brane; this mountain–and–crater structure is the reason why one often refers to it as the 'volcano potential'.

The form of (4.23) is very peculiar:

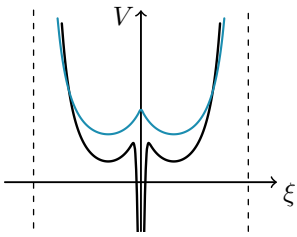
$$\left(-\frac{\mathrm{d}}{\mathrm{d}\xi} + \frac{3}{2}\frac{A'}{A}\right)\left(\frac{\mathrm{d}}{\mathrm{d}\xi} + \frac{3}{2}\frac{A'}{A}\right)\Psi = m^2\Psi. \tag{4.24}$$

This makes it transparent that a massless zero modes always exist that satisfies the boundary condition imposed by the brane:

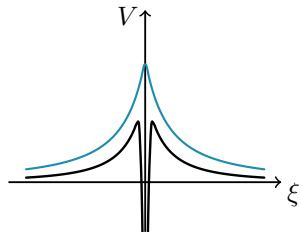
$$\Psi_{m=0}(\xi) \propto A^{3/2}(\xi). \tag{4.25}$$

This zero mode is also shown in figure 4.2 for the different braneworlds. In all three cases, this zero mode exhibits a 'peak' close to the brane. Yet, the behavior of the zero mode in the AdS_4 braneworld is different from its counterpart in the MkW_4 and dS_4 braneworlds away from the brane.

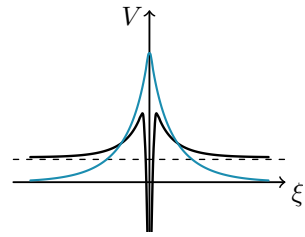
For flat and positively curved branes, the zero mode (4.25) is normalizable. It peaks around the location of the brane and completely decays as $\xi \to \pm\infty$; this bound state is interpreted as the *localization* of gravity [17,159]. This bound state is ultimately responsible for the recovery of four–dimensional gravity on the brane. Besides it, there also exists a continuum of modes –akin to KK states– with $m^2 > 0$ that is separated from the zero mode by a mass gap of $\triangle m^2 = 9K^2/4$, as indicated by the horizontal dashed line in figure 4.2c. Note that this mass gap vanishes in the limit of a Minkowski brane. They do not spoil the low–energy gravitational theory because of their weak coupling to fields localized on the brane [161].



(a) For an AdS_4 brane, the potential and zero mode diverge near the boundary of AdS (dashed).



(b) For a MkW_4 brane, gravity is localized and there is no mass gap.



(c) For a dS4 brane, gravity is localized and the potential asymptotes to a constant (dashed) near the de Sitter horizon.

Figure 4.2: Plot of the potential (black) and zero mode (blue) for the Randall–Sundrum and Karch–Randall braneworlds in terms of the conformal coordinate ξ. The brane is located at $\xi = 0$. The corresponding delta–function is, for illustrative purposes, exaggerated.

For the AdS_4 braneworld, the range of the conformal coordinate is limited, as indicated by the vertical dashed line in figure 4.2a. This effectively creates a box–type potential, which results in a discrete spectrum of massive modes. The boundary of this box corresponds to the boundary of AdS_5. Crucially different from above, the zero mode (4.25) is non–normalizable and cannot localize gravity. Instead, it has been shown that, when the brane tension is slightly subcritical (i.e. $K/k \ll 1$), there exists a bound state with small, but non–zero mass $m^2 \sim (K/k)^2 K^2$, which is responsible for four–dimensional gravity [159, 168–170]. It is only in this limit that gravity effectively appears four–dimensional on the brane.

Localization of gravity on its own is obviously not sufficient to have a theory of gravity on the brane that, in the low–energy limit, reduces to conventional four–dimensional gravity as we know it. With the perturbative approach outlined above, one could examine the effective gravitational potential on the brane; gravity arises as an exchange of all the graviton modes. For phenomenological reasons, this should be Newtonian at low energy. For simplicity, we restrict the discussion to the Randall–Sundrum model. Adding a small amount of matter on the brane

causes it to bend [171,172]. For a conceptual illustration of brane bending due to a localized mass, the reader is referred ahead to the left portion of figure 4.6. When brane bending is properly taken into account, the gravitational potential of some mass M on the brane can be shown to be

$$V(r) = \frac{\kappa_4^2 M}{4\pi r}\left[1 + \mathcal{O}\left(\frac{1}{(kr)^2}\right)\right], \tag{4.26}$$

where we identified the four–dimensional gravitational constant as

$$\kappa_4^2 = \kappa_5^2 k. \tag{4.27}$$

The first term in (4.26) arises from the exchange of the localized zero mode and coincides with four–dimensional Newtonian gravity. The correction terms arise from the exchange of the massive KK modes, whose contribution is suppressed at short distances compared to k^{-1}; at energies below the AdS_5 scale, gravity effectively appears Newtonian.

Newtonian gravity at low energies is not sufficient to ultimately claim that the model is phenomenologically viable; what one truly wants is relativistic gravity. This goes beyond the linearized treatment presented so far. By projecting the bulk Einstein equation onto the brane, with the aid of the Gauss–Codazzi equation, it has been verified that Einstein's gravity indeed persists on the braneworld at low energy [173]. Corrections typically involve contributions from the square of the brane energy–momentum tensor and non–local contributions from the gravitational field outside the braneworld. Because it would go beyond the scope of this thesis, we do not further discuss the issues of brane bending or the covariant reduction of the Einstein equation; we leave it on these short notes.

4.3 Dark Bubbles: a way out of the swampland?

Although string theory naively seems to present the ingredients that are necessary to construct the Randall–Sundrum and Karch–Randall braneworlds, it is unclear whether a phenomenologically viable model can be obtained at all. One of the obstructions is that the required branes are typically not the ones one gets from

string theory [174, 175]. Realizing de Sitter space, even by braneworld scenarios, seems to remain a difficult topic (see e.g. [176] and the discussion in [158]). It could therefore be important –or relevant– to consider other alternatives. Braneworld scenarios might offer yet another possibility, as suggested in [150]. In the remainder of this chapter, we focus on this particular model, later dubbed the Dark Bubble model. This is explored and discussed in a variety of papers of which (parts of) the following exposition are based [150, 177, 178].

4.3.1 Following stringy and swampy signposts

To motivate the Dark Bubble, we briefly present some elements of the swampland program that serve as 'signposts' towards it. We do not go into depth on the swampland constraints, but merely state and briefly discuss them. Arguments for their support can be found in the relevant references, or in the reviews on the topic (see e.g. [26, 179, 180]). We then use these signposts to argue in favor of the Dark Bubble model. We conceptually outline its key ingredients, which are then elaborated in the remainder of this section.

Signpost I: the de Sitter conjecture. [28, 29, 148, 149]
Given the difficulty in finding explicit dS solutions within string theory [24], it has been conjectured that simply no such constructions exist. We have to mention that this is not a fully proven no–go theorem about the existence of dS vacua in string theory and that the conjecture is not free of controversy (although it is better understood in certain controlled regimes). We remain agnostic about this contemporary fiery debate; we merely use it as a motivation to look for possible ways around the conjecture.

Signpost II: abundance of anti–de Sitter vacua.
In sharp contrast to dS vacua, AdS vacua seem to be rather ubiquitous in string theory. For instance, $\mathrm{AdS}_5 \times S^5$ is a well–understood background for any flux value [130]. Typically, AdS vacua do not feature scale separation [140].

Signpost III: the (electric) weak gravity conjecture. [181, 182]

In simple[6] terms, the weak gravity conjecture asserts that, in a gauge theory consistently coupled to gravity, there must exist an elementary charged object for which the gauge repulsion exceeds the gravitational attraction; in that sense, gravity is 'the weakest force' on at least one object in the theory. The nomenclature 'elementary' refers to particles or, more generally, to membranes. In general, for a p–form gauge field coupled to D–dimensional gravity, the conjecture is that there *must* exist a brane for which the tension σ is bounded by its charge q:

$$\frac{p(D-p-2)}{D-2}\sigma^2 \leqslant \left(\frac{q}{\kappa_D}\right)^2 . \tag{4.28}$$

A further sharpening of the conjecture claims that the inequality can only be saturated in supersymmetric theories [185]. Phrased differently, when the strict inequality holds, the weak gravity conjecture predicts a *superextremal*[7] object. The lowest possible p for which the weak gravity conjecture applies is $p = 0$. This is a priori not present in (4.28), but it can be argued that also in this case superextremal objects must exist. In this $p = 0$ case, the gauge field under consideration is an axion field and the charged objects are (-1)–branes, i.e. instantons. We briefly come back to this in the discussion of chapter 5.

Signpost IV: no non–supersymmetric AdS conjecture. [185, 186]

Vacuum decay is a well–known phenomenon in quantum field theory (coupled to gravity), which, under certain circumstances, constitutes a non–perturbative instability of the system. For a consistent theory of quantum gravity, it has been conjectured that non–supersymmetric AdS vacua that are –in part– supported by fluxes are at most metastable. In the context of flux vacua, the instability can

[6] As formulated here, the (electric) weak gravity conjecture takes its simplest form owing to the presence of only one gauge field. Nevertheless, more complicated scenarios arise as well, such as the case where multiple U(1) gauge fields are present [183], or when scalar fields are present [184]. The given formulation, however, is more than sufficient for the purpose of this thesis; generalizations of the weak gravity conjecture are not further discussed, and the reader is referred to the appropriate literature.

[7] States that saturate (4.28) are called extremal. It could be instructive to consider the $D = 4$, $p = 1$ case, which corresponds to four–dimensional Einstein–Maxwell theory. The weak gravity conjecture then predicts the existence of a particle with mass m and charge q, satisfying the bound $m < \sqrt{2}q/\kappa_4$. Naively speaking, this would be a *superextremal* Reissner–Nordström black hole. In the general case, the weak gravity conjecture implies the existence of a brane whose charge–to–tension ratio violates the extremality bound for the black brane solution [182].

proceed through the nucleation of a spherical codimension–one brane with less flux inside [137, 138, 187]. Whether or not such a bubble can nucleate, depends on the charge–to–tension ratio of the membranes in the theory. This is intuitively clear: the brane tension (gravity) typically forces the bubble to contract, while fluxes (gauge force) force the bubble to expand (cf. the discussion in section 3.1). If there exists a brane whose tension is sufficiently low, bubble nucleation constitutes an instability of AdS: gravity loses from the gauge force and the bubble inevitably expands, thereby changing the amount of flux (i.e. the cosmological constant). Following the weak gravity conjecture, such a brane must exist if AdS is not supersymmetric. This is not directly obvious from (4.28) with $p = D - 1$, but it has been conjectured in [177] to generalize the weak gravity conjecture by the replacement $(D - p - 2) \to |D - p - 2|$.

Dark Bubbles: the road out of the swamp?

The braneworld scenario proposed in [150] provides a realization of (phenomenologically relevant) de Sitter space in a string–inspired scenario where instabilities –that typically push a theory towards the swampland– are not considered problematic, but are rather turned into a virtue. Since the interest in this thesis is four–dimensional cosmology, the following discussion is adapted to the realization of a dS_4 cosmology (although it most likely generalizes to other dimensionalities as well).

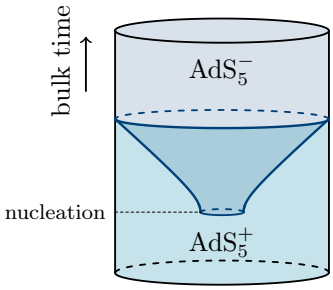
The proposal starts from a five–dimensional time–independent flux compactification of string theory that has (at least) two non–degenerate AdS_5 vacua. As argued below, scale separation is *not* an essential ingredient. Crucially, however, the vacuum with the larger (less negative) cosmological constant is required to be non–supersymmetric. This state is put forward as the 'starting point' for the model. If the presumed landscape of non–supersymmetric AdS vacua is sufficiently rich, it seems likely that such set–up is obtainable from string theory [151, 152].

In light of the weak gravity conjecture and, relatedly, the no non–supersymmetric AdS conjecture, this state is merely 'a false vacuum' and condemned to decay. One possible decay channel for vacua supported by fluxes is the nucleation of a bubble of another AdS vacuum with a different cosmological constant. The

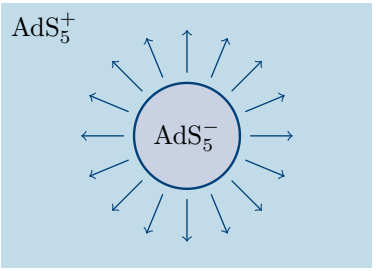
Dark Bubble model thus emerges naturally whenever there is a codimension–one brane in the theory that can mediate the decay of AdS_5. In principle, this could constitute a whole 'cascade' of decays which ultimately ends with the nucleation of a supersymmetric AdS. This, however, does not complicate the Dark Bubble model, since multiple nucleated bubbles will never collide; it is, therefore, sufficient to specialize to a single nucleation event. This is discussed in section 4.3.2. Although an unstable AdS_5 space whose primary decay proceeds through bubble nucleation is the only requirement for the model, it seems rather difficult to find a vacuum that decays in this way (see however [151, 152]).

The key insight of the Dark Bubble model is that it realizes a four–dimensional de Sitter cosmology on top of the expanding brane mediating the decay: expansion of the bubble in AdS_5 bulk corresponds to the accelerated expansion of the four–dimensional universe for observers confined to the wall. A conceptual illustration can be found in figure 4.3. A closed universe is automatic in this construction. Although most phenomenologically relevant scenarios involve universes with a flat topology, this is in fact included in the Dark Bubble model as well: when the bubble is sufficiently evolved, the brane appears flat for a local observer (the spatial curvature is unobservable within the observer's Hubble radius at late times). The name 'Dark Bubble' finds its origin in the fact that it is the bubble which gives rise to the dark energy perceived in four dimensions. How this exactly works, is outlined in section 4.3.3. To make contact with cosmology, the model should also allow for the inclusion of matter and radiation in the four–dimensional world. As explained in section 4.3.4, this can be achieved by introducing features in the bulk AdS space. This is one of the interesting aspects of the model: all features in the four–dimensional cosmology find an interpretation in the bulk. Another application of this 'higher–dimensional view' is provided in section 4.4.

The Dark Bubble set–up is similar to, but also crucially different from, Randall–Sundrum and Karch–Randall braneworlds. The nucleation event requires a brane construction with an inside and an outside. This should be contrasted with the $\mathbb{Z}_2$–symmetric braneworlds having two insides. Asymmetric braneworlds have been considered before and this is briefly commented on in section 4.3.5.



(a) Nucleation of a bubble converts all of AdS_5^+ into AdS_5^- in a finite amount of (bulk) time.



(b) Constant time slice of the panel (a). The bubble boundary realizes a cosmological braneworld scenario.

Figure 4.3: Schematic representation of a brane that has nucleated in a background, metastable AdS_5^+ with cosmological constant Λ_+. The interior of the bubble contains an AdS_5^- space with a lower cosmological constant $\Lambda_- < \Lambda_+$. After nucleating at rest the bubble expands outwards towards the conformal boundary of AdS_5^+, thereby realizing a dS_4 cosmology on top.

The Dark Bubble could thus potentially provide a new way to realize de Sitter space in string theory. It radically differs from other, more conventional attempts. Firstly, it is not a fundamental, scale–separated, time–independent de Sitter vacuum in the strict sense; rather, it is a time–*dependent* construction. Secondly, the extra dimension present in the model is not compact, but infinite in extent. Although the way in which de Sitter is realized is radically different than originally envisioned, a low–energy observer cannot distinguish between a fundamental de Sitter vacuum and the de Sitter space realized on top of Dark Bubble.

4.3.2 The decay of AdS_5

One possible decay channel of AdS vacua that are –at least in part– supported by fluxes is through bubble nucleation à la Brown and Teitelboim [137,138]. Intuitively, this is a higher–dimensional generalization of the Schwinger pair production in two–dimensional electromagnetism in which the role of the electric field is played

by higher–form gauge fields and the role of particles by membranes. This is a non–perturbative instability of AdS that can occur whenever the following conditions are met:

- a spacetime filling D–form flux is present in AdS_D;
- there exists a $(D-2)$–brane that sources this flux;
- the brane tension is 'subcritical'.

The first two requirements ensure that the cosmological constant assumes a dynamical role, making it sensitive to instabilities. The cosmological constant is partly determined by the flux which jumps discontinuously across a charged brane. A brane thus divides the entire spacetime into two regions with different cosmological constants. The last condition implies that a membrane can be created quantum mechanically. In the semi–classical approximation, this creation event may be described by an instanton.

If the aforementioned conditions are satisfied, the AdS background is unstable against membrane creation. A crucial restriction on their formation is energy conservation. It implies that only compact branes ('bubbles') can nucleate, since it costs infinite energy to nucleate a brane that is infinite in extent. Furthermore, in order to form, the energy cost of creating a brane with non–zero tension should be exactly compensated by a reduction of the energy density of the interior region. A bubble thus nucleates with a fixed radius and, after nucleation, it quickly accelerates outwards from zero velocity to 'eat up' all of AdS in a finite amount of (bulk) time.

This is in some respects analogous to bubble nucleation à la Coleman–de Luccia [136, 188, 189], but also different in others. Both correspond to a first–order phase transition, converting a false vacuum to a true (or 'more true') one. Yet membrane creation can occur repeatedly, while a Coleman–de Luccia transition typically occurs once. Brown–Teitelboim instantons in AdS thus provide a mechanism to continuously lower the cosmological constant until some minimal value is reached,

at which AdS is supersymmetric. Besides the fields supporting the background fluxes, it does not require any additional scalar fields like the Coleman–de Luccia mechanism.

We now briefly review Brown–Teitelboim instantons that describe the decay of a cosmological constant through membrane nucleation. We closely follow [137, 138], but quickly focus on the decay of AdS_5. The crucial ingredient is the existence of a 5–form field strength $F = \mathrm{d}A$ and a brane, whose tension is denoted σ, that is charged under the 4–form gauge field A. Although in principle multiple branes could nucleate, a Brown–Teitelboim instanton by definition describes the quantum creation of a *single*, closed membrane.

We thus aim at finding a solution to the Euclidean equations of motion in which one charged brane is present. We adopt a similar notation as in appendix C: bulk (brane) coordinates are denoted x^μ (y^m) and we specify the brane parametrically by the embedding $x^\mu(y)$. The Jacobian matrix is denoted e^μ_m and the brane has an induced metric h. The Euclidean action is given by

$$\begin{aligned} S_{\mathrm{E}} = \int \mathrm{d}^5x\sqrt{|g|}\left[-\frac{1}{2\kappa_5^2}\left({}^{(5)}\mathcal{R} - 2\Lambda_5\right) - \frac{1}{240}F^{\mu_1\mu_2\mu_3\mu_4\mu_5}F_{\mu_1\mu_2\mu_3\mu_4\mu_5}\right] \\ + \sigma\int \mathrm{d}^4y\sqrt{|h|} + \frac{q}{24}\int \mathrm{d}^4y\, A_{\mu_1\mu_2\mu_3\mu_4}e^{\mu_1}_{m_1}e^{\mu_2}_{m_2}e^{\mu_3}_{m_3}e^{\mu_4}_{m_4}\epsilon^{m_1m_2m_3m_4} \\ + \frac{1}{\kappa_5^2}\int \mathrm{d}^4x\sqrt{|\zeta|}\mathcal{K} + \frac{1}{24}\int \mathrm{d}^5x\sqrt{|g|}\nabla_\mu\left(F^{\mu\mu_1\mu_2\mu_3\mu_4}A_{\mu_1\mu_2\mu_3\mu_4}\right) \end{aligned} \tag{4.29}$$

The first line describes conventional five–dimensional gravity sourced by a 'bare' cosmological constant Λ_5 and the 5–form field strength F. The second line describes a brane with tension σ and charge q. The last line contains boundary terms that ensure the well–definedness of the variational principle (ζ being the boundary metric). Since F is of top degree, it is proportional to the volume form:

$$F_{\mu_1\mu_2\mu_3\mu_4\mu_5} = E(x)\varepsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}, \tag{4.30}$$

for some scalar function $E(x)$. However, the equation of motion for A shows that $E(x)$ is in fact constant away from brane sources and that it jumps discontinuously by one charge unit q when crossing the brane:

$$\partial_\mu E\, \varepsilon^{\mu\mu_1\mu_2\mu_3\mu_4} = -q \int \mathrm{d}^4 y\, \delta\left(x^\alpha - x^\alpha(y)\right) e^{\mu_1}_{m_1} e^{\mu_2}_{m_2} e^{\mu_3}_{m_3} e^{\mu_4}_{m_4} \epsilon^{m_1 m_2 m_3 m_4}, \qquad (4.31)$$

where ϵ is the anti–symmetric symbol. Inserting (4.30) into the action (4.29) demonstrates that F contributes to the cosmological constant; we define an effective bulk cosmological constant Λ by

$$\Lambda \equiv \Lambda_5 + \frac{1}{2}\kappa_5^2 E^2. \qquad (4.32)$$

Since E jumps discontinuously across a brane, the set–up consists of a conjunction of two *different* maximally symmetric spaces with *different* cosmological constants $\Lambda_\pm$ on either side of the membrane. The discontinuity in the cosmological constant is denoted

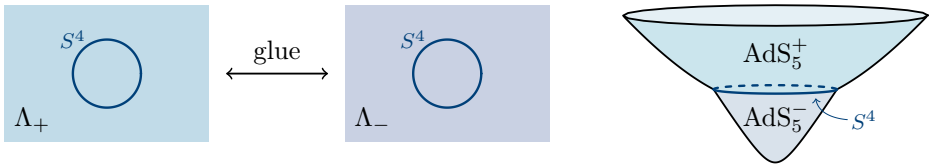
$$\triangle\Lambda \equiv \Lambda_+ - \Lambda_- = \kappa_5^2 \left(\epsilon E_+ q - \frac{1}{2} q^2\right), \qquad (4.33)$$

where $\epsilon = +1\ (-1)$ depending on whether E_+ is larger (smaller) than E_-. For convenience, we let quantities indexed by $+$ $(-)$ refer to quantities on the 'outside' ('inside') of the brane, although this nomenclature could possibly be counter–intuitive, ambiguous, or seemingly ill–defined. Many circumstances allow for analysis without an explicit mentioning of the outside/inside label; when clear, we therefore omit the $\pm$ label for notational simplicity. As in Lorentzian signature, figure 4.3, 'outside' refers to the configuration prior to membrane creation.

Since it is argued that spherically symmetric instantons minimize the action [190], the dominant instantons are geometries of the form

$$\mathrm{d}s^2 = N^2(r)\, \mathrm{d}r^2 + a^2(r)\, \mathrm{d}\Omega_4^2, \qquad (4.34)$$

where N is a lapse function and $\mathrm{d}\Omega_4^2 = \gamma_{mn}\, \mathrm{d}y^m\, \mathrm{d}y^n$ is the metric on the unit S^4. To respect spherical symmetry, the brane must have an S^4 topology and thus be located at some specific value of the r coordinate. The brane glues together two portions of different (Euclidean) spaces with cosmological constants $\Lambda_\pm$ as conceptually depicted in figure 4.4a. The first junction condition implies that the two spaces are connected across an S^4 with the *same* proper radius a. The brane–



(a) Pictorial representation of the construction of the Brown–Teitelboim instanton. One portion of the left space (i.e. one side of the S^4) is connected to one portion of the right space across an S^4 brane.

(b) Instanton that mediates the decay of AdS_5^+ to AdS_5^- with $\Lambda_- < \Lambda_+$.

Figure 4.4: The Brown–Teitelboim instanton.

induced metric is therefore given by

$$h_{mn} = a^2 \gamma_{mn}. \tag{4.35}$$

There are a priori four possibilities regarding which portion of one space is glued to the other. This translates into a choice of outward pointing normal (in the direction of increasing or decreasing S^4 area), which ultimately propagates into a sign of the extrinsic curvature of the brane:

$$K^{\pm}_{mn} = s_{\pm} \frac{a}{N} \frac{\mathrm{d}a}{\mathrm{d}r} \gamma_{mn}, \tag{4.36}$$

where $s_{\pm} = +1\,(-1)$ if the AdS_5^+ outwards pointing normal points in the direction of increasing (decreasing) area of S^4.

The Euclidean equation of motion is the Einstein equation away from the brane

$$\left(\frac{1}{N} \frac{\mathrm{d}a}{\mathrm{d}r} \right)^2 = 1 - \frac{\Lambda}{6} a^2, \tag{4.37}$$

which is supplemented by the Israel equation that relates the brane tension to the jump in extrinsic curvature

$$\triangle K_{mn} = \frac{\kappa_5^2}{3} \sigma a^2 \gamma_{mn}. \tag{4.38}$$

By combining (4.36) and (4.37), this junction condition can explicitly be solved for the proper radius a of the S^4 brane:

$$\frac{1}{a^2} = \frac{1}{6}\Lambda_+ + \frac{1}{\kappa_5^4\sigma^2}\left[\frac{\kappa_5^4\sigma^2}{6} - \frac{1}{4}\triangle\Lambda\right]^2 . \tag{4.39}$$

This is exactly the radius of the bubble that can be created with a zero net energy cost. Indeed, energy cost due to the tension is compensated by the energy freed up by reducing the cosmological constant. Note that the expression above is always well–defined if $\Lambda_+ > 0$. However, this is not true for the case of interest, the decay of AdS, and (4.39) may be ill–defined. The reality condition on a thus entails a constraint on the parameters in order for bubble nucleation in AdS to occur at all. This is commented on below.

Further analysis of the junction condition rules out constructions that have $(s_+, s_-) = (-,+)$ as well as constructions with $(s_+, s_-) = (-,-)$ and simultaneously $\Lambda_- \geqslant 0$, since these would yield an unphysical brane of negative tension. A priori it thus seems that there exist five different instantons (to be listed and analyzed below) that could describe the decay of AdS. However, it can be argued that there is only *one* possible decay channel with non–zero decay probability. In the semi–classical approximation, the decay probability per unit spacetime volume is given by

$$P \sim e^{-B}, \tag{4.40a}$$

where B is called the 'bounce', measuring the difference in Euclidean action

$$B \equiv S_{\mathrm{E}}[\text{instanton}] - S_{\mathrm{E}}[\text{background}], \tag{4.40b}$$

where 'background' refers to the entire AdS_5^+ space, prior to membrane creation. The decay probabilities of the five possible instantons can now be analyzed.

- The decay to dS ($\Lambda_- \geqslant 0$) requires $s_+ = +1$. Regardless of the sign of s_-, this instanton is compact and thus differs with the background over an infinite region. Consequently, the bounce diverges $B = +\infty$ yielding a vanishing decay probability.

- The instanton with $(s_+, s_-) = (+, -)$ is always compact and, by the same reasoning, has $P = 0$.
- The decay to another AdS space, or to MkW, with a less negative cosmological constant ($\Lambda_+ < \Lambda_-$) requires $(s_+, s_-) = (+, +)$ and is not compact. It still differs from the background over an infinite region. By the same token, this does neither describe a possible decay channel of AdS.
- The only possible instanton that describes the decay of AdS is non–compact and differs from the background only over a finite region of space. It corresponds to an $(s_+, s_-) = (-, -)$ construction which has $\Lambda_- < \Lambda_+$. This instanton is depicted in figure 4.4b.

Vacuum decay of AdS thus has an irreversible and recurring character: through spontaneous creation of membranes, the cosmological constant is always –and continuously– lowered to a more negative value (in the notation of (4.1): $k_- > k_+$). From the reality requirement on (4.39), it follows that a nucleation event can only occur if the brane tension is strictly smaller than a critical value:

$$\sigma < \sigma_{\rm cr} \equiv \frac{3}{\kappa_5^2} \left(k_- - k_+\right). \tag{4.41}$$

The obstruction for supercritical branes to nucleate makes sense: for larger brane tension, the radius has to increase to compensate in the energy balance. However in AdS, the area and volume of the brane grow alike at large radial coordinate such that the energy liberated from an increase in volume can no longer compensate for the increase in energy due to the tension.

The critical tension can also be written in terms of the brane charge q, in which case the condition (4.41) becomes

$$\sigma < \sigma_{\rm cr} = \frac{\sqrt{3}}{2\kappa_5} q, \tag{4.42}$$

with is perfectly in line with the (proposed generalization of) the weak gravity conjecture (4.28). This also shows when the decay of AdS stops: whenever the underlying microscopic theory does not admit subcritical branes.

The probability to decay (per unit spacetime volume) can be estimated from the bounce:

$$B = \Omega_4 \sigma a^4 - \triangle \left[\frac{2\Lambda}{3\kappa_5^2} V_5 - \frac{4}{\kappa_5^2} \Omega_4 a^4 \sqrt{\frac{1}{a^2} - \frac{\Lambda}{6}} \right], \tag{4.43a}$$

where Ω_4 is the volume of the unit S^4 and V_5 represents the volume of the compact region in which the instanton and background solution differ

$$V_5 \equiv \int_0^a \mathrm{d}^5 x \sqrt{|g|}, \tag{4.43b}$$

with a given[8] in (4.39). An explicit calculation yields the bounce

$$\frac{\kappa_5^2}{\Omega_4} B = \kappa_5^2 \sigma a^4 - \mathcal{B}(k_+) + \mathcal{B}(k_-), \tag{4.44a}$$

with $\mathcal{B}(k)$ given by

$$\mathcal{B}(k) = \frac{3}{2k^3} \left[\sinh^{-1}(ka) - ka \left(2k^2 a^2 + 1\right) \sqrt{k^2 a^2 + 1} \right] \tag{4.44b}$$

The nucleation probability $P \sim e^{-B}$ is thus a rather complicated function of the parameters k_-, k_+ and σ. It is instructive to consider its dependence on the brane tension in the limit cases $\sigma \ll \sigma_{\mathrm{cr}}$ and $\sigma \to \sigma_{\mathrm{cr}}$. In the former case, the nucleation radius (4.39) is small compared to the AdS scales ($ka \ll 1$), which implies a relatively high nucleation probability $P \to 1$. When the brane tension increases, the nucleation probability decreases [137, 138]. When the brane tension is near criticality, the nucleation radius is large compared to the AdS scales ($ka \gg 1$), entailing a small nucleation probability $P \to 0$.

4.3.3 . . . lets a de Sitter universe emerge

Once a spherical bubble has nucleated, it quickly accelerates outwards. We now analyze the motion of this bubble in the bulk in Lorentzian signature. As before, + (−) denotes quantities on the outside (inside) of the bubble, where outside (inside) is defined as the region where the area of a radial slice decreases (increases) when moving towards the brane. From the previous part it should be clear that the Dark Bubble is an *inside–outside* construction where the AdS scale on the inside is larger

[8] In fact, it is an independent check of (4.39) that this value maximizes the bounce (4.44).

than the on the outside:

$$k_+ < k_-. \tag{4.45}$$

Furthermore the tension of the brane should be subcritical (4.41). Both inequalities are required by the nucleation event itself. The following discussion is based (mainly) on [150, 177].

The brane separates two $\mathrm{AdS}_5^{\pm}$ geometries, as illustrated in figure 4.3. In global coordinates, the outside and inside metrics are given by

$$\mathrm{d}s_\pm^2 = g_{\mu\nu}^\pm \,\mathrm{d}x^\mu \,\mathrm{d}x^\nu = -f_\pm(r)\,\mathrm{d}t_\pm^2 + \frac{\mathrm{d}r^2}{f_\pm(r)} + r^2\,\mathrm{d}\Omega_3^2, \tag{4.46a}$$

where $\mathrm{d}\Omega_3^2 = \gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ the metric on the unit S^3 and the function f for a pure AdS bulk is given by

$$f_\pm(r) = 1 + k_\pm^2 r^2. \tag{4.46b}$$

Usually, the time coordinate in the global coordinate chart of AdS has the full range $-\infty < t < \infty$. This, however, is not be realized for proposed construction: one has to take into account the inside–outside composition and the non–eternity of the bubble. The time domain of the outside geometry is $-\infty < t_+ < 0$, where the limit $t_+ \to 0$ corresponds to the bubble having eaten all of AdS_5^+. The inside geometry is only present once the bubble has nucleated, after which it persists forever. Therefore $t_\mathrm{N} < t_- < +\infty$, where t_N is the moment of nucleation. This is equally clear from figure 4.3a.

By introducing some time coordinate λ on the brane, the location of the brane can be described by the radial slice $r = a(\lambda)$, and the induced metric takes the FLRW form

$$\mathrm{d}s_\mathrm{ind}^2 = h_{mn}\,\mathrm{d}y^m\,\mathrm{d}y^n = -N^2(\lambda)\,\mathrm{d}\lambda^2 + a^2(\lambda)\,\mathrm{d}\Omega_3^2, \tag{4.47}$$

where N is an arbitrary lapse function that ensures λ–reparametrization invariance. The relation between bulk time t and brane time λ is given by

$$N^2(\lambda) = f(a)\dot{t}^2 - \frac{\dot{a}^2}{f(a)}, \tag{4.48}$$

where a dot denotes a derivative with respect to λ. The behaviour of the bubble in the bulk is dictated by the Israel equation[9]; the extrinsic curvature is explicitly given by

$$K_{\lambda\lambda} = \frac{N^2}{\dot{a}} \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[\frac{1}{N} \sqrt{f(a)N^2 + \dot{a}^2} \right], \tag{4.49a}$$

$$K_{ij} = -\frac{a}{N} \sqrt{f(a)N^2 + \dot{a}^2} \gamma_{ij}, \tag{4.49b}$$

where γ_{ij} is the metric on the unit S^3. The space–space component of the (trace–reversed) Israel equation is

$$\sqrt{\frac{f_-(a)}{a^2} + H^2} - \sqrt{\frac{f_+(a)}{a^2} + H^2} = \frac{1}{3}\kappa_5^2 \sigma, \tag{4.50a}$$

where we recall that $H \equiv \dot{a}/aN$ is the Hubble rate associated with cosmology on the bubble, cf. equation (2.3b). This is simply the first Friedmann equation in disguise:

$$H^2 = \frac{\kappa_5^4 \sigma^2}{36} - \frac{f_-(a) + f_+(a)}{2a^2} + \left[\frac{3}{2} \frac{f_-(a) - f_+(a)}{\kappa_5^2 \sigma a^2} \right]^2 \tag{4.50b}$$

The (redundant) second Friedmann equation follows from the time–time component of the junction condition.

For a pure AdS bulk, with f given by (4.46b), the Friedmann equation clarifies that the bubble describes a closed FLRW cosmology with a cosmological constant only:

$$H^2 = -\frac{1}{a^2} + \frac{1}{3}\Lambda_4, \tag{4.51a}$$

where the four–dimensional cosmological constant Λ_4 is explicitly given by the complicated, non–linear expression

$$\Lambda_4 = \frac{\kappa_5^4 \sigma^2}{12} - \frac{3}{2}\left(k_-^2 + k_+^2\right) + \frac{27}{4}\left(\frac{k_-^2 - k_+^2}{\kappa_5^2 \sigma} \right)^2. \tag{4.51b}$$

The dependence of Λ_4 on the brane tension is plotted in figure 4.5. Crucially, the Dark Bubble model has a positive four–dimensional cosmological constant; this is built in from the start! It immediately follows from the requirement that the brane

[9] Directly applying the Israel equation by computing the extrinsic curvature is the quickest option to analyze the dynamics of the bubble. Another –obviously equivalent– procedure is based on an action principle. This is presented in section 4.4, where it is demonstrated how the junction condition arises from λ–reparametrization invariance.

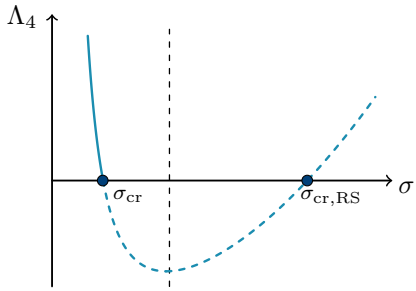


Figure 4.5: The dependence of the effective four–dimensional cosmological constant Λ_4 on the brane tension σ. Two possible brane constructions exist, separated by the vertical line. The region on the left corresponds to the inside–outside construction, inherent to the Dark Bubble model. The region on the right corresponds to an inside–inside construction of generalized asymmetric version of the Randall–Sundum and Karch–Randall braneworlds. Since only subcritical branes (4.41) can nucleate, only the solid line is realizable by the Dark Bubble model. In the limiting case where $k_+ \to k_-$, the region on the left disappears and one recovers the $\mathbb{Z}_2$ symmetry, cf. figure 4.1. One even recognizes the critical tension of the Randall–Sundrum world (4.13), denoted here by $\sigma_{\rm cr,RS}$.

tension is subcritical (4.41). Note that this is equally clear from the Friedmann equation (4.51a), which does not admit a solution if $\Lambda_4 \leqslant 0$. Therefore, any observer confined to the bubble experiences a de Sitter universe with positive spatial curvature.

We should revisit the comment at the end of section 4.3.2. This seems to suggest that a near–critical Dark Bubble with a phenomenologically small cosmological constant is much less likely to nucleate, compared to a bubble with a Planckian value for Λ_4. However, this claim does not really have a meaning since the nucleation event depends on which branes exist in the underlying microscopic theory (a fact that we have ignored so far). Indeed, from a UV perspective, one should not regard the brane tension as a free variable.

Akin to $\mathbb{Z}_2$–symmetric braneworlds, discussed in section 4.2, the brane appears flat ($\Lambda_4 = 0$) at the critical tension[10]. In contrast, MkW_4 or AdS_4 branes cannot be realized in the Dark Bubble scenario; only *sub*critical branes exist and these realize de Sitter. This is different than in Karch–Randall braneworlds where the tension has to be *super*critical to realize de Sitter, see (4.14).

For phenomenological purposes, we are interested in Dark Bubble cosmologies that resemble our late–time universe. We therefore introduce the 'phenomenological limit' of the Dark Bubble scenario which amounts to imposing that

- a four–dimensional observer does not see any spatial curvature;
- neither does is any bulk curvature observed;
- the induced cosmological constant is small in Planck units.

The former simply requires that the curvature contribution has sufficiently diluted in the Friedmann equation (4.51a). Physically, this means it is unobservable within the observer's Hubble radius, i.e. $Ka \gg 1$. The second requirement is realized by insisting that the four–dimensional Hubble scale is smaller than both AdS scales:

$$H \leqslant K \ll k. \tag{4.52}$$

This means that one can think of the AdS scale k as some 'UV scale'. Combining this with the first condition implies

$$1 \ll ka. \tag{4.53}$$

The last phenomenological demand implements that the universe has a built–in cosmological constant problem. This is achieved if the brane is tuned to a near subcritical tension, in which case the induced cosmological constant is by construction 'small'. We therefore take

$$\sigma = (1 - \epsilon)\sigma_{\mathrm{cr}}, \tag{4.54}$$

[10]The critical tension in the Randall–Sundrum braneworld (4.13) is recoverable form the critical tension in the Dark Bubble model (4.41) by formally flipping the sign of k_+ and setting $k_+ = k_-$. The sign flip corresponds to flipping the sign of the normal to the brane boundary of AdS_5^+, while imposing $k_+ = k_-$ corresponds to imposing $\mathbb{Z}_2$–symmetry.

with $\epsilon \ll 1$. Expanding (4.51b) in small ϵ yields

$$\Lambda_4 = \frac{2k_+k_-}{k_- - k_+}\kappa_5^2\left(\sigma_{\rm cr} - \sigma\right) + \mathcal{O}\left(\epsilon^2\right). \tag{4.55}$$

This allows one to identify[11] the four–dimensional gravitational constant κ_4^2 with

$$\kappa_4^2 \equiv \frac{2k_-k_+}{k_- - k_+}\kappa_5^2, \tag{4.56}$$

as well as the four–dimensional vacuum energy $\rho_\Lambda = \Lambda_4/\kappa_4^2$ with

$$\rho_\Lambda \equiv \sigma_{\rm cr} - \sigma. \tag{4.57}$$

If the 'naturalness assumption' is made that there is no large hierarchy between the AdS scales $k_\pm$, the dark energy density on the brane can be roughly estimated to be

$$\kappa_4^4\rho_\Lambda \sim \kappa_5^2 k^3\epsilon. \tag{4.58}$$

If, furthermore, the AdS vacua are assumed to be natural with k is comparable to the five–dimensional Planck scale, a tuning of $\epsilon \sim 10^{-120}$ is needed to fulfill the last phenomenological requirement.

Two comments are in order. Firstly, a natural AdS_5 vacuum would imply no scale separation between the AdS_5 scale and the KK scale associated with the extra dimensions. In the discussion above, we were tacit about the existence of possible dimensions. We, however, do not expect that the analysis changes much when these are included. Secondly, it could seem worrisome that the brane tension σ contributes *negatively* to the energy density perceived on the bubble (4.57). This is contrary to Karch–Randall scenarios where it seems to contribute positively to the four–dimensional energy density, see equation (4.11). We comment on this apparent issue in section 4.6 and argue that this is nothing to worry about.

[11] From the perspective of dimensional reduction, cf. equation (4.4), it seems that κ_4^2 would be infinite, since the integration over the warp factor diverges for the present setup–up. For a more careful analysis that yields this result, see [191, 192].

4.3.4 . . . that seems to be empty

For the Dark Bubble model to make contact with Standard Cosmology, it must –at least– allow for the inclusion of matter and radiation in the effective four–dimensional cosmology that is realized on top of the brane. It has long been known that the expansion rate of a 'brane cosmology' not only depends on the energy–momentum on the brane but also on the properties of the bulk geometry (see e.g. [163]). Indeed, features in the bulk backreact on brane through the Israel equation. This possibility has also been explored in the Dark Bubble scenario in [150, 177].

For $\mathbb{Z}_2$–symmetric braneworlds, it was quickly realized that generalizing the pure AdS_5 bulk to Schwarzschild–AdS_5 realizes a radiation component in the cosmic fluid [174]. This does not require any additional energy–momentum in the bulk; Schwarzschild–AdS is an exact solution to the Einstein equation with a cosmological constant only. In a similar vein, one can consider a Dark Bubble scenario that glues two different Schwarzschild–$AdS_5^{\pm}$ spaces together across a three–brane. This amounts to generalizing the function $f(r)$ appearing in the metric (4.46) to

$$f(r) = 1 - \frac{\kappa_5^2 M}{3\pi^2 r^2} + k^2 r^2, \tag{4.59}$$

where M is the ADM mass of the black hole. Note that M, as k, can take different values on the inside and outside of the bubble. The infinite AdS throat has now been replaced by the event horizon of the black hole. The junction condition (4.50) informs us how the brane responds to the presence of five–dimensional matter in the bulk. This is encapsulated in the first Friedmann equation. We focus on a brane that is moving well outside of the black hole horizons, i.e. $f(a) > 0$. In the limit $k^2 M \ll 1$, we find (see also [193])

$$H^2 = -\frac{1}{a^2} + \frac{\kappa_4^2}{3}\rho_\Lambda + \frac{\kappa_4^2}{3}\rho_{\rm r} a^{-4}, \tag{4.60a}$$

where κ_4 and ρ_Λ remain given by, respectively, (4.56) and (4.57) and where the effective radiation density $\rho_{\rm r}$ is given by

$$\rho_{\rm r} = \frac{1}{2\pi^2}\left(\frac{M_+}{k_+} - \frac{M_-}{k_-}\right) \tag{4.60b}$$

It has to be emphasized that this is merely an *effective* form of radiation; the expansion of the bubble slows down due to the black hole in the bulk in the same way as an ordinary four–dimensional cosmology would decelerate due to the presence of radiation. The black hole does not realize a true form of radiation on the bubble cosmology, for instance in the form of photons or gravitational waves.

Realizing matter on the brane is not as 'for free' as radiation; it requires an additional source of energy in the bulk. It turns out that a cloud of strings in AdS [194] can realize a dust component in the cosmic fluid. Denoting the string tension by α, the function $f(r)$ is for a 'string–AdS$_5$' space given by

$$f(r) = 1 - \frac{\kappa_5^2 \alpha}{4\pi r} + k^2 r^2. \tag{4.61}$$

The junction condition (4.50) now yields a Friedmann equation with an effective pressureless matter component:

$$H^2 = -\frac{1}{a^2} + \frac{\kappa_4^2}{3}\rho_\Lambda + \frac{\kappa_4^2}{3}\rho_{\mathrm{m}} a^{-3}, \tag{4.62a}$$

with the effective matter energy density ρ_{m} given by

$$\rho_{\mathrm{m}} = \frac{3}{8\pi}\left(\frac{\alpha_+}{k_+} - \frac{\alpha_-}{k_-}\right). \tag{4.62b}$$

Similar as before, it is assumed that $k\alpha \ll 1$. The foregoing discussion demonstrates how to make contact with cosmology: a combination of a bulk black hole and a cloud of strings induces isotropic radiation and matter in the four–dimensional world.

It has also been examined how localized sources on the Dark Bubble should be interpreted from the perspective of the bulk:

- A bulk particle that 'rides' the bubble, contributes as radiation [177]. Indeed, the corresponding energy density dilutes due to two causes, which combine to yield the typical radiation–like $\sim 1/a^4$ dilution: the expansion of the bubble dilutes the particle density ($\sim 1/a^3$) while climbing out of the gravitational well of AdS redshifts the masses ($\sim 1/a$).

- Matter on the bubble is identified with the endpoints of hanging strings [191]. This is illustrated in the right portion of figure 4.6. As the bubble climbs out of the well, it 'eats' the string. This supplies the matter particles with exactly the amount of energy to counterbalance the redshift of the mass that bulk particles are sensitive to. This already indicates the crucial role of strings in the Dark Bubble model.

4.3.5 . . . and asymmetric

A crucial aspect in which the Dark Bubble scenario differs from the Randall–Sundrum or Karch–Randall set–up is the absence of a $\mathbb{Z}_2$ symmetry across the brane. This particular inside–outside construction is inherited from the nucleation event. In the language of section 4.2, the bulk metric can be well–approximated by (4.7) with $\gamma_{mn} = \eta_{mn}$, when the bubble has sufficiently evolved such that its curvature is negligible. However, the warp factor would now rather take the *asymmetric* form

$$A(z) = \begin{cases} e^{k_- z} & \text{if } z < 0 \\ e^{k_+ z} & \text{if } z > 0 \end{cases}, \tag{4.63}$$

instead of the manifestly $\mathbb{Z}_2$ symmetric from (4.13). Although asymmetric braneworld cosmologies have been considered before (see e.g. [193, 195–201]), we argue that the Dark Bubble requires a different interpretation.

One concern about asymmetric braneworlds was raised in [199], where it was argued that four–dimensional Einstein gravity can be recovered on the brane, but only under specific circumstances. Nevertheless, the best case scenario would predict that gravity is modified in the UV *and* in the IR. This can of course be acceptable; it just depends on how much faith one has that gravity should be 'Einsteinian' at very high energy (which we do not really believe, gravity should be modified in the UV) *and* at very low energy (have we really tested Einstein's gravity at extremely large distances?). For the Dark Bubble model, this does not seem to matter; in fact, a naive application of [199] to the Dark Bubble model seems to suggest that four–

dimensional Einstein gravity can*not* be recovered on the bubble. An inspection of (4.25) shows another disturbing feature: the graviton zero mode does not peak at the location of the brane. Gravity is *not* localized in the Dark Bubble model.

How then does gravity on the Dark Bubble 'work'? An enlightening discussion has been given in [158]; we highlight the main aspects below and refer the reader to the relevant literature for the details. One option is to mimic [173] and study gravity on the bubble by projecting the five–dimensional Einstein equation onto it. This was pioneered in [177]: by combining the Gauss–Codazzi equations[12] and the junction condition, this projection procedure effectively yields the four–dimensional Einstein equation:

$$\begin{aligned} {}^{(4)}G_{mn} = \left(\kappa_4^2\sigma - 3k_+k_-\right) h_{mn} - \kappa_4^2\tau_{mn} \\ + \frac{k_+k_-}{k_- - k_+}\triangle\left[\frac{1}{k}\mathcal{J}_{mn} - \frac{1}{2k}\mathcal{J}h_{mn}\right] + \dots , \end{aligned} \tag{4.64a}$$

where τ_{mn} is the energy–momentum tensor of sources that would be localized on the brane (apart from the brane tension) and $\mathcal{J} \equiv h^{mn}\mathcal{J}_{mn}$ with $\mathcal{J}_{mn}$ the tensor

$$\mathcal{J}_{mn} \equiv e^\mu_m e^\nu_n \left({}^{(5)}R_{\mu\nu} - {}^{(5)}R_{\alpha\mu\beta\nu}n^\alpha n^\beta\right), \tag{4.64b}$$

where n^μ is the normal to the brane. The four–dimensional geometry is sourced by two easily identifiable contributions (the ellipses denote high–energy corrections):

- There is a contribution from the bulk geometry (second line) whose information is encoded in $\mathcal{J}_{mn}$. This demonstrates how the bulk geometry effectively induces features in the effective four–dimensional theory. For instance, in the case where the bulk is Schwarzschild–AdS with a uniform dense could of strings (as in section 4.3.4) and where the brane–induced metric is FLRW, the effective Einstein equation (4.64) produces the known Friedmann equation in the presence of curvature, a cosmological constant, radiation, and matter.

[12] See appendix C, in particular equation (C.16).

- There is a contribution from localized, four–dimensional sources, described by the brane energy–momentum tensor τ_{mn}. However, it enters the effective Einstein equation with the 'wrong' sign! Indeed, this is how localized sources seem to affect the four–dimensional cosmology: as a *negative* energy density. However, this is *not* how one should realize four–dimensional matter in the Dark Bubble model; one has to take the backreaction into account. Phrased differently, features on the brane should have an 'uplifted' interpretation in the bulk. The prime example hereof is ordinary matter which is realized by hanging strings, as mentioned at the end of section 4.3.4. The tensor τ_{mn} should thus not be present in (4.64a).

This analysis indeed seems to suggest that one can recover Einstein gravity in four dimensions. Why does this not contradict the analysis of [199]? The answer has been elaborated at a technical level in [191], but we discuss it conceptually. The upshot is that there is one crucial difference between the Dark Bubble and the 'conventional' braneworlds, namely the way that matter is realized. In the latter scenarios, this happens usually through sources that are *localized* onto the brane. As we have emphasized, matter in the Dark Bubble scenario is realized by the endpoints of hanging *strings*. In both cases, there is a localized source on the brane (a true matter particle versus the endpoint of a string), but in the Dark Bubble model the source further extends 'holographically' into the bulk. The presence of such sources causes the bane to bend, as illustrated in figure 4.6:

- A localized matter source deforms the brane inwards; it contributes effectively as a negative mass particle. This resonantes with the negative sign in front of the energy–momentum tensor τ_{mn} in four–dimensional Einstein equation (4.64). Since this is not how one should interpret matter on the bubble, it is of no concern.

- A string that is pulling from the inside of the bubble also causes the brane to sag. The endpoint of such a string would correspond to a negative mass particle. This can also be seen from (4.62b).

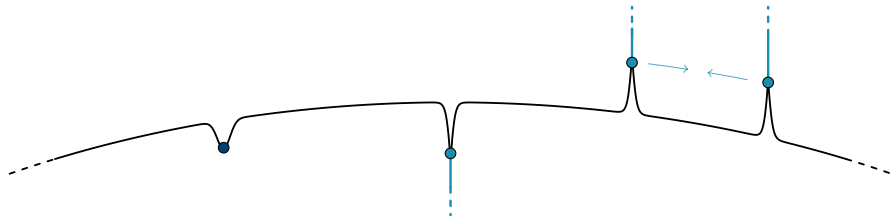


Figure 4.6: Matter on the Dark Bubble. Localized matter (left), as well as a string pulling downwards (middle), causes the brane to 'sag', yielding an effective negative energy density on the brane. A string pulling upwards (right) contributes as a positive energy density on the brane. The gravitational interaction between two such strings, indicated by the blue arrows, descends on the brane as four–dimensional Einstein gravity between the corresponding endpoints.

- A string stretching outwards corresponds to a positive mass particle, as could indeed be expected from (4.62b). If one wants to examine whether Einstein's gravity is recovered at the brane, one should examine the gravitational attraction between these strings. In [192] it has been shown that two such strings gravitationally attract each other in such a way that the endpoints follow the usual, four–dimensional geodesics on the bubble; the endpoints thus mimic gravitating matter on the bubble.

This also highlights the main difference with the scenario considered in [199]. The presence of strings stretching outwards allows for non–normalizable modes in the graviton propagator [191], which ultimately yields a different result: one recovers four–dimensional Einstein gravity on the bubble at energies below the AdS scale. This resonates with the fact that gravity is not localized in the Dark Bubble model; the graviton mode that is responsible for four–dimensional gravity is non–normalizable and extends along the strings.

4.4 Higher–dimensional view on quantum cosmology

In the framework of quantum cosmology, it seems rather natural that a whole four–dimensional universe can nucleate quantum mechanically. This has been discussed as a tunneling–from–nothing event in the 'vanilla model' of section 2.3.2, as well

as tunneling–from–a–cosmic–egg state in the eponymous scenario of section 3.3.2. In the former model, the choice of boundary conditions to be imposed on the wave function of the universe Ψ can seriously be debated and one can reach various conclusions. Indeed, the amplitudes of the Hartle–Hawking and Vilenkin wave function have a vastly different dependence on the four–dimensional cosmological constant, schematically

$$\Psi_{\mathrm{HH}} \sim e^{+\frac{c}{\Lambda_4}}, \qquad \Psi_{\mathrm{V}} \sim e^{-\frac{c}{\Lambda_4}}, \tag{4.65}$$

with c some constant (for details, compare with equation (2.60)).

We find it natural that –in the end– a UV completion of gravity decides which boundary conditions are to be imposed near the 'Big Bang phase' by providing additional physical input that informs us what the Hamiltonian at large curvatures looks like. This was completely absent in section 2.3.2, where the choice of boundary condition seemed completely arbitrary. In chapter 3, however, the idea of 'additional physics' was first apparent: the divergence of the minisuperspace potential at vanishing scale factor pointed towards the Hartle–Hawking weighting.

Since the Dark Bubble model provides a model of four–dimensional cosmology, it is interesting to consider the possibility that the Dark Bubble could give us a 'higher–dimensional view' on (quantum) cosmology. In particular, since the nucleation of a four–dimensional universe is built in the Dark Bubble, it is interesting to address the choice of weighting in this model of cosmology. The following treatment was presented [3], which closely followed and extended previous works [202, 203]. We follow the lines of section 2.3.2 and first consider the classical dynamics and then quantize canonically.

The starting point for describing the dynamics of a shell in a five–dimension bulk is (4.5), where it is understood that the Einstein–Hilbert term integrates over the inside as well as the outside of the shell (being different spacetimes, in contrast to the Randall–Sundrum and Karch–Randall braneworlds). The set–up is equivalent to the set–up of section 4.3.3: the bulk Einstein equation yields the five–dimensional metric (4.46) and the metric induced on the shell at $r = a(\lambda)$ is given by (4.47). The purpose is to obtain the dynamics from an action principle by reducing the

action (4.5) to an effective action that only depends on the radius of the shell $a(\lambda)$ and the lapse function $N(\lambda)$. For further convenience, we introduce the notation

$$\beta \equiv \sqrt{f(a)N^2 + \dot{a}^2}, \tag{4.66}$$

where an inside/outside–index is understood. The effective action receives three contributions, from each term in (4.5) separately:

- The 'bulk' contribution. Given that ${}^{(5)}\mathcal{R} = \frac{10}{3}\Lambda_5$, the Einstein–Hilbert action reduces to a volume integral

$$\begin{aligned} S_{\text{EH}} &= \frac{1}{2\kappa_5^2}\int \mathrm{d}^5x\sqrt{|g|}\left({}^{(5)}\mathcal{R} - 2\Lambda_5\right) \\ &= \frac{\Lambda_5\Omega_3}{6\kappa_5^2}\int \mathrm{d}\lambda\,\frac{\beta}{f(a)}a^4 \end{aligned} \tag{4.67}$$

  There are two such contributions with opposite sign, one coming from the interior of the bubble and one from the exterior.

- The 'brane curvature' contribution. Using the expression (4.49) for the extrinsic curvature, its trace can be computed and inserted into the Gibbons–Hawking–York term to yield

$$\begin{aligned} S_{\text{GHY}} &= \frac{1}{\kappa_5^2}\int \mathrm{d}^4y\sqrt{|h|}\,\mathcal{K} \\ &= \frac{\Omega_3}{\kappa_5^2}\int \mathrm{d}\lambda\,\triangle\left[-3\beta a^2 - \frac{a^3}{\beta}\left(\ddot{a} - \frac{\dot{a}\dot{N}}{N} + \frac{N^2}{2}f'(a)\right)\right]. \end{aligned} \tag{4.68}$$

  The appearance of the second derivative $\ddot{a}$ is undesirable since it makes the quantization procedure less straightforward. Since it appears linearly, however, it can be eliminated at the expense of introducing a boundary term [202]. In particular, for any function $h(a, \dot{a}, N)$ it holds that

$$-\frac{\partial h}{\partial \dot{a}}\ddot{a} = \frac{\partial h}{\partial a}\dot{a} + \frac{\partial h}{\partial N}\dot{N} - \frac{\mathrm{d}h}{\mathrm{d}\lambda}, \tag{4.69}$$

where the last term is a total derivative. By defining the function h by $h(a, \dot{a}, N) \equiv a^3 \tanh^{-1}(\dot{a}/\beta)$, it becomes clear how to eliminate the second derivative:

$$-\frac{a^3}{\beta}\ddot{a} = 3a^2\dot{a}\tanh^{-1}\left(\frac{\dot{a}}{\beta}\right) - \frac{a^3\dot{a}^2 f'(a)}{2f(a)\beta} - \frac{a^3\dot{a}\dot{N}}{N\beta} + \frac{\mathrm{d}h}{\mathrm{d}\lambda}. \tag{4.70}$$

Substituting this into the action (4.68) yields

$$S'_{\mathrm{GHY}} = \frac{\Omega_3}{\kappa_5^2}\int \mathrm{d}\lambda \triangle \left[-3\beta a^2 + 3a^2\dot{a}\tanh^{-1}\left(\frac{\dot{a}}{\beta}\right) - \frac{a^3 f'(a)}{2f(a)}\beta\right], \tag{4.71}$$

where the prime indicates the boundary term $\triangle h$ has been dropped (therefore S'_{GHY} is strictly speaking not equal to S_{GHY} in (4.68); the inferred dynamics, however, is).

- The 'brane' contribution, coming from its energy–momentum:

$$S_{\mathrm{brane}} = -\sigma\int \mathrm{d}^4y\sqrt{|h|} = \Omega_3\int \mathrm{d}\lambda\left(-\sigma N a^3\right) \tag{4.72}$$

Summing all contributions yields an effective action that describes the dynamics of the shell:

$$S = \frac{\Omega_3}{\kappa_5^2}\int \mathrm{d}\lambda\left\{\triangle\left[3a^2\dot{a}\tanh^{-1}\left(\frac{\dot{a}}{\beta}\right) - 3\beta a^2\right] - \kappa_5^2\sigma N a^3\right\}. \tag{4.73}$$

Although it is not manifest in the above formulae, it is verifiable that the volume terms (4.67) cancel against the last term in (4.71) to eventually produce the action above. In the phenomenological limit where $ka \gg 1$ and $k \gg H$, this reduces to (2.49) upto a sign[13]. This demonstrates that our five–dimensional UV completion gives full control over an infinite tower of corrections to the four–dimensional cosmological model.

As a consistency check, it can be shown how the Israel equation (4.50a) emerges from an action principle. The lapse function N is non–dynamical and acts as a Lagrange multiplier enforcing the vanishing of the Hamiltonian, as expected from λ–reparametrization invariance. The momentum conjugate π_a to a is given

[13]The overall sign of the Lorentzian action does not matter for the dynamics of the system. We have not yet established where the sign difference comes from.

implicitly[14] by

$$\cosh\left(\frac{\kappa_5^2\pi_a}{3\Omega_3 a^2}\right) = \frac{\beta_+\beta_- - \dot{a}^2}{N^2\sqrt{f_+(a)f_-(a)}}. \tag{4.74}$$

As could be anticipated, the explicit expression for π_a reduces to (2.50), upto a sign, in the above limit. The Hamiltonian[15] is given by

$$\mathcal{H} = \Omega_3 a^3\left[\sigma + \frac{3}{\kappa_5^2}\frac{1}{aN}\triangle\beta\right] \tag{4.75a}$$

$$= \Omega_3 a^3\sigma - \frac{3\Omega_3}{\kappa_5^2}a^2\sqrt{f_+(a) + f_-(a) - 2\sqrt{f_+(a)f_-(a)}\cosh\left(\frac{\kappa_5^2\pi_a}{3\Omega_3 a^2}\right)} \tag{4.75b}$$

where first line expresses the Hamiltonian in terms of the variables $(a, \dot{a})$ and the second line in terms of the canonical variables (a, π_a). The first equation demonstrates that the Hamiltonian constraint $\mathcal{H} = 0$ is equivalent to the junction condition (4.50a). We note that the equation of motion obtained by varying the action with respect to a is equivalent to the time–time component of the junction condition. This is a redundant equation that can be cast into the form $\dot{\mathcal{H}} = 0$.

Now that the action and Hamiltonian provide us with an 'higher–dimensional view' on our four–dimensional bubble cosmology, we can proceed and address the issue of weighting in quantum cosmology. We note that when quantizing the system by turning π_a into an appropriate operator, the Wheeler–DeWitt equation $\mathcal{H}\Psi = 0$ is of infinite order in the momentum. Instead, we focus on the phenomenological limit of small momentum, where the Hamiltonian becomes quadratic in π_a, in which case it reduces to (2.51), upto an overall sign. We thus recover the quantum cosmology of section 2.3.2 in the low–energy limit.

Recall that the choice of weighting was left unspecified in this 'vanilla' model, we now argue that the 'higher–dimensional view' unambiguously fixes this. There are two different approaches to calculate the nucleation probability $P \sim e^{-B}$ that both yield the same result:

[14]The sign ambiguity is resolved by requiring that $\pi_a > 0$, which follows from (4.73).

[15]Recall from the discussion in section 2.3 that $\mathcal{H}$ is related to the ordinary Hamiltonian H through multiplication by the lapse: $H = N\mathcal{H}$.

- By using the WKB wall penetration probability we have

$$B = 2 \int \mathrm{d}\lambda \, \pi_a \approx \frac{24\pi^2}{\kappa_4^2 \Lambda_4}. \tag{4.76}$$

- Applying the method of Brown and Teitelboim, outlined in section 4.3.2, to the Dark Bubble in the phenomenological limit, amounts to analysing the complicated expression (4.44) when $ka \gg 1$. For (4.44b) this implies

$$\mathcal{B}(k) \approx -\frac{3}{k} \left(k^2 a^4 + a^2 \right). \tag{4.77}$$

  It then follows that the bounce is given by

$$B \approx \Omega_4 \left[(\sigma - \sigma_{\mathrm{cr}}) \, a^4 + \frac{3a^2(k_- - k_+)}{\kappa_5^2 k_+ k_-} \right] \tag{4.78}$$

  By furthermore using that the nucleation radius is given by $a^2 = 3/\Lambda_4$, as well as the relations (4.56) and (4.57), we ultimately find

$$B \approx \frac{9\Omega_4}{\kappa_4^2 \Lambda_4} = \frac{24\pi^2}{\kappa_4^2 \Lambda_4}. \tag{4.79}$$

In the Dark Bubble picture, the nucleation of a four–dimensional cosmology seems to point towards the Vilenkin weighting, see (2.60). Indeed, as expected from the physical picture of tunneling, this weighting exactly matches the well–known amplitudes of bubble nucleation. Our understanding of bubble nucleation thus makes it possible to address the the choice of boundary conditions in quantum cosmology. We are left to conclude that the Dark Bubble provides a clean 'higher–dimensional view' on standard four–dimensional (quantum) cosmology.

In this model, there is no Big Bang *singularity* to worry about. This would correspond to the fiducial zero size of the bubble, which is not a physical solution to worry about. Admittedly, neither should one worry about this in the 'vanilla model' of section 2.3.2 for the same reason. The Dark Bubble's higher–dimensional view turns out to be useful in two regards. Firstly, it sheds light on 'cosmological nucleation'. From the four–dimensional point of view, the nucleation of a closed universe is a creation event out of nothing, while, from the five–dimensional point of view, it is simply bubble nucleation. We note that this merely replaces the need

for explaining 'nothing' from a four–dimensional point of view by explaining the origin of the unstable AdS_5 space. This is currently not addressed in the Dark Bubble model. Secondly, all physics at all the length scales that are involved in the process of nucleation and expansion is essentially understood: it is semi–classical gravity in five dimensions, which provides our UV completion of four–dimensional cosmology.

4.5 Gravitational waves on and around the Dark Bubble

In this section, based on [4], we discuss gravitational fluctuations permeating the Dark Bubble set–up. This is interesting to study for two reasons. Firstly, as we already mentioned, the brane tension seems to act as a *negative* energy density in the four–dimensional cosmology; at least, this is what is suggested (4.64a) or, equivalently, by the appearance of the minus sign in (4.57). Do the fluctuations of the brane itself, that naively seem to add energy to the tension, then correspond to an instability of the Dark Bubble? Secondly, the previous section argued in favor of the tunneling wave function, which has recently been subject of debate when perturbations were included. In particular, in [119, 204–207] it was argued that perturbations are unbounded and would ultimately destroy Vilenkin's quantum cosmology (and thus the Dark Bubble?). However, in [106, 120, 208, 209] it was argued that this problem can be avoided by imposing suitable boundary conditions on the perturbation amplitudes. Our embedding of Vilenkin into the Dark Bubble model suggests that such instability cannot be present, since there is no reason why the nucleation event would be unphysical. Nevertheless, it would be interesting to develop a map between the minisuperspace model and the Dark Bubble model with perturbations included.

We do not attempt to construct such a map; rather, we make the first step in this direction. In particular, we consider the most natural fluctuations from the point of view of the Dark Bubble, namely the gravitational ones, and we refrain from introducing any other kinds of fields. To ultimately map gravitational waves in the

four–dimensional (quantum) cosmology to gravitational perturbations in the Dark Bubble model, we construct the uplift of four–dimensional gravitational waves to five–dimensional ones[16]. We are only interested in the simplest case where the bulk remains pure anti–de Sitter and the bubble realizes a pure de Sitter cosmology (i.e. we do not include contributions from radiation or matter).

How gravitational waves in the bulk affect the evolution of the bubble, is encapsulated in the Israel equation. To write down this junction condition, one is obliged to find the backreaction of the gravitational waves on the bulk geometry. This is a highly non–trivial task that we tackle perturbatively in the metric:

$$g_{\mu\nu} \to g_{\mu\nu} + \epsilon\, \delta g_{\mu\nu} + \epsilon^2\, \delta^2 g_{\mu\nu} + \mathcal{O}(\epsilon^3), \tag{4.80}$$

where ϵ is a formal expansion parameter that keeps track of the order in perturbation theory. The conventional procedure is to solve the Einstein equation order by order in ϵ:

$$\begin{aligned} G_{\mu\nu} + \Lambda_5 g_{\mu\nu} \to (G_{\mu\nu}[g] + \Lambda_5 g_{\mu\nu}) + \epsilon\, (\delta G_{\mu\nu}[\delta g] + \Lambda_5 \delta g) \\ + \epsilon^2 \left(\delta^2 G_{\mu\nu}[\delta g] + \delta G_{\mu\nu}[\delta^2 g] + \Lambda_5 \delta^2 g_{\mu\nu}\right) + \mathcal{O}(\epsilon^3) = 0, \end{aligned} \tag{4.81}$$

where $\delta^i G_{\mu\nu}[\delta^j g]$ denotes the i–the order variation of the Einstein tensor evaluated on the j–th order metric perturbation. This quantity is formally of order $\max\{i, j\}$.

At zeroth order, the Einstein equation simply yields the background metric $g_{\mu\nu}$. In the present case, this will be pure AdS_5. Gravitational waves appear at first order in ϵ through the linearized Einstein equation (the gravitational wave equation):

$$\delta G_{\mu\nu}[\delta g] + \Lambda_5 \delta g_{\mu\nu} = 0. \tag{4.82}$$

The gravitational waves in the Dark Bubble model must satisfy specific requirements, given that the bulk induces a four–dimensional metric constrained by the junction conditions.

[16]There exist metric perturbations in AdS_5 that do not correspond to gravitational waves on the bubble, but we do not discuss these.

The second order Einstein equation is conventionally written as

$$\delta G_{\mu\nu}[\delta^2 g] + \Lambda_5 \delta^2 g_{\mu\nu} = -\delta^2 G_{\mu\nu}[\delta g]. \tag{4.83}$$

Although tedious, this equation can in principle be solved for $\delta^2 g_{\mu\nu}$, which describes how the geometry reacts to the presence of the gravitational wave $\delta g_{\mu\nu}$. The right–hand side can be interpreted as an effective energy–momentum tensor, quadratic in $\delta g_{\mu\nu}$:

$$\langle T_{\mu\nu} \rangle \equiv -\kappa_D^{-2} \left\langle \delta^2 G_{\mu\nu}[\delta g] \right\rangle , \tag{4.84}$$

where κ_D^2 is the gravitational constant in D dimensions. The angular brackets $\langle \cdot \rangle$ denote an averaging procedure that is required for a proper interpretation of the tensor, see e.g. [210]. Observe the overall minus sign in the definition above; as argued below, this is of fundamental relevance.

4.5.1 Waves on the bubble

In this section, we briefly review gravitational waves in a four–dimensional FLRW cosmology with a flat or spherical topology. When the wavelength is sufficiently small, waves in a flat universe serve as a proxy for those in a spherical universe. Indeed, high–frequency waves only probe small regions of space and do not feel the curvature at larger scales. For simplicity, we restrict a pure dS_4 universe with a cosmological constant $\Lambda_4 = 3K^2$. To establish the uplift to five dimensions, the spherical waves are more natural starting points. We also discuss the backreaction of the gravitational waves onto the FLRW geometry. This is more complicated for spherical waves; for simplicity, we restrict to the gravitational waves in the flat universe.

Gravitational waves are described by TT fluctuations to the metric (4.47). In the conformal time gauge, these are

$$\begin{aligned} \mathrm{d}s^2 &= (g_{\mu\nu} + \epsilon\, \delta g_{\mu\nu})\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu \\ &= a^2(\eta) \left[-\mathrm{d}\eta^2 + (\gamma_{ij} + \epsilon\, \psi_{ij}(\eta, x))\, \mathrm{d}x^i\, \mathrm{d}x^j \right] , \end{aligned} \tag{4.85}$$

where η is conformal time, γ_{ij} is the metric on a spatial slice in x^i–coordinates, and ψ_{ij} represents a TT perturbation. For further convenience, we define the conformal Hubble rate

$$\mathcal{H} \equiv \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\eta}. \tag{4.86}$$

Flat universe

The scale factor of a flat dS_4 universe is given by

$$a(\eta) = -\frac{1}{K\eta}, \tag{4.87}$$

with $-\infty < \eta < 0$, where $\eta \to 0$ corresponds to the infinite future. For concreteness, we consider a gravitational wave travelling in the x^1–direction[17] with either a + or $\times$ polarization. By furthermore expanding the perturbation into harmonics on the spatial manifold (i.e. a Fourier decomposition), the Ansatz for the gravitational wave is

$$\psi_{ij}(\eta, x) = \psi_{4\mathrm{D}}(\eta) e^{iqx^1} \mathrm{e}_{ij}, \tag{4.88}$$

where e_{ij} is a polarization tensor and q is a (continuous) wave number. The amplitude $\psi_{4\mathrm{D}}$ is in principle also dependent on the wave number q, however, for notational simplicity, this labelling is omitted. The linearized Einstein equation yields a wave equation for each mode separately:

$$\frac{\mathrm{d}^2\psi_{4\mathrm{D}}}{\mathrm{d}\eta^2} + 2\mathcal{H}\frac{\mathrm{d}\psi_{4\mathrm{D}}}{\mathrm{d}\eta} + q^2\psi_{4\mathrm{D}} = 0, \tag{4.89}$$

with the conformal Hubble rate $\mathcal{H} = -1/\eta$. The solutions are given by

$$\psi_{4\mathrm{D}}(\eta) = -\eta\cos(q\eta + \phi_0) + \frac{1}{q}\sin(q\eta + \phi_0), \tag{4.90}$$

where ϕ_0 is an arbitrary phase. The wave $\psi_{4\mathrm{D}}$ freezes out to a constant at late–times (which happens to be zero if $\phi_0 = 0$).

[17] It is a trivial modification to the given analysis to consider a wave travelling in any other direction.

Closed universe

A spherical dS_4 universe is a bouncing cosmology, with the scale factor given by

$$a(\eta) = -\frac{1}{K \sin \eta}, \tag{4.91}$$

with η in the domain $-\pi/2 \leqslant \eta < 0$. The moment $\eta = -\pi/2$ is the bounce. This coincides with the moment of nucleation. In a similar vein as above, one can expand the perturbations in TT harmonics[18] on S^3:

$$\psi_{ij}(\eta, x) = \psi_{4\mathrm{D}}(\eta) Y_{ij}(x). \tag{4.92}$$

As before, mode indices are suppressed. The gravitational wave equation for a single mode, labelled by the quantum number n, is

$$\frac{\mathrm{d}^2 \psi_{4\mathrm{D}}}{\mathrm{d}\eta^2} + 2\mathcal{H} \frac{\mathrm{d}\psi_{4\mathrm{D}}}{\mathrm{d}\eta} + \left(n^2 - 1\right) \psi_{4\mathrm{D}} = 0, \tag{4.93}$$

with the conformal Hubble rate $\mathcal{H} = -\cot \eta$.

To find solutions, it is useful to redefine the conformal time coordinate to

$$v = \cos \eta, \tag{4.94}$$

with $v \in [0, 1)$. The gravitational wave equation then becomes

$$\left(1 - v^2\right) \frac{\mathrm{d}^2 \psi_{4\mathrm{D}}}{\mathrm{d}v^2} + v \frac{\mathrm{d}\psi_{4\mathrm{D}}}{\mathrm{d}v} + \left(n^2 - 1\right) \psi_{4\mathrm{D}} = 0. \tag{4.95}$$

This differential equation has three regular singular points ($v = -1, +1, \infty$). It is well–known that such differential equations can be converted to the hypergeometric kind. The solutions are thus given in terms of these functions:

$$\psi_{4\mathrm{D}}(\eta) = {}_2F_1 \left(-\frac{n+1}{2}, \frac{n-1}{2}, -\frac{1}{2}; 1 - v^2 \right), \tag{4.96a}$$

$$\tilde{\psi}_{4\mathrm{D}}(\eta) = \left(1 - v^2\right)^{3/2} {}_2F_1 \left(1 - \frac{n}{2}, 1 + \frac{n}{2}, \frac{5}{2}; 1 - v^2 \right). \tag{4.96b}$$

[18] We adopt the convention that the S^3 transverse–tracefree harmonics Y_{ij} satisfy $\triangle Y_{ij} = -(n^2 - 3) Y_{ij}$, with $n \geqslant 3$ and $\triangle$ the Laplacian on S^3. Note that i and j are tensor indices and that mode indices are suppressed. See e.g. [211, 212].

At late times ($v \to 1$), the former freezes out to a constant value, while the latter decays completely. Since n is an integer, the hypergeometrics take a simpler form and can be rewritten in terms of the Chebyshev polynomials[19] [213]:

$$\psi_{4\mathrm{D}}(\eta) = vT_n(v) - \frac{n}{n+1}T_{n+1}(v), \tag{4.97a}$$

$$\tilde{\psi}_{4\mathrm{D}}(\eta) = \sqrt{1-v^2}\left[vU_{n-1}(v) - \frac{n}{n+1}U_n(v)\right], \tag{4.97b}$$

where T_n and U_n are the Chebyshev polynomials of, respectively, the first and second kind. One can furthermore simplify these expressions:

$$\psi_{4\mathrm{D}}(\eta) = \frac{1}{n+1}\cos\left((n+1)\eta\right) + \sin\eta\sin(n\eta) \tag{4.97c}$$

$$\tilde{\psi}_{4\mathrm{D}}(\eta) = \frac{1}{n+1}\sin\left((n+1)\eta\right) - \sin\eta\cos(n\eta). \tag{4.97d}$$

At late times, high–frequency gravitational waves (large n) reduce to (4.90) upon the identification $q = \sqrt{n^2-1} \approx n$. This furthermore requires taking the limit $\eta \to 0$, while keeping ηn finite. Physically, this corresponds to a late–time observer being able to see gravitational fluctuations within his Hubble radius. Note that $\psi_{4\mathrm{D}}$ reduces to a wave with phase $\phi_0 = 0$, while $\tilde{\psi}_{4\mathrm{D}}$ will have a phase $\phi_0 = \pi/2$.

Energy–momentum tensor

The presence of these gravitational waves affects the background by sourcing an effective energy–momentum tensor. For phenomenological reasons, we are interested in the limiting case where the bubble appears flat (i.e. large wave number and late times), and where the gravitational radiation respects the FLRW symmetries (i.e. being distributed homogeneously and isotropically throughout the universe). We therefore restrict the following analysis to the waves found in the flat universe (4.90).

[19] Chebyshev polynomials of the first kind are defined by $T_n(\cos\eta) = \cos(n\eta)$, and Chebyshev polynomials of the second kind are defined by $U_n(\cos\eta)\sin\eta = \sin\left((n+1)\eta\right)$.

By following the averaging procedure (integrating over all the phases x^1), one finds the energy–momentum tensor

$$\langle T^{\mu}{}_{\nu} \rangle = \frac{K^2\eta^2}{8\kappa_4^2} \begin{pmatrix} 7-2q^2\eta^2 & \pm 2q^2\eta^2 & 0 & 0 \\ \mp 2q^2\eta^2 & 5+2q^2\eta^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.98}$$

where the $\pm$ sign represents waves travelling in opposite x^1–directions. A uniform, isotropic background of gravitational radiation is realized by averaging this tensor over all sorts of waves travelling in all possible x^1–, x^2–, and x^3–directions with all possible polarizations. The energy–momentum tensor that describes this uniform background of gravitational waves is thus given by, in terms of the scale factor,

$$\langle T^{\mu}{}_{\nu} \rangle_{\text{iso}} = \frac{7}{8\kappa_4^2}\frac{1}{a^2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} + \frac{q^2}{4\kappa_4^2 K^2}\frac{1}{a^4} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{4.99}$$

The first term corresponds to a form of energy with equation of state $p = -\rho/3$ whose energy dilutes as $\rho \sim 1/a^2$. This behaves like curvature in the Friedmann equation. The second term has the equation of state $p = \rho/3$ and a dilution $\rho \sim 1/a^4$, which corresponds to radiation. When the wavelength of the gravitational waves is larger than the Hubble radius, it becomes frozen and the curvature component is all that remains.

Backreaction

To address the problem of backreaction, we make an Ansatz of what the backreacted geometry should look like. Given that the geometry is sourced by a perturbative amount of radiation and curvature without any spatial anisotropies or inhomogeneities, cf. (4.99), we put forward the Ansatz

$$\begin{aligned} \mathrm{d}s^2_{\text{back}} &= \left(g_{\mu\nu} + \epsilon^2\delta^2 g_{\mu\nu}\right)\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu} \\ &= a^2(\eta)\left[-\left(1+\epsilon^2 Q(\eta)\right)\mathrm{d}\eta^2 + \gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\right], \end{aligned} \tag{4.100}$$

where $Q(\eta)$ represents the backreaction and $a(\eta)$ is as given above. The second–order Einstein equation fixes Q:

$$Q(\eta) = \frac{7}{24}\eta^2 - \frac{q^2}{12}\eta^4. \tag{4.101}$$

By introducing a new time coordinate $\bar{\eta}$

$$\mathrm{d}\bar{\eta} \equiv \sqrt{1+\epsilon^2 Q(\eta)}\,\mathrm{d}\eta \approx \left(1 + \frac{\epsilon^2}{2}Q(\eta)\right)\mathrm{d}\eta, \tag{4.102}$$

the backreacted metric (4.100) is of the FLRW form with a new conformal time parameter $\bar{\eta}$. By expanding in small ϵ, the old and new time coordinate are related by

$$\eta \approx \bar{\eta} + \epsilon^2\left(-\frac{7}{144}\bar{\eta}^3 + \frac{q^2}{120}\bar{\eta}^5\right). \tag{4.103}$$

The Hubble rate H can now be computed to display the Friedmann equation:

$$H^2 = \left(\frac{1}{a^2}\frac{\mathrm{d}a}{\mathrm{d}\bar{\eta}}\right)^2 \approx K^2 + \epsilon^2\left(-\frac{7}{24}\frac{1}{a^2} + \frac{q^2}{12K^2}\frac{1}{a^4}\right). \tag{4.104}$$

One easily recognizes –beyond the cosmological constant– the contributions from a perturbative amount of curvature and radiation, as expected from the presence of gravitational waves.

4.5.2 Waves around the bubble

We are interested in perturbations propagating in the AdS_5 bulk that correspond to gravitational waves on the bubble. Clearly, there are different possible fluctuations of the AdS_5 geometry and not all of them meet the aforementioned requirement. The ones relevant to the present discussion are TT perturbation to S^3. These enter in the metric as

$$\begin{aligned}\mathrm{d}s^2 &= (g_{\mu\nu} + \epsilon\,\delta g_{\mu\nu})\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \\ &= -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\gamma_{ij} + \epsilon\,\psi_{ij}(t,r,x)\right)\mathrm{d}x^i\,\mathrm{d}x^j,\end{aligned} \tag{4.105}$$

where f is given by (4.46b), γ_{ij} is the metric on S^3 in an x^i–coordinate chart, and ψ_{ij} represents a TT perturbation. It is easily checked that the induced metric on the brane, in conformal coordinates, corresponds to (4.85).

Uplifting four–dimensional gravitational waves

The TT perturbations can be decomposed into S^3 harmonics, and for a single mode (secretly) labelled by a discrete wave number n

$$\psi_{ij}(t,r,x) = \psi_{5\mathrm{D}}(\eta,r)Y_{ij}(x), \tag{4.106}$$

the linearized Einstein equation becomes

$$\frac{\partial^2\psi_{5\mathrm{D}}}{\partial t^2} - f^2\frac{\partial^2\psi_{5\mathrm{D}}}{\partial r^2} - \frac{1}{r}f\left(2+4k^2z^2+f\right)\frac{\partial\psi_{5\mathrm{D}}}{\partial r} + \frac{n^2-1}{r^2}f\psi_{5\mathrm{D}} = 0. \tag{4.107}$$

This equation determines the evolution of the gravitational wave throughout the AdS_5 bulk. In order to find solutions, it is useful to work with a coordinate that provides the uplift of (4.94):

$$w \equiv \cos(kt) = \frac{\cos\eta}{\sqrt{1+\left(\frac{K}{k}\right)^2\sin^2\eta}} = \cos\eta + \mathcal{O}\left(\left(\frac{K}{k}\right)^2\sin^2\eta\right), \tag{4.108}$$

where the relation between bulk and brane time (4.48) has been used. This means that in the limit relevant for four–dimensional General Relativity, i.e. $K/k \ll 1$, one has $w \approx v$. In particular,

$$kt = \eta + \mathcal{O}\left(\left(\frac{K}{k}\right)^2\sin(2\eta)\right). \tag{4.109}$$

It is important to note that this relation is only meaningful once the bubble has nucleated. This occurs at conformal time $\eta = -\pi/2$, as alluded to in the four–dimensional discussion, and corresponds to a bulk time $t_{\mathrm{N}} \approx -\pi/2k$. In terms of the new time coordinate w, the gravitational wave equation in the bulk becomes

$$\begin{aligned} k^2\left[\left(1-w^2\right)\frac{\partial^2\psi_{5\mathrm{D}}}{\partial w^2} - w\frac{\partial\psi_{5\mathrm{D}}}{\partial w}\right] - f^2\frac{\partial\psi_{5\mathrm{D}}}{\partial r^2} \\ - \frac{1}{r}f\left(2+4k^2r^2+f\right)\frac{\partial\psi_{5\mathrm{D}}}{\partial r} + \frac{n^2-1}{r^2}f\psi_{5\mathrm{D}} = 0. \end{aligned} \tag{4.110}$$

To solve this wave equation uniquely, it needs to be supplemented by suitable boundary conditions. In particular, when $\psi_{5\mathrm{D}}$ is restricted to the brane (this will be called the 'induced wave' ψ_{ind}), we require that the induced wave ψ_{ind} coincides with the four–dimensional gravitational waves (4.97) to leading order in K/k. This amounts to imposing the following boundary conditions at the location of the bubble $r = a(w)$, assuming $ka \gg 1$,

$$\psi_{\mathrm{ind}} \equiv \psi_{5\mathrm{D}}\left(w, \frac{-1}{K\sqrt{1-w^2}}\right) = \psi_{4\mathrm{D}}(v) + \mathcal{O}\left(\left(\frac{K}{k}\right)^2\right) \tag{4.111a}$$

$$\lim_{r\to 0} \psi_{5\mathrm{D}}(w, r) = 0. \tag{4.111b}$$

The last condition is the requirement that there are no sources inside the bubble. One might remark that in principle there are *two* different waves: one inside and one outside the bubble. It would therefore seem natural to insist that *only* the inside wave decays as $r \to 0$ and that *only* the outside wave remains finite as $r \to \infty$. However note that their evolution is governed by the same wave equation (up to a difference in k) and that, furthermore, the boundary condition at the location of the brane must be imposed for both of them. Since the boundary condition uniquely fixes the inside *and* outside wave, this means that if the outside wave would be extrapolated in the would–be limit $r \to 0$, it would still vanish (and similarly for the inside wave for $r \to \infty$).

As far as the uplift is concerned, one may verify that the following waves meet the proposed requirements

$$\psi_{5\mathrm{D}}(w, r) = \frac{(kr)^{n-1}}{\left(1+k^2r^2\right)^{(n-1)/2}}\left[wT_n(w) - \frac{n\left(n+1+2k^2r^2\right)}{2(n+1)\left(1+k^2r^2\right)}T_{n+1}(w)\right], \tag{4.112a}$$

$$\tilde{\psi}_{5\mathrm{D}}(w, r) = \frac{(kr)^{n-1}\sqrt{1-w^2}}{\left(1+k^2r^2\right)^{(n-1)/2}}\left[wU_{n-1}(w) - \frac{n\left(n+1+2k^2r^2\right)}{2(n+1)\left(1+k^2r^2\right)}U_n(w)\right], \tag{4.112b}$$

which can be rewritten as

$$\psi_{5\mathrm{D}}(t,r) = \frac{(kr)^{n-1}}{(1+k^2r^2)^{(n-1)/2}}\left[\frac{\frac{1}{2}(1+n)(2-n)+k^2r^2}{(1+n)\,(1+k^2r^2)}\cos\left((n+1)kt\right)+\sin(kt)\sin(nkt)\right] \tag{4.112c}$$

$$\tilde{\psi}_{5\mathrm{D}}(t,r) = \frac{(kr)^{n-1}}{(1+k^2r^2)^{(n-1)/2}}\left[\frac{\frac{1}{2}(1+n)(2-n)+k^2r^2}{(1+n)\,(1+k^2r^2)}\sin\left((n+1)kt\right)-\sin(kt)\cos(nkt)\right]. \tag{4.112d}$$

It is important to emphasize that this is an *exact* solution to the five–dimensional wave equation (4.107). The approximation $K/k \ll 1$ comes in when considering the induced wave: the identification $kt \approx \eta$ only holds in this limit (4.109), see also (4.111).

Just as for the four–dimensional waves, one could be tempted to take the large n, late time ($t \to 0$) limit in order to find an uplift for the gravitational waves in a flat dS_4 universe (4.90). However, one must also take into account that, to reach this flat limit, the wave must be considered in a region *near* the bubble. This means it is also required to take the limit in which kr is large. The following waves represent the correct uplift of (4.90), with phases $\phi_0 = 0$ and $\phi_0 = \pi/2$ respectively

$$\psi_{5\mathrm{D}}(t,r) = \frac{-\frac{1}{2}n^2+k^2r^2}{nk^2r^2}\cos(nkt)+kt\sin(nkt), \tag{4.113a}$$

$$\tilde{\psi}_{5\mathrm{D}}(t,r) = \frac{-\frac{1}{2}n^2+k^2r^2}{nk^2r^2}\sin(nkt)-kt\cos(nkt), \tag{4.113b}$$

where we identified $q = \sqrt{n^2-1} \approx n$.

For large n, there are interesting corrections beyond the four–dimensional gravitational waves of General Relativity at high momentum. At fixed kr, there is a competition between n and kr in the first term of (4.113). The first modification to the four–dimensional waves comes into play when $kr \sim n$. This translates into $p \sim n/r \sim k$, where p is the proper momentum of the wave. We therefore conclude that the AdS scale k represents a UV scale where new physics is introduced beyond four–dimensional gravity on the brane.

Energy–momentum tensor

In the same spirit as the four–dimensional treatment, we compute the energy–momentum tensor associated with the waves found above. We again restrict to waves for which the four–dimensional cosmology effectively appears flat, i.e. the waves given in (4.113). By an average over several wavelengths, polarizations, and propagation directions, one can obtain an isotropic tensor $\langle T^{\mu}{}_{\nu} \rangle_{\text{iso}}$. This superposition of waves represents uniform gravitational radiation filling the bulk geometry. The isotropic energy–momentum tensor $\langle T^{\mu}{}_{\nu} \rangle_{\text{iso}}$ consists of three identifiable pieces (respectively curvature, radiation, and flux):

$$\langle T^{\mu}{}_{\nu} \rangle_{\text{iso}} = \langle T^{\mu}{}_{\nu} \rangle_{\text{c}} + \langle T^{\mu}{}_{\nu} \rangle_{\text{r}} + \langle T^{\mu}{}_{\nu} \rangle_{\text{f}}, \tag{4.114a}$$

where each component is given by (to make the split between the bulk and the bubble more apparent, we ordered the bulk coordinates as $x^{\mu} = (t, x^{i}, r)$)

$$\langle T^{\mu}{}_{\nu} \rangle_{\text{c}} = \frac{1}{8\kappa_5^2 r^2} \left(7 - \frac{n^4}{2k^4 r^4} \right) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} + \dots & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} + \dots & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} + \dots & 0 \\ 0 & 0 & 0 & 0 & 1 + \dots \end{pmatrix}, \tag{4.114b}$$

$$\langle T^{\mu}{}_{\nu} \rangle_{\text{r}} = \frac{k^2 n^2 t^2}{4\kappa_5^2 r^2} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.114c}$$

$$\langle T^{\mu}{}_{\nu} \rangle_{\text{f}} = \frac{n^2}{8\kappa_5^2 r^2} \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{2t}{k^2 r^3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2k^2 tr & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.114d}$$

where the ellipses denote corrections of order $\sim \mathcal{O}\left(n^2/k^2 r^2\right)$. Recalling that the momentum of the wave is $p \sim n/r$, we see how the UV corrections we commented on earlier enter in the curvature contribution, but not in the radiation piece. The flux contribution shows how the gravitational waves represent a net flow of energy

in the positive r direction. Physically, this means that we have a cloud of radiation bound inside the AdS throat, that expands towards the maximum dilution (at $t = 0$) and then recollapses. Actually, it takes an infinite amount of proper time for the bubble to enter the recollapsing phase; it will therefore not occur in the cosmology we study.

Backreaction

Following the same spirit as in the four–dimensional treatment, we compute the response of the bulk geometry due to the presence of the gravitational waves (4.113). The presence of the energy–momentum tensor (4.114) generates a deformation in the bulk geometry, which can be described by a backreacted metric accounting for it. Since the gravitational wave background was made isotropic, it is convenient to adopt the global coordinate system for the backreacted bulk geometry. We make the Ansatz

$$\begin{aligned} \mathrm{d}s^2_{\mathrm{back}} &= \left(g_{\mu\nu} + \epsilon^2\delta^2 g_{\mu\nu}\right)\mathrm{d}x^\mu\,\mathrm{d}x^\nu \\ &= -\left[f(r) + \epsilon^2 Q_1(t)\right]\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)+\epsilon^2 Q_2(t)} + r^2\,\mathrm{d}\Omega_3. \end{aligned} \tag{4.115}$$

The backreaction piece $g^{(2)}_{\mu\nu}$ is given by the ϵ^2 coefficient in the small ϵ expansion:

$$g^{(2)}_{tt} = -Q_1(t), \qquad\qquad g^{(2)}_{rr} = -\frac{Q_2(t)}{f^2(r)}. \tag{4.116}$$

The functions Q_1 and Q_2 are fixed by the second–order Einstein equation. Since it is sourced by the energy–momentum tensor corresponding to the waves (4.113), which serve as a proxy for (4.112) only in the limit where $kr \gg 1$, the metric (4.115) only describes the backreaction in that particular limit. Taking this small caveat into account, one readily finds

$$Q_1(t) = \frac{7}{24} - \frac{1}{6}n^2k^2t^2, \qquad\qquad Q_2(t) = \frac{7}{24} - \frac{1}{12}n^2k^2t^2. \tag{4.117}$$

This change in the bulk geometry affects the evolution of the bubble wall through the junction condition. This can be analyzed using the four–dimensional Einstein equation (4.64), from which it is clear that this change in bulk geometry sources

the four–dimensional geometry through an effective energy–momentum tensor. It is verifiable that this coincides with the energy–momentum tensor found before (4.99). Equivalently, from (4.64) one recovers the modified Friedmann equation (4.104) on the bubble. Another, equivalent approach, is to compute the induced metric on the bubble; this coincides with (4.100). This demonstrates the self–consistency of the Dark Bubble model; it implies that the junction condition has taken care of the gravitational perturbations in the bulk, providing a clear connection between the physics in the bulk and on the boundary of the bubble.

4.6 Brane dynamics versus Einstein gravity

The Friedmann equation on the brane (4.51a), possibly generalized to include radiation or matter as discussed in section 4.3.4, is a manifestation of four–dimensional gravity on the bubble, which is dictated by (4.64). It is important to examine how the energy–momentum of the brane contributes to the Friedmann equation; from (4.64), it seems to do so with the 'wrong sign'. We focus on the brane tension only (see also [177]). For near–critical branes, it is obvious how the brane tension acts as a *negative* energy density; this is immediate from equation (4.57). Note that this is also present –although more obscure– in the full, non–linear expression (4.51b) since Λ_4 is a decreasing function of the brane tension in the inside–outside branch (figure 4.5). Although it might seem unconventional that σ contributes *negatively* to the four–dimensional energy density, this is physically correct: increasing σ eventually brings it above its critical value, yielding a negative four–dimensional cosmological constant that prevents the bubble from nucleating.

What could be worrisome, are the brane fluctuations. These would seem to add energy to the tension, thus contributing as a *negative* energy density in the four–dimensional energy–momentum tensor. This naively signals an instability. As we argue below, such fluctuations are already taken care of by the junction conditions.

To see this, one needs to recognize that there are two equivalent ways to describe the motion and fluctuations of the brane. From the five–dimensional perspective, one studies the brane equations of motion. These make sure that the backreaction

on the bulk geometry is taken into account. It is physically clear that there are no instabilities in this system –beyond the accelerated expansion of the bubble itself– and that all other perturbations will cost energy. On the other hand, one can use the Israel equation, where the motion of the brane is captured by four–dimensional Einstein gravity (the Friedmann equation). This latter approach has been the dominant one in the foregoing exposition. Eventually, the result of both approaches should be the same.

Without gravitational fluctuations, the junction condition (4.50a) can be rewritten in a more suggestive form as

$$\begin{aligned}\frac{\sigma}{2}\left[\sqrt{k_-^2+H^2+\frac{\Delta_-(a)}{a^2}}+\sqrt{k_+^2+H^2+\frac{\Delta_+(a)}{a^2}}\right]\\ =\frac{3}{2\kappa_5^2}\left(k_-^2-k_+^2+\Delta_-(a)-\Delta_+(a)\right),\end{aligned} \tag{4.118}$$

with the metric factor written as $f(r)=k^2r^2+\Delta(r)$. By multiplying both sides of the equation by the volume of the bubble (proportional to a^4), this can be interpreted as an energy conservation equation, comparing the cases with and without the bubble. The energy of the brane is simply set equal to the energy difference between the two vacua. Following Brown–Teitelboim, we note that the energy of the brane itself is given by the average of the energy obtained from the two sides of the brane. This is dictated by the junction condition. In terms of time coordinate in the bulk, (4.118) can be expressed as

$$\begin{aligned}\frac{\sigma}{2}\left[\frac{f_-(a)}{a\sqrt{f_-(a)-\frac{1}{f_-(a)}\left(\frac{\mathrm{d}a}{\mathrm{d}t_-}\right)^2}}+\frac{f_+(a)}{a\sqrt{f_+(a)-\frac{1}{f_+(a)}\left(\frac{\mathrm{d}a}{\mathrm{d}t_+}\right)^2}}\right]\\ =\frac{3}{2\kappa_5^2}\left(k_-^2-k_+^2+\Delta_-(a)-\Delta_+(a)\right).\end{aligned} \tag{4.119}$$

One can recognize the relativistic energy of a brane in a curved background[20]. Hence we see how the junction condition, interpreted as four–dimensional gravity, is equivalent to the equations of motion for the transverse brane degree of freedom. There are no issues with any wrong sign kinetic term; there is no sign of an instability.

If one considers a bulk geometry that induces matter with a positive energy density on the bubble, see e.g. (4.64), it requires a response from the four–dimensional Einstein tensor with the same positive sign to satisfy the Einstein equations. The brane gives rise to such a geometric contribution, that enters into the Einstein equation through the four–dimensional Einstein tensor. However, one can also view the brane as a contribution to the energy–momentum tensor by formally moving this geometric contribution to the other side of the Einstein equation, thereby picking up a sign. If it is to account for adding matter with a *positive* energy density, the geometric response should appear as a *negative* energy density. This is precisely what an increase in the brane energy will accomplish! The sign only looks wrong if one –incorrectly– interprets the brane term to be associated with four–dimensional matter. Instead, it is precisely this physical behaviour of the brane that is responsible for four–dimensional gravity. In the case of vibrational modes in the bulk, the brane starts to vibrate and increases its energy as a response. These vibrations are, through the junction conditions, encoded into the four–dimensional Einstein tensor and identical to the response of gravity to a matter source.

The gravitational waves we have studied provide a nice illustration of the same phenomena. To see this, consider an oscillator exposed to a gravitational wave that functions as an antenna and absorbs energy. In particular, one can have a situation where the oscillator is in sync with the with and sits in an excited state. This is how to think of the brane in the Dark Bubble model: in the presence of the five–dimensional gravitational wave, the brane becomes excited. This is exactly how the gravitational wave backreacts on the brane. Hence, there are contributions to the energy of the brane corresponding to such excitations that contribute *negatively* in the effective energy density in four dimensions, for the simple reason that they *add* to the tension. However, these are precisely the terms needed to cancel the

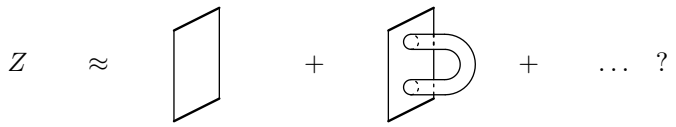
[20]Compare with the energy of a relativistic particle $m/\sqrt{1-v^2}$.

induced energy–momentum tensor on the bubble from the five–dimensional waves through the junction conditions. From the perspective of the backreactions is is also clear how to interpret these terms: if we move them to the other side of the Einstein equation, they represent the response of four–dimensional gravity to the presence of the waves.

Chapter 5

Wormholes

Does a consistent theory of quantum gravity allow for topology change? A change of topology is a transition where the spatial geometry, by some quantum mechanical process, evolves in a non–diffeomorphic way. To give an example, one could imagine our universe Σ giving birth to an S^3 'baby universe'. This would correspond to the topology–changing process $\Sigma \to \Sigma \oplus S^3$. This question is well–posed in the context of Euclidean quantum gravity, where it a priori seems unnatural to not include manifolds with non–trivial topology in the path integral. However, the inclusion of such manifolds equally poses several paradoxes [214–216]. This is the particularly long–standing debate on the saddle point expansion of the path integral of Euclidean quantum gravity: should one, or should one not, include saddle points of various topologies? Pictorially,



Each 'term' in this sum represents a gravitational instanton that connects certain boundary conditions. The second term has a non–trivial, 'wormhole' topology and corresponds to a certain topology–changing process. The fundamental question is

whether saddles of this wormhole–type contribute to the semi–classical expansion. In this chapter, we shed some light on this issue by consider one specific type of wormhole.

In section 5.1, we first outline how to examine whether certain saddles do –or do not– contribute to the path integral. We then focus on a very particular type of saddle point, namely wormholes sourced by an axion field (known as Giddings–Strominger wormholes [38]). These objects are briefly reviewed in section 5.2. Their perturbative stability as a saddle point is examined in section 5.3 in two different manners and we reach the same conclusion. The (perturbative) calculations were performed using the Mathematica packages xPand, xTensor, xPert, and xAct [217–219].

5.1 Instantons and path integrals

In quantum mechanics, tunneling is an archetypal example of a non–perturbative effect. Tautologically, non–perturbative contributions to the path integral are completely invisible to perturbation theory; rather, they are inherently non–analytical. These contributions are conventionally described using the language of instantons [220] (see also [221]). Instantons are solutions to the *Euclidean* equations of motion and have a profound effect on the vacuum structure of the theory.

A useful tool to compute the ground state energy is the Euclidean path integral

$$Z = \langle x_{\mathrm{f}} | e^{-H \triangle t} | x_{\mathrm{i}} \rangle, \tag{5.1a}$$

$$= \int_{x(t_{\mathrm{i}})=x_{\mathrm{i}}}^{x(t_{\mathrm{f}})=x_{\mathrm{f}}} \mathcal{D}x \, e^{-S_{\mathrm{E}}[x]}, \tag{5.1b}$$

which computes the imaginary–time propagator from the position eigenstate $|x_{\mathrm{i}}\rangle$ to $|x_{\mathrm{f}}\rangle$ by summing over all paths that connect these Dirichelet–type boundary conditions within a fixed Euclidean time span $\triangle t = t_{\mathrm{f}} - t_{\mathrm{i}}$. In the above expressions,

H is the Hamiltonian and $S_{\rm E}$ the Euclidean action, for simplicity taken to be

$$S_{\rm E} = \int {\rm d}t \left[\frac{1}{2}\dot{x}^2 + V(x)\right], \tag{5.2}$$

with $V(x)$ being some potential[1]. Although it is possible to generalize this, we stick to this simplest case of a single particle subjected to a potential V in one dimension. Inserting a complete set of energy eigenstates $|n\rangle$ into (5.1a) yields

$$Z = \sum_n e^{-E_n \triangle t} \psi_n(x_{\rm f}) \psi_n^\dagger(x_{\rm i}), \tag{5.3}$$

where E_n is the energy of the n–th energy eigenstate, whose wave function is identified as $\psi_n(x) = \langle x|n\rangle$. In the limit $\triangle t \to \infty$, the dominant contribution to the summation comes from the ground state; the path integral (5.1b) thus provides a tool to compute the ground state energy. An exact calculation of the path integral is in most cases analytically impossible and one often relies on perturbative and (semi–classical) non–perturbative methods. We focus on the latter.

The semi–classical approximation amounts to a (functional generalization of) the saddle point approximation[2]: the dominant contribution to the integral comes from the region around the *minima* of $S_{\rm E}$. These minima, denoted $x_0(t)$, are called instantons (or saddle points) and solve the classical Euclidean equations of motion:

$$\left.\frac{\delta S_{\rm E}}{\delta x}\right|_{x=x_0(t)} = 0. \tag{5.4}$$

Although the path integral receives contributions from any path subjected to the boundary conditions, the most dominant contributions come from the paths that are 'close to' the classical ones. To approximate the path integral, the action of the region around an instanton is approximated to second order in the deviation from

[1] Note that the Euclidean action describes the motion of a particle in the *inverted* potential.

[2] The semi–classical limit corresponds to $\hbar \to 0$. When reinstating units of $\hbar$ into the functional integral, $Z = \int \mathcal{D}x\, e^{-S_{\rm E}[x]/\hbar}$, it becomes clear that the semi–classical approximation corresponds to the (functional version of the) saddle point approximation. Its non–perturbative nature follows from the non–analyticity in $\hbar$.

the classical solution $\mathcal{X} \equiv x - x_0$:

$$S_{\mathrm{E}}[x_0 + \mathcal{X}] \approx S_{\mathrm{E},0} + \frac{1}{2} \left.\frac{\delta^2 S_{\mathrm{E}}}{\delta x^2}\right|_{x=x_0(t)} \mathcal{X}^2, \tag{5.5a}$$

where $S_{\mathrm{E},0} = S_{\mathrm{E}}[x_0]$ is the instanton action and the second functional derivative represents the quadratic operator (up to a boundary term)

$$\left.\frac{\delta^2 S_{\mathrm{E}}}{\delta x^2}\right|_{x=x_0(t)} \mathcal{X}^2 = \int \mathrm{d}t\, \mathcal{X} M \mathcal{X}, \tag{5.5b}$$

with M the self–adjoint differential operator

$$M \equiv -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + V''(x_0(t)). \tag{5.5c}$$

Because of the Dirichlet boundary conditions on x, we require that $\mathcal{X} \to 0$ at the Euclidean–time boundary; this is dictated by the path integral (5.1a). The Taylor expansion above turns the path integral (5.1b) into a manageable Gaussian path integral over the fluctuation $\mathcal{X}$. By formally solving the Sturm–Liouville problem

$$M\mathcal{X}_n = \lambda_n \mathcal{X}_n, \tag{5.6}$$

with Dirichlet boundary conditions on $\mathcal{X}_n$, a complete set of orthonormal eigenfunctions $\mathcal{X}_n$ can be obtained in which the fluctuation $\mathcal{X}$ can be expanded. The path integral (5.1b) is then, in this semi–classical limit, approximated by

$$Z \approx \sum_{\text{saddles}} \frac{e^{-S_{\mathrm{E},0}}}{\sqrt{\det M}}. \tag{5.7}$$

The determinant, as written here, is a suitably regularized version of the product of all the eigenvalues λ_n, sometimes written as $\det' M$. This is needed in the first place since the eigenvalues generically form a set of ever–increasing numbers $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$, that may be discrete or continuous, without an upper bound. Their product thus generically diverges, requiring a regularization scheme. Secondly, there could exist eigenvalues that are zero or negative, making (5.7) a priori ill–defined. An exact calculation of the determinant is not of interest in this thesis and will not be discussed further. The existence of zero or negative

modes, on the contrary, is of particular relevance, as this occurs in a variety of quantum mechanical systems [222]. We therefore briefly discuss the role of these modes, determining the *stability* of a saddle, in the dynamical systems considered by Coleman [222] (these are generalizations of (5.2) to include multiple coordinates x^i).

- There always exists one zero mode, namely $\mathcal{X}_0 = \dot{x}_0$, which is a consequence of the time–translational invariance of the action. The divergence originating from the zero mode can be easily understood and cured in the so–called dilute instanton gas approximation. The upshot of this is that zero eigenvalues can be omitted from the determinant. If this is the mode with the lowest eigenvalue, which typically happens if V' vanishes at the turning point, the instanton is stable and corresponds to a true minimum of the action. Physically, it describes a lift of degeneracy of the ground state energy. For non–Coleman systems, there may even exist a multitude of zero modes.

- It is possible that a negative mode exists. This typically happens if V' does not vanish at the turning point and the instanton describes a 'bounce off' a potential barrier. Because of the square root in (5.7), this yields an imaginary contribution to the energy. The ground state is thus a resonant state with a finite lifetime, rather than a true bound state of the potential. The vacuum is thus unstable and the instability is interpreted as quantum tunneling: the particle can penetrate the potential barrier and escape.

- Instantons with multiple negative modes do not describe the tunneling process and should not be taken into account in the saddle point expansion. Indeed, these instantons do not represent true minima of the action and we call them 'unstable'. A multitude of negative modes does not occur in the Coleman systems but might occur in others.

This outlines a general procedure for examining the vacuum structure of a theory in a non–perturbative way: one examines how instantons contribute to the path integral through a mode analysis. This reveals the eigenvalues of the quadratic operator and therefore the ultimate fate of the instanton: lifting a degeneracy, signalling an instability, or being 'unphysical'. This procedure is straightforwardly

extended to field theories. The most common example is the decay of a false vacuum in a scalar field theory [188]. In this case, it has been demonstrated that the bounce has exactly one negative mode, as required for a tunneling interpretation [189]. For field theories coupled to gravity, the situation is much more subtle because of gauge issues and the conformal factor problem [103, 104]. False vacuum decay with the inclusion of gravity is described by the Coleman–de Luccia instanton [136], see e.g. the discussion in [223]. Another example of a gravitational instanton is the Brown–Teitelboim instanton of section 4.3.2. It has been demonstrated that this instanton exhibits exactly one negative mode [224], such that it can indeed be interpreted as mediating the decay of AdS. Ultimately, a statement about the stability of an instanton is a statement about the physics and cannot depend on any choice of gauge. It is therefore desirable to present a stability analysis in terms of gauge–invariant degrees of freedom.

Wormholes are a particular type of gravitational instantons, for which it is extremely interesting to study the (non–)existence of negative modes. This would ultimately answer the fundamental question of whether or not they correspond to true minima of the action and, consequently, whether or not they contribute to the path integral. If they do, the implications for cosmology are rather rich, as reviewed in [216]. In this chapter, we focus on a particular type of Euclidean wormholes, namely wormholes sourced by axion fields. These fields often arise in string compactifications, making the question about their stability inevitable in string theory. Indeed, several embeddings of axion wormholes have been found [215, 225–227].

5.2 Euclidean axion wormholes

In contrast to phenomenological backgrounds of string theory, where it is presumed that all moduli are stabilized, there equally exist interesting backgrounds that do not share this property. These backgrounds have (some amount of) unbroken supersymmetry and find their usefulness in the fact that they constitute trustworthy vacua with a known UV completion. At low energy, these string compactifications

typically have actions that can be consistently truncated to the form where multiple (unstabilized) moduli fields are present:

$$S = \int \left[\frac{1}{2\kappa_D^2} \star (\mathcal{R} - 2\Lambda) - \frac{1}{2} G_{ij}(\phi) \star \mathrm{d}\phi^i \wedge \mathrm{d}\phi^j \right] + \text{boundary terms}, \tag{5.8}$$

where, for further convenience, we adopt the form notation. The many massless scalar fields ϕ^i can be coupled to one another through a positive–definite metric $G_{ij}(\phi)$ on the moduli space. These scalars often come in pairs, which are conventionally called axions and saxions. Axion fields share the crucial property that they enjoy a shift symmetry at the level of the classical action. In this section, we focus on the simplest system of the form (5.8): four–dimensional gravity coupled to a single axion field.

5.2.1 Axion fields and Hodge duality

The fact that (scalar) axion fields have a shift symmetry means that they can be dualized to a two–form gauge field through Hodge duality [228]. We already encountered this form–field formulation of the axion in chapter 3, where it was used to model matter in a cosmic egg scenario. We regard this form–field formulation as fundamental. The Euclidean action for gravity coupled to an axion three–form field strength $H = \mathrm{d}B$, with B the two–form gauge potential, is

$$S_{\mathrm{E}} = \int \left[-\frac{1}{2\kappa_4^2} \star (\mathcal{R} - 2\Lambda) + \frac{1}{2} \star H \wedge H + \chi \, \mathrm{d}H \right], \tag{5.9}$$

where χ is a (scalar) Lagrange multiplier that enforces that the axion field strength H is closed, $\mathrm{d}H = 0$. This is required if the field strength H is viewed as a fundamental field, rather than the two–form gauge potential. Integrating the last term by parts produces a crucial total derivative:

$$\chi \, \mathrm{d}H = \mathrm{d}\left(\chi H\right) - \mathrm{d}\chi \wedge H. \tag{5.10}$$

After this integration by parts, H becomes a Lagrange multiplier, whose equation of motion is given by

$$H = - \star \mathrm{d}\chi. \tag{5.11}$$

When substituted into the action, it becomes

$$S_{\mathrm{E}} = \int \left[-\frac{1}{2\kappa_4^2} \star (\mathcal{R} - 2\Lambda) - \frac{1}{2} \star \mathrm{d}\chi \wedge \mathrm{d}\chi \right] - \chi \star \mathrm{d}\chi|_{\partial}\,, \tag{5.12}$$

where the notation $\cdot|_{\partial}$ means the form is to be integrated over the boundary, as follows from Stokes' theorem. The actions (5.9) and (5.12) are completely equivalent on–shell; the degrees of freedom are simply grouped differently. The crucial difference is that the former has a manifestly positive kinetic energy for the three–form H, while the latter contains negative kinetic energy for the scalar field χ. This is an immediate consequence of working in Euclidean signature[3]. In this sense, one could say that axion scalars are Wick rotated[4] $\chi \to i\chi$ when passing to Euclidean signature. What ensures that both actions are equivalent on–shell, and that the negative kinetic energy in χ does not render the theory unstable, is the boundary term in (5.12). Given the one–to–one correspondence between boundary terms in the action and boundary conditions on the equations of motion, the difference between the dynamics of (5.9) and (5.12) is that H satisfies a Dirichlet boundary condition,

$$\delta H|_{\partial} = 0, \tag{5.13a}$$

while χ satisfies a Neumann condition,

$$\delta \star \mathrm{d}\chi|_{\partial} = 0. \tag{5.13b}$$

[3]The cause of this can be traced back to the double application of the Hodge operator, see equation (A.23). In a Lorentzian signature, the form–field as well as the scalar field formulation would have the correct sign kinetic term.

[4]Alternatively, one could choose to *not* Wick rotate the axion field along, thereby preserving the correct sign for the kinetic energy. Wormholes still arise if the dual three–form is real, but this comes at the expense of having complex saddle points, in particular a complex scalar field. In this regard, the Wick rotation and dualization commute (although the presented discussion seems to convey the contrary). [228, 229]

Whenever multiple scalars are present, such as in (5.8), the nomenclature axion versus saxion is used to delineate the fields that can be dualized versus the ones that cannot. Physically, this prescribes which boundary conditions are to be imposed on the fields: axions (saxions) have Neumann (Dirichlet) boundary conditions. In passing to the Euclidean signature, this nomenclature equally distinguishes the fields that are Wick rotated from the ones that are not.

5.2.2 Wormhole backgrounds

Although it seems conventional –or optional– to regard the form–field formulation for the axions as fundamental, it is exactly the negative energy in the dual scalar field that allows wormholes to exist. If one wants to examine their fate as instantons in the path integral, one is instructed to consider (5.9) as fundamental. We now describe how these wormholes arise as spherically symmetric, finite–action saddle points. For simplicity, we restrict to wormholes sourced by a single axion field and a vanishing cosmological constant, known as Giddings–Strominger wormholes [38]. We note that the extension to include multiple scalar fields is easily made (see e.g. [215, 230]).

The metric Ansatz that respects spherical symmetry is

$$\mathrm{d}s^2 = N^2(r)\,\mathrm{d}r^2 + a^2(r)\,\mathrm{d}\Omega_3^2. \tag{5.14a}$$

For the scalar field χ, or equivalently for the three–form H, we take the Ansatz

$$\chi = \chi(r), \qquad\qquad H = f(r)\mathcal{E}, \tag{5.14b}$$

where $\mathcal{E}$ is the volume form of the unit S^3. The scalar field equation of motion, $\Box\chi = 0$, can be written in terms of the radial harmonic function $h(r)$, defined by $\mathrm{d}h = Na^{-3}\,\mathrm{d}r$:

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}h^2} = 0. \tag{5.15}$$

This implies

$$\chi(r) = Qh(r), \tag{5.16}$$

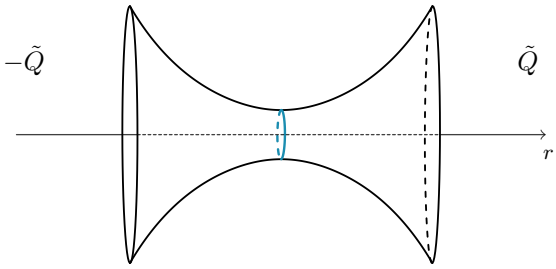


Figure 5.1: Pictorial representation of a Euclidean wormhole with an axion dipole charge $\tilde{Q}$. The throat is indicated by the blue circle, which represents an S^3.

with Q some integration constant. Completely equivalent, the Bianchi identity, $\mathrm{d}H = 0$, implies

$$\frac{\mathrm{d}f}{\mathrm{d}r} = 0, \tag{5.17}$$

such that the field strength is

$$H = Q\mathcal{E}. \tag{5.18}$$

We note that Q is the same integration constant as in (5.16), on account of (5.11). The gravitational equation of motion is a constraint equation (Euclidean Friedmann equation):

$$\left(\frac{\dot{a}}{N}\right)^2 = 1 - \frac{\kappa_4^2 Q^2}{6a^4}, \tag{5.19}$$

where a dot denotes a derivative with respect to r. It is easy to see that this equation admits wormhole solutions, i.e. solutions that connect two asymptotically flat regions. These exhibit the feature that the scale factor a first decreases, reaches a minimal value (conventionally at $r = 0$) at the neck of the wormhole, and increases again. The geometry is thus everywhere regular. This is depicted in figure 5.1.

The integration constant Q is related to the axion charge $\tilde{Q}$, defined by

$$\tilde{Q} = \int_{S^3} H. \tag{5.20a}$$

Clearly $\tilde{Q} = \Omega_3 Q$, with Ω_3 is the volume of the unit S^3. In light of (5.11), the axion charge can be equally expressed in terms of the scalar field χ as

$$\tilde{Q} = \int_{S^3} \mathrm{d}^3x \sqrt{|\gamma|}\, \pi_\chi, \tag{5.20b}$$

where $\pi_\chi = N^{-1}a^3\dot{\chi}$ is the momentum conjugate to χ and γ is the metric on the unit S^3. It is also verifiable that $\tilde{Q}$ corresponds to the Noether charge associated with the shift symmetry of χ. It should be noted that one cannot view wormholes as objects carrying a net axion charge $\tilde{Q}$, since wormholes are everywhere regular objects. Instead, they carry a dipole charge, as depicted in figure 5.1: a charge measurement on one end of the wormhole is of opposite sign than a charge measurement on the other end. This is immediate from (5.20a), where the S^3 is of opposite orientation on both sides of the wormhole neck, or from the multi–valuedness inherent to (5.19). Wormholes are, therefore, rather 'charge conduits' [182].

So far we have demonstrated that wormholes naturally arise as gravitational instantons in Euclidean gravity coupled to an axion field. It is also readily verified that they have finite action:

$$S_{\mathrm{E},0} = \frac{\sqrt{6}\pi\tilde{Q}}{2\kappa_4}. \tag{5.21}$$

It remains to discuss between what states they compute the transition amplitude in the path integral (cf. (5.1a)). In fact, this question has already been answered by the boundary conditions (5.13): a Dirichlet condition on H, or a Neumann condition on χ, keeps the axion charge fixed at the wormhole ends. Schematically, wormholes are saddles that relate to transition amplitudes between charge eigenstates:

$$Z = \langle \tilde{Q}_\mathrm{f} | e^{-H\triangle r} | \tilde{Q}_\mathrm{i} \rangle. \tag{5.22}$$

Clearly wormholes impose $\tilde{Q}_\mathrm{i} = -\tilde{Q}_\mathrm{f}$. If the action in the path integral is taken to be (5.9) that involves the three–form, this represents a conventional configuration space path integral. Indeed, fixing charge is a Dirichlet boundary condition on the three–form (5.13a). However, if the action is taken to be in the scalar field formulation (5.12), this is a momentum space path integral [228,231]. Indeed, fixing

charge is a Neumann condition on the scalar (5.13b). The 'wrong'–sign kinetic term for χ thus finds its origin in the Euclidean path integral that computes a transition amplitude between its momentum eigenstates [228].

For the purpose of this chapter, it is convenient to switch to the conformal gauge $N = a$, in which the wormhole solution can be presented analytically:

$$a(r) = \sqrt{\frac{\kappa_4 Q}{\sqrt{6}} \cosh(2r)}, \qquad \dot{\chi} = \frac{Q}{a^2}. \tag{5.23}$$

The minimal radius of the three–sphere occurs at the wormhole neck when $\dot{a} = 0$. Its proper radius a_0 is explicitly given by

$$a_0 \equiv \sqrt{\frac{\kappa_4 Q}{\sqrt{6}}}. \tag{5.24}$$

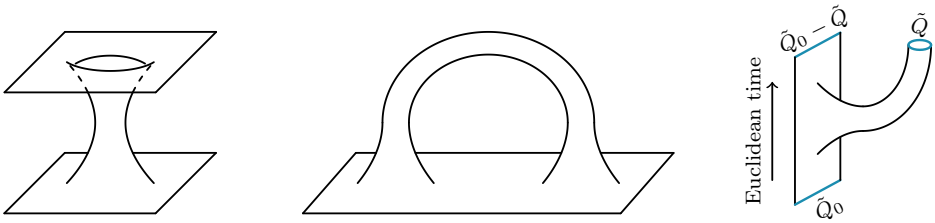
Macroscopic wormholes, which are everywhere in the semi–classical regime, have $a_0 \gg \kappa_4$. These objects possess a sufficient amount of axion charge:

$$Q \gg \kappa_4. \tag{5.25}$$

In the saddle point expansion, these correspond to exponentially suppressed contributions compared to to microscopic, Planckian–sized wormholes.

5.2.3 Wormhole physics

Like any instanton, a wormhole –if it contributes to the path integral– has the potential to drastically influence the vacuum structure of the theory. It is therefore important to examine the physical effects of wormholes, and how they should be interpreted as instantons. We give a rather brief and concise account of this; more information can be found in the excellent review [216]. Although the foregoing discussion, as well as the remainder of this chapter, seemingly restricts to to axionic wormholes of the Giddings–Strominger type, most of the physics presented in this section is independent of the type of wormhole.



(a) A wormhole that connects two different universes.

(b) A wormhole connecting the same universe, thereby creating a 'loophole'.

(c) Nucleation of a baby universe, described by a semi–wormhole.

Figure 5.2: Pictorial representation of Euclidean wormholes.

The Giddings–Strominger wormhole is, by construction, a wormhole that connects two different asymptotically flat spaces (compare figures 5.1 and 5.2a). This is an exact solution to the equations of motion. Another –topologically distinct– type of wormhole connects two asymptotically flat regions that are part of the same universe, as illustrated in figure 5.2b. Strictly speaking, these are not truly Giddings–Strominger wormholes. Nevertheless, they are still approximate solutions, valid if the separation distance between the wormhole ends is sufficiently large compared to the wormhole throat. As we argue below, wormholes have a peculiar effect on the low–energy physics.

Since a wormhole is everywhere Euclidean, an interpretation is not immediate. The situation, however, is no different when discussing, for instance, the de Sitter instanton in section 2.3.2. In a similar vein, an interpretation can be provided by slicing the instanton in half at the moment of time–symmetry and joining it to an appropriate Lorentzian manifold. Such 'semi–wormhole' is sketched in figure 5.2c: it represents an instanton that interpolates between a state with one FLRW universe with axionic charge $\tilde{Q}_0$ and a state with two FLRW universes with axionic charges $\tilde{Q}_0 - \tilde{Q}$ and $\tilde{Q}$. This semi–wormhole thus represents the nucleation of a 'baby universe' from a parent FLRW universe; it is a topology–changing process $\mathbb{R}^3 \to \mathbb{R}^3 \oplus S^3$, where the final geometry has two disconnected components. The converse, the absorption of a baby universe, can equally occur.

If this interpretation is correct, an observer in the parent universe will see a certain amount of axion charge disappear when a baby universe nucleates. Euclidean wormholes are therefore argued to be instantons that violate[5] charge conservation.

This interpretation also connects with the cosmic eggs of chapter 3. The Giddings–Strominger wormhole offers cosmic eggs the possibility to 'exchange' charge or to 'give birth' to other baby eggs. Phrased differently, they 'interconnect' the landscape of cosmic eggs.

For several years, it has been believed that wormholes would lead to a loss of quantum coherence[6] [38, 232–234]. At the heart of this lies a combination of two facts, namely that an observer is always confined to *one* universe and can therefore *never* measure the state of any baby universe, and that baby universes can carry –ungauged– quantum numbers away from the parent universe. This has been countered by Coleman [214] (see also [235]) by suggesting a different interpretation: wormholes do not cause a loss of coherence, rather they give rise to an intrinsic randomness of the constants of nature [236–238]

The upshot is that if microscopic wormholes of the type sketched in figure 5.2b contribute to the path integral, they introduce a bilocal expression to the effective action [236, 237]

$$S_{\text{eff}} = S_{\text{E}} - \frac{1}{2}\sum_{ij}\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\, C_{ij}\mathcal{O}_i(x_1)\mathcal{O}_j(x_2), \tag{5.26}$$

where $\{\mathcal{O}_i\}$ constitutes a suitable basis of local operators and C is a real symmetric matrix whose entries are constant and proportional to $\sim e^{-S_{\text{E}},0}$. By expressing $e^{-S_{\text{eff}}}$ as a Gaussian integral, locality can be restored at the expense of introducing a new set of parameters α_i:

$$e^{-S_{\text{eff}}} = \int \prod_i \mathrm{d}\alpha_i\, \exp\left[-\frac{1}{2}\sum_{ij}\left(C^{-1}\right)_{ij}\alpha_i\alpha_j - \sum_i \int \mathrm{d}^4x\,\alpha_i\mathcal{O}_i(x) - S_{\text{E}}\right]. \tag{5.27}$$

[5] From the perspective of the parent universe; nucleation of baby universes can never violate the known conservation laws (energy, momentum, charge, etc.) when taking baby universes into account.

[6] Pure quantum states evolving into mixed states.

By writing the original action S_{E}, where the effects of wormholes are not included, in terms of the local operators $\mathcal{O}_i$ as

$$S_{\mathrm{E}} = \sum_i \int \mathrm{d}^4x \; g_i \mathcal{O}_i(x), \tag{5.28}$$

with g_i denoting the coupling constant associated with $\mathcal{O}_i$, the effect of including wormholes becomes clear: they shift the value of the coupling constants g_i to $\tilde{g}_i$, defined by

$$\tilde{g}_i(\alpha) = g_i + \alpha_i. \tag{5.29}$$

This suggests defining a 'shifted' action that gathers the last two terms in the exponent of (5.27):

$$\tilde{S}_{\mathrm{E}} \equiv \sum_i \int \mathrm{d}^4x \; \tilde{g}_i \mathcal{O}_i(x). \tag{5.30}$$

A path integral with wormhole effects included now takes the form of a 'conventional' path integral using the shifted action as the weighting function that *also* averages over the α–parameters with a Gaussian weight:

$$Z = \int \prod_i \mathrm{d}\alpha_i \; \rho(\alpha) \int \mathcal{D}\phi \, e^{-\tilde{S}_{\mathrm{E}}[\phi]}, \tag{5.31a}$$

where ϕ represents the collection of fields and $\rho(\alpha)$ is defined by

$$\rho(\alpha) = \exp\left[-\frac{1}{2} \sum_{ij} \left(C^{-1}\right)_{ij} \alpha_i \alpha_j \right]. \tag{5.31b}$$

Wormholes thus entail a multiverse–like scenario where the coupling constants of nature are random variables $\tilde{g}_i(\alpha)$ with an in principle calculable probability function $\rho(\alpha)$. One could interpret $\rho(\alpha)$ as being a distribution of universes labeled by the α–parameters; we are simply in one specific universe with a specific set of α–parameters. This is how quantum coherence is 'regained': wormholes are integrated out, but the price to pay is an intrinsic randomness of the constants of nature.

This randomness seems to interfere with the 'promise' of string theory that the coupling constants should be calculable from first principles; rather, they would give rise to an imprint of other vacua onto others [215]. However, Coleman [239] has also argued that wormholes bear on the cosmological constant problem; without going into the details of the argument (which is generally not believed to be a valid solution to the cosmological constant problem [216]), Coleman has claimed that a vanishing cosmological constant is statistically favored. Given the embeddings [215, 225–227], it is a false hope that string theory by itself resolves the paradoxes posed by the wormholes. An evident way out is if wormholes turn out to be unstable saddle points. To that extent, a stability analysis is required. Such analysis is pursued in the remainder of this chapter for the Giddings–Strominger wormhole.

5.3 Stability analysis

To understand the role of Giddings–Strominger wormholes in the path integral, one has to analyze the quadratic action associated with (5.9) or (5.12). As could be anticipated, it does not matter which action is used for the mode analysis. This is demonstrated explicitly in the first part of this section. To preserve four–dimensional covariance, we first write the fluctuations as

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \qquad H_{\mu\nu\rho} \to H_{\mu\nu\nu} + \delta H_{\mu\nu\rho}, \qquad \chi \to \chi + \delta\chi. \tag{5.32}$$

Greek indices are raised and lowered with the metric g and we define the trace of the metric fluctuation as $\delta g \equiv g^{\mu\nu}\delta g_{\mu\nu}$. The field perturbations transform under a change of coordinates (gauge transformation). In the framework of perturbation theory, it is sufficient to consider infinitesimal gauge transformations

$$x^\mu \to x^\mu - \xi^\mu, \tag{5.33a}$$

under which the perturbations transform as

$$\delta g_{\mu\to} \to \delta g_{\mu\nu} + \pounds_\xi g_{\mu\nu}, \quad \delta H_{\mu\nu\rho} \to \delta H_{\mu\nu\rho} + \pounds_\xi H_{\mu\nu\rho}, \quad \delta\chi \to \delta\chi + \pounds_\xi \chi. \tag{5.33b}$$

To avoid difficulties or subtleties regarding gauge fixation, the stability analysis of the Giddings–Strominger wormhole will be carried out in terms of gauge–invariant variables. For an appropriate choice, this even has the advantage of avoiding the conformal factor problem. We will only be interested in the (non–)existence of negative modes. In the present model, it turns out that we do not have to perform an explicit mode analysis to this extent.

Perturbation theory around a wormhole background (5.14a) ultimately amounts to cosmological perturbation theory (see e.g. [240]), but in Euclidean signature. This perturbation theory is pursued in the rest of this section. It is well–known that one can decompose the metric fluctuations $\delta g_{\mu\nu}$ into scalars, divergence–free vectors and a transverse–tracefree tensor defined on the radial slices of the geometry. Since axion fields only have one scalar degree of freedom, we just need to consider the scalar fluctuations of gravity (vector and tensor perturbations decouple). This is most easily analyzed in the conformal gauge

$$\begin{aligned} \mathrm{d}s^2 &= (g_{\mu\nu} + \delta g_{\mu\nu})\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \\ &= a^2\left[(1+2A)\,\mathrm{d}r^2 + 2D_iB\,\mathrm{d}x^i\,\mathrm{d}r + ((1-2\psi)\gamma_{ij} + 2D_iD_jE)\,\mathrm{d}x^i\,\mathrm{d}x^j\right], \end{aligned} \tag{5.34}$$

where γ is the metric on the unit S^3 and D_i the corresponding covariant derivative. In the following, Latin indices are raised and lowered with this metric. All the scalar fluctuations are only sensitive to gauge transformations of the scalar type

$$\xi^\mu = \left(\xi^0, D^i\xi\right), \tag{5.35}$$

with ξ^0 and ξ two scalar functions. The four gravitational degrees of freedom transform as

$$A \to A + \dot{\xi}^0 + \mathcal{H}\xi^0, \quad B \to B + \xi^0 + \dot{\xi}, \quad \psi \to \psi - \mathcal{H}\xi^0, \quad E \to E + \xi, \tag{5.36}$$

where $\mathcal{H} = \dot{a}/a$ is the conformal Hubble rate, given by the right–hand side of equation (5.19).

The stability of any instanton depends on the quadratic operator *and* on the boundary condition that is imposed on the fluctuation. This is not to be chosen at will, but dictated by the physical process the instanton corresponds to. We reemphasize that the path integral (5.22) decides: the physical boundary condition is keeping the axion charge fixed at the wormhole ends

$$\delta\tilde{Q}\big|_{\partial} = 0. \tag{5.37}$$

In this case, the notation $\cdot|_{\partial}$ means the respective quantity is evaluated at the boundary ($r \to \pm\infty$). We assume it is clear to the reader when to integrate a form over and evaluate a quantity at the boundary, hence the same notation. We wish to emphasize that fluctuations of the charge are gauge–invariant; under a gauge transformation (5.33), the fluctuation of the charge (5.20) transforms as

$$\delta\tilde{Q} \to \int_{S^3} \mathrm{d}^3x\sqrt{|\gamma|}\, Q\triangle\xi, \tag{5.38}$$

which vanishes since the integrand is a total derivative on S^3. Consequently, (5.37) is a gauge–invariant statement; it is the only meaningful thing that can be said about the boundary conditions!

5.3.1 Hodge duality of the fluctuations

The aim of this section is to show that the quadratic action associated with (5.9) is exactly equivalent to the one associated with (5.12). Since the gravitational part is equivalent, we do not explicitly write it down. In components, the action (5.9) is

$$S_{\mathrm{E}} = \int \mathrm{d}^4x\sqrt{|g|}\left[\frac{1}{12}H^{\mu\nu\rho}H_{\mu\nu\rho} + \frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}\chi\nabla_\mu H_{\nu\rho\sigma}\right] + \text{gravity}. \tag{5.39}$$

The associated quadratic action is given by

$$\begin{aligned}\delta^2 S_{\rm E} = \int {\rm d}^4x\sqrt{|g|}\Big[&\frac{1}{12}\delta H^{\mu\nu\rho}\delta H_{\mu\nu\rho} + \frac{1}{12}\delta g\, H^{\mu\nu\rho}\delta H_{\mu\nu\rho} \\ &- \frac{1}{2}\delta g^{\alpha\beta} H^{\mu\nu}{}_{\alpha}\delta H_{\mu\nu\beta} + \frac{1}{4}\delta g^{\alpha\beta}\delta g^{\kappa\lambda} H^{\mu}{}_{\alpha\kappa} H_{\mu\beta\lambda} \\ &+ \frac{1}{4}\left(\delta g^{\alpha\sigma}\delta g_{\sigma}{}^{\beta} - \frac{1}{2}\delta g\,\delta g^{\alpha\beta}\right) H^{\mu\nu}{}_{\alpha} H_{\mu\nu\beta} \\ &+ \frac{1}{48}\left(\frac{1}{2}\delta g^2 - \delta g^{\alpha\beta}\delta g_{\alpha\beta}\right) H^{\mu\nu\rho} H_{\mu\nu\rho} \\ &+ \frac{1}{6}\varepsilon^{\alpha\mu\nu\rho}\delta\chi\nabla_{\alpha}\delta H_{\mu\nu\rho}\Big] + \text{gravity}.\end{aligned} \tag{5.40}$$

Proceeding along similar lines as in 5.2.1, the last term is integrated by parts and the equation of motion for the field strength fluctuation is imposed. The Hodge duality relation (5.11) expanded to first order in the perturbation is

$$H_{\mu\nu\rho} = \varepsilon_{\alpha\mu\nu\rho}\partial^{\alpha}\chi \tag{5.41a}$$

$$\delta H_{\mu\nu\rho} = \varepsilon_{\alpha\mu\nu\rho}\left(\partial^{\alpha}\delta\chi + \frac{1}{2}\delta g\,\partial^{\alpha}\chi - \delta g^{\alpha\beta}\partial_{\beta}\chi\right) \tag{5.41b}$$

The first order piece (5.41b) contains the same information as the equation of motion for δH derived from (5.40). When this is inserted into the action, one obtains –up to the famous boundary term– the quadratic action associated with a scalar field with the wrong sign kinetic term:

$$\begin{aligned}\delta^2 S_{\rm E} = -\int {\rm d}^4x\sqrt{|g|}\Big[&\frac{1}{2}\partial_{\mu}\delta\chi\partial^{\mu}\delta\chi + \frac{1}{2}\delta g\,\partial_{\mu}\chi\partial^{\mu}\delta\chi + \delta g^{\mu\nu}\partial_{\mu}\chi\partial_{\nu}\delta\chi \\ &+ \frac{1}{8}\left(\frac{1}{2}\delta g^2 - \delta g^{\alpha\beta}\delta g_{\alpha\beta}\right)\partial_{\mu}\chi\partial^{\mu}\chi \\ &+ \frac{1}{2}\left(\delta g^{\mu\alpha}\delta g_{\alpha}{}^{\nu} - \frac{1}{2}\delta g\,\delta g^{\mu\nu}\right)\partial_{\mu}\chi\partial_{\nu}\chi \\ &- \nabla_{\mu}\left(\delta\chi\partial^{\mu}\delta\chi + \frac{1}{2}\delta g\,\delta\chi\partial^{\mu}\chi - \delta g^{\mu\nu}\delta\chi\partial_{\nu}\chi\right)\Big] + \text{gravity}.\end{aligned} \tag{5.42}$$

The presented argument demonstrates that one can adopt either the three–form formulation or the scalar field formulation to analyze the stability of the Giddings–Strominger wormhole. This is pursued in the rest of this section.

5.3.2 Three–form fluctuations

A stability analysis based on the three–form formulation was first presented in [241]. We closely follow their arguments, but, in contrast to their calculation, we present a treatment that is Euclidean from the start. That is, we start directly from the Euclidean action (5.40) (instead of from a Lorentzian one, as done in [241]). We parametrize the three–form fluctuation with auxiliary fields f and w as

$$\delta H = \frac{1}{6} f \mathcal{E}_{ijk}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k + \frac{1}{2} \mathcal{E}_{ijk} D^k w\; \mathrm{d}r \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j, \tag{5.43}$$

where, as before, $\mathcal{E}$ is the volume form on the unit S^3. The closure of δH implies

$$\dot{f} = \triangle w, \tag{5.44}$$

where $\triangle$ is the Laplacian on S^3. Under the gauge transformation (5.33), the fluctuations transform as

$$f \to f + Q \triangle \xi, \qquad\qquad w \to w + Q\dot{\xi}. \tag{5.45}$$

When the metric fluctuations (5.34) and the three–form fluctuations (5.43) are inserted into the action (5.40), a tedious calculation yields

$$\begin{aligned}
\delta^2 S_{\mathrm{E}} = \int \mathrm{d}^4x \sqrt{|\gamma|} \Big\{ & \frac{1}{2a^2} \left(f^2 - w \triangle w \right) + \frac{Q}{a^2} \left[(A + 3\psi - \triangle E)\, f + B \triangle w \right] \\
& + \frac{Q^2}{2a^2} \left[-B \triangle B + (A + 3\psi - \triangle E)^2 \right] + \left(\dot{f} - \triangle w \right) \delta\chi \\
& + \frac{a^2}{\kappa_4^2} \Big[-3\dot{\psi}^2 + \psi \triangle (\psi - 2A) - 6\mathcal{H} A \dot{\psi} - 2 \left(\mathcal{H} A + \dot{\psi} \right) \triangle \left(B - \dot{E} \right) \\
& + k \left((B - \dot{E}) \triangle (B - \dot{E}) - 3A^2 - 6A\psi + 3\psi^2 \right) \Big] \Big\}
\end{aligned} \tag{5.46}$$

The first two lines represent the matter fluctuations (coupled to gravity) and the last two lines the purely gravitational ones. Note that we inserted explicit factors of the three–space curvature k (although in practice we set it to one). This action includes *seven* scalar degrees of freedom $(\delta\chi, f, w, A, B, \psi, E)$. These are not all dynamical variables and three of them (namely $\delta\chi, A, B$) act as Lagrange multipliers. The constraints that they enforce, are a direct consequence of, respectively, the closure of the three–form field strength, Euclidean time–reparametrization invariance, and the arbitrariness of choosing a three–space coordinate system.

The Hamiltonian approach (see e.g. [242, 243]) is well–suited to treat constrained systems. This amounts to rewriting the action in a first–order form, schematically

$$\delta^2 S_{\mathrm{E}} = \int \mathrm{d}r \left[\pi_i \dot{q}^i - H(\pi, q) - N^i C_i\right], \tag{5.47}$$

where q^i represent the dynamical variables with momentum π_i, $H(\pi, q)$ the Hamiltonian, and C_i the constraints enforced by the Lagrange multiplies N^i. To this extent, we expand all the fields in S^3 harmonics[7], which allows for a reduction of the action by effectively integrating out the S^3 coordinates. In order not to over–complicate the notation, we suppress the mode indices, as well as the expansion sum (the actions written below can thus be understood as the action for one particular mode). We also do not distinguish the notation between the Laplacian operator $\triangle$ and its eigenvalue $-n^2 + 1$. There are some subtleties regarding the $n = 1$ homogeneous mode ($\triangle = 0$) and the $n = 2$ mode ($\triangle = -3k$). Nevertheless, we proceed by treating all modes simultaneously and we will comment on this caveat later.

[7]The scalar S^3 harmonics satisfy $\triangle Y = -(n^2 - 1)Y$ with $n \in \mathbb{N}$ and where mode indices are suppressed.

The gauge–invariant constraint (5.44) is now easily imposed[8] and the field w can be eliminated from the action at the expense of introducing $\dot{f}$. This gives an action in terms of the three dynamical variables ψ, E, f, whose momenta are given by

$$\pi_\psi = -\frac{2a^2}{\kappa_4^2}\left[3\dot{\psi} + 3\mathcal{H}A + \triangle\left(B - \dot{E}\right)\right] \tag{5.48a}$$

$$\pi_E = \frac{2a^2}{\kappa_4^2}\triangle\left[\dot{\psi} + \mathcal{H}A - k\left(B - \dot{E}\right)\right] \tag{5.48b}$$

$$\pi_f = \frac{1}{\triangle a^2}\left(Q\triangle B - \dot{f}\right) \tag{5.48c}$$

This allows a rewriting of the action in the first–order formalism:

$$\begin{aligned}\delta^2 S_{\mathrm{E}} = \int \mathrm{d}r\Big\{ & \pi_\psi\dot{\psi} + \pi_E\dot{E} + \pi_f\dot{f} - \frac{3\kappa_4^2}{4\triangle(\triangle+3k)a^2}\pi_E^2 + \frac{\kappa_4^2 k}{4(\triangle+3k)a^2}\pi_\psi^2, \\ & - \frac{\kappa_4^2}{2(\triangle+3k)a^2}\pi_E\pi_\psi + \frac{1}{2}a^2\triangle\pi_f^2 + \frac{a^2}{\kappa_4^2}(\triangle+3k)\psi^2 \\ & + \frac{1}{2a^2}\left[f + Q(3\psi - \triangle E)\right]^2 - B\left[\pi_E + Q\triangle\pi_f\right] \\ & - A\left[\left(\frac{2a^2}{\kappa_4^2}\left(\triangle+3k\right) - \frac{3Q^2}{a^2}\right)\psi - \mathcal{H}\pi_\psi + \frac{Q}{a^2}\left(Q\triangle E - f\right)\right]\Big\}.\end{aligned} \tag{5.49}$$

This makes the constraints imposed by A and B transparent. After introducing the gauge–invariant variable $\mathcal{F}$ and its conjugate momentum $\Pi_{\mathcal{F}}$, defined by

$$\mathcal{F} = f - Q\triangle E, \qquad\qquad \Pi_{\mathcal{F}} = \pi_f + \frac{Q}{a^2\mathcal{H}}\psi, \tag{5.50}$$

the action eventually becomes manifestly gauge–invariant:

$$\delta^2 S_{\mathrm{E}} = \int \mathrm{d}r\left[\Pi_{\mathcal{F}}\dot{\mathcal{F}} + \alpha\Pi_{\mathcal{F}}^2 + \beta\Pi_{\mathcal{F}}\mathcal{F} + \gamma\mathcal{F}^2\right], \tag{5.51a}$$

[8]Imposing constraints can be interpreted in the path–integral formulation as performing the path integral over the Lagrange multiplier. This results in a δ–functional that effectively imposes the constraint.

with the coefficients given by the background–dependent functions

$$\alpha = \frac{1}{2}\triangle a^2 - \frac{3\kappa_4^2 Q^2 \triangle}{4a^2(\triangle + 3k)}, \tag{5.51b}$$

$$\beta = -\frac{\kappa_4^2 Q^2 \triangle}{2a^4(\triangle + 3k)\mathcal{H}}, \tag{5.51c}$$

$$\gamma = \frac{\triangle}{2a^2(\triangle + 3k)} + \frac{3k^2}{2a^2\mathcal{H}^2}. \tag{5.51d}$$

The fields ψ and E have disappeared from the action. This is a direct consequence of working with gauge–invariant variables. In the path–integral formulation, the path integral over them would yield a gauge–orbit contribution. Before analyzing the stability, it is useful to discuss some subtleties of this action.

- The homogeneous mode ($n = 1$) is non–dynamical. Indeed, setting $\triangle = 0$ results in the vanishing of α and β, such that only the $\mathcal{F}^2$–term in the action survives. The constraint enforced by the Lagrange multiplier $\Pi_{\mathcal{F}}$ then implies $\dot{\mathcal{F}} = 0$. This is a direct consequence of the constraint equation (5.19). Since the conformal factor problem only manifests itself in the homogeneous sector [244], we do not encounter it.

- The action is not well–defined for $n = 2$ mode ($\triangle = -3k$), since this would naively imply a divergence of α, β, and γ. For these modes, the momenta π_ψ and π_E are equal, and ψ and E combine to form one dynamical field $\psi + E$. When carefully analyzed, one is left to conclude that this mode is also non–dynamical [241]. The same observation was made in [244].

- At the wormhole neck, the conformal Hubble rate $\mathcal{H}$ vanishes. This is problematic for β and γ. In order to have a finite action, one must impose that the fluctuations vanish at the neck, $\mathcal{F}(r = 0) = 0$. An explicit mode analysis, performed in [241], shows that this is indeed the case: only odd fluctuations have finite action and therefore correspond to physical fluctuations.

- As a (mild) sanity check, one can easily verify that the correct free–field theory is obtained when setting $\kappa_4 \to 0$ and $\mathcal{H} \to 1$.

To analyze the stability of the wormhole, one must only concentrate on modes with $n \geqslant 3$. It is crucial to impose the correct boundary condition on the fluctuations $\mathcal{F}$ and $\Pi_{\mathcal{F}}$ that correspond to these modes. Recall that the axion charge must remain fixed at the boundary, cf. (5.37). Since fluctuation of the charge can be expressed in terms of the gauge–invariant variables as

$$\delta\tilde{Q} = \int_{S^3} \mathrm{d}^3x \sqrt{|\gamma|}\, f = \int_{S^3} \mathrm{d}^3x \sqrt{|\gamma|}\, \mathcal{F}, \tag{5.52}$$

the boundary condition $\delta\tilde{Q}\big|_{\partial} = 0$ can be translated as a *Dirichlet* boundary condition on $\mathcal{F}$. This can be imposed by integrating out $\Pi_{\mathcal{F}}$, which turns (5.51) into the familiar Lagrangian form:

$$\delta^2 S_{\mathrm{E}} = \int \mathrm{d}r \left[K\dot{\mathcal{F}}^2 + V\mathcal{F}^2\right] - B\mathcal{F}^2\big|_{\partial}\,, \tag{5.53a}$$

with the background functions explicitly given by (setting $k = 1$ and using (5.23))

$$K = \frac{1}{\kappa_4 Q}\sqrt{\frac{3}{2}}\frac{n^2-4}{n^2-1}\frac{\cosh(2r)}{(n^2-4)\cosh^2(2r)+9} \tag{5.53b}$$

$$V = \frac{n^2-4}{\kappa_4 Q}\sqrt{\frac{3}{2}}\frac{\left[\left(n^2-4\right)\cosh^2(2r)-3\right]\cosh(2r)}{\left[(n^2-4)\cosh^2(2r)+9\right]^2} \tag{5.53c}$$

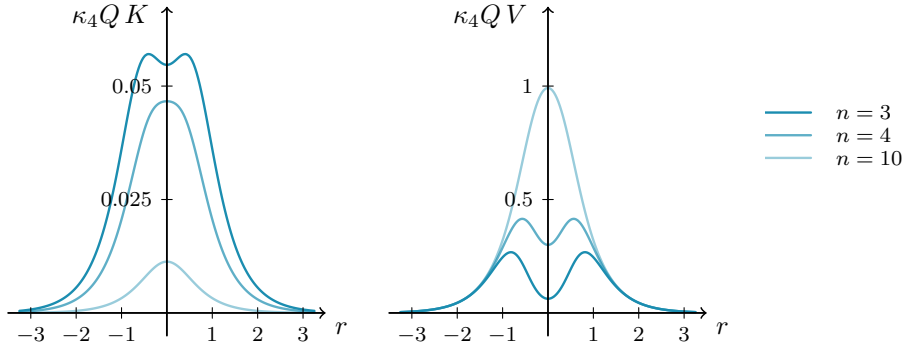
and where we integrated by parts, producing the boundary term

$$B = \frac{1}{\kappa_4 Q}\sqrt{\frac{3}{2}}\frac{3\,\mathrm{csch}(2r)}{(n^2-4)\cosh^2(2r)+9}. \tag{5.53d}$$

Since at the boundary $B \to 0$, and simultaneously $\mathcal{F} \to 0$, the boundary term vanishes and is of no concern. The functions K and V are plotted in figure 5.3 for the modes $n = 3, 4, 10$. For all n, both functions are everywhere positive on the real axis. It must therefore be concluded that all fluctuations obeying the boundary conditions cannot lower the action:

$$\delta^2 S_{\mathrm{E}} \geqslant 0. \tag{5.54}$$

The Giddings–Strominger wormhole is thus perturbatively stable!



(a) The 'kinetic' function K. **(b)** The 'potential' function V.

Figure 5.3: Plot of the background–dependent functions K and V that appear in the quadratic action (5.53) for the $n = 3, 4, 10$ modes. All other modes ($n \geqslant 3$) have a qualitatively similar behavior.

5.3.3 Scalar field fluctuations

In the past, there have been several attempts to perform a stability analysis in the scalar field formulation of the axion, none of which have been fully correct or satisfactory [245–247]. We now present a treatment that is fully Euclidean from the start and that imposes the correct boundary conditions on the scalar field fluctuations. Since it has been argued that the three–form formulation is identical to the scalar field formulation, the conclusion of the previous section should remain unaltered.

Under the gauge transformation (5.33), the scalar field fluctuation transforms as

$$\delta\chi \to \delta\chi + \xi^0 \dot{\chi}. \tag{5.55}$$

The advantage of the scalar field formulation is that the fluctuation theory has been discussed before in different contexts (e.g. in the context of cosmology [240, 244, 248–251]). The results obtained below can therefore be compared –after a suitable analytic continuation– with the literature, and we indeed find an agreement.

Inserting the metric fluctuations (5.34) into the action (5.42) yields

$$\begin{aligned}\delta^2 S_{\rm E} = \int {\rm d}^4x\sqrt{|\gamma|}a^2\Big\{ &-\frac{1}{2}\dot{\delta\chi}^2+\frac{1}{2}\delta\chi\triangle\delta\chi+A\dot\chi\dot{\delta\chi}-\left[3\dot\psi+\triangle\left(B-\dot E\right)\right]\dot\chi\delta\chi\\ &+\frac{1}{\kappa_4^2}\Big[-3\dot\psi^2+\psi\triangle(\psi-2A)-6\mathcal{H}A\dot\psi-2\left(\mathcal{H}A+\dot\psi\right)\triangle\left(B-\dot E\right)\\ &+k\left((B-\dot E)\triangle(B-\dot E)-3A^2-6A\psi+3\psi^2\right)\Big]\Big\},\end{aligned} \tag{5.56}$$

where the first line contains the gravitationally coupled matter fluctuations and the last two lines are the purely gravitational fluctuations. Proceeding as before, all fields are expanded in S^3 harmonics. The constrained system is best analyzed in the Hamiltonian formulation; the momenta conjugate to the dynamical variables ψ, E, and $\delta\chi$ are given by

$$\pi_\psi = -\frac{2a^2}{\kappa_4^2}\left[3\dot\psi+3\mathcal{H}A+\triangle\left(B-\dot E\right)+\frac{3}{2}\kappa_4^2\dot\chi\delta\chi\right] \tag{5.57a}$$

$$\pi_E = \frac{2a^2}{\kappa_4^2}\triangle\left[\dot\psi+\mathcal{H}A-k\left(B-\dot E\right)+\frac{1}{2}\kappa_4^2\dot\chi\delta\chi\right] \tag{5.57b}$$

$$\pi_{\delta\chi} = a^2\left(A\dot\chi-\dot{\delta\chi}\right) \tag{5.57c}$$

The action in Hamiltonian form eventually becomes

$$\begin{aligned}\delta^2 S_{\rm E} = \int {\rm d}r\Big\{ &\pi_\psi\dot\psi+\pi_E\dot E+\pi_{\delta\chi}\dot{\delta\chi}-\frac{3\kappa_4^2}{4\triangle\left(\triangle+3k\right)a^2}\pi_E^2\\ &+\frac{\kappa_4^2 k}{4\left(\triangle+3k\right)a^2}\pi_\psi^2-\frac{\kappa_4^2}{2\left(\triangle+3k\right)a^2}\pi_E\pi_\psi+\frac{1}{2a^2}\pi_{\delta\chi}^2\\ &+\frac{a^2}{\kappa_4^2}\left(\triangle+3k\right)\psi^2+\frac{1}{2}\kappa_4^2\dot\chi\pi_\psi\delta\chi+\frac{1}{2}a^2\left(\triangle+\frac{3}{2}\kappa_4^2\dot\chi^2\right)\delta\chi^2\\ &-B\pi_E-A\left[\pi_{\delta\chi}\dot\chi-\mathcal{H}\pi_\psi-3a^2\mathcal{H}\dot\chi\delta\chi+\frac{2a^2}{\kappa_4^2}\left(\triangle+3k\right)\psi\right]\Big\}.\end{aligned} \tag{5.58}$$

We now introduce the well–known gauge–invariant Mukhanov variable $\mathcal{X}$ and its conjugate momentum $\Pi_{\mathcal{X}}$:

$$\mathcal{X} = \delta\chi + \frac{\dot{\chi}}{\mathcal{H}}\psi, \qquad \Pi_{\mathcal{X}} = \pi_{\delta\chi} + 3a^2\dot{\chi}\psi. \tag{5.59}$$

This turns the action into the gauge–invariant form

$$\delta^2 S_{\mathrm{E}} = \int \mathrm{d}r \left[\Pi_{\mathcal{X}}\dot{\mathcal{X}} + \gamma\Pi_{\mathcal{X}}^2 - \beta\Pi_{\mathcal{X}}\mathcal{X} + \alpha\mathcal{X}^2\right] - \Pi_{\mathcal{X}}\mathcal{X}|_{\partial}\,, \tag{5.60}$$

where the background functions α, β, γ are again given by (5.51b)–(5.51d). The same remarks as above are in order: only the $n \geqslant 3$ modes are dynamical and one must –at least– insist that $\Pi_{\mathcal{X}}(r=0) = 0$ to have a finite action. Admittedly, we have been a bit sloppy about keeping track of boundary terms. We, however, argue that the non–gauge invariant boundary term in (5.42) pairs up with gravitational terms to ultimately produce the infamous gauge–invariant $\Pi_{\mathcal{X}}\mathcal{X}|_{\partial}$ boundary term. This boundary term enforces a Dirichlet boundary condition on the momentum:

$$\Pi_{\mathcal{X}}|_{\partial} = 0, \tag{5.61}$$

or equivalently, a *Neumann* condition on $\mathcal{X}$. This indeed provides the correct boundary condition that keeps the charge fixed:

$$\delta\tilde{Q} = \int \mathrm{d}^3x\sqrt{|\gamma|}\left[a^2\delta\dot{\chi} + (\triangle E - A - 3\psi)\,a^2\dot{\chi}\right] = -\int \mathrm{d}^3x\sqrt{|\gamma|}\,\Pi_{\mathcal{X}}. \tag{5.62}$$

Imposing the Neumann condition on $\mathcal{X}$ can be achieved by integrating out $\mathcal{X}$ in the action (5.60). This gives

$$\delta S_{\mathrm{E}} = \int \mathrm{d}r \left[K\dot{\Pi}_{\mathcal{X}}^2 + V\Pi_{\mathcal{X}}^2\right] + B\Pi_{\mathcal{X}}|_{\partial}\,, \tag{5.63}$$

with K and V given by (5.53b) and (5.53c). By the same argument, we are again led to conclude that Giddings–Strominger wormholes are stable: $\delta^2 S_{\mathrm{E}} \geqslant 0$.

5.3.4 Comments on the literature

The treatment presented above is everywhere Euclidean and fully gauge–invariant. This is not encountered anywhere in the literature on the topic and different conclusions can be found. In [245] it was demonstrated that there exists –at least– one negative mode around the Giddings–Strominger wormhole, found in the homogeneous sector. Their approach, however, was only based on the Lagrangian formalism and was non–gauge invariant. Since, as argued above, the homogeneous mode should be non–dynamical, this conclusion is obviously incorrect. In particular, they inconsistently truncated the perturbations.

A gauge–invariant approach has been presented in [241, 246, 247], but none of them performed Euclidean perturbation theory. Instead, they used (known) results from Lorentzian, cosmological perturbation theory and Wick–rotated the quadratic action. Confusingly, they reached different conclusions. The analysis presented in [246] claimed an absence of negative modes. However, they were completely careless about boundary conditions: within the Hamiltonian formulation, they performed symplectic transformations on the gauge–invariant variable and its momentum with the motivation to eventually be able to explicitly diagonalize the quadratic operator. This is obviously incorrect since symplectic transformations affect the boundary conditions.

In trying to correct this, [247] insisted on imposing a Neumann boundary condition on the scalar field fluctuation. They found negative modes in every inhomogeneous sector, concluding the Giddings–Strominger wormholes would not contribute to the path integral. This claim, however, is incorrect: although claimed differently, they eventually imposed the wrong boundary conditions. Since this is a very subtle issue, we must scrutinize it. Instead of using the Mukhanov variable (5.59), they used the rescaled version

$$\mathcal{Y} = \frac{\mathcal{H}}{\dot{\chi}}\mathcal{X}, \qquad\qquad \Pi_{\mathcal{Y}} = \frac{\dot{\chi}}{\mathcal{H}}\Pi_{\mathcal{X}}. \tag{5.64}$$

This is in principle not a problem and a correct result could still be obtained. The gauge–invariant quadratic action (5.60) becomes

$$\delta^2 S_{\rm E} = \int {\rm d}r \left[\Pi_{\mathcal{Y}} \dot{\mathcal{Y}} + \frac{\mathcal{H}^2 a^4}{Q^2} \gamma \Pi_{\mathcal{Y}}^2 - \left(\beta + \frac{2k}{\mathcal{H}} \right) \Pi_{\mathcal{Y}} \mathcal{Y} + \frac{Q^2}{\mathcal{H}^2 a^4} \alpha \mathcal{Y}^2 \right] - \Pi_{\mathcal{Y}} \mathcal{Y}|_{\partial} \,. \tag{5.65}$$

Imposing a Neumann condition on $\mathcal{Y}$ by integrating it out of this action, keeping the action in terms of $\Pi_{\mathcal{Y}}$, entails a subtlety that is apparent from (5.62):

$$\delta \tilde{Q}\big|_{\partial} \propto \int_{S^3} {\rm d}^3 x \sqrt{|\gamma|}\, a^2 \Pi_{\mathcal{Y}}. \tag{5.66}$$

The correct boundary condition is therefore that $\Pi_{\mathcal{Y}} \to 0$ faster than $\sim e^{2|r|}$ as $r \to \pm\infty$. This, however, was not implemented in [247]. To make matters worse, they used a result in the existing literature [251] which presented the action already in the Lagrangian formalism. The crux is that $\Pi_{\mathcal{Y}}$ is integrated out (5.65) *and* integration by parts is performed to yield an action of the form

$$\delta^2 S_{\rm E} = \int {\rm d}r \left[\tilde{K} \dot{\mathcal{Y}}^2 + \tilde{V} \mathcal{Y}^2 \right] + \tilde{B} \mathcal{Y}^2 \big|_{\partial} \,. \tag{5.67}$$

In [247], the boundary term was omitted, which is clearly incorrect for the wormhole case! Proceeding ignorantly about this issue, they claimed to impose a Neumann condition on $\mathcal{Y}$ by performing a 'symplectic flip' of this action. This amounts to defining the momentum conjugate $\Pi = 2\tilde{K}\dot{\mathcal{Y}}$, flipping it with $\mathcal{Y}$ in the Hamiltonian, and integrating out $\mathcal{Y}$. This results in the action

$$\delta^2 S_{\rm E} = -\int {\rm d}r \left[\frac{1}{4} \tilde{V}^{-1} \dot{\Pi}^2 + \frac{1}{4} \tilde{K}^{-1} \Pi^2 \right]. \tag{5.68}$$

Crucially, the new momentum Π is manifestly different from $\Pi_{\mathcal{Y}}$:

$$\Pi = \Pi_{\mathcal{Y}} - 2\tilde{B}\mathcal{Y}, \tag{5.69}$$

which is clearly a consequence of the omitted boundary term. The conclusion of [247] is that there exists an infinite amount of fluctuations with $\Pi \to 0$ at the boundary that lower this action. These fluctuations, however, do not fix the wormhole charge at the boundary; rather, they impose a Robin boundary condition on $\mathcal{Y}$ instead of a Neumann one.

The boundary conditions of [247] were also criticized in [241], but –unfortunately– incorrectly. The latter claim that the correct boundary conditions are Dirichlet for the metric fluctuations and Neumann for the scalar field ones, or equivalently, also Dirichlet for the three–form fluctuations. This lies at the basis of their motivation for studying three–form fluctuations: they argued that it is more transparent to impose a Dirichlet on $\mathcal{F}$ (5.50), which would correspond to a Dirichlet on the metric and the three–form fluctuation. Working with the Mukhanov variable $\mathcal{Y}$ (5.64), they argued, mixes metric and axion fluctuations, which would make it unclear how to impose a Dirichlet for the metric and a Neumann for the axion scalar. This claim, however, has no meaning: boundary conditions are to be imposed on gauge–invariant quantities only. For Giddings–Strominger wormholes, this is the charge. We demonstrated in equation (5.52) that the boundary condition of [241] indeed leaves the charge fixed at the wormhole ends. We also indicated that this is not at all unclear if one wishes to work with the scalar field formulation, see equation (5.62).

5.4 Discussion

This chapter provided the technical evidence that Giddings–Strominger wormholes should be viewed as genuine minima of the action and thus contribute to the saddle point expansion. The absence of a negative mode suggests that a tunneling interpretation is not appropriate. Rather, wormholes are instantons that lift a degeneracy. In particular, they would compute a non–perturbative correction to the energy of degenerate flux vacua with flux $\pm\tilde{Q}$ [241].

In section 4.3.1 it was mentioned that the weak gravity conjecture also applies to axions; this is called the axion weak gravity conjecture [182]. Likewise, it predicts the existence of *superextremal* objects; instead of a superextremal charge–to–tension ratio, the axion weak gravity conjecture predicts (-1)–branes with a superextremal charge–to–*action* ratio. How this applies to wormholes can best be understood from instanton solutions to (5.8), which fall in three categories [215]:

- wormhole solutions, possessing no net charge;
- extremal solutions where the metric is everywhere flat, but the field profiles are non–trivial;
- singular solutions, called 'cored' instantons, that can carry a net charge.

Inspection of these solutions reveals that wormholes have *lower* action than the extremal objects [230]; they are the superextremal objects required by the weak gravity conjecture. This supports the view that wormholes are relevant saddle points.

Although the analysis is carried out in asymptotically–flat space, the result likely carries over to wormholes in de Sitter or anti–de Sitter. This would mean that one has to face the problems presented by the wormholes. Of course, the Giddings–Strominger wormhole is merely a toy model and the aforementioned problems are only to be faced when wormholes are realizable in string theory. It is a false hope that string theory would naturally avoid these; on the contrary, explicit embeddings

have been found [215,225–227]. An obvious way out would be that the inclusion of other fields, such as a saxion, drastically changes the perturbative stability. This remains an interesting topic for future research.

Chapter 6

Espice, Adspice, Prospice

In this thesis, we have analyzed three semi–classical, string–inspired toy models to gain insight into some quantum aspects of gravity in a cosmological setting.

- The question of why our universe appears four–dimensional led us to consider cosmic egg scenarios in chapter 3.
- Motivated by the difficulty of finding explicit de Sitter solutions in string theory, we discussed the Dark Bubble model in chapter 4 and explored its applicability to quantum cosmology, as well as the apparent negative energy contributions in the four–dimensional Einstein equation.
- The paradoxes posed by Euclidean wormholes provided the motivation for chapter 5, where we analyzed their perturbative stability as saddle points of the Euclidean path integral.

In this last chapter, we conclude, present the key findings, and give an outlook for possible future research.

In chapter 3, we have argued that scale separation already feels unnatural from a cosmological perspective and that the cosmological constant problem motivates the study of cosmological scenarios in which all spatial dimensions are initially of comparable, small size. That is, cosmology encourages one to consider *de*compactification (such as string gas cosmology [121]).

We have proposed a new toy model in this spirit in which a universe with three large spatial dimensions is seen to emerge in a two–step process. First, we conceive of a metastable, classical, fully compactified configuration in the Hartle–Hawking state. These effectively one–dimensional cosmic egg configurations are neither de Sitter nor anti-de Sitter but breathing, i.e. they have an oscillating volume modulus. The absence of scale separation means that cosmic eggs do not suffer from a cosmological hierarchy problem and are thus plausibly natural early universe configurations.

When the cosmological constant term is positive but below the Einstein static value in the model, we have identified a decompactification channel in which the egg decays through tunneling into an expanding universe with three large spatial dimensions. According to our interpretation, this is not quite a creation out of nothing, but rather a creation out of something, i.e. a cosmic egg. In a landscape consisting of eggs with different parameters the characteristic Hartle–Hawking weighting favors the nucleation of large universes with a low value of the four–dimensional cosmological constant. This casts the cosmological constant problem in a new light: a seemingly unnaturally small vacuum energy is seen to arise dynamically from natural initial conditions.

In constructing our cosmic egg model, we have focused on the creation of four–dimensional universes with a fixed value for the cosmological constant. These two limitations could be further examined in upcoming research. Firstly, it would be interesting to see an embedding of cosmic eggs into quintessence scenarios, which would make a cosmic egg scenario independent of the existence of a de Sitter landscape. We expect, however, that much of our reasoning carries over: the eggs decay to a large expanding universe with a rolling value of the vacuum energy. It would be equally interesting to see higher–dimensional universes being born

from cosmic eggs and to compare the nucleation rates of four–dimensional versus higher–dimensional universes. Our expectation is that it is very likely that the decay towards a four–dimensional universe is on statistical grounds preferred.

Even more exciting would be to embed our cosmic egg scenario into a –somewhat– realistic model of the early universe. One possible route would be to consider the breathing eggs as a pre–inflationary phase, with the decay process leading to a period of inflation that generates a large universe filled with matter and primordial perturbations.

Chapter 4 discussed the Dark Bubble model, a novel realization of de Sitter space in a string–inspired set–up. It proposes that a de Sitter universe lives on top of an expanding Brown–Teitelboim bubble in a five–dimensional anti–de Sitter space. Given our current understanding of the landscape and the swampland, it seems plausible that the model can be embedded into string theory. There is presently no top–down construction; this constitutes an interesting direction for further research. It is distinct from Randall–Sundrum since neither gravity nor matter is localized to the brane, but rather extends 'holographically' into the bulk. The model seems viable for late–time cosmology; early universe aspects, however, have not been addressed and it would be interesting to examine whether, for instance, inflation can be realized.

We argued that the Dark bubble is a realization of Vilenkin's quantum cosmology; the Vilenkin amplitude can be understood as the nucleation probability of a Brown–Teitelboim bubble. According to our interpretation, the nucleation of the four–dimensional cosmology is not quite a creation out of nothing, but rather a creation out of something, i.e. the already present bulk spacetime. Our understanding of the physics of bubble nucleation translates to an understanding of the more involved issues in quantum cosmology, such as the choice of boundary conditions. From the five–dimensional point of view, one still needs to explain its origin, which relates to the aforementioned research question of finding an explicit embedding.

Cosmic eggs and the Dark Bubble substantiate our claim that the boundary conditions in quantum cosmology are to be decided by the UV completion of gravity. In the former case, the effective minisuperspace potential diverges near vanishing

scale factor, which selects the Hartle–Hawking proposal. The latter scenario rather selects the Vilenkin proposal by demanding consistency with the known physics of bubble nucleation.

Our higher–dimensional view on the Vilenkin proposal intuitively explains why the instabilities argued for in [204, 206] can not be there on physical grounds. To put this on a firm footing, it is needed to include, for instance, gravitational perturbations into the minisuperspace model and it would be interesting to see whether the 'higher–dimensional view' could shed some new light on this. This would require the full five–dimensional uplift of the gravitational waves. We have provided this uplift, but we leave the exact application to quantum cosmology for further research.

The waves we constructed extend outwards as well as inwards from the bubble. The waves remain finite everywhere and, in particular, go to zero at the center of the bubble. We have shown that the gravitational waves in the bulk backreact on the bubble in the same manner as a four–dimensional wave would do in a four–dimensional cosmology. This demonstrates the consistency of the model.

Finally, we have also commented on the negative sign kinetic terms of brane excitations in the Dark Bubble model. This sign is a direct consequence of the dark bubble having an inside and an outside, contrary to Randall–Sundrum. This changes the physics in a dramatic way. Using the gravitational waves, we have shown how the excitations should not be interpreted as four–dimensional matter but instead as the four–dimension gravitational response mediated by the brane.

In chapter 5 we have confirmed the recent result of [241] that Giddings–Strominger wormholes are perturbatively stable saddle points in the three–form picture. Using a gauge–invariant formulation, the same conclusion was reached in the scalar field picture. A correct application of the boundary conditions on the field fluctuations is crucial to reach these conclusions; physically, as we substantiated, this corresponds to fixing the axion charge at the wormhole ends.

It would be interesting to apply the same perturbative analysis for wormholes sourced by multiple axion as well as saxion fields. The scalar field formulation of the axion is possibly the most transparent approach since a wormhole solution corresponds to a geodesic in scalar field space. This suggests adopting Fermi normal coordinates [252], or a suitable generalization thereof [253], to parameterize the field fluctuations. This is currently work in progress and we hope to obtain some results in the near future. Equally interesting would be to consider wormholes embedded in (anti–)de Sitter. First inspection seems to suggest that the stability is unchanged in the case of Giddings–Strominger wormholes.

Appendix A

Conventions, differential forms, and useful formulae

This appendix outlines the basic notations, conventions, and some formulae used 'behind the scenes' in this thesis regarding differential geometry. Let $\mathcal{M}$ be a D–dimensional manifold equipped with (pseudo–)Riemannian metric g and let x^μ be a smooth coordinate chart.

We denote the symmetrization of a tensor $T_{\mu_1 \dots \mu_p}$ by

$$T_{(\mu_1 \dots \mu_p)} = \frac{1}{p!} \left(T_{\mu_1 \dots \mu_p} + \text{even permutations} + \text{odd permutations} \right), \tag{A.1a}$$

and the anti–symmetrization by

$$T_{[\mu_1 \dots \mu_p]} = \frac{1}{p!} \left(T_{\mu_1 \dots \mu_p} + \text{even permutations} - \text{odd permutations} \right). \tag{A.1b}$$

A.1 The metric

The metric can be written in the x^μ–coordinate chart as

$$g = g_{\mu\nu}(x)\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu. \tag{A.2}$$

The Lorentzian signature is taken to be mostly plus

$$g = \mathrm{diag}\,(-,+,+,\ldots,+)\,. \qquad \text{(Lorentzian)} \tag{A.3}$$

The Wick rotation transforms the time coordinate x^0 into a *Euclidean* time coordinate $x^0_{\mathrm{E}} \equiv ix^0$. The metric signature then changes to Euclidean[1]:

$$g = \mathrm{diag}\,(+,+,+,\ldots,+)\,. \qquad \text{(Euclidean)} \tag{A.4}$$

For notational convenience, we introduce the notation

$$\mathrm{sgn}\; g_{00} = (-)^t, \tag{A.5}$$

where t is the number of time dimensions ($t = 1$ for Lorentzian, $t = 0$ for Euclidean).

A.2 Curvature symbols

The unique, torsion–free, metric compatible connection is called the Levi–Civita or Christoffel connection:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right). \tag{A.6}$$

From this, the following curvature tensors are constructed:

[1] Strictly speaking, the metric is not necessarily (flat) Euclidean, but rather (curved) Riemannian. Nevertheless, 'Euclidean' is the conventional nomenclature.

- Riemann tensor

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} - \Gamma^\alpha_{\nu\sigma}\Gamma^\mu_{\alpha\rho} - \Gamma^\alpha_{\nu\rho}\Gamma^\mu_{\alpha\sigma}, \tag{A.7a}$$

- Ricci tensor

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}, \tag{A.7b}$$

- Ricci scalar

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu}, \tag{A.7c}$$

- Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu}, \tag{A.7d}$$

- Kretschmann scalar

$$K = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}. \tag{A.7e}$$

When confusion is possible, e.g. in the context of hypersurfaces or for factorizable metrics, we introduce a prescipt on the curvature symbols to denote the dimensionality of the manifold to which they refer.

In this thesis, three types of metrics often arise. It is useful to present their general form, as well as their scalar curvature:

- Cosmological metrics of the form

$$\mathrm{d}s^2 = (-)^t N^2(\lambda)\,\mathrm{d}\lambda^2 + a^2(\lambda)\,\mathrm{d}\Omega^2_{D-1}, \tag{A.8}$$

  where $\mathrm{d}\Omega^2_{D-1}$ is the metric on a Euclidean $(D-1)$–space with constant curvature $k \in \{-1, 0, +1\}$. The Ricci scalar is given by

$$\mathcal{R} = -(-)^t (D-1)\left[(D-2)\left(\frac{1}{N^2}\frac{\dot{a}^2}{a^2} - (-)^t\frac{k}{a^2}\right) + 2\left(\frac{1}{N^2}\frac{\ddot{a}}{a} - \frac{\dot{a}\dot{N}}{aN^3}\right)\right], \tag{A.9}$$

  where a dot denotes a derivative with respect to λ.

- Another type of metrics of interest are of the Kaluza–Klein type where n dimensions are compactified

$$\mathrm{d}s^2 = \mathrm{d}s^2_{D-n} + \mathrm{d}s^2_n = e^{2\alpha\phi(x)} g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + e^{2\beta\phi(x)}\gamma_{ij}(\theta)\,\mathrm{d}\theta^i\,\mathrm{d}\theta^j. \quad \text{(A.10)}$$

  The associated Ricci scalar is

$$\begin{aligned} {}^{(D)}\mathcal{R} = e^{-2\beta\phi}\,{}^{(n)}\mathcal{R} + e^{-2\alpha\phi}\Big[{}^{(D-n)}\mathcal{R} - 2\left[(D-n-1)\alpha + n\beta\right]\Box\phi \\ - \left[(D-n-2)(D-n-1)\alpha^2 + 2n(D-n-2)\alpha\beta \right. \\ \left. + n(n+1)\beta^2\right]\partial_\mu\phi\partial^\mu\phi\Big] \end{aligned} \quad \text{(A.11)}$$

- Metrics where the geometry is warped, as in the context of braneworlds:

$$\mathrm{d}s^2 = A^2(z)\gamma_{mn}(y)\,\mathrm{d}y^m\,\mathrm{d}y^n + \mathrm{d}z^2, \quad \text{(A.12)}$$

  where $A(z)$ is called the warp factor. The normalization is often chosen such that γ is the induced metric on the brane. The Ricci scalar is given by

$${}^{(D)}\mathcal{R} = \frac{1}{A^2}\,{}^{(D-1)}\mathcal{R} - 2(D-1)\frac{A''}{A} - (D-1)(D-2)\left(\frac{A'}{A}\right)^2, \quad \text{(A.13)}$$

  where ${}^{(D-1)}\mathcal{R}$ is the Ricci scalar on the brane, obtained from the metric γ, and where a prime denotes a derivative with respect to z.

A.3 Differential forms

A differential p–form $\omega^{(p)}$ on $\mathcal{M}$ is an alternating tensor field: a map that sends p vector fields onto one real–valued function on $\mathcal{M}$. Alternating means that it flips its sign if any two arguments are interchanged. Locally, in the coordinate chart x^μ, a p–form can be expressed in components as

$$\omega^{(p)} = \frac{1}{p!}\omega^{(p)}_{\mu_1\ldots\mu_p}\,\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}. \quad \text{(A.14)}$$

The wedge product between a p–form and a q–form produces a $(p+q)$–form, with $p+q \leqslant D$. It is defined to be totally anti–symmetric and it, consequently, satisfies

$$\omega^{(p)} \wedge \omega^{(q)} = (-)^{pq} \omega^{(q)} \wedge \omega^{(p)}. \tag{A.15}$$

The exterior derivative d is a linear map from p–forms to $(p+1)$–forms that introduces the notion of exterior calculus. It is defined by

$$\mathrm{d}\omega^{(p)} = \frac{1}{p!} \partial_{[\alpha} \, \omega^{(p)}_{\mu_1 \ldots \mu_p]} \, \mathrm{d}x^{\alpha} \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}. \tag{A.16}$$

Note that the anti–symmetrization guarantees that $\partial_{[\alpha} \, \omega^{(p)}_{\mu_1 \ldots \mu_p]}$ transforms as a tensor, so no covariant derivative is needed. We furthermore have the Leibniz rule

$$\mathrm{d}\left(\omega^{(p)} \wedge \omega^{(q)}\right) = \mathrm{d}\omega^{(p)} \wedge \omega^{(q)} + (-)^{p} \omega^{(p)} \wedge \mathrm{d}\omega^{(q)}. \tag{A.17}$$

Differential p–forms for which $\mathrm{d}\omega^{(p)} = 0$ are called *closed*, and p–forms $\omega^{(p)}$ for which there exists a $(p-1)$–form $\omega^{(p-1)}$ such that $\omega^{(p)} = \mathrm{d}\omega^{(p-1)}$ are called *exact*.

To measure the volume of $\mathcal{M}$, we adopt the *canonical volume form*. To introduce it, we first define the Levi–Civita *symbol* as the totally anti–symmetric symbol $\epsilon_{\mu_1 \mu_2 \ldots \mu_D}$ with $\epsilon_{01\ldots D} = +1$, explicitly

$$\epsilon_{\mu_1 \mu_2 \ldots \mu_D} = \begin{cases} +1 & \text{if } \mu_1 \mu_2 \ldots \mu_D \text{ is an even permutation of } 01 \ldots D \\ -1 & \text{if } \mu_1 \mu_2 \ldots \mu_D \text{ is an odd permutation of } 01 \ldots D \\ 0 & \text{else} \end{cases} . \tag{A.18}$$

Although it is defined in the x^{μ} coordinate system, the symbol does not transform as a tensor. To reconcile this, we define the Levi–Civita *tensor* as

$$\varepsilon_{\mu_1 \ldots \mu_D} \equiv \sqrt{|g|} \epsilon_{\mu_1 \ldots \mu_D}, \qquad \varepsilon^{\mu_1 \ldots \mu_D} \equiv \frac{(-)^t}{\sqrt{|g|}} \epsilon^{\mu_1 \ldots \mu_D}, \tag{A.19}$$

where g is the metric determinant. Contractions obey

$$\varepsilon_{\alpha_1 \ldots \alpha_r \mu_{r+1} \ldots \mu_D} \varepsilon^{\alpha_1 \ldots \alpha_r \nu_{r+1} \ldots \nu_D} = (-)^t r! \delta^{\nu_{r+1} \ldots \nu_D}_{\mu_{r+1} \ldots \mu_D}, \tag{A.20}$$

where $\delta^{\nu_1\dots\nu_k}_{\mu_1\dots\mu_k}$ is the generalized Kronecker delta. The canonical volume form ε is

$$\varepsilon = \frac{1}{D!}\varepsilon_{\mu_1\dots\mu_D}\,\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_D} = \sqrt{|g|}\,\mathrm{d}^D x, \tag{A.21}$$

where we identified $\mathrm{d}^D x \equiv \mathrm{d}x^0\wedge\ldots\wedge\mathrm{d}x^D$.

Using the Levi–Civita tensor, we can define a linear, one–to–one map between p–forms and $(D-p)$–forms. This map, denoted $\star$, is called the *Hodge dual* or *Hodge operator*. On basis p–forms it is defined as

$$\star\,(\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_p}) = \frac{1}{(D-p)!}\varepsilon_{\nu_1\dots\nu_{D-p}}{}^{\mu_1\dots\mu_p}\,\mathrm{d}x^{\nu_1}\wedge\ldots\wedge\mathrm{d}x^{\nu_{D-p}}. \tag{A.22}$$

Note that $\varepsilon = \star 1$. Since the map is one–to–one, $\omega^{(p)}$ and $\star\,\omega^{(p)}$ contain exactly the same information; both forms thus consitute an identical, but dual, description of the problem at hand.

Two useful identities of the Hodge star are

- applying the Hodge operator twice to a p–form:

$$\star\star\,\omega^{(p)} = (-)^{p(D-p)+t}\omega^{(p)}. \tag{A.23}$$

- construction of a volume form from a p–form

$$\star\,\omega^{(p)}\wedge\tilde{\omega}^{(p)} = \star\,\tilde{\omega}^{(p)}\wedge\omega^{(p)} = \frac{1}{p!}\omega^{(p)}_{\mu_1\dots\mu_p}\tilde{\omega}^{(p)}_{\nu_1\dots\nu_p}g^{\mu_1\nu_1}\dots g^{\mu_p\nu_p}\,\varepsilon. \tag{A.24}$$

Appendix B

Cosmic egg wave function

In this appendix, we derive an approximate analytical solution of the wave function in the cosmic egg model beyond the WKB approximation that fails in the small volume regime. Specifically we first solve the Wheeler–DeWitt equation approximately in three different potential regions and then match the solutions across these and eventually connect it with the WKB form at large x. Figure B.1 shows the subdivision in regions for a potential of the form (3.30).

As discussed in the text, the existence of metastable cosmic eggs relies on (1) on a sufficient amount of axion flux compared to the contributions coming from quantum corrections and (2) the smallness of the cosmological constant. We assume this incarnates in the following requirements: the quantum corrections are sufficiently 'strong' at small x, say

$$\alpha \geqslant \frac{5}{2}, \tag{B.1}$$

the cosmic egg pocket is sufficiently deep in the sense that

$$Q > \frac{\kappa_4}{\sqrt{6}\Omega_3}, \tag{B.2}$$

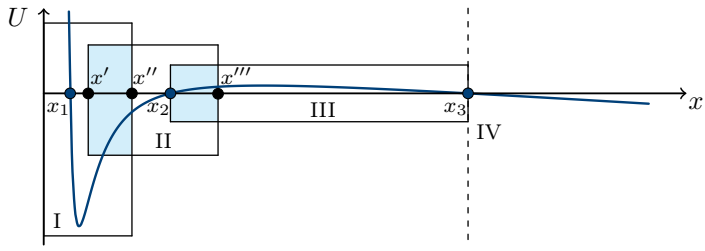


Figure B.1: The different regions of the potential in which the Wheeler–DeWitt equation is solved. Solutions in region I, II and III are matched within the overlap (shaded) and connected to region IV with the WKB connection formula.

and (1) and (2) are realised by

$$\kappa_4 Q \gg 3A^{2/\alpha} \tag{B.3a}$$

$$\Lambda \lesssim \frac{\sqrt{8}}{\kappa_4 Q} \tag{B.3b}$$

Under these assumptions, the potential has three turning points given by[1]

$$x_1 = \left(\frac{2A^2}{\kappa_4^2 Q^2}\right)^{\frac{1}{\alpha-2}} \left[1 + \mathcal{O}\left(\left(\frac{A^2}{(\kappa_4 Q)^\alpha}\right)^{\frac{2}{\alpha-2}}\right) + \mathcal{O}\left(\kappa_4 \Lambda Q\right)\right] \tag{B.4a}$$

$$x_2 = \frac{\kappa_4 Q}{\sqrt{6}} \left[1 + \mathcal{O}\left(\left(\alpha - 6^{1-\frac{\alpha}{2}} \frac{(\kappa_4 Q)^\alpha}{A^2}\right)^{-1}\right) + \mathcal{O}\left(\kappa_4 \Lambda Q\right)\right] \tag{B.4b}$$

$$x_3 = \frac{3}{\Lambda}\left[1 + \mathcal{O}((\kappa_4 Q \Lambda)^\alpha)\right] \tag{B.4c}$$

Region I is a 'quantum gravity' region. It encompasses the repulsive barrier and a large portion of the cosmic egg region. It corresponds roughly to $x < x'$ with

$$x' = \sqrt{\frac{2}{3}} x_2 \tag{B.5}$$

[1] With a slight abuse of notation. The notation $\mathcal{O}(X)$ denotes the order of magnitude of the correction terms, with the bounds (B.3) these satisfy $X \ll 1$. The notation $\mathcal{O}(X)$ denotes that the correction terms are always much smaller than X where X is found using one of the bounds of (B.3).

where the potential is approximated by

$$U(x) \approx \frac{A^2}{x^\alpha} - \frac{Q^2}{2x^2}. \qquad \text{(region I)} \tag{B.6}$$

Consequently, the Wheeler–DeWitt equation is approximately solved by

$$\Psi_{\mathrm{I}}(x) \approx \mathcal{A}\sqrt{x}K_\nu\left(\frac{\tilde{A}}{x^{\frac{\alpha}{2}-1}}\right) + \mathcal{B}\sqrt{x}I_\nu\left(\frac{\tilde{A}}{x^{\frac{\alpha}{2}-1}}\right) \tag{B.7}$$

where I_ν and K_ν are the modified Bessel functions of the first and second kind respectively and

$$\nu = \frac{1}{\alpha-2}\sqrt{1 - \frac{6\Omega_3^2}{\kappa_4^2}Q^2}, \qquad \tilde{A} = \frac{2\sqrt{3}\Omega_3}{(\alpha-2)\kappa_4^2}A. \tag{B.8}$$

The order ν is purely imaginary by (B.2). This means the wave function displays oscillations in the potential well that are characteristic to a classical spacetime. The Wheeler–DeWitt equation thus admits two types of solutions: one that diverges and one that decays as $x \to 0$. The repulsive barrier suggests that the wave function should decay towards the origin. This implies a choice of boundary conditions that is akin to the Hartle–Hawking one:

$$\mathcal{A} = 1, \qquad \mathcal{B} = 0. \tag{B.9}$$

In region II, any quantum effects $\sim A^2/x^\alpha$ are damped out and the curvature becomes relevant while the cosmological constant term remains negligible. This corresponds to $x'' < x < x'''$ with

$$x'' = \left(\frac{\alpha(\alpha+1)}{6}\right)^{\frac{1}{\alpha-2}} x_1, \qquad x''' = \left(\frac{\kappa_4^2 Q^2}{2\Lambda}\right)^{1/3} \tag{B.10}$$

The potential in this region can be approximated by

$$U(x) \approx -\frac{Q^2}{2x^2} + 3 \qquad \text{(region II)}. \tag{B.11}$$

The wave function is region is then well–approximated by

$$\Psi_{\rm II}(x) \approx a\sqrt{x}K_\mu\left(\frac{3\Omega_3}{\kappa_4^2}x\right) + b\sqrt{x}M_\mu\left(\frac{3\Omega_3}{\kappa_4^2}x\right), \tag{B.12}$$

where a and b are complex constants and $M_\mu(x)$ is defined to be

$$M_\mu(x) \equiv I_\mu(x) + I_{-\mu}(x), \tag{B.13}$$

with the order μ of the Bessel functions given by

$$\mu = \left(\frac{\alpha}{2} - 1\right)\nu. \tag{B.14}$$

The choice of writing M_μ instead of I_μ is to simplify the interpretation: if a and b are real, so is $\Psi_{\rm II}$. These coefficients are determined by the behaviour of the wave function in region I. These can be found by a matching procedure in the overlap region between x' and x''. In particular we find the asymptotic form

$$\begin{aligned}\Psi_{\rm I}(x) \sim & \frac{\tilde{A}^\nu\Gamma(-\nu)}{2^{1+\nu}}x^{-\left(\frac{\alpha}{2}-1\right)\nu+\frac{1}{2}} + \frac{\tilde{A}^{-\nu}\Gamma(\nu)}{2^{1-\nu}}x^{\left(\frac{\alpha}{2}-1\right)\nu+\frac{1}{2}} \\ & + \mathcal{O}\left(\left(\frac{x_1}{x}\right)^{\alpha-\frac{5}{2}\pm\left(\frac{\alpha}{2}-1\right)\nu}\right)\end{aligned} \tag{B.15a}$$

$$\begin{aligned}\Psi_{\rm II}(x) \sim & \left(\frac{3\Omega_3}{2\kappa_4^2}\right)^\mu\left(\frac{a}{2}\Gamma(-\mu) + \frac{b}{\Gamma(1+\mu)}\right)x^{\frac{1}{2}+\mu} \\ & + \left(\frac{3\Omega_3}{2\kappa_4^2}\right)^{-\mu}\left(\frac{a}{2}\Gamma(\mu) + \frac{b}{\Gamma(1-\mu)}\right)x^{\frac{1}{2}-\mu} + \mathcal{O}\left(\left(\frac{x}{x_2}\right)^{\frac{5}{2}\pm\mu}\right)\end{aligned} \tag{B.15b}$$

Thus the matching condition is

$$a = \left(\frac{3\Omega_3}{2\kappa_4^2}\right)^\mu\left(\frac{\tilde{A}}{2}\right)^\nu\frac{\Gamma(1-\mu)\Gamma(\nu)}{\Gamma(1-\mu)\Gamma(\mu) - \Gamma(-\mu)\Gamma(1+\mu)} + \text{c.c.}, \tag{B.16a}$$

$$b = \frac{1}{2}\left(\frac{3\Omega_3}{2\kappa_4^2}\right)^\mu\left(\frac{\tilde{A}}{2}\right)^\nu\frac{\Gamma(1-\mu)\Gamma(1+\mu)\Gamma(-\mu)\Gamma(-\nu)}{\Gamma(1-\mu)\Gamma(\mu) - \Gamma(-\mu)\Gamma(1+\mu)} + \text{c.c.}, \tag{B.16b}$$

where c.c. denotes complex conjugate. Hence it is obvious that a and b are real and, consequently, $\Psi_{\rm II}$ is real.

Within region II it is to be expected that the WKB approximation becomes increasingly more accurate for larger x. We therefore aim at giving WKB solutions in region III (overlapping with the right portion of region II) and eventually in region IV.

Region III is the underbarrier region between the turning points $x_2 < x < x_3$ in which the wave function takes the WKB form

$$\Psi_{\mathrm{III}}(x) \sim \left(\frac{2\kappa_4^4}{3\Omega_3^2}\right)^{1/4} \frac{1}{|2U(x)|^{1/4}} \left(ce^{-S(x,x_2)} + de^{S(x,x_2)}\right), \tag{B.17a}$$

where $S(x, x_{\mathrm{i}})$ is defined as usual by

$$S(x, x_{\mathrm{i}}) \equiv \sqrt{3}\frac{\Omega_3}{\kappa_4^2} \int_{x_{\mathrm{i}}}^{x} \mathrm{d}x' \sqrt{U(x')} \tag{B.17b}$$

As before, the amplitudes c and d are to be connected with the coefficients a and b that specify the behaviour of the wave function in the first part of the underbarrier regime (region of overlap between II and III) where the cosmological constant term can still be neglected and the potential is again approximated by (B.11). One has

$$S \approx \frac{3\Omega_3}{\kappa_4^2} x_2 \left[\sqrt{\left(\frac{x}{x_2}\right)^2 - 1} - \arctan\left(\sqrt{\left(\frac{x}{x_2}\right)^2 - 1}\right)\right], \tag{B.18}$$

with the asymptotic form away from the turning point (ignoring supressed Λ terms)

$$S \sim \frac{3\Omega_3}{\kappa_4^2} x - \frac{\pi}{2}\sqrt{\frac{3}{2}}\frac{\Omega_3}{\kappa_4} Q + \mathcal{O}\left(\frac{x_2}{x}\right). \tag{B.19}$$

The asymptotic form of the wave function in region III near x_2 thus is

$$\Psi_{\mathrm{III}}(x) \sim c\sqrt{\frac{\kappa_4^2}{3\Omega_3}} e^{\frac{\pi}{2}\sqrt{\frac{3}{2}}\frac{\Omega_3}{\kappa_4^2}Q} e^{-\frac{3\Omega_3}{\kappa_4}x} + d\sqrt{\frac{\kappa_4^2}{3\Omega_3}} e^{-\frac{\pi}{2}\sqrt{\frac{3}{2}}\frac{\Omega_3}{\kappa_4}Q} e^{\frac{3\Omega_3}{\kappa_4^2}x}. \tag{B.20}$$

This is to be matched to the asymptotic form of Ψ_{II} in the region of overlap

$$\Psi_{\mathrm{II}}(x) \sim \left[a\sqrt{\frac{\pi\kappa_4^2}{6\Omega_3}} e^{-\frac{3\Omega_3}{\kappa_4^2}x} + b\sqrt{\frac{2\kappa_4^2}{3\Omega_3\pi}} e^{\frac{3\Omega_3}{\kappa_4^2}x}\right] \left[1 + \mathcal{O}\left(\left(\frac{x_2}{x}\right)^{3/2}\right)\right] \tag{B.21}$$

Hence the amplitudes c and d are given by

$$c = \sqrt{\frac{\pi}{2}} e^{-\frac{\pi}{2}\sqrt{\frac{3}{2}}\frac{\Omega_3}{\kappa_4}Q} a, \qquad d = \sqrt{\frac{2}{\pi}} e^{\frac{\pi}{2}\sqrt{\frac{3}{2}}\frac{\Omega_3}{\kappa_4}Q} b. \tag{B.22}$$

To make the transition to region IV, $x > x_3$, one simply uses the standard WKB connection formulae. The wave function in this large x regime is given by

$$\Psi_{\rm IV}(x) \sim \left(\frac{2\kappa_4^4}{3\Omega_3^2}\right)^{1/4} \frac{1}{|2U(x)|^{1/4}} \left(\mathcal{C} e^{iS(x,x_3)} + \mathcal{D} e^{-iS(x,x_3)}\right), \tag{B.23}$$

where, by using the standard WKB connection formulae,

$$\mathcal{C} = \frac{1}{2} c e^{-S_{23}+i\frac{\pi}{4}} + d e^{S_{23}-i\frac{\pi}{4}}, \tag{B.24a}$$

$$\mathcal{D} = \frac{1}{2} c e^{-S_{23}-i\frac{\pi}{4}} + d e^{S_{23}+i\frac{\pi}{4}}, \tag{B.24b}$$

where $S_{23} \equiv S(x_3, x_2)$.

As stated in the text, the connection formulae for this specific cosmic egg model are of the general form

$$\mathcal{C} = e^{S_{23}-i\frac{\pi}{4}} \mathcal{H}(A,\alpha,Q) + e^{-S_{23}+i\frac{\pi}{4}} \mathcal{V}(A,\alpha,Q), \tag{B.25a}$$

$$\mathcal{D} = e^{S_{23}+i\frac{\pi}{4}} \mathcal{H}(A,\alpha,Q) + e^{-S_{23}-i\frac{\pi}{4}} \mathcal{V}(A,\alpha,Q), \tag{B.25b}$$

where $\mathcal{H}$ and $\mathcal{V}$ depend on the quantum contributions to the potential as well as the scalar flux. In principle it also has a (weak) dependence on the cosmological constant, yet this is negligible in many circumstances. Using equations (B.16),

(B.22) and (B.24), $\mathcal{H}$ and $\mathcal{V}$ are explicitly given by

$$\mathcal{H} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\exp\left(\frac{\pi}{2}\sqrt{\frac{3}{2}}\frac{\Omega_3}{\kappa_4}Q\right) \times \mathrm{Re}\left[\left(\frac{3\Omega_3}{2\kappa_4}\right)^{\mu}\left(\frac{\tilde{A}}{2}\right)^{\nu}\frac{\Gamma(1-\mu)\Gamma(1+\mu)\Gamma(-\mu)\Gamma(-\nu)}{\Gamma(1-\mu)\Gamma(\mu)-\Gamma(-\mu)\Gamma(1+\mu)}\right] \tag{B.26a}$$

$$\mathcal{V} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\exp\left(-\frac{\pi}{2}\sqrt{\frac{3}{2}}\frac{\Omega_3}{\kappa_4}Q\right) \times \mathrm{Re}\left[\left(\frac{3\Omega_3}{2\kappa_4}\right)^{\mu}\left(\frac{\tilde{A}}{2}\right)^{\nu}\frac{\Gamma(1-\mu)\Gamma(\nu)}{\Gamma(1-\mu)\Gamma(\mu)-\Gamma(-\mu)\Gamma(1+\mu)}\right] \tag{B.26b}$$

Appendix C

Shells in General Relativity

Extended objects often arise as non–perturbative features in physical theories. For instance D–branes in string theory, nucleated bubbles in semiclassical theories of gravity, domain walls, etc. They can be described in terms of submanifolds embedded in the ambient spacetime, provided they are sufficiently 'thin'. That is, they are strongly localized in some directions while being extended in others. For the scope of this thesis, it is sufficient to consider hypersurfaces only. This appendix will outline the basic machinery needed to treat them in a general relativistic setting. The material can be found in, for instance, [100, 254–256]. We aim to set up notation, establish definitions and present the necessary equations rather than being pedagogical or rigorous. Throughout this appendix $\mathcal{M}$ denotes a D dimensional manifold with metric g and we work locally in a set of coordinates x^μ, and Σ denotes a $d \equiv D-1$ dimensional manifold with coordinates y^m. Greek (Latin) indices refer to quantities defined on $\mathcal{M}$ (Σ).

C.1 Formalism and definitions

A hypersurface is a codimension one submanifold that can be defined, together with appropriate smoothness conditions, as a subset

$$S = \{x^\alpha \in \mathcal{M} \mid f(x^\alpha) = 0\} \subset \mathcal{M}, \tag{C.1}$$

for some function $f : \mathcal{M} \to \mathbb{R}$ or, parametrically, by an embedding

$$\Phi : \Sigma \hookrightarrow \Phi(\Sigma) \subset \mathcal{M} : y^m \mapsto x^\mu(y). \tag{C.2}$$

Strictly speaking, Σ exists as an independent manifold separate from $\mathcal{M}$. The map Φ, however, contains the information on how to embed Σ into $\mathcal{M}$. Its image $\Phi(\Sigma) = S$ is said to be an embedded submanifold. This is illustrated in figure C.1. It is, however, convenient not to view Σ as an external entity by itself and we will therefore not make the distinction between Σ and $\Phi(\Sigma)$. We simply refer to it as the hypersurface Σ.

The embedding Φ can be used to push forward vectors defined on Σ to $\mathcal{M}$ or to pull back differential forms defined $\mathcal{M}$ to Σ. In particular,

$$\frac{\partial}{\partial y^m} \mapsto \Phi_* \left(\frac{\partial}{\partial y^m} \right) = e^\mu_m \frac{\partial}{\partial x^\mu}, \qquad \mathrm{d}y^m \mapsto \Phi^* \left(\mathrm{d}y^m \right) = e^m_\mu \, \mathrm{d}x^\mu, \tag{C.3}$$

where e^μ_m is the Jacobian

$$e^\mu_m = \frac{\partial x^\mu}{\partial y^m}. \tag{C.4}$$

The m rows of the Jacobian constitute a set of d linearly independent vectors e^μ_m that are tangent to the embedded hypersurface Σ. Consequently, the unit normal vector n^μ to Σ in $\mathcal{M}$ satisfies

$$n_\mu e^\mu_m = 0. \tag{C.5}$$

The norm of the normal will be denoted by $\xi \equiv n^\mu n_\mu$. The hypersurface Σ is called spacelike (timelike) if $\xi = -(+)1$. The case of null hypersurfaces ($\xi = 0$) is not of interest in this thesis and will not be considered. Under the characterization (C.1), the unit normal vector field n^μ is be proportional to $\partial^\mu f$. For non–null

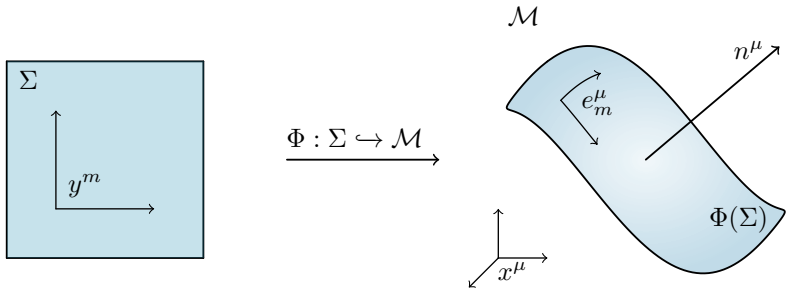


Figure C.1: Embedding a manifold Σ into $\mathcal{M}$.

hypersurfaces there is a sign ambiguity in defining the normal, as well as in the overall sign in the defining function f. We will adopt the convention that n^μ points in the direction of increasing f

$$n^\mu \partial_\mu f > 0. \tag{C.6}$$

If Σ corresponds to the boundary of the manifold $\mathcal{M}$, this convention implies that n^μ is pointing outwards (inwards) for non–positive (non–negative) f. In this case we will adopt the convention that n^μ always points *outwards*, this fixes an overall sign in the defining function f.

Using the normal, one can construct the projection operator

$$P_{\mu\nu} = g_{\mu\nu} - \xi n_\mu n_\nu \tag{C.7}$$

that projects onto directions tangent to Σ. Indeed, one can verify its idempotence and the fact that projects out normal components

$$P_\mu{}^\alpha P_{\alpha\nu} = P_{\mu\nu}, \qquad\qquad P_{\mu\nu} n^\nu = 0. \tag{C.8}$$

Tangent vectors v^μ thus satisfy $P^\mu{}_\nu v^\nu = v^\mu$. Any tensor $T^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s}$ on $\mathcal{M}$ can now be projected to a new tensor having only components tangent to Σ and the Jacobian can then subsequently be used to define a tensor on Σ:

$$T^{m_1\ldots m_r}{}_{n_1\ldots n_s} = e^{m_1}_{\mu_1}\ldots e^{m_r}_{\mu_r} e^{\nu_1}_{n_1}\ldots e^{\nu_s}_{n_s} T^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s} \tag{C.9}$$

In this way, structures on $\mathcal{M}$ descend to structures on Σ.

First and foremost, Σ inherits a metric h_{mn} from $\mathcal{M}$ that completely characterizes its intrinsic geometry. This is called the *induced* metric

$$\mathrm{d}s^2_{\mathrm{ind}} = h_{mn}\,\mathrm{d}y^m\,\mathrm{d}y^n, \tag{C.10}$$

and it is simply the pull back of the metric on $\mathcal{M}$:

$$h_{mn} = (\Phi^* g)_{mn} = e^\mu_m e^\nu_n g_{\mu\nu} = e^\mu_m e^\nu_n P_{\mu\nu}. \tag{C.11}$$

This shows that h_{mn} essentially contains the same information as $P_{\mu\nu}$. The difference between the two is the manifold on which they are defined and the fact that h_{mn} is non–degenerate while $P_{\mu\nu}$ is. We therefore refrain from calling $P_{\mu\nu}$ the induced metric on Σ.

Using the induced metric one can define the Levi–Civita covariant derivative D_m on Σ that is compatible with h_{mn}. On the other hand, there already exists a covariant derivative ∇_μ on the ambient space $\mathcal{M}$ that might be used to differentiate tensors tangent to Σ. Both definitions are compatible and, in particular, if v^μ is a tangent vector to Σ, one has

$$D_m v_n = e^\mu_m e^\nu_n \nabla_\mu v_\nu. \tag{C.12}$$

The curvature symbols derived from the induced metric, contain all information about the (local) *intrinsic* geometry of Σ. However, the geometry of hypersurfaces is not fully characterized by the intrinsic geometry alone; one also requires information on how Σ is 'curved' into $\mathcal{M}$. This is known as the *extrinsic* geometry of Σ and one defines the extrinsic curvature of Σ by

$$K_{\mu\nu} \equiv \frac{1}{2}\pounds_n P_{\mu\nu} = P_\mu{}^\alpha P_\nu{}^\beta \nabla_{(\alpha} n_{\beta)}. \tag{C.13}$$

Note that the sign ambiguity in defining the normal descends to a sign ambiguity in the extrinsic curvature. This tensor is tangent to the hypersurface Σ such that it can be pulled back without loss of information

$$K_{mn} = e^\mu_m e^\nu_n K_{\mu\nu}. \tag{C.14}$$

One also defines the trace of the extrinsic curvature

$$\mathcal{K} \equiv g^{\mu\nu} K_{\mu\nu} = h^{mn} K_{mn}. \tag{C.15}$$

The curvature of Σ is determined by the (projection of the) curvature of the ambient space $\mathcal{M}$ as well as the curvature coming from the embedding itself. This is captured by the *Gauss equation*

$${}^{(d)}R_{mnrs} = e^{\mu}_{m} e^{\nu}_{n} e^{\rho}_{r} e^{\sigma}_{s} \, {}^{(D)}R_{\mu\nu\rho\sigma} + \xi \left(K_{mr} K_{ns} - K_{ms} K_{nr} \right), \tag{C.16a}$$

which expresses the intrinsic curvature of Σ in terms of the projected components of the intrinsic curvature of $\mathcal{M}$ and extrinsic curvature tensor of Σ. Contractions hereof express the Ricci tensor/scalar on $\mathcal{M}$ in terms of the Ricci tensor/scalar and extrinsic curvature on Σ:

$${}^{(d)}R_{mn} = e^{\mu}_{m} e^{\nu}_{n} \, {}^{(D)}R_{\mu\nu} + \xi \left(\mathcal{K} K_{mn} - K_{ms} K^{s}{}_{n} - e^{\mu}_{m} e^{\nu}_{n} \, {}^{(D)}R_{\mu\rho\nu\sigma} n^{\rho} n^{\sigma} \right) \tag{C.16b}$$

$${}^{(d)}\mathcal{R} = {}^{(D)}\mathcal{R} + \xi \left(\mathcal{K}^2 - K^{mn} K_{mn} - 2 \, {}^{(D)}R_{\mu\nu} n^{\mu} n^{\nu} \right) \tag{C.16c}$$

The *Codazzi equation* expresses the Riemann tensor with one normal component in terms of the extrinsic curvature

$${}^{(D)}R_{\mu\nu\rho\sigma} n^{\sigma} = \nabla_{\mu} K_{\nu\rho} - \nabla_{\nu} K_{\mu\rho}. \tag{C.17}$$

The induced metric also defines a canonical volume form $\mathcal{E}$ on Σ that is inherited from the canonical volume form ε on $\mathcal{M}$. In components,

$$\mathcal{E}_{m_1 \ldots m_d} = n^{\alpha} \varepsilon_{\alpha \mu_1 \ldots \mu_d} e^{\mu_1}_{m_1} \ldots e^{\mu_d}_{m_d}. \tag{C.18}$$

This is simply the Levi–Civita tensor on Σ

$$\mathcal{E}_{m_1 \ldots m_d} = \sqrt{|h|} \epsilon_{m_1 \ldots m_d}. \tag{C.19}$$

Integration over hypersurfaces typically arises in the context of Stoke's theorem asserting that

$$\int_{\mathcal{M}} \mathrm{d}^D x \sqrt{|g|}\, \nabla_\mu v^\mu = \xi \int_{\partial\mathcal{M}} \mathrm{d}^d y \sqrt{|h|}\, n_\mu v^\mu, \tag{C.20}$$

where the boundary of $\mathcal{M}$ is denoted by $\partial\mathcal{M}$. The appearance of ξ on the right hand side is due to the above convention that n^μ always points outwards.

In many situations it is convenient to introduce a set of coordinates in $\mathcal{M}$, at least in a small neighbourhood around Σ, that is naturally adapted to the hypersurface at hand. These are called *Gaussian normal coordinates* and are constructed as follows. Given a point P with coordinates y^m on Σ, with $m = 1, \ldots, d$, one constructs the unique geodesic $\gamma(\lambda)$ originating from P with initial velocity n^μ. Note that since n^μ is normalised, the coordinate λ measures the proper distance away from Σ. By construction Σ is located at $\lambda = 0$. If Σ corresponds to the boundary of a manifold $\mathcal{M}$, then $\mathcal{M}$ is located at $\lambda \leqslant 0$, since n^μ pointed outwards by convention. Any point $Q \in \gamma$ can now be given the coordinates

$$x^0 = \lambda, \qquad\qquad x^m = y^m. \tag{C.21}$$

This is a well–defined coordinate system in a neighbourhood around Σ where the emanating geodesics do not intersect. The metric in these Gaussian normal coordinates takes a very simple form:

$$\mathrm{d}s^2 = \xi\, \mathrm{d}\lambda^2 + \gamma_{mn}(\lambda, y)\, \mathrm{d}y^m\, \mathrm{d}y^n, \tag{C.22}$$

where $\gamma_{mn}(0, y) = h_{mn}$. The unit normal in this coordinate system is explicitly given by

$$n^\lambda = 1, \qquad\qquad n^m = 0. \tag{C.23}$$

The extrinsic curvature tensor is also easily computed

$$K_{mn} = -\Gamma^\lambda_{mn} = \frac{1}{2}\, \frac{\partial \gamma_{mn}}{\partial \lambda}\bigg|_{\lambda=0}. \tag{C.24}$$

C.2 The physical picture

Physically, one is interested in spacetime manifolds $\mathcal{M}$ that satisfy the D–dimensional Einstein equation. This gravitational equation can be obtained from the Einstein–Hilbert action supplemented by the Gibbons–Hawking–York boundary term (if $\mathcal{M}$ possesses a boundary $\partial\mathcal{M}$). In agreement with the previous notations and conventions, the boundary $\partial\mathcal{M}$ is a hypersurface with coordinates y^m, induced metric h_{mn}, and with the unit normal n^μ taken to be outward pointing with a norm $\xi = n_\mu n^\mu$. The total action is given by

$$\begin{aligned} S &= S_{\rm EH} + S_{\rm GHY} + S_{\rm m} \\ &= \frac{1}{2\kappa_D^2}\int_{\mathcal{M}} \mathrm{d}^D x\sqrt{|g|}\,(\mathcal{R}-2\Lambda) + \frac{\xi}{\kappa_D^2}\int_{\partial\mathcal{M}} \mathrm{d}^d y\sqrt{|h|}\mathcal{K} + S_{\rm m}, \end{aligned} \tag{C.25}$$

where $S_{\rm m}$ is the action that contains all matter fields.

The Gibbons-Hawking-York boundary term is required for the variational principle to be properly defined since the Ricci scalar $\mathcal{R}$ is constructed from second derivatives of the metric. It is instructive to outline the variation of the action (C.25) with respect to the inverse metric. From the decomposition (C.7) of the metric $g_{\mu\nu}$, it is clear that a metric variation $\delta g^{\mu\nu}$ relates to a variation of the induced metric (the projection operator) and a variation of the normal. In particular,

$$\delta g^{\mu\nu} = \delta P^{\mu\nu} + 2\xi n^{(\mu}\delta n^{\nu)}. \tag{C.26}$$

The variation of the terms in (C.25) is given by

$$\delta S_{\mathrm{EH}} = \frac{1}{2\kappa_D^2}\int_{\mathcal{M}} \mathrm{d}^D x\sqrt{|g|}\Big[\,(G_{\mu\nu}+\Lambda g_{\mu\nu})\,\delta g^{\mu\nu} + \nabla_\mu\left(g_{\alpha\beta}\nabla^\mu\delta g^{\alpha\beta} - \nabla_\alpha\delta g^{\alpha\mu}\right)\Big] \tag{C.27a}$$

$$S_{\mathrm{GHY}} = \frac{\xi}{2\kappa_D^2}\int_{\partial\mathcal{M}} \mathrm{d}^d y\sqrt{|h|}\Big[-\mathcal{K}h_{mn}\delta h^{mn} + \delta g^{\alpha\beta}\nabla_\alpha n_\beta + n_\mu\left(\nabla_\alpha\delta g^{\alpha\mu} - g_{\alpha\beta}\nabla^\mu\delta g^{\alpha\beta}\right)\Big] \tag{C.27b}$$

$$\delta S_{\mathrm{m}} = -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^D x\sqrt{|g|}T_{\mu\nu}\delta g^{\mu\nu} \tag{C.27c}$$

where $T_{\mu\nu}$ is the energy–momentum tensor defined by

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{|g|}}\frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}}. \tag{C.28}$$

By applying Stoke's theorem (C.20), the boundary terms of the Einstein–Hilbert action (C.27a) cancel against the two boundary terms of the Gibbons–Hawking–York term (C.27b) on the second line. The boundary terms that are left, can be rewritten in terms of the extrinsic curvature and its trace such that the total variation of the action is given by

$$\delta S = \frac{1}{2\kappa_D^2}\int_{\mathcal{M}} \mathrm{d}^D x\sqrt{|g|}\left[G_{\mu\nu}+\Lambda g_{\mu\nu}-\kappa_D^2 T_{\mu\nu}\right]\delta g^{\mu\nu} + \frac{\xi}{2\kappa_D^2}\int_{\partial\mathcal{M}} \mathrm{d}^d y\sqrt{|h|}\left[K_{mn}-\mathcal{K}h_{mn}\right]\delta h^{mn} \tag{C.29}$$

The standard lore of Euler–Lagrange variations is to fix the field variations at the boundary to obtain the field equation in $\mathcal{M}$. This amounts to setting $\delta h^{mn} = 0$. It is then clear that the variational principle is well–defined and that the action is stationary if the Einstein equation in $\mathcal{M}$ is satisfied.

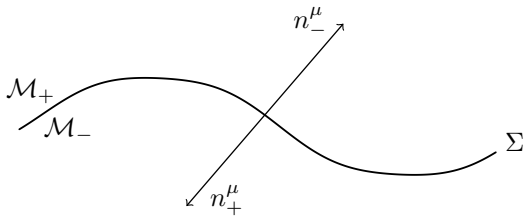


Figure C.2: Conjunction of two manifolds $\mathcal{M}_+$ and $\mathcal{M}_-$ across a hypersurface Σ.

We now consider a situation as depicted in figure C.2 where two different spacetimes $\mathcal{M}_+$ and $\mathcal{M}_-$ are connected across a surface layer Σ. This hypersurface coincides with the boundaries[1] of the two manifolds $\Sigma = \partial\mathcal{M}_+ = \partial\mathcal{M}_-$. The hypersurface Σ now inherits structures from $\mathcal{M}_+$ as well as from $\mathcal{M}_-$. Note that the inherited normal from $\mathcal{M}_+$ points in the opposite direction as the normal inherited from $\mathcal{M}_-$. For convention, we will define the normal to Σ to be n^μ_+, that is, the outward pointing normal to $\partial\mathcal{M}_+$. Furthermore, Σ inherits a metric from $\mathcal{M}_+$ as well as from $\mathcal{M}_-$. For the geometry on Σ to be properly defined, one thus requires

$$h^+_{mn} = h^-_{mn}, \tag{C.30}$$

in a suitable coordinate system in a neighborhood around Σ, where the notation $A_\pm$ means the limit where the quantity A approaches Σ from $\mathcal{M}_\pm$. This is known as the *first junction condition*: the metric is continuous at Σ.

The dynamics of this configuration is described by (C.25) with $\mathcal{M} = \mathcal{M}_+ \cup \mathcal{M}_-$. By varying the action one finds

$$\begin{aligned}\delta S = &\frac{1}{2\kappa_D^2}\int_{\mathcal{M}_+} \mathrm{d}^D x\sqrt{|g|}\left[G_{\mu\nu} + \Lambda g_{\mu\nu} - \kappa_D^2 T_{\mu\nu}\right]\delta g^{\mu\nu} \\ &+ \frac{1}{2\kappa_D^2}\int_{\mathcal{M}_-} \mathrm{d}^D x\sqrt{|g|}\left[G_{\mu\nu} + \Lambda g_{\mu\nu} - \kappa_D^2 T_{\mu\nu}\right]\delta g^{\mu\nu} \\ &+ \frac{\xi}{2\kappa_D^2}\int_\Sigma \mathrm{d}^d y\sqrt{|h|}\left[\left(K^+_{mn} - \mathcal{K}^+ h_{mn}\right) - \left(K^-_{mn} - \mathcal{K}^- h_{mn}\right) - \kappa_D^2\xi S_{mn}\right]\delta h^{mn}\end{aligned} \tag{C.31}$$

[1] In fact, Σ could also be a part of $\partial\mathcal{M}_\pm$. For the present discussion this is irrelevant, however.

where S_{mn} is the energy–momentum tensor on the surface layer. This can be defined using Gaussian normal coordinates

$$S_{mn} \equiv \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \mathrm{d}n \, e_m^\mu e_n^\nu T_{\mu\nu} = \frac{-2}{\sqrt{|h|}} \frac{\delta S_{\mathrm{m}}}{\delta h^{mn}}. \tag{C.32}$$

The first two lines in (C.31) show that the Einstein equation should be satisfied in $\mathcal{M}_+$ and $\mathcal{M}_-$ separately. The third line considers the boundary variation. In order for the joint configuration $\mathcal{M}_+ \cup \mathcal{M}_- \cup \Sigma$ to be a solution to the full Einstein equation, this term should vanish. In contrast to the case above, note that cannot simply set $\delta h^{mn} = 0$ since the boundary can be dynamical. The first two terms arise from the variation of the Gibbons–Hawking–York boundary term as in (C.29). The relative sign comes from the fact that the we have flipped the sign of the n_-^μ normal. Importantly, the extrinsic curvature of $\partial\mathcal{M}_-$ has to be computed with the chosen n_+^μ normal (and not with the conventional outward pointing normal). The third line thus vanishes only if the following equation is satisfied

$$\triangle K_{mn} - \triangle \mathcal{K} h_{mn} = \kappa_D^2 \xi S_{mn}, \tag{C.33}$$

where $\triangle A \equiv A_+ - A_-$ measures the difference of a certain quantity A across Σ. This is known as the (*second*) *junction condition* or as the *Isreal equation*. The energy–momentum tensor S_{mn} describes the matter fields that are localized on Σ. Indeed, it can only be non–zero if $T_{\mu\nu}$ contains a δ–function singularity that peaks near Σ. Any energy–momentum present on Σ thus sources a jump discontinuity in the extrinsic curvature. We close this appendix by mentioning that different resources might display the Israel equation (C.33) with a different definition of $\triangle$ or with an minus sign at the right hand side. To avoid any ambiguity it is important to fix and understand the conventions; in the case presented the normal n^μ always points *outwards* of $\mathcal{M}_+$ and $\triangle A \equiv A_+ - A_-$, i.e. the jump discontinuity is measured *in the direction* of the normal.

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