NEW APPROACH TO THE RATIONAL INTERPOLATION PROBLEM

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Abstract

We shall reformulate the classical Newton-Padé rational interpolation problem (NPRIP) to take away several drawbacks of the Newton-Padé approach. A new recursive algorithm, not reordering the interpolation points, will be designed, enabling us to give a parametrization of all solutions, not only for the linearized RIP but also for the proper RIP. This paper generalizes our earlier work where the Padé approximation problem, i.e. when all the interpolation points coincide at the origin, was solved by reformulating it as a minimal partial realization problem (Van Barel 1989), Van Barel and Bultheel, 1989).

1 Motivation

The interpolation problem, where one has to fit some given data points $(z_i, f_i/e_i)$; $f_i \neq 0$ or $e_i \neq 0$, $i = 1, 2, \ldots, p$, by a function of some specific class, is a very important one in applied mathematics and numerical analysis. If the given data points do not exhibit a "polynomial behaviour", we have to look for function classes different from the set of polynomials. A good alternative is the set of rational functions.

In the literature, a lot of attention has been given to the Newton-Padé rational interpolation problem (NPRIP). For an overview of the development of the basic results, we refer to Meinguet [8]. In the NPRIP we look for a rational function $n(z)/d(z)$, satisfying the linearized rational interpolation conditions (1) given below, where the maximum allowable degrees β of the numerator and α of the denominator are given. These are directly coupled with the number of data points by $p = \alpha + \beta + 1$. This approach is a generalization of the Padé approximation problem (PAP), i.e. the rational interpolation problem in confluent interpolation points [2]. As an advantage of this problem setting, we indicate the fact that the NPRIP always has a solution and that all the solutions are unique up to common factors in the numerator $n(z)$ and denominator polynomial $d(z)$. However, if we take $n(z)/d(z) = n'(z)/d'(z)$, where $n'(z)$ and $d'(z)$ have no common factors, the rational function $n'(z)/d'(z)$, called the (β, α) Newton-Padé approximant (NPA), will not necessarily satisfy the proper rational interpolation conditions, i.e. it is possible that $n'(z_i)/d'(z_i) \neq f_i/e_i$, with $1 \leq i \leq p$. In this case, we call z_i an unattainable point. As for the PAP, a Newton-Padé table can be defined, where the (β, α) NPA is placed at row β and column α. If the same rational function appears more than once, we have a nonnormal Newton-Padé table. In a

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normal one, all the entries are different. The set of positions in the table, containing the same rational function, is called a block. If we do not consider the special cases of an approximant equal to 0 or ∞, the Padé table can only contain square blocks, whereas the Newton-Padé table can have blocks that are unions of squares [4], [6]. As far as we know, all the algorithms to compute recursively the (β, α) NPA follow a path in the Newton-Padé table, connected with the sequence of data points $(z_1, f_1/e_1)$, $(z_2, f_2/e_2), \ldots$ possibly reordered[1]. These algorithms reorder the given data points to transform the original Newton-Padé table into one for which the blocks are such that the algorithm is able to compute a path through or around the blocks. Since they essentially rely upon reordering, these algorithms do not allow in general that further data points are added. However, when all the data points are known from the start, reordering is very useful to stabilize the algorithm [10]. In the following, we shall not be concerned with numerical stability problems.

Our main motivation will be to give a problem setting and a theoretical algorithm, which take away the drawbacks of the Newton-Padé approach. Instead of fixing the degree of the numerator and denominator polynomial and connecting these with the number of data points, we only set the relative importance of these two degrees. Within this set of rational functions, our theoretical algorithm not only computes an element having least complexity and satisfying the linearized rational interpolation conditions (1), but also the proper rational interpolation conditions. Moreover, we shall give a parametrization of all the solutions. Our algorithm does not reorder the data points. Therefore, it is possible to add further data points and to proceed with the algorithm. Considering the PAP, we have established several connections between minimal partial realizations (MPRs), arising in linear system theory, and Padé approximants [9]. Currently, we are investigating possible definitions and the properties of vector and matrix Padé approximants based on the minimal partial realization approach [11]. Similar to the MPR problem, we shall define the rational interpolation problem (RIP) in section 2. In section 3, we shall solve the linearized RIP and, in section 4, the proper RIP. The theoretical algorithm, given there, is very similar to the Berlekamp-Massey algorithm, applied in a MPR context. In section 5, we shall establish a parametrization of all solutions. Section 6 will indicate how to solve the confluent case. A small example will be shown in section 7. Due to space limitations, we shall not investigate the deeper connections between the solutions generated by our algorithm and NPAs. Similar to the minimal partial realization approach, we could easily extend the approach of this paper to the vector and matrix RIP. In subsequent publications, these subjects will be explained in detail.

2 The rational interpolation problem (RIP)

The field, finite or infinite, we are working with, is denoted by K. The set of polynomials with coefficients of the field K, is denoted by $K[z]$. Suppose we want to interpolate the given data f_i/e_i; $f_i, e_i \in K$, in the interpolation points $z_i \in K, i = 1, 2, \ldots, p$, by a rational function $n(z)/d(z)$ with $n(z), d(z) \in K[z]$. Note that we include the possibility of a data value ∞ by setting $e_i = 0$ and $f_i \neq 0$. The data couple (f_i, e_i) is always different from $(0, 0)$ and $(d(z), n(z)) \neq (0, 0)$. Without loss of generality we assume that all the interpolation points

[1] As one of the referees pointed out, [5] describes a recursive algorithm without reordering the data points. Of course, this algorithm has the same drawback as all the other algorithms, i.e. it computes the solution satisfying the linearized rational interpolation conditions and not the proper ones.

z_i, $i = 1, 2, \ldots, p$, are different. The confluent case is considered in section 6. Note that the previous problem is equivalent to the problem of finding a polynomial couple $(n(z), d(z))$ not divisible by $z - z_i$, $i = 1, 2, \ldots, p$, i.e. $n(z)$ is not divisible by $z - z_i$ or $d(z)$ is not divisible by $z - z_i$, satisfying $f_i d(z_i) = e_i n(z_i)$, $i = 1, 2, \ldots, p$. Similar to the polynomial interpolation problem, we want to restrict the complexity of the rational function $n(z)/d(z)$. The complexity of a polynomial can be indicated by its degree. We shall base our definition of the complexity of a polynomial couple $(n(z), d(z))$ on the degrees of the polynomials $n(z)$ and $d(z)$. To indicate the relative importance of the two degrees, the shift parameter $s \in \mathbb{Z}$ is given. We define the s-degree α of $(n(z), d(z))$ as $\alpha = \max(\deg d(z), \deg n(z) - s)$ (We define $\deg 0 = -\infty$). Given the shift parameter $s \in \mathbb{Z}$, we take the s-degree as the complexity of the polynomial couple $(n(z), d(z))$. We define the following two RIPs.

The *linearized rational interpolation problem* (LRIP): The interpolation points $z_i \in K$ and data $(f_i, e_i) \neq (0, 0)$, $i = 1, 2, \ldots, p$ and the shift parameter s are given. We look for a polynomial couple $(n(z), d(z))$ of minimal s-degree, satisfying the linearized rational interpolation conditions

$$f_i d(z_i) = e_i n(z_i), \quad i = 1, 2, \ldots, p. \tag{1}$$

The *proper rational interpolation problem* (PRIP): This is the linearized rational interpolation problem with the additional condition that $(n(z), d(z))$ is not divisible by $z - z_i$, $i = 1, 2, \ldots, p$.

3 The solution of the LRIP

We can write (1) as a set of linear homogeneous equations:

$$R_i v(z) = 0, \quad i = 1, 2, \ldots, p \tag{2}$$

where $R_i v(z)$ is the i-th residual of the polynomial couple $v(z) = (n(z), d(z))$ defined as $R_i v(z) = f_i d(z_i) - e_i n(z_i)$. We denote by $S_{p,\alpha}$ the vectorspace of all polynomial couples $(n(z), d(z))$ having s-degree $\leq \alpha$ and satisfying (1) or, what is the same, solving (2). Note that if $s \geq 0$ and

if $\alpha < -s$, then $S_{p,\alpha} = \{(0, 0)\}$,
if $-s \leq \alpha < 0$, then $S_{0,\alpha} = \{(n(z), 0) : \deg n(z) \leq \alpha + s\}$,
if $0 \leq \alpha$, then $S_{0,\alpha} = \{(n(z), d(z)) : \deg n(z) \leq \alpha + s, \deg d(z) \leq \alpha\}$,

and if $s \leq 0$ and

if $\alpha < 0$, then $S_{p,\alpha} = \{(0, 0)\}$,
if $0 \leq \alpha < -s$, then $S_{0,\alpha} = \{(0, d(z)) : \deg d(z) \leq \alpha\}$,
if $-s \leq \alpha$, then $S_{0,\alpha} = \{(n(z), d(z)) : \deg n(z) \leq \alpha + s, \deg(z) \leq \alpha\}$.

It is easy to see that a basis $BS_{p,\alpha}$ for the vectorspace $S_{p,\alpha}$, $-\infty < \alpha < +\infty$, can be based on two polynomial couples $v_p(z) = (n_{p,v}(z), d_{p,v}(z))$ and $w_p(z) = (n_{p,w}(z), d_{p,w}(z))$ as will be shown in the sequel. We need some notation now. We define the multiplication of a polynomial couple $x(z)$ with a polynomial $p(z)$ as $p(z)x(z) = p(z)(n_x(z), d_x(z)) = (p(z)n_x(z), p(z)d_x(z))$. Dividing a polynomial couple $x(z) = (n_x(z), d_x(z))$ by a polynomial $p(z)$ leads in a similar way to $x(z)/p(z) = (n_x(z)/p(z), d_x(z)/p(z))$, not necessarily a polynomial couple. If $n_x(z)$ and $d_x(z)$ are both divisible by $p(z)$, $x(z)/p(z)$ reduces to a polynomial couple and we say that the polynomial couple $x(z)$ is divisible by $p(z)$. Given an integer number α and a polynomial couple $x(z)$ having s-degree α_x, we define the set of polynomial

couples $\{x(z)\}^\alpha$ as follows:

$$\{x(z)\}^\alpha = \varnothing, \quad \text{if} \quad \alpha < \alpha_x$$
$$\{x(z)\}^\alpha = \{x(z), zx(z), \ldots, z^{\alpha-\alpha_x}x(z)\}, \quad \text{if} \quad \alpha \geq \alpha_x.$$

We shall prove the following theorem:

Theorem 1 *For each $p \geq 0$, there exist two polynomial couples $v_p(z)$ and $w_p(z)$ such that for each α, $|\alpha| < \infty$, a basis $BS_{p,\alpha}$ for $S_{p,\alpha}$, is given by*

$$BS_{p,\alpha} = V_{p,\alpha} \cup W_{p,\alpha} \quad \textit{with} \quad V_{p,\alpha} = \{v_p(z)\}^\alpha \quad \textit{and} \quad W_{p,\alpha} = \{w_p(z)\}^\alpha.$$

Proof. We shall prove the theorem by induction. It is clear that the theorem is true for $p = 0$ and, e.g., $v_0(z) = (1, 0)$ and $w_0(z) = (0, 1)$. We assume that the theorem is true for a certain value for p. Without loss of generality we can assume that $v_p(z)$ and $w_p(z)$ are ordered such that the s-degree $\alpha_{p,v}$ of $v_p(z)$ is smaller than or equal to the s-degree $\alpha_{p,w}$ of $w_p(z)$.

The proof is constructive because an algorithm is given to compute a possible choice for $v_{p+1}(z)$ and $w_{p+1}(z)$, given $v_p(z)$ and $w_p(z)$. First of all we shall describe how to compute a polynomial couple $a_{p+1}(z) \neq (0, 0) \in S_{p+1,\alpha}$, not divisible by $z - z_{p+1}$, which always exists, having minimal s-degree α. This polynomial couple $a_{p+1}(z)$ together with another polynomial couple $b_{p+1}(z)$ (up to reordering), will give us $v_{p+1}(z)$ and $w_{p+1}(z)$. Because $a_{p+1}(z)$ has to be in $S_{p+1,\alpha} \subset S_{p,\alpha}$, we can write $a_{p+1}(z)$ as a linear polynomial combination of $v_p(z)$ and $w_p(z)$. Because $S_{p,\alpha} = \{(0, 0)\}$ with $\alpha < \alpha_{p,v}$, we first try $\alpha = \alpha_{p,v}$.

(1) If $v_p(z)$ satisfies the $(p+1)$-st interpolation condition, i.e., $R_{p+1}v_p(z) = 0$, then $v_p(z) \neq (0, 0)$ is an element of $S_{p+1,\alpha}$ with minimal s-degree. We can take $a_{p+1}(z) = v_p(z)$, because $v_p(z)$ is not divisible by $z - z_{p+1}$. If $v_p(z)$ would be divisible by $z - z_{p+1}$, the polynomial couple $(n_{p,v}(z)/(z - z_{p+1}), d_{p,v}(z)/(z - z_{p+1})) \neq (0, 0)$ would be an element of $S_{p,\alpha_{p,v}-1}$ but $S_{p,\alpha_{p,v}-1} = \{(0, 0)\}$. We choose $b_{p+1}(z) = (z - z_{p+1})w_p(z)$.

(2) If $v_p(z)$ does not satisfy the $(p+1)$-st interpolation condition, i.e. $R_{p+1}v_p(z) \neq 0$, then $a_{p+1}(z) = p(z)v_p(z) + q(z)w_p(z)$ with $q \neq 0$. Otherwise $a_{p+1}(z)$ could be written as $a_{p+1}(z) = p(z)v_p(z)$ with $p(z)$ a polynomial. Since $a_{p+1}(z)$ should satisfy the $(p+1)$-st interpolation condition, i.e.,

$$\begin{aligned} 0 &= f_{p+1}(p(z_{p+1})d_{p,v}(z_{p+1})) - e_{p+1}(p(z_{p+1})n_{p,v}(z_{p+1})) \\ &= p(z_{p+1})(f_{p+1}d_{p,v}(z_{p+1}) - e_{p+1}n_{p,v}(z_{p+1})) \\ &= p(z_{p+1})R_{p+1}v_p(z) \quad \text{with} \quad R_{p+1}v_p(z) \neq 0, \end{aligned}$$

it follows that $p(z)$ is divisible by $z - z_{p+1}$. Hence $a_{p+1}(z)$ is divisible by $z - z_{p+1}$. Because $a_{p+1}(z)$ must not be divisible by $z - z_{p+1}$, $a_{p+1}(z)$ should have a component $w_p(z)$. The smallest α for which $a_{p+1}(z)$ can have a component $w_p(z)$ is $\alpha_{p,w}$. We shall prove now that a possible choice for $a_{p+1}(z)$ is $w_p(z) - h_p^v(z)$ with $h \in K$ computed as follows:

$$\begin{aligned} &f_{p+1}(d_{p,w}(z_{p+1}) - hd_{p,v}(z_{p+1})) - e_{p+1}(n_{p,w}(z_{p+1}) - hn_{p,v}(z_{p+1})) = 0 \quad \text{or} \\ &f_{p+1}d_{p,w}(z_{p+1}) - e_{p+1}n_{p,w}(z_{p+1}) - h(f_{p+1}d_{p,v}(z_{p+1}) - e_{p+1}n_{p,v}(z_{p+1})) = 0. \end{aligned}$$

We get that $h = R_{p+1}w_p(z)/R_{p+1}v_p(z) \neq \infty$. If $a_{p+1}(z)$ would be divisible by $z - z_{p+1}$, then $a'(z) = a_{p+1}(z)/(z - z_{p+1}) \in S_{p,\alpha-1}$ or $a'(z) = p(z)v_p(z)$. Thus $a_{p+1}(z) = (z -$

$z_{p+1})a'(z) = (z - z_{p+1})p(z)v_p(z)$ so that $a_{p+1}(z)$ is written as a linear polynomial combination of the polynomial couple $v_p(z)$, without using $w_p(z)$ as a component. This leads to a contradiction because $a_{p+1}(z)$ has a component $w_p(z)$. Hence, $a_{p+1}(z)$ is not divisible by $z - z_{p+1}$. In this second case, $b_{p+1}(z)$ is chosen as $b_{p+1}(z) = (z - z_{p+1})v_p(z)$.

$a_{p+1}(z)$ and $b_{p+1}(z)$ are a possible choice for $v_{p+1}(z)$ and $w_{p+1}(z)$ if we can prove that for each α a basis for $S_{p+1,\alpha}$ is given by $BS_{p+1,\alpha} = A_{p+1,\alpha} \cup B_{p+1,\alpha} = \{a_{p+1}(z)\}^\alpha \cup \{b_{p+1}(z)\}^\alpha$. Take $a(z) \in S_{p+1,\alpha}$ then $a(z)$ can be written as $a(z) = ca_{p+1}(z) + a'(z)$ with $a'(z)$ divisible by $z - z_{p+1}$. If $a(z)$ has s-degree smaller than the s-degree of $a_{p+1}(z)$, then $a(z)$ is divisible by $z - z_{p+1}$. Hence $c = 0$. Otherwise, $a_{p+1}(z)$ would not have minimal s-degree. When the s-degree of $a(z)$ is greater than or equal to the s-degree of $a_{p+1}(z)$, we can always choose c such that $a(z) - ca_{p+1}(z)$ is divisible by $z - z_{p+1}$, i.e. such that

$$d_a(z_{p+1}) - c^d_{p+1,a}(z_{p+1}) = 0 \quad \text{and} \quad n_a(z_{p+1}) - c^n_{p+1,a}(z_{p+1}) = 0. \tag{3}$$

Because $a_{p+1}(z)$ is by construction not divisible by $z - z_{p+1}$, either $d_{p+1,a}(z_{p+1}) \neq 0$ or $n_{p+1,a}(z_{p+1}) \neq 0$. Suppose $d_{p+1,a}(z_{p+1}) \neq 0$, the other case is similar. We can determine c as $c = d_a(z_{p+1})/d_{p+1,a}(z_{p+1})$. Because $R_{p+1}a(z) = 0$ and $R_{p+1}a_{p+1}(z) = 0$, we obtain:

$$0 = f_{p+1}(d_a(z^{p+1}) - cd_{p+1,a}(z_{p+1})) - e_{p+1}(n_a(z_{p+1}) - cn_{p+1,a}(z_{p+1})).$$

$e_{p+1} \neq 0$ because $e_{p+1} = 0$ implies $f_{p+1} \neq 0$ so that $d_{p+1,a}(z_{p+1}) = 0$, which contradicts our assumption. Hence, $e_{p+1} \neq 0$ and therefore (3) is true.

From $a(z) = c^a_{p+1}(z) + a'(z)$ with $a'(z)$ divisible by $z - z_{p+1}$, it follows that $a(z)$ can be written in a unique way as $a(z) = ca_{p+1}(z) + (z - z_{p+1})a''(z)$ with $a''(z) \in S_{p,\alpha-1}$. Using the basis for $S_{p,\alpha-1}$ based on $v_p(z)$ and $w_p(z)$, it is easy to see that $A_{p+1,\alpha} \cup B_{p+1,\alpha} = V_{p+1,\alpha} \cup W_{p+1,\alpha}$ forms a possible basis for $S_{p+1,\alpha}$.

□

Notes:

(1) The proof of the theorem contains an algorithm to compute the two polynomial couples $v_p(z)$ and $w_p(z)$ at each level p knowing $v_{p-1}(z)$ and $w_{p-1}(z)$ of the previous level. We can summarize the algorithm as follows:

{Initialization}

If $s \geq 0$ **then** $[v_0(z)\ w_0(z)] = \begin{bmatrix} n_{0,v}(z) & n_{0,w}(z) \\ d_{0,v}(z) & d_{0,w}(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Else $[v_0(z)\ w_0(z)] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

For $p = 0, 1, 2, \ldots$

If $R_{p+1}v_p(z) = 0$ **then** $V_{p+1}(z) = \begin{bmatrix} 1 & 0 \\ 0 & z - z_{p+1} \end{bmatrix}$

Else $V'_{p+1}(z) = \begin{bmatrix} z - z_{p+1} & -h \\ 0 & 1 \end{bmatrix}$ with $h = R_{p+1}w_p(z)/R_{p+1}v_p(z)$.

If the s-degree of $v_p(z)$ is equal to the s-degree of $w_p(z)$ **then**

$$V_{p+1}(z) = V'_{p+1}(z) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
Else $V_{p+1}(z) = V'_{p+1}(z)$.
$$[v_{p+1}(z)\ w_{p+1}(z)] = [v_p(z)\ w_p(z)]V_{p+1}(z)\ .$$

(2) It is clear that we can write:

$$[v_p(z)\ w_p(z)] = V_0(z)V_1(z)\dots V_p(z) \quad \text{with}$$

$$\begin{aligned} V_0(z) &= \begin{bmatrix} x_0(z) & t_0(z) \\ y_0(z) & u_0(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{if} \quad s \geq 0 \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{if} \quad s < 0 \end{aligned}$$

and

$$\begin{aligned} V_i(z) &= \begin{bmatrix} x_i(z) & t_i(z) \\ y_i(z) & u_i(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & z - z_i \end{bmatrix} \quad \text{or} \\ &= \begin{bmatrix} z - z_i & -h \\ 0 & 1 \end{bmatrix} \quad \text{or} \\ &= \begin{bmatrix} -h & z - z_i \\ 1 & 0 \end{bmatrix} \quad \text{with} \quad i \geq 1. \end{aligned}$$

It follows that $n_{p,v}(z)/d_{p,v}(z)$, the solution of the LRIP, is the p-th convergent of the generalized continued fraction:

$$x_0(z) + t_0(z)\cfrac{x_1(z) + t_1(z)\cfrac{\cdots}{\cdots}}{y_1(z) + u_1(z)\cfrac{\cdots}{\cdots}} \Bigg/ y_0(z) + u_0(z)\cfrac{x_1(z) + t_1(z)\cfrac{\cdots}{\cdots}}{y_1(z) + u_1(z)\cfrac{\cdots}{\cdots}}. \tag{4}$$

Note that we can also write a similar continued fraction whose p-th convergent is $n_{p,a}(z)/d_{p,a}(z)$. Remember that the polynomial couple $a_p(z)$ is a solution of the LRIP with the additional condition that this polynomial couple is not divisible by $z - z_p$.

(3) The algorithm, described above, can be implemented in several ways. The linearized rational interpolation conditions (1) can be written in terms of the classical basis $1, z, z^2, z^3, \dots$, involving classical Vandermonde matrices. In this case, the multiplication of a polynomial by $z - z_{p+1}$, needed in the algorithm, is straightforward. Another possibility is to write the interpolation conditions (1) in terms of a basis of orthogonal polynomials $\phi_0(z), \phi_1(z), \phi_2(z), \dots$, involving generalized Vandermonde matrices. The multiplication by $z - z_{p+1}$ of a polynomial written in terms of this basis of orthogonal polynomials can be carried out using the recurrence relation for orthogonal polynomials: $\phi_k(z) = \lambda_k(z - \alpha_k)\phi_{k-1}(z) - \beta_k\phi_{k-2}(z)$. More generally, we can use any basis as long as the shift operation, i.e., the multiplication by z of a polynomial written in terms of this basis, can be carried out in a simple and efficient way.

(4) If we consider $[v_p(z)\ w_p(z)] = V_0(z)V_1(z)V_2(z)\dots V_p(z)$ as an abstract layered medium, built up by the layers $V_i(z)$, each step of the algorithm, given above, computes the residuals $R_{p+1}v_p(z)$ and $R_{p+1}w_p(z)$, needed to compute the next layer $V_{p+1}(z)$, and connects this new layer to the layered medium. So we can call this a layer adjoining algorithm. However, it is not necessary to obtain the layered medium $[v_p(z)\ w_p(z)]$ explicitly. If we know the separate layers $V_i(z)$, we can represent the layered medium by the continued fraction (4). To compute the layer $V_p(z)$, we need to know the residuals $R_pv_{p-1}(z)$ and $R_pw_{p-1}(z)$. Suppose we know all residuals $R_jv_{p-1}(z)$ and $R_jw_{p-1}(z)$ of level $p-1$, with $j \geq p$. It is easy to design an algorithm computing the next layer $V_p(z)$ and transforming all the residuals of the previous step into the residuals of level p, using the formula

$$[R_lv_p(z)\ R_lw_p(z)] = [R_lv_{p-1}(z)\ R_lw_{p-1}(z)]\ V_p(z_l).$$

We call this algorithm a layer peeling one, because at each step p we peel off the layer $V_p(z)$. Compared to the layer adjoining algorithm, the layer peeling one has the drawback that we have to know all the residuals $R_lv_0(z)$ and $R_lw_0(z)$, with $l \geq 1$ at the beginning of the algorithm. As an advantage, we mention that the layer peeling algorithm is easier to parallelize than the other one. From the previous explanation, it should be clear that also a mixture of a layer adjoining and layer peeling algorithm is possible. For more information, we refer to [3].

4 The solution of the PRIP

Theorem 2 *Let $v_p(z)$ and $w_p(z)$ be defined by the theorem of the previous section. If $v_p(z)$ is not a solution of the PRIP, then $a_vv_p(z) + a_ww_p(z)$ is, with $(a_v, a_w) \neq (0,0)$ and $a_v/a_w \neq a_{i,v}/a_{i,w}$, $i = 1, 2, \dots, p$. $(a_{i,v}, a_{i,w})$ is the only couple such that $a_{i,v}v_p(z)+a_{i,w}w_p(z)$ is divisible by $z - z_i$.*

Proof. When $v_p(z)$ is divisible by $z - z_i$ for a certain i, $1 \leq i \leq p$, then $v_p(z)$ is a solution of the LRIP but not of the PRIP. It is clear that $V_i(z)$ has rank 2 if $z \neq z_i$ and has rank 1 if $z = z_i$. Hence, $[v_p(z)\ w_p(z)] = V_0(z)V_1(z)\dots V_p(z)$ has rank 2 if $z \neq z_i$ and has rank 1 if $z = z_i$. Therefore, $a_{i,v}v_p(z) + a_{i,w}w_p(z)$, with $(a_{i,v}, a_{i,w}) \neq (0,0) \in K^2$, is divisible by $z - z_i$ for only one specific value of the fraction $a_{i,v}/a_{i,w}$. □

5 A parametrization of all solutions of the LRIP and PRIP

Theorem 3 *If the s-degree alpha$_{p,v}$ of $v_p(z)$ is smaller than the s-degree $\alpha_{p,w}$ of $w_p(z)$, then there is one and only one rational function $n_{p,v}(z)/d_{p,v}(z)$ which solves the LRIP. Otherwise a parametrization of all solutions is given by:*

$$\begin{bmatrix} n_p^a(z) \\ d_p^a(z) \end{bmatrix} = \begin{bmatrix} n_{p,v}(z) & n_{p,w}(z) \\ d_{p,v}(z) & d_{p,w}(z) \end{bmatrix} \begin{bmatrix} a_v \\ a_w \end{bmatrix} \quad \textit{with} \quad a = (a_v, a_w) \neq (0,0) \in K^2.$$

Proof. For different values of a_v/a_w, we get different rational functions $n_p^a(z)/d_p^a(z)$, because, if $n_p^a(z)/d_p^a(z) = n_p^{a'}(z)/d_p^{a'}(z)$ with $a_v/a_w \neq a'_v/a'_w$, we get:

$$\begin{aligned} 0 &= n_p^a(z)d_p^{a'}(z) - d_p^a(z)n_p^{a'}(z) \\ &= (n_{p,v}(z)d_{p,w}(z) - d_{p,v}(z)n_{p,w}(z))(a_v a'_w - a_w a'_v) \end{aligned}$$

or

$$\begin{bmatrix} n_{p,v}(z) & n_{p,w}(z) \\ d_{p,v}(z) & d_{p,w}(z) \end{bmatrix}$$

would be of rank < 2 for all possible values of z. Because $V_i(z)$ has rank < 2 only when $z = z_i$, this leads to a contradiction. □

Theorem 4 *Concerning a parametrization of all solutions of the PRIP, we have the following. If $v_p(z)$ is also a solution of the PRIP, the two cases of the previous theorem also apply here, with the only exception in the second case, $a_v/a_w \neq a_{i,v}/a_{i,w}$, $i = 1, 2, \ldots, p$. If $v_p(z)$ is not a solution of the PRIP, a parametrization of all solutions of the problem is given by:*

$$\begin{bmatrix} n_p^a(z) \\ d_p^a(z) \end{bmatrix} = \begin{bmatrix} n_{p,v}(z) & n_{p,w}(z) \\ d_{p,v}(z) & d_{p,w}(z) \end{bmatrix} \begin{bmatrix} a(z) \\ 1 \end{bmatrix}$$

with $\deg a(z) \leq \alpha_{p,w} - \alpha_{p,v}$ *and* $a(z_i) \neq a_{i,v}/a_{i,w}$. *As before we can show that for different polynomials $a(z)$, we get different rational functions.*

6 The confluent case

When for certain interpolation points z_i, not only the function value f_i/e_i is given but also the values of one or several consecutive derivatives, we get the confluent LRIP and PRIP. If we consider the interpolation data for the "confluent" points in the natural ordering, i.e., function value, first derivative value, second derivative value, ..., then we can prove in a similar way that the same algorithm can be used to compute the polynomial couples $v_p(z)$ and $w_p(z)$ of the confluent problem. The parametrization of all solutions of the LRIP and PRIP is also valid in this case. Due to space limitations, we shall work this out in a future publication.

7 A small example

We take the data points from example 2 of Graves-Morris and Hopkins [7]: $\{(0, 1/1), (1, 2/1), (2, 2/1)\}$. We use the Chebyshev polynomials $T_k(z)$ as a basis for the set of polynomials: $T_0(z) = 1$, $T_1(z) = z$ and $T_{k+1}(z) = 2zT_k(z) - T_{k-1}(z)$ for $k \geq 1$. Hence, the multiplication of $T_k(z)$ by $z - z_{p+1}$ can be written as $(z - z_{p+1})T_0(z) = T_1(z) - z_{p+1}T_0(z)$ and $(z - z_{p+1})T_k(z) = \frac{1}{2}T_{k+1}(z) - z_{p+1}T_k(z) + \frac{1}{2}T_{k-1}(z)$ for $k \geq 1$. The algorithm of section 3 with the shift parameter $s = 0$ generates the following $V_i(z)$-matrices:

$$V_1(z) = \begin{bmatrix} 1 & z \\ 1 & 0 \end{bmatrix}, \quad V_2(z) = \begin{bmatrix} z-1 & 1 \\ 0 & 1 \end{bmatrix}, \quad V_3(z) = \begin{bmatrix} 1 & z-2 \\ 1 & 0 \end{bmatrix}.$$

Whereas Graves-Morris and Hopkins do not give a solution, satisfying the proper rational interpolation conditions (the (1,1) NPA has an unattainable point at $z_1 = 0$), our result of section 5 gives all solutions of the PRIP having a minimal s-degree 2, with $\deg a(z) \leq 1$ and $a(1) \neq 1$ and $a(2) \neq 0$, as

$$[v_3(z) \;\; w_3(z)] \begin{bmatrix} a(z) \\ 1 \end{bmatrix}, \quad \text{with} \quad [v_3(z) \;\; w_3(z)] = \begin{bmatrix} 2T_1(z) & T_2(z)/2 - T_1(z) - 3T_0(z)/2 \\ T_1(z) & T_1(z) - 2T_0(z) \end{bmatrix}.$$

8 Conclusion

Instead of taking the classical Newton-Padé approach, we have redefined the RIP. To get a parametrization of all solutions of the new problem, we have designed a new and recursive algorithm, not reordering the given data points. In future work, we shall indicate the deep connection between the Newton-Padé problem setting, the work of Antoulas and Anderson in linear system theory [1] and our approach, and show that the generalization to the vector and matrix case is easy within our framework.

References

[1] A.C. Antoulas and B.D.O. Anderson. State-space and polynomial approaches to rational interpolation. In A.C.M. Ran M.A. Kaashoek and J.H. van Schuppen, editors, *Progress in Systems and Control Theory III: Realization and Modeling in System Theory*, pages 73–82. Birkhäuser Verlag, 1990.

[2] G.A. Baker, Jr. and P.R. Graves-Morris. *Padé Approximants. Part II: Extensions and Applications*, volume 14 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley, Reading, MA, 1981.

[3] A.M. Bruckstein and T. Kailath. An inverse scattering framework for several problems in signal processing. *IEEE ASSP magazine*, pages 6–20, 1987.

[4] G. Claessens. On the structure of the Newton-Padé table. *J. Approx. Theory*, 22:304–319, 1978.

[5] F. Cordellier. Utilisation de l'invariance homographique dans les algorithmes de losanges. In H.J. Bünger H. Werner, editor, *Padé Approximation and its Applications*, volume 1071 of *Lecture Notes in Math.*, pages 62–94, Bad Honnef, 1983.

[6] W.B. Gragg. The Padé table and its relation to certain algorithms of numerical analysis. *SIAM Rev.*, 14:1–62, 1972.

[7] P.R. Graves-Morris and T.R. Hopkins. Reliable rational interpolation. *Numer. Math.*, 36:111–128, 1981.

[8] J. Meinguet. On the solubility of the Cauchy interpolation problem. In A. Talbot, editor, *Approximation theory*, pages 137–163, London and New York, 1970. Academic Press.

[9] M. Van Barel. *Nested Minimal Partial Realizations and Related Matrix Rational Approximants*. PhD thesis, K.U. Leuven, January 1989.

[10] H. Werner. Algorithm 51 : A reliable and numerically stable program for rational interpolation of Lagrange data. *Computing*, 31:269–286, 1983.

[11] Guo-liang Xu and A. Bultheel. The problem of matrix Padé approximation. Technical Report TW116, Department of Computer Science, K.U. Leuven, November 1988.