WHY IS THERE NO SIMPLE ANALOGON OF SZEGÖ'S THEORY ON THE UNIT CIRCLE FOR ORTHOGONAL POLYNOMIALS ON JULIA SETS ?

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Ortogonality on Julia sets and formal orthogonal polynomials

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nonlinear complex polynomial

$$P(z) = \sum_0^n p_i z^i \in \mathbf{C}[z], \quad n \geq 2$$

$J = J(P)$: Julia set

μ : (log) equilibrium measure on J

$(f, g)_C = \int \overline{f} g d\mu$: inner product

$\mu_{ij} = (z^i, z^j)_C$: moments

moment generating function

$$\begin{aligned} g_\mu &= \int \frac{1}{z - \zeta} d\mu(\zeta) \\ &= \sum_{j=0}^{\infty} \frac{\mu_{0,j}}{z^{j+1}} \end{aligned}$$

$\mu_{0j} = (1, z^j)_C = \int z^j d\mu$

iterates of $P = \sum_{i=0}^{n} p_i z^i$ are $P^{\circ k}$

$$a_\nu = P^{\circ k} + \frac{p_{n-1}}{n p_n},$$

$$\deg a_\nu = \nu = n^k$$

are orthogonal polynomials

$$(z^i, a_\nu(z))_C = 0, \text{ for } i = 0, \ldots, n^k - 1$$

[Pitcher & Kinney - 1968]

construction of measure

$$\mu_0 = \delta_c, \quad \mu_{i+1} = \tau(\mu_i) \to \mu$$

construction of moment generating function

$$g_0 = \frac{1}{z-c}, \quad g_{i+1} = \tilde{\tau}(g_i) \to g_\mu$$

$$\tilde{\tau}(g) = \frac{1}{n}(g \circ P)P'$$

rational approximants

$$g_i = \frac{c_\nu}{a_\nu} = \sum_{j=0}^{\infty} \frac{\mu_j^i}{z^{j+1}}$$

$$\mu_j^i = \mu_{0,j}, \quad j = 0, 1, \ldots, n^i$$

$$\deg a_\nu = \nu = n^i, \deg c_\nu = \nu - 1$$

c_ν polynomial of second kind

$$c_\nu(z) = \int \frac{a_\nu(z) - a_\nu(\zeta)}{z-\zeta} d\mu(\zeta)$$

If $P \in \mathbf{R}$ and $J(P) \subset \mathbf{R}$ then for real fncts

$$(f,g)_C = \int \overline{f} g d\mu = (f,g)_R = \int f g d\mu$$

For $(\cdot,\cdot)_R$ holds

$$\mu_{ij} = (z^i, z^j)_R = \int z^{i+j} d\mu(z) = \mu_{i+j+1}$$

$\Rightarrow M$ is Hankel

$\Rightarrow$ 3-term recurrence

$\Rightarrow$ Jacobi matrix

$\Rightarrow$ Padé approximants,...

BUT

$M = [\mu_{i+j+1}]$ not strongly regular!

$a_\nu = p^{ok} + \frac{p_{n-1}}{n p_n}$ still $\perp$ for $(\cdot,\cdot)_R$

D. Bessis used

$$(f_1, f_2) = \frac{1}{2\pi j} \oint_K f_1(z) f_2(z) g_\mu(z) dz$$

IF M strongly regular THEN

$$a_{\nu+1}(z) = (z - \gamma_{\nu+1}) a_\nu(z) - \lambda_{\nu+1} a_{\nu-1}(z)$$

Jacobi matrix

$$T = \begin{bmatrix} \gamma_1 & \lambda_2 & 0 & 0 & \\ 1 & \gamma_2 & \lambda_3 & 0 & \\ 0 & 1 & \gamma_3 & \lambda_4 & \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix}.$$

$$\sigma(T) = J$$

$g_i =$ Padé approximant of type $[n^i - 1/n^i]_{g_\mu}$

O.P. may not exist:

e.g. $P(z) = z^2 - 1$

$\rightarrow \mu_0 = \mu_2 = 1, \mu_4 = 2, \mu_6 = 4, \mu_8 = 9, \cdots$

$\mu_1 = \mu_3 = \mu_5 = \mu_7 = \mu_9 = 0$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 4 \\ 0 & 2 & 0 & 4 & 0 \\ 2 & 0 & 4 & 0 & 9 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = - \begin{bmatrix} 0 \\ 4 \\ 0 \\ 9 \\ 0 \end{bmatrix}$$

$\rightarrow a_5$ does not exist in usual sense

what realy holds

BLOCKS of polynomials

$$\begin{aligned} \mathbf{a}_i &= [a_{\nu_i}, \ldots, a_{\nu_i+\alpha_{i+1}-1}] \\ &= [a_i^1, \ldots, a_i^{\alpha_i+1}] \end{aligned}$$

ν_i start of block i

$\leftrightarrow$

regular submatrix M_i of M

α_{i+1} is number of elements in a block.

$$\nu_k = \sum_{i=1}^{k} \alpha_i$$

$$(\mathbf{a}_i, \mathbf{a}_j)_R = [(a_i^k, a_j^l)_R] = \delta_{ij} D_{jj}$$

D_{jj} regular matrix of size α_{j+1}

3 term recurrence

$$a^1_{i+1}(z) = \gamma_{i+1}(z)a^1_i(z) + \lambda_{i+1}a^1_{i-1}(z)$$

$\gamma_{i+1}(z)$ monic of degree α_{i+1}

$$F_n A_n = A_n T_n$$

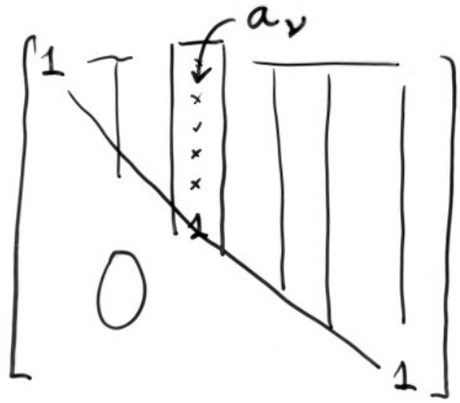
$F_n = F(a^1_{n+1})$ companion matrix

A_n unit upper triangular

$$T_n = \begin{bmatrix} T_{0,0} & T_{0,1} & & & \\ T_{1,0} & T_{1,1} & T_{1,2} & & \\ & T_{2,1} & T_{2,2} & \ddots & \\ & & \ddots & \ddots & T_{n-1,n} \\ & & & T_{n,n-1} & T_{n,n} \end{bmatrix}$$

$$a^1_{n+1}(z) = \det(zI - T_n)$$

$A_n =$



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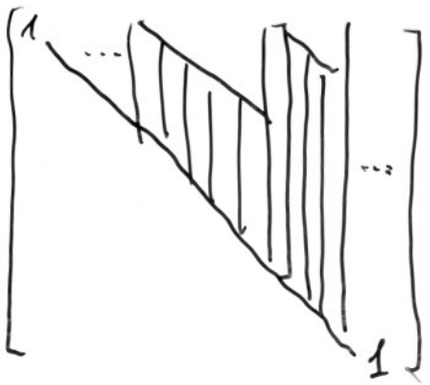
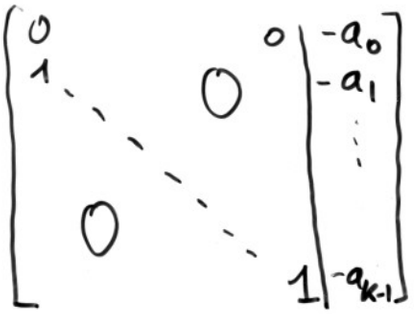
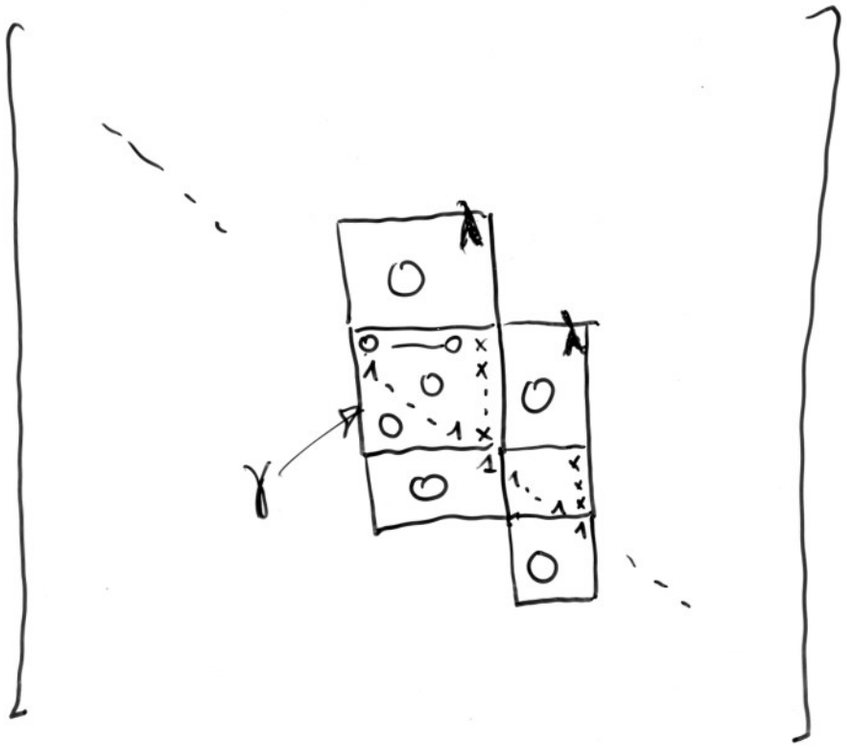


$F_{a_k} =$



$a_k(z) = a_0 + a_1 z + \cdots a_{k-1} z^{k-1} + z^k$

$T_n =$



rational approximants

$$\begin{aligned} g_n(z) &= e_0^T M_n (z M_n - M_n^<)^{-1} M_n e_0 \\ &= \sum_{i=0}^{\infty} \mu_i^n z^{-i} \end{aligned}$$

$$M_n = [\mu_{i+j+1}], M_n^< = [\mu_{i+j+2}] = M_n F_n$$

$$i, j = 0, \ldots, \nu_n - 1$$

g_n approximates $g_\mu = \sum \mu_k z^{-k}$ by

$$\mu_k = \mu_k^n, \quad k = 1, \ldots, 2\nu_{n+1} + \alpha_{n+2}$$

in general NO 3 term recurrence

$$F_n A_n = A_n T_n$$

$$T_n = \begin{bmatrix} T_{0,0} & T_{0,1} & T_{0,2} & \cdots & T_{0,n} \\ T_{1,0} & T_{1,1} & T_{1,2} & & T_{1,n} \\ & T_{2,1} & T_{2,2} & \ddots & T_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & & T_{n-1,n} \\ & & & T_{n,n-1} & T_{n,n} \end{bmatrix}$$

$$a^1_{n+1}(z) = \det(zI - T_n)$$

rational approximants

$$g^{[i]}(z) = \sum_{k=0}^{\infty} \frac{\mu_{i,k}}{z^{k+1}}, \quad i = 0, 1, 2, \ldots$$

$$\begin{aligned} g_n^{[i]}(z) &= e_i^T M_n (z M_n - M_n^{<})^{-1} M_n e_0 \\ &= c_{n+1}^{[i]}(z) / a_{n+1}^{1}(z) \\ &= \sum_{k=0}^{\infty} \frac{\mu_{i,k}^{n}}{z^{k+1}} \end{aligned}$$

polynomial of second kind

$$c_{n+1}^{[i]}(z) = (\zeta^i, \frac{a_{n+1}^{1}(z) - a_{n+1}^{1}(\zeta)}{z - \zeta}).$$

$g_n^{[i]}$ approximates $g^{[i]}$ by

$\mu_{i,k} = \mu_{i,k}^{n}$ for $k = 0, 1, \ldots, \nu_{n+1}$.

VECTOR Padé approximant

what if moment matrix Toeplitz?

T_n block upper Hessenberg, but $T_n = S_n U_n^{-1}$

$$F_n A_n = A_n T_n \Leftrightarrow F_n A_n U_n = A_n S_n$$

$$a^1_{n+1}(z) = \det(zU_n - S_n)$$

approximation

$$\begin{aligned} g^-(z) &= \mu_1 z^{-1} + \mu_{\mathbf{1}} z^{-2} + \cdots \\ g^+(z) &= \mu_0 + \mu_{-1} z + \mu_{-\mathbf{1}} z^2 + \cdots \end{aligned}$$

$$\begin{aligned} g^-(z) - \frac{c_n(z)}{a^1_n(z)} &= *z^{-\nu_n - \alpha^-_{n+1} - 1} + \text{l.o.t} \\ g^+(z) - \frac{c_n(z)}{a^1_n(z)} &= *z^{\nu_n - \alpha^+_{n+1}} + \text{h.o.t.} \end{aligned}$$

$$\nu_n = \alpha_1 + \cdots + \alpha_n$$
$$\alpha_{n+1} = \alpha^+_{n+1} + \alpha^-_{n+1} + 1$$

TWO POINT Padé approximant

$$T_n = \begin{bmatrix} X & X & X & \cdots & X & X \\ X & X & X & & \vdots & \vdots \\ & X & X & & \vdots & \vdots \\ & & X & \ddots & \vdots & \vdots \\ & 0 & & \ddots & X & X \end{bmatrix} = S_n U_n^{-1}$$

$$S_n = \begin{bmatrix} X & & & & & 0 \\ X & \ddots & & & & \\ & \ddots & \boxed{\begin{matrix} 1 & & X \\ & \ddots & \vdots \\ & & X \\ 0 & & 1 \end{matrix}} & & & \\ & & & \ddots & & \\ & 0 & & & X & X \end{bmatrix}$$

$$U_n = \begin{bmatrix} X & X & & & & 0 \\ & \ddots & \ddots & & & \\ & & \boxed{\begin{matrix} \cdot & & \\ 1 & \ddots & X \\ & & 1 \end{matrix}} & \ddots & & \\ & & & \ddots & \ddots & \\ & 0 & & & \ddots & X \\ & & & & & X \end{bmatrix}$$

what inner product?

$$(f_1, f_2)_T = \int \overline{f_1(1/\overline{z})} f_2(z) d\mu(z)$$

$$\mu_{ij} = (z^j, z^i)_T = \int z^{i-j} d\mu(z) = \mu_{i-j}$$

$$J(P) \subset \mathbf{T} \Rightarrow (f, g)_C = (f, g)_T$$

BUT

nonsymmetric & non-hermitian

$$J \subset \mathbf{T} \Rightarrow J = \mathbf{T}$$

$(\cdot, \cdot)_T$ may not be defined!!

e.g. $0 \in J = \text{supp}(\mu)$ as for $P(z) = z^2 - 2$

$$\mu_k = \int z^k d\mu(z) = ?, \quad k < 0$$

$C_\ell(J) = |p_n|^{\frac{-1}{n-1}}$

J = Jordan curve



FRACTAL !
(unless $J = \mathbb{T}$)

$$(f,g)_S = \frac{1}{L} \oint_J f(z)\,\overline{g(z)}\,w(z)\,dz$$

$$\downarrow$$

$$\int f(z)\,\overline{g(z)}\,d\tilde{\mu}(z) \qquad \left(\frac{d\tilde{\mu}}{d\mu} = w\right)$$

$z \in ext(J)$

$$\rightarrow a_i(z) \sim \sqrt{\frac{L}{2\pi}}\,\Delta(z)^{-1}\sqrt{\beta'(z)}\,(\beta(z))^i \qquad i \to \infty$$

β : Böttcher f. $\quad z \in ext(J)$

$$\downarrow$$

$$a_i(z) \sim \beta^i / \Delta$$

$\mu = \tilde{\mu}$

$$\downarrow$$

Stahl - Totik

$$\lim_{i\to\infty} \sqrt[i]{|a_i|} = |\beta|$$

Bessis

$$\beta(z) = \lim_{i\to\infty} \left[P^{oi}(z)\right]^{n^{-i}} \qquad (p_{n-1} = 0)$$

$z \in int(J)$

$a_i \to 0$



$$\Rightarrow \quad \overbrace{P^{oi}(z) + \frac{p_{n-1}}{n p_n}}^{a_{ni}(z)} \longrightarrow z_0 + \frac{p_{n-1}}{n p_n}$$

if $z_0 \neq -\frac{p_{n-1}}{n p_n} \longrightarrow \neq 0$