Computation of Rational Szegő-Lobatto Quadrature Formulas

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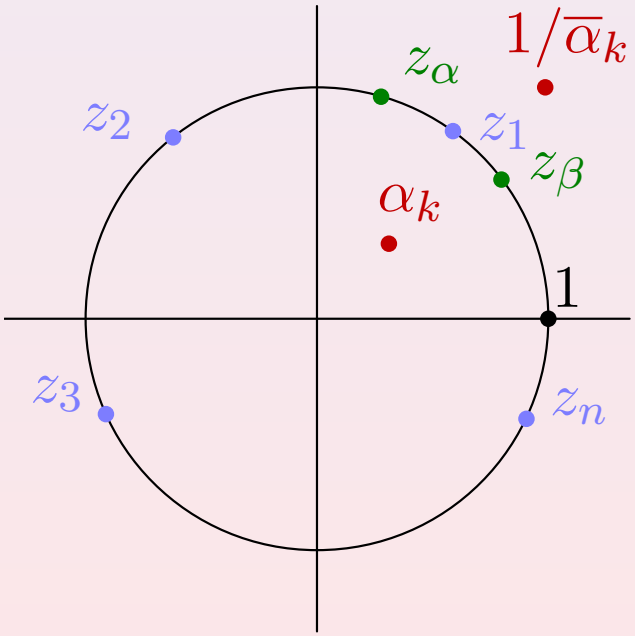
General aim

Approximate calculation of integrals on the unit circle.

$$I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta \approx Af(z_\alpha)+Bf(z_\beta)+\sum_{j=1}^{n}\lambda_j f(z_j) = I_{n+2}(f)$$



$$A, B, \lambda_j > 0$$

$$z_j \neq z_k,\ j \neq k$$

$$z_j \notin \{z_\alpha, z_\beta\}$$

$I_\omega(f) = I_{n+2}(f)$ with f in a subspace of rational functions with prescribed poles at α_k and $1/\overline{\alpha}_k$ of as high dimension as possible.

Contents

I Introduction: Gauss, Gauss-Radau and Gauss-Lobatto formulas.

II Periodic integrands: Szegő-type quadrature formulas.

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IV Rational Szegő-Lobatto formulas.

V Error and convergence.

Gauss, Gauss-Radau and Gauss-Lobatto formulas

$$I_\sigma(f) = \int_a^b f(x)\sigma(x),\ -\infty \leq a < b \leq +\infty$$

There exist n distinct nodes $\{x_j\}_{j=1}^n \subset (a,b)$ and positive weights $\{A_j\}_{j=1}^n$ such that,

$$I_\sigma(P) = I_n(P) = \sum_{j=1}^{n} A_j P(x_j), \forall P \in \mathbb{P}_{2n-1}$$

$$\mathbb{P}_{2n-1} = span\{x^k : 0 \leq k \leq 2n-1\},\ \mathbb{P} = span\{x^k, k = 0, 1, \ldots\}$$

$$\Downarrow$$

Gauss, Gauss-Christtophel or Gaussian quadrature formulas.

- Maximal domain of validity
- Positivity of the weights $\Rightarrow$ Stability and convergence.
- Efficient computation in terms of an eigenvalue problem involving certain tridiagonal Jacobi matrices.

Gauss, Gauss-Radau and Gauss-Lobatto formulas

$[a,b]$ finite. $[a,b]=[-1,1]$.

$$I_\sigma(f)=\int_{-1}^{+1} f(x)\sigma(x)dx$$

Theorem

Given $r,s\in\{0,1\}$ *there exist positive weights* $A_+^{(r,s)}, A_-^{(r,s)}$ *and* $\{A_j^{(r,s)}\}_{j=1}^n$ *along with* n *distinct nodes* $\{x_j^{(r,s)}\}\subset(-1,1)$ *such that,*

$$I_{n+r+s}(f)=rA_+^{(r,s)}f(1)+sA_-^{(r,s)}f(-1)+\sum_{j=1}^{n}A_j^{(r,s)}f(x_j^{(r,s)})=I_\sigma(f)$$

$\forall f\in\mathbb{P}_{2n+r+s-1}$

- $r+s=0\Rightarrow$ *Gauss Formulas.*
- $r+s=1\Rightarrow$ *Gauss-Radau Formulas.*
- $r+s=2\Rightarrow$ *Gauss-Lobatto Formulas.*

$$\Downarrow$$

Gauss type Formulas

Szegő-type quadratures

$$I_\omega(f) = \int_{-\pi}^{\pi} f(\theta)\omega(\theta)d\theta$$

f: 2π-periodic. $\omega(\theta)$: 2π-periodic weighted function.

$$I_n(f) = \sum_{j=1}^{n} \lambda_j f(\theta_j);\ \{\theta_j\} \subset [-\pi, \pi),\ \theta_j \neq \theta_k \text{ if } j \neq k$$

$$I_\omega(T) = I_n(T):\ T(\theta) = \sum_{k=0}^{N} (a_k \cos k\theta + bk \sin k\theta)$$

T: trigonometric polynomial with as high degree as possible.

$$\boxed{N \leq n-1}$$

$N = n-1 \Rightarrow$ Quadrature formulas with the highest trigonometric precision degree.

(Bi-orthogonal systems of trigonometric polynomials)

Szegő-type quadratures

Alternative approach

Passing to the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Some notations

$\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}$.

$p, q \in \mathbb{Z}$, $p \leq q$, $\Lambda_{p,q} = span\{z^k : p \leq k \leq q\}$ (Laurent polynomials)

$\Lambda \equiv$ space of all Laurent polynomials.

$$T(\theta) = \sum_{k=0}^{m} (a_k \cos k\theta + b_k \sin k\theta) = L(e^{i\theta}),\ L \in \Lambda_{-m,m}$$

$$I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta \approx I_n(f) = \sum_{j=1}^{n} \lambda_j f(z_j)$$

$z_j \in \mathbb{T}$, $z_j \neq z_k$ if $j \neq k$. $z_j = e^{i\theta_j}$, $\theta_j \in [-\pi, \pi)$, $\theta_j \neq \theta_k$.

$I_\omega(f) = I_n(f)$, $\forall f \in \Lambda_{-(n-1),(n-1)}$.

Szegő-type quadratures

Some notations

$\{\rho_k\}_0^\infty$: monic Szegő polynomials associated with $\omega(\theta)$, i.e.
$\rho_0(z) = 1, \rho_n(z) = z^n + \cdots + \delta_n$, $n = 1.2\ldots$. $z = e^{i\theta}$, $n \geq 1$,
$\langle \rho_n(z), z^k \rangle_\omega = \int_{-\pi}^{\pi} \rho_n(z)\overline{z^k}\omega(\theta)d\theta = 0$, $k = 0, 1, \ldots, n-1$.
$\rho_n^*(z) = z^n\overline{\rho_n(1/\overline{z})}$ (Reverse polynomial)

Theorem

Given $I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta$, *then* $I_n(f) = \sum_{j=1}^n \lambda_j f(z_j) =$
$= I_\omega(f)$, $\forall f \in \Lambda_{-(n-1),(n-1)}$, *if and only if,*

1. *The nodes* $\{z_j\}_{j=1}^n$ *are the zeros of:*

$$B_n(z,\tau) = z\rho_{n-1}(z) + \tau\rho_{n-1}^*(z), \ |\tau| = 1.$$

2. $$\lambda_j = \left[\sum_{k=0}^{n-1} \frac{|\rho_k(z_j)|^2}{\langle \rho_k, \rho_k \rangle_\omega}\right]^{-1}, \ j = 1, \ldots, n.$$

Szegő-type quadratures

- Szegő quadrature formulas (W. B. Grags (1982,1991), Jones et als (1989))
- Efficient computation in terms of an eigenvalue problem involving Hessemberg Matrices whose entries essentially depend on the numbers:

$$\delta_k = \rho_k(0),\ \rho_0 = 1,\ |\delta_k| < 1,\ k = 1, 2, \ldots$$

(Verblunsky coefficients, Schur parameters, Reflection coefficients).

Szegő-Lobatto Quadrature rules

C. Jagels, L. Reichel (2007)

A. Bultheel, L. Daruis, P. G-V (2009)

$$I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta$$

Given: $z_\alpha \neq z_\beta$ on $\mathbb{T}$, find n distinct nodes $\{z_j\}_{j=1}^n \subset \mathbb{T}$, $z_j \notin \{z_\alpha, z_\beta\}$ and $n+2$ positive weights A, B and $\{\lambda_j\}_{j=1}^n$ such that:

$$I_\omega(f) = I_{n+2}(f) = Af(z_\alpha) + Bf(z_\beta) + \sum_{j=1}^{n} \lambda_j f(z_j),\ \forall f \in \Lambda_{-n,n}.$$

Szegő-type quadratures

Basic approach

$\omega(\theta) \rightsquigarrow \{\rho_k\}_0^\infty$

Available information: $\mu_k = \int_{-\pi}^{\pi} e^{-ik\theta}\omega(\theta)d\theta$, $k = 0, \pm 1, \pm 2, \ldots$

Recursion: $\rho_k(z) = z\rho_{k-1}(z) + \delta_k\rho_{k-1}^*(z)$, $\rho_0 = 1$, $\delta_k = \rho_k(0)$, $k = 1, 2, \ldots$. (Levinson algorithm).

From $\rho_n(z)$ and given $\tilde{\delta}_{n+1} \in \mathbb{D}$. $(|\tilde{\delta}_{n+1}| < 1)$.

Define $\tilde{\rho}_{n+1}(z) = z\rho_n(z) + \tilde{\delta}_{n+1}\rho_n^*(z)$.

Now take $\tilde{\tau}_{n+2} \in \mathbb{T}$, $(|\tau_{n+2}| = 1)$ and consider

$$\tilde{B}_{n+2}(z) = z\tilde{\rho}_{n+1}(z) + \tilde{\tau}_{n+2}\tilde{\rho}_{n+1}^*(z)$$

$\tilde{B}_{n+2}(z)$ has $n+2$ distinct zeros on $\mathbb{T}$.

AIM

Determine $\tilde{\delta}_{n+1} \in \mathbb{D}$ and $\tilde{\tau}_{n+2} \in \mathbb{T}$ such that

$$\tilde{B}_{n+2}(z_\alpha) = \tilde{B}_{n+2}(z_\beta) = 0$$

Rational Quadrature Formulas on the Unit Circle

- Gauss-type Formulas: Exact integration of polynomials (all the poles at infinity)
- Szegő-type formulas: Exact integration of Laurent polynomials (all the poles at the origin and infinity)

$$\Downarrow$$

Given $\int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta \Rightarrow$ quadrature formulas exactly integrating more general rational functions with prescribed poles not on $\mathbb{T}$.

The poles: $\{\alpha_k\} \subset \mathbb{D}$; $\{1/\overline{\alpha}_k\} \subset \mathbb{E}$

$$\Downarrow$$

Theory on Orthogonal Rational Functions
(A. Bultheel, E. Hendriksen, P. G-V, O. Njastad)

Rational Quadrature Formulas on the Unit Circle

The spaces of rational functions

$\{\alpha_k\}_1^\infty \subset \mathbb{D}$

Blaschke Factors

$\xi_i(z) = \eta_i \frac{\alpha_i - z}{1-\overline{\alpha}_i z}$; $\eta_i = \frac{\alpha_i}{|\alpha_i|}$ if $\alpha_i \neq 0$; $\eta_i = -1$ if $\alpha_i = 0$, i=1,2,...

Blaschke Products

$B_0 = 1$, $B_n = B_{n-1}\xi_n$, $n \geq 1$.

Set $\mathcal{L}_n = span\{B_k : k = 0, 1, \ldots, n\}$

$$\omega_0 = \pi_0 = 1;\ \omega_k = \prod_{j=1}^{k}(z - \alpha_j);\ \pi_k = \prod_{j=1}^{k}(1 - \overline{\alpha}_j z),\ k = 1, 2, \ldots$$

$$B_k(z) = \gamma_k \frac{\omega_k(z)}{\pi_k(z)};\ \gamma_k = (-1)^k \prod_{j=1}^{k} \eta_j;$$

$$\mathcal{L} = \left\{ R = \frac{P}{\pi_n} : P \in \mathbb{P}_n \right\} = \left\{ R(z) = \frac{P(z)}{(1 - \overline{\alpha}_1 z) \cdots (1 - \overline{\alpha}_n z)} \right\}$$

The spaces of rational functions

$$f_*(z) = \overline{f(1/\overline{z})} \Rightarrow \mathcal{L}_{n*} = \{f : f_* \in \mathcal{L}_n\} = \{f = \frac{Q}{\omega_n}, Q \in \mathbb{P}_n\}$$

$$= \left\{ f = \frac{Q(z)}{(z-\alpha_1)\cdots(z-\alpha_n)} \right\}$$

$$\mathcal{R}_{p,q} = \mathcal{L}_{p*} + \mathcal{L}_q = \left\{ f = \frac{P}{\omega_p \pi_q} : P \in \mathbb{P}_{p+q} \right\}$$

$$= span\{B_k, k = -p, \ldots, -1, 0, 1, \ldots, q\}$$

$$B_{-k} = B_{k*} = \frac{1}{B_k}$$

$\alpha_k = 0,\ k = 1, 2, \ldots,\ \mathcal{R}_{p,q} = \Lambda_{-p,q}.$

Rational Quadrature Formulas on the Unit Circle

$$I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta$$

$$I_n(f) = \sum_{j=1}^{n} \lambda_j f(z_j),\ z_j \in \mathbb{T},\ z_j \neq z_k \text{ if } j \neq k$$

$I_\omega(R) = I_n(R), \forall R \in \mathcal{R}_{p(n),p(n)}$, $p(n)$ as large as possible.

$$\boxed{p(n) \leq n-1} \qquad p(n) = n-1?$$

Orthogonal Rational Functions

$n \geq 1$, $\{\varphi_j\}_{j=0}^{n}$, $\varphi_j \in \mathcal{L}_j \backslash \mathcal{L}_{n-1}$ an orthogonal basis for $\mathcal{L}_n$

(Gram-Schmidt process on $\{B_k\}_0^n$)

$$\varphi_k(z) = \sum_{j=0}^{k} a_j B_j(z),\ a_k \neq 0 \text{ leading coefficient}$$

$a_k = 1 \Rightarrow \varphi_k(z) \equiv$ monic.

When the process is repeated $\forall n \geq 1 \Rightarrow \{\varphi_k\}_0^\infty$ a sequence of ORF w. r. t. $\omega(\theta)$ and the poles $\{\alpha_k\}_1^\infty$. (Rational Szegő Functions)

Drawback

The zeros of $\varphi_n(z)$ lie in $\mathbb{D}$.

Theorem (Rational Szegő Quadratures)

Let $\{\varphi_k\}_0^\infty$ *be the sequence of monic Rational Szegő Functions. For each* $n \geq 1$, *take* $\tau \in \mathbb{T}$ *and set*

$X_n(z) = X_n(z,\tau) = (z-\alpha_{n-1})\varphi_{n-1}(z) + \tau(1-\overline{\alpha}_{n-1}z)\varphi^*_{n-1}(z)$,

$(\varphi^*_k(z) = B_k(z)\varphi_{k*}(z) = B_k(z)\overline{\varphi_k(1/\overline{z})})$. *Then,*

1. $X_n(z)$ *has exactly* n *distinct nodes* $z_1,\ldots,z_n$ *which lie on* $\mathbb{T}$.
2. *There exist positive numbers* $\lambda_1,\ldots,\lambda_n$ *such that*

$$I_n(f) = \sum_{j=1}^{n} \lambda_j f(z_j) = I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta,\ \forall f \in \mathcal{R}_{n-1,n-1}$$

3. $\lambda_j = \sum_{k=0}^{n-1} \frac{|\varphi_k(z_j)|^2}{\|\varphi_k\|^2_\omega},\ j=1,\ldots,n,$

   $\|\varphi_k\|^2_\omega = \langle \varphi_k, \varphi_k\rangle_\omega = \int_{-\pi}^{\pi} \varphi_k(z)\overline{\varphi_k(z)}\omega(\theta d\theta),\ (z = e^{i\theta})$
4. *There can not exist an* n*-point quadrature rule with nodes on* $\mathbb{T}$ *to be exact either in* $\mathcal{R}_{n,n-1}$ *or* $\mathcal{R}_{n-1,n}$.

An Illustrative Example

$\omega(\theta) = \frac{1}{2\pi}$, $\theta \in [-\pi, \pi]$, $\{\alpha_k\} \subset \mathbb{D}$.
$\{\varphi\}_0^\infty$: the Orthogonal Sequence of monic Rational Szegő Function.
$\langle \varphi_k, \varphi_k \rangle_\omega = 1$, $k = 0, 1, 2, \ldots$ $\varphi_n(z) = \kappa_n \frac{zB_n(z)}{z-\alpha_n}$, $\kappa_n \neq 0$, $n = 1, 2, \ldots$

$$\tau \in \mathbb{T} \Rightarrow I_n(f) = \sum_{j=1}^{n} \lambda_j f(z_j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) d\theta, \; \forall f \in \mathcal{R}_{n-1,n-1}$$

$$\boxed{z_j : \; z\omega_{n-1}(z) + \tau \Pi_{n-1}(z) = 0}$$

$$\omega_{n-1}(z) = \prod_{k=1}^{n-1} (z - \alpha_k); \; \pi_{n-1}(z) = \prod_{k=1}^{n-1} (1 - \overline{\alpha}_k z)$$

$$\boxed{\lambda_j = \frac{1}{1 + \sum_{k=1}^{n-1} \frac{1-|\alpha_k|^2}{|z_j - \alpha_k|^2}}, \; j = 1, \ldots, n}$$

$\alpha_k = 0, k = 1, 2, \ldots \Rightarrow z_j : z^n + \tau = 0$, $\lambda_j = \frac{1}{n}$, $j = 1, \ldots, n$.

Rational Szegő-Lobatto Formulas

Given $x_1, x_2 \in \mathbb{T}$, $x_1 \neq x_2$, $n \geq 1$. Find n distinct nodes $z_1, \ldots, z_n$ on $\mathbb{T}$, $z_j \notin \{x_1, x_2\}$ and positive weights $A_1, A_2, \lambda_1, \ldots, \lambda_n$ such that:

$$\tilde{I}_{n+2}(f) = A_1 f(x_1) + A_2 f(x_2) + \sum_{j=1}^{n} \lambda_j f(z_j) = I_\omega(f)$$

$\forall f \in \mathcal{R}_{p(n),p(n)}$, $p(n)$ as large as possible.
$n+2$ nodes on $\mathbb{T} \Rightarrow p(n) \leq n+1$.

Rational Szegő-Lobatto Formulas

- $p(n) = n + 1 \Leftrightarrow \tilde{I}_{n+2}(f)$ a Rational Szegő Quadrature Formula$\Leftrightarrow \gamma_n(x_1) = \gamma_n(x_2)$ with

$$\gamma_n(x_j) = \frac{(\alpha_{n-1} - x_j)\varphi_{n-1}(x_j)}{(1 - \overline{\alpha}_{n-1}x_j)\varphi^*_{n-1}(x_j)} \in \mathbb{T},\ j = 1, 2.$$

  $\varphi_{n-1}(z)$: $(n-1)$-th monic Rational Szegő Function

- $\gamma_n(x_1) \neq \gamma_n(x_2) \Rightarrow p(n) \leq n$

  If $p(n) = n \Rightarrow \tilde{I}_{n+2}(f)$ Rational Szegő-Lobatto Quadrature Formula.

Rational Szegő-Lobatto Formulas

The Approach

μ: positive Borel Measure on $\mathbb{T}$;
$\{\alpha_k\}_1^\infty \subset \mathbb{D}$: The poles of the rational functions.
$\{\varphi_k\}_0^\infty$: Sequence of Rational Szegő Functions.

Recurrence Relation

$$\varphi_n(z) = \varepsilon_n \left[\left(\frac{z-\alpha_{n-1}}{1-\overline{\alpha}_n z} \right) \varphi_{n-1}(z) + \left(\frac{\delta_n}{\varepsilon_n} \right) \left(\frac{1-\overline{\alpha}_{n-1} z}{1-\overline{\alpha}_n z} \right) \varphi_{n-1}^*(z) \right]$$

$$\varphi_n^*(z) = -\eta_n \overline{\varepsilon}_n \left[\overline{\left(\frac{\delta_n}{\varepsilon_n} \right)} \left(\frac{z-\alpha_{n-1}}{1-\overline{\alpha}_n z} \right) \varphi_{n-1}(z) + \left(\frac{1-\overline{\alpha}_{n-1} z}{1-\overline{\alpha}_n z} \right) \varphi_{n-1}(z) \right]$$

$$\varphi_n^*(z) = B_n(z)\varphi_{n*}(z) = B_n(z)\overline{\varphi_n(1/\overline{z})},\ n \geq 1.$$

$$B_0(z) = 1,\ B_k(z) = (-1)^k \prod_{j=1}^{k} \eta_j \frac{(z-\alpha_1)\cdots(z-\alpha_k)}{(1-\overline{\alpha}_1 z)\cdots(1-\overline{\alpha}_k z)},\ \eta_j = \begin{cases} \frac{\overline{\alpha}_j}{|\alpha_j|}, \alpha_j \neq 0 \\ -1,\ \alpha_j = 0 \end{cases}$$

$$\varepsilon_n \neq 0;\ |\delta_n| < |\varepsilon_n|,\ \beta_n = \frac{\delta_n}{\varepsilon_n},\ |\beta_n| < 1.$$

Theorem

Let μ be a positive Borel measure on $\mathbb{T}$, $\{\varphi_n\}_0^\infty$ being its sequence of monic Rational Szegő Functions with respect to the fixed sequence of poles $\{\alpha_k\}_1^\infty \subset \mathbb{D}$. Then, given $\beta_{n+1} \in \mathbb{D}$, there exists a positive Borel measure $\tilde{\mu}$ on $\mathbb{T}$ such that if $\{\tilde{\varphi}_k\}_0^\infty$ represents its monic sequence, it holds that

- $\tilde{\varphi}_k(z) = \varphi_k(z), k = 0, 1, \ldots, n.$
- $\tilde{\varphi}_{n+1}(z) = \tilde{\varepsilon}_{n+1}\left[\left(\frac{z-\alpha_n}{1-\overline{\alpha}_{n+1}z}\right)\varphi_n(z) + \beta_{n+1}\left(\frac{1-\overline{\alpha}_n z}{1-\overline{\alpha}_{n+1}z}\right)\varphi_n^*(z)\right]$, $\tilde{\varepsilon}_{n+1} \neq 0.$

Taking $d\mu(\theta) = \omega(\theta)d\theta$; $d\tilde{\mu}(\theta) = \tilde{\omega}(\theta)d\theta$;

$$\Rightarrow I_\omega(R) = I_{\tilde{\omega}}(R),\ \forall R \in \mathcal{R}_{n,n}$$

Rational Szegő-Lobatto Formulas

Take $\tau_{n+2} \in \mathbb{T}$ and set

$$X_{n+2}(z) = (z - \alpha_{n+1})\tilde{\varphi}_{n+1}(z) + (1 - \overline{\alpha}_{n+1}z)\tilde{\varphi}^*_{n+1}(z)$$

$X_{n+2}(z)$ has $n+2$ distinct zeros $\tilde{z}_1, \ldots, \tilde{z}_{n+2}$ on $\mathbb{T}$ and there exist positive numbers $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n+2}$ such that

$$\tilde{I}_{n+2}(f) = \sum_{j=1}^{n+2} \tilde{\lambda}_j f(\tilde{z}_j) = I_{\tilde{\omega}}(f), \ \forall f \in \mathcal{R}_{n+1,n+1}.$$

Since

$$I_\omega(R) = I_{\tilde{\omega}}(R), \forall f \in \mathcal{R}_{n,n} \Rightarrow \boxed{\tilde{I}_{n+2}(R) = I_\omega(R), \forall R \in \mathcal{R}_{n,n}}$$

$X_{n+2}(z) \in \mathcal{L}_{n+2}$ depends on two arbitrary parameters $\beta_{n+1} \in \mathbb{D}$ and $\tau_{n+2} \in \mathbb{T}$.

Theorem (Main Result)

Given a natural number n and two distinct complex numbers $x_1, x_2 \in \mathbb{T}$ along with a weight function $\omega(\theta)$ on $\mathbb{T}$ there exist $\tilde{\beta}_{n+1} \in \mathbb{D}$ and $\tilde{\tau}_{n+2} \in \mathbb{T}$ such that:

$$B_{n+2}(x_1) = B_{n+2}(x_2) = 0$$

where

$$B_{n+2}(z) = (z - \alpha_{n+1})\tilde{\varphi}_{n+1}(z) + (1 - \overline{\alpha}_{n+1}z)\tilde{\tau}_{n+2}\tilde{\varphi}^*_{n+1}(z)$$

and

$$\tilde{\varphi}_{n+1}(z) = \left(\frac{z - \alpha_n}{1 - \overline{\alpha}_{n+1}z}\right)\varphi_n(z) + \tilde{\beta}_{n+1}\left(\frac{1 - \overline{\alpha}_n z}{1 - \overline{\alpha}_{n+1}z}\right)\varphi^*_n(z)$$

$\varphi_n(z)$ being the n-th monic Rational Szegő Function for $\omega(\theta)$ and the given sequence of poles $\{\alpha_k\}_1^\infty \subset \mathbb{D}$.

Corollary

Given x_1 *and* $x_2 \in \mathbb{T}$ $(x_1 \neq x_2)$ *there exist* n *distinct nodes* $\tilde{z}_1, \ldots, \tilde{z}_n$ *on* $\mathbb{T}$ *with* $\tilde{z}_j \notin \{x_1, x_2\}$, $j = 1, \ldots, n$ *and* $n+2$ *positive weights* A_1, A_2 *and* λ_j, $j = 1, \ldots, n$ *such that*

$$\tilde{I}_{n+2}(f) = A_1 f(x_1) + A_2 f(x_2) + \sum_{j=1}^{n} \tilde{\lambda}_j f(\tilde{z}_j) = I_\omega(f)$$

$$\forall f \in \mathcal{R}_{n,n} = \left\{ \frac{P(z)}{(z-\alpha_1)\cdots(z-\alpha_n)(1-\overline{\alpha}_1 z)\cdots(1-\overline{\alpha}_n z)};\ P \in \mathbb{P}_{2n} \right\}$$

Assume $x_1 \neq x_2$ both on $\mathbb{T}$. Consider

$$\tilde{I}_{n+2}(f) = A_1 f(x_1) + A_2 f(x_2) + \sum_{j=1}^{n} \tilde{\lambda}_j f(\tilde{z}_j)$$

and set $E_{n+2}(f) = I_\omega(f) - \tilde{I}_{n+2}(f)$. Suppose f analytic on a neighborhood of $\mathbb{T}$ and let U be the domain of analiticity of $\omega_n(z)\pi_n(z)f(z)$. Then,

$$|E_{n+2}(f)| \leq 2\|\omega_n \pi_n f\|_{\gamma_r \cup \gamma_R} \left(\frac{r^{n+1}}{1-r^2} g_n(r) + \frac{R^{1-n}}{R^2-1} g_n\left(\frac{1}{R}\right) \right)$$

where $r < 1 < R$ such that $\gamma_r \cup \gamma_R \subset U$ and

$$g_n(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{j=1}^{n} \frac{1}{|1 - s e^{i\theta} \alpha_j|^2} d\theta, \; |s| < 1$$

$a > 0$; $\gamma_a = \{z \in \mathbb{C} : |z| = a\}$, $\|f\|_K = \max_{z \in K}\{|f(z)|\}$

For each $n \geq 1$ take $x_{1,n}, x_{2,n} \in \mathbb{T}$, $x_{1,n} \neq x_{2,n}$

$$\tilde{I}_{n+2}(f) = A_{1,n} f(x_{1,n}) + A_{2,n} f(x_{2,n}) + \sum_{j=1}^{n} \tilde{\lambda}_{j,n} f(\tilde{z}_{j,n})$$

(Rational Szegő-Lobatto formula)
Recall: $\tilde{I}_{n+2} = I_\omega(f)$, $\forall f \in \mathcal{R}_{n-n}$, $\lim_{n\to\infty} \tilde{I}_{n+2}(f) = I_\omega(f)$?

Lemma

Set $\mathcal{R} = \cup_{n=0}^{\infty} \mathcal{R}_{n,n}$, then $\mathcal{R}$ is dense in the class $\mathcal{C}(\mathbb{T})$ of continuous functions on $\mathbb{T}$, iff $\sum_{k=1}^{\infty}(1 - |\alpha_k|) = +\infty$

Theorem

Set $R_\omega(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C} : f\omega \text{ integrable on } \mathbb{T}\}$; *then*

$$\lim_{n\to\infty} \tilde{I}_{n+2}(f) = I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta, \forall f \in R_\omega(\mathbb{T})$$

if $\sum_{k=1}^{\infty}(1 - |\alpha_k|) = +\infty$