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Ion Acoustic Instability due to Collisional Energy Transfer

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The friction and the consequent energy transfer in plasmas consisting of two species with different temperatures is discussed. In a quasi-neutral plasma the friction between the two species is shown to have no effect on the ion acoustic mode, while using the Poisson equation instead of the quasi-neutrality reveals the possibility for an instability driven by the collisional energy transfer. The different starting temperatures of the two species imply an evolving background. It is shown that the relaxation time of the background electron-ion plasma is, in fact, always shorter than the growth time. Therefore the mentioned instability is unlikely to develop. The results obtained here should contribute to the clarification of some contradictory results obtained in the past.

Keywords: friction, collisions, energy transfer, ion acoustic mode

1. Introduction

Plasmas are frequently in the state of partial thermodynamic equilibrium, i.e., with an initial temperature disparity of the plasma constituents. Collisions in such plasmas will after some time eventually result in equal temperatures of the species, implying an evolving plasma. In Ref. [1] it is claimed that the corresponding energy transfer may result in the instability of the acoustic mode within the quasi-neutrality limit. The *necessary* instability condition for an electron-ion plasma appears very easily satisfied because it requires $T_e > 4T_i/3$. This instability condition is obtained by using the energy equations including the source/sink terms originating from the collisional transfer, together with the corresponding friction force terms in the momentum equations. The *sufficient* instability condition is stronger than the *necessary* one because of additional dissipative effects, like viscosity and thermal conductivity. However, the current-less instability described in Ref. [1] is based on a model which disregards the same temperature disparity in the description of the spatially homogeneous background, which, due to the same reasons, must be time evolving. In other words, the effects of collisions in the background plasma have been explicitly neglected. Note that here and further in the text, because of the time evolution, the term background is used instead of the equilibrium. These effects of collisions have been discussed in Ref. [2] for the same quasi-neutrality case. It is claimed that there is no instability for any temperature ratio of the two plasma components, and moreover, that this holds even in a current-carrying plasma, as long as the difference between the electron and ion equilibrium velocity remains below the sound speed. All that was needed to come to that conclusion was to let the background plasma evolve freely in the presence of the given temperature difference. However, Ref. [2] has remained almost unnoticed by researchers, in contrast to the widely cited Ref. [1].

This controversy is revisited in the present work for any two-component plasma. Essential for the problem is the energy equation describing the temperature variation of the species a, which, following Braginskii [3] is given in the form:

$$\frac{3}{2}n_a\frac{\partial T_a}{\partial t}+n_aT_a\nabla\cdot\vec{v}_a+\frac{3}{2}n_a(\vec{v}_a\cdot\nabla)T_a=Q_a. \quad (1)$$

The corresponding equation for the species b has the same shape, but with a minus sign on the right-hand side. We use the Landau formula [4] for the energy transfer source/sink term, $Q_a = 3m_b\nu_{ba}n_b(T_b - T_a)/m_a$, where [5]

$$\nu_{ba}=4\left(\frac{2\pi}{m_b}\right)^{1/2}\left(\frac{q_aq_b}{4\pi\varepsilon_0}\right)^2\frac{n_aL_{ba}}{3(T_b+T_am_b/m_a)^{3/2}}. \quad (2)$$

The Coulomb logarithm is given by $L_{ba} = \log[r_d/b_0]$, $r_d = r_{da}r_{db}/(r_{da}^2 + r_{db}^2)^{1/2}$, $r_{dj} = v_{Tj}/\omega_{pj}$, and $b_0 = [|q_aq_b|/(4\pi\varepsilon_0)]/[3(T_a + T_b)]$ is the impact parameter.

The other terms, due to viscosity and thermal conductivity, are omitted only for the sake of clarity, i.e., in order to demonstrate more clearly the effect of the disputed collisional energy transfer term. The effect of these omitted terms is easily predictable.

2. Non-Evolving Background

In Ref. [1] the collisions in the equilibrium were explicitly ignored. In that case, the two perturbed energy equations without the evolving background effects read:

$$\frac{\partial T_{(a,b)1}}{\partial t} + \frac{2}{3}T_{(a,b)0}\nabla\cdot\vec{v}_{(a,b)1} =$$

$$\pm 2\frac{m_b}{m_a}\nu_{ba}(T_{b1}-T_{a1}) \pm 2\nu_{ba}\frac{m_b}{m_a}(T_{b0}-T_{a0})\frac{n_{b1}}{n_0}.$$

Here, the minus sign applies to the species b.

The momentum equation which we use throughout the text for the species a is of the form $m_a n_a \partial\vec{v}_a/\partial t = -q_a n_a\nabla\phi - \nabla(n_a T_a) - m_a n_a \nu_{ab}(\vec{v}_a - \vec{v}_b)$, and the continuity equation has its standard form. Similar equations are used for the species b, where the friction term is of the form $\vec{F}_{fb} = -m_b n_b \nu_{ba}(\vec{v}_b - \vec{v}_a)$.

In the case of quasi-neutral perturbations, the two number densities $n_{(a,b)1}$ are calculated from the continuity equations and are made equal, like in Refs. [1, 2] (this is typically valid when dealing with wavelengths that are much longer than the Debye length). The dispersion equation in that case reads:

$$\left(\omega + \frac{i4m_b\nu_{ba}}{m_a}\right)\left(\omega^2 - \frac{5k^2}{3}\frac{T_{a0}+T_{b0}}{m_a+m_b}\right) = 0. \qquad (3)$$

Hence, even using the same model as in Ref. [1], we conclude that there is neither an instability nor damping of the acoustic mode, regardless of the ratio T_{a0}/T_{b0}.

Note that the momentum conservation condition $\nu_{ab} = m_b n_b \nu_{ba}/(m_a n_a)$ is nowhere used in the derivation of Eq. (3). This is because the friction terms vanish in any case. In fact, from the two continuity equations we have the velocities $v_{a1} - v_{b1} = (\omega/k)(n_{a1}/n_{a0} - n_{b1}/n_{b0})$. Using $q_a n_a = q_b n_b$, and assuming the constant charge on the two species a and b, we have $v_{a1} - v_{b1} = 0$. Hence, the assumption of quasi-neutrality cancels the friction completely. Otherwise, for a varying charge we would have $v_{a1} - v_{b1} = [\omega n_{a1}/(kn_{a0})][1 - z_{a1}z_{b0}/(z_{b1}z_{a0})]$, where z_j denotes the charge number, and therefore the friction effects would remain. This would require dealing with additional charge variation equations, however this is beyond the scope of the present work. Further in the text we assume singly charged species.

3. Isothermal case

The derivations are now repeated for isothermal quasi-neutral perturbations. In addition, the energy equation may be omitted in the equilibrium also assuming that the relaxation time for the equilibrium temperature is much longer than the period of wave oscillations. Keeping the full friction force $\vec{F}_f$ in both momentum equations, and within the same quasi-neutrality limit, yields a real dispersion equation $\omega^2 = k^2(T_{a0}+T_{b0})/(m_a+m_b)$. Hence, in the given limit the collisions (through friction) do not affect the isothermal ion acoustic mode.

This fact is usually overlooked in the literature where, in the limit $m_a \gg m_b$, a typical mistake is that the friction is kept for the lighter species only, in the incomplete simplified form $\vec{F}_{fb} = -m_b n_b \nu_{ba}\vec{v}_b$. For an electron-ion plasma this gives the incorrect result $\omega = \pm k(c_s^2 + v_{Ti}^2)^{1/2} - \nu_{ei}/2$. We stress again that the collisions appear in Eq. (3) only from the energy equations, yet they do not affect the IA mode.

Using the Poisson equation instead of quasi-neutrality, for isothermal perturbations we obtain coupled and damped IA and Langmuir waves

$$\omega^4 + i(\nu_{ab}+\nu_{ba})\omega^3 - \left[k^2\left(v_{Ta}^2 + v_{Tb}^2\right) + \omega_{pa}^2\right.$$

$$\left.+\omega_{pb}^2\right]\omega^2 - ik^2\left(\nu_{ab}v_{Tb}^2 + \nu_{ba}v_{Ta}^2\right)\omega$$

$$+k^4 v_{Ta}^2 v_{Tb}^2 + k^2\left(v_{Ta}^2\omega_{pb}^2 + v_{Tb}^2\omega_{pa}^2\right) = 0. \qquad (4)$$

In the collision-less limit the two modes (4) decouple by setting $T_a = T_b$, though in some situations this may have no sense because in this case the acoustic mode may lose its electrostatic nature, especially in pair-plasmas. For a pair (pair-ion, electron-positron) collision-less plasma the solutions are $\omega^2 = \omega_p^2 + k^2(v_{Ta}^2 + v_{Tb}^2)/2 \pm [\omega_p^4 + k^4(v_{Ta}^2 - v_{Tb}^2)/4]^{1/2}$.

In the low frequency limit $\omega \ll \omega_{p(a,b)}$ and for an e-i plasma, from Eq. (4) we have $\omega^2 = k^2 v_s^2 - i2\nu_{ei}\omega m_e r_{de}^2 k^2/m_i$, so that the IA mode is damped

$$\omega = \pm k v_s\left(1 - r_{de}^2 k^2\frac{\nu_{ie}^2 r_{de}^2}{v_s^2}\right)^{1/2} - i\nu_{ie}r_{de}^2k^2. \qquad (5)$$

Here, $v_s^2 = c_s^2 + v_{Ti}^2$ and we have used the momentum conservation $\nu_{ie} = m_e\nu_{ei}/m_i$. Note that the damping is k-dependent.

4. Evolving Background Plasma

From Eq. (1) it is seen that in a quasi-neutral homogeneous background, without flows/currents, *the background temperature is also evolving in time* as

$$\frac{\partial T_{(a,b)0}}{\partial t} = \pm 2\frac{m_b}{m_a}\nu_{ba}(T_{b0} - T_{a0}). \qquad (6)$$

Keeping the collision frequencies constant (the approximation discussed below), this gives the temperatures for the two species

$$T_{(a,b)0}(t) = \frac{1}{2}\left[\widehat{T}_{(a,b)0}(1+\exp(-4\nu_{ab}t))\right.$$

$$\left.+\widehat{T}_{(a,b)0}(1-\exp(-4\nu_{ab}t))\right].$$

Here, $\widehat{T}_{(a,b)0}$ are the starting values of the temperature for the two species in some moment which we set to be $t = 0$. It is seen that they evolve towards the common value $(\widehat{T}_{a0} + \widehat{T}_{b0})/2$.

On the other hand, solving Eq. (6) numerically, with time dependent collision frequencies (2) gives a

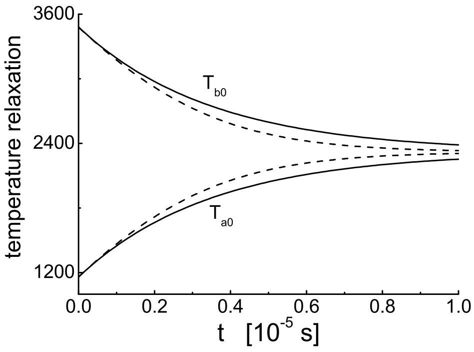

Fig. 1 Approximative (full lines), and exact relaxation with time-dependent collision frequencies (dashed lines) of the background plasma temperatures (6).

slightly faster relaxation for the two temperatures. To get a feeling on the relaxation time scale, this is presented in Fig. 1 by taking $n_0 = 10^{18}$ m^{-3} and $\widehat{T}_{a0} = 0.1$ eV, $\widehat{T}_{b0} = 3\widehat{T}_{a0}$.

Eq. (6) is to be used in the linearization of Eq. (1), which in the case $n_{a0} = n_{b0} = n_0$ yields:

$$\frac{\partial T_{a1}}{\partial t} + \frac{2}{3}T_{a0}\nabla\cdot\vec{v}_{a1} = +2\frac{m_b}{m_a}\nu_{ba}\left(T_{b1} - T_{a1}\right)$$
$$-2\nu_{ba}\frac{m_b}{m_a}\frac{n_{a1} - n_{b1}}{n_0}\left(T_{b0} - T_{a0}\right). \qquad (7)$$

The corresponding equation for the component b is

$$\frac{\partial T_{b1}}{\partial t} + \frac{2}{3}T_{b0}\nabla\cdot\vec{v}_{b1} = -2\frac{m_b}{m_a}\nu_{ba}(T_{b1} - T_{a1}). \quad (8)$$

Here, in the process of linearization yielding Eq. (8), the term $(3/2)n_{b1}\partial T_{b0}/\partial t$ on the left-hand side, cancels out with the term $-(m_b/m_a)\nu_{ba}(T_{b0} - T_{a0})n_{b1}$ on the right-hand side after using the background equation (6) for the species b.

Hence, both Eqs. (7) and (8) are obtained taking into account the evolution of the background. There appears an additional asymmetry between the two energy equations (apart from the opposite signs of the first term on the right-hand side), due to the last term in Eq. (7). This extra asymmetry is a consequence of the fact that the internal energy of the two species may also change due to the presence of the new ingredient in the system, i.e., the perturbed electric field $n_{a1} - n_{b1} = \varepsilon_0\nabla\cdot\vec{E}_1/e$ (in the presence of the necessary collisions of course). However, it vanishes if the quasi-neutrality condition is used on the right-hand side in Eq. (7). This may sometimes be permissible in higher order terms but not in general, for example assuming that the source/sink term in the energy equations gives only small imaginary corrections to the frequency.

However, regardless of the fact that the last term in Eq. (7) is used or not, the effects of the evolving background remain within Eq. (7) in both cases. Note also that the cancelation of the terms in the equation for the species b (which is due to evolving background as described above) remains intact regardless if the quasi-neutrality is used or not.

We stress that Eq. (3) is obtained also by using Eqs. (7, 8) in the quasi-neutral limit (implying that the last term in Eq. (7) is omitted). Hence, the IA mode appears unaffected by friction in the quasi-neutral limit even if the energy equations are used, and if the background is described correctly as evolving.

We now use the two energy equations (7, 8) *with the Poisson equation.* The dispersion equation becomes

$$\omega^6 + i\nu_{ba}\left(1+\frac{5m_b}{m_a}\right)\omega^5 - \omega^4\left[\frac{5}{3}k^2\left(v_{Ta}^2+v_{Tb}^2\right)\right.$$
$$\left.+\omega_{pa}^2 + \omega_{pb}^2 + 4\nu_{ba}^2\frac{m_b}{m_a}\left(1+\frac{m_b}{m_a}\right)\right]$$
$$+i\omega^3\nu_{ba}\left[k^2v_{Tb}^2\frac{2m_b^2}{m_a^2}k^2v_{Ta}^2\left(\frac{5}{3}+\frac{22m_b}{3m_a}\right)\right.$$
$$\left.-4\omega_{pa}^2\left(1+\frac{m_b}{m_a}\right) - 7k^2v_{Tb}^2\frac{m_b}{m_a}\right]$$
$$+\omega^2\left[\frac{25}{9}k^4v_{Ta}^2v_{Tb}^2 + \frac{5k^2}{3}\left(v_{Ta}^2\omega_{pb}^2 + v_{Tb}^2\omega_{pa}^2\right)\right.$$
$$\left.+4k^2\nu_{ba}^2\frac{m_b}{m_a}\left(v_{Ta}^2\left(\frac{2}{3}+\frac{m_b}{m_a}\right)+v_{Tb}^2\frac{m_b^2}{m_a^2}\left(\frac{8}{3}-\frac{m_b}{m_a}\right)\right)\right]$$
$$+i10\omega\nu_{ba}k^2\left[k^2v_{Tb}^2\frac{m_b}{m_a}\left(v_{Ta}^2 - \frac{v_{Tb}^2}{3}\frac{m_b}{m_a}\right)\right.$$
$$\left.+\frac{2}{3}\omega_{pa}^2\left(v_{Ta}^2 + v_{Tb}^2\frac{m_b}{m_a}\right)\right]$$
$$+4k^4\nu_{ba}^2\frac{m_b}{m_a}\left(v_{Ta}^2 - v_{Tb}^2\frac{m_b}{m_a}\right)^2 = 0. \qquad (9)$$

Solving for the IA mode yields approximately

$$\omega_{IA}^2 = \frac{5k^2(\omega_{pb}^2v_{Ta}^2 + \omega_{pa}^2v_{Tb}^2 + 5k^2v_{Ta}^2v_{Tb}^2/3)}{3\left[\omega_{pa}^2 + \omega_{pb}^2 + (5/3)k^2(v_{Ta}^2 + v_{Tb}^2)\right]}. \quad (10)$$

The corresponding growth rate is:

$$\gamma = \frac{1}{2\left[\omega_{pa}^2 + \omega_{pb}^2 + (5/3)k^2(v_{Ta}^2 + v_{Tb}^2) - 2\omega_{IA}^2\right]}$$
$$\times\left\{\nu_{ab}\left(\omega_{IA}^2 - (5/3)k^2v_{Tb}^2\right)\left(1 + (4/3)k^2v_{Ta}^2/\omega_{IA}^2\right)\right.$$
$$+\nu_{ba}\left(\omega_{IA}^2 - (5/3)k^2v_{Ta}^2\right)\left[1\right.$$
$$\left.+(4/3)\nu_{ab}k^2v_{Tb}^2/(\nu_{ba}\omega_{IA}^2)\right]$$
$$+2(1 - T_{a0}/T_{b0})\left(\nu_{ab}k^2c_s^2/\omega_{IA}^2\right)\left[\omega_{IA}^2\right.$$
$$-(5/3)k^2v_{Tb}^2 - \left(8\nu_{ab}\nu_{ba}/\omega_{IA}^2\right)\left(\omega_{IA}^2 - k^2v_{Ta}^2\right)$$

$$+(8\nu_{ab}^2/\omega_{IA}^2)\left(\omega_{IA}^2 - k^2 v_{Tb}^2\right)]\}. \qquad (11)$$

In principle, Eq. (11) reveals the possibility for a growing IA mode if the Poisson equation is used instead of the quasi-neutrality in a time-evolving background plasma. For example, this can be easily demonstrated in the limit of negligible terms originating from the last term in Eq. (7). This is permissible on condition $|(T_{a1} - T_{b1})/(n_{a1} - n_{b1})| \gg |T_{a0} - T_{b0}|/n_0$, or in an alternative form, $|(T_{a1} - T_{b1})/(T_{a0} - T_{b0})| \gg r_{db}^2 k^2 |q_b \phi_1|/T_{b0}$. In that limit, the numerical solution of Eq. (9) yields the growth-rate of the IA mode in an electron-ion plasma.

However, we stress that the system evolves in time. Thus, in order to have a reasonably fast growth of the perturbations, the following condition must be satisfied [cf. Eq. (6)] :

$$\gamma_r \equiv 2(m_b/m_a)\nu_{ba} \ll \gamma. \qquad (12)$$

Here, γ_r^{-1} determines the relaxation time for the background. Taking the electron-ion case like in Ref. [1] and the corresponding self-evident conditions $m_b \ll m_a$, $T_{a0} < T_{b0}$, $k^2 v_{Ta}^2 < \omega_{IA}^2 < \omega_{pa}^2 < \omega_{pb}^2$, from Eq. (11) to the leading order terms we obtain

$$\gamma - \gamma_r \simeq -\frac{\nu_{ba}}{2\left(\omega_{pb}^2 + 5k^2 v_{Tb}^2/3\right)}\left\{4\omega_{pa}^2 - \omega_{IA}^2\right.$$

$$\left. + \frac{2k^2 c_s^2}{\omega_{IA}^2}\left[k^2 v_{Ta}^2\left(\frac{5}{3} + \frac{8\nu_{ab}^2}{\omega_{IA}^2}\right) + 8\nu_{ab}^2\right]\right\}. \qquad (13)$$

Hence, because always $\omega_{pa} \geq \omega_{IA}$, here we have

$$\gamma < \gamma_r, \qquad (14)$$

i.e., the system relaxes on a time scale that is (much) shorter than the eventual growth time of the perturbations. Consequently the assumed instability actually can not develop. We note that this is in agreement with some experiments, e.g., in a Q-machine plasma where the instability has never been observed even by cooling the ions to near room temperature while keeping various temperatures for the electrons.

5. Summary

The long existing controversy dealing with the stability of the ion acoustic mode in plasmas in the state of partial thermodynamic equilibrium has been revisited. The results obtained here can be summarized as follows. i) The friction does not affect the IA mode in the limit of quasi-neutral perturbations. ii) Even using the non-evolving model equivalent to Ref. [1], there is no instability of the IA mode, contrary to claims from Ref. [1]. iii) When the background plasma is properly described as evolving in time, and as long as the quasi-neutrality is used, collisions do not produce a growth of the ion acoustic mode. iv) When the Poisson equation is used instead of quasi-neutrality, in principle there is a possibility for a positive growth-rate of the IA mode. It appears as a combined effect of the breakdown of the charge neutrality from one side (introduced by the Poisson equation), and the heat transfer (the compressibility and advection in energy equation) from the other side, all within the background of a time-evolving plasma.

However, as the equilibrium plasma evolves in time, with the relaxation time τ_r given in Eq. (6), the obtained growth time must be (much) shorter than the relaxation time. Yet, this shows to be impossible and we conclude that there is no instability in the electron-ion plasma with an initial temperature disparity if the plasma evolves freely.

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