Quadrature formulas on an arc of the unit circle

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`http://nalag.cs.kuleuven.be/papers/ade/NM65`

What you should *NOT* do and why

Quadrature on an arc $[a, b] = [e^{i\alpha}, e^{i\beta}] \subset \mathbb{T}$

Find $\int_a^b f(t)d\mu(\theta) = \int_\alpha^\beta F(\theta)d\mu(\theta)$, $t = e^{i\theta}$,
μ positive Borel measure on $[\alpha, \beta]$,

- Choose $\{\theta_{k,n}\}_{k=1}^n \subset [\alpha, \beta]$
  Lagrange Interpolation $L_{n-1}(\theta) = \sum_{k=1}^n F(\theta_{k,n})\ell_{k,n}(\theta)$
- Integrate $\int_a^b f(t)d\mu \approx \sum_{k=1}^n F(\theta_{k,n})w_{k,n}$, $w_{k,n} = \int_a^b \ell_{k,n}(\theta)d\mu$
- exact $\forall f(e^{i\theta}) = F(\theta) \in \mathbb{P}_{n-1} \rightarrow$ Gauss quadrature
  - Choose $\{\theta_{k,n}\} = \mathcal{Z}(\phi_n)$, $\phi_n =$ OPRL for $[\alpha, \beta]$
  - $t_{k,n} \in (a, b)$ and $w_{k,n} > 0$, $k = 1, \ldots, n$
  - quadrature exact on $\mathbb{P}_{2n-1}$ polynomials in θ
- $f(e^{i\theta})$ not well approximated by polynomials in θ!!
  For example $f(t) = t^n = \cos(n\theta) + i\sin(n\theta)$ is oscillating
  $\cos(n\theta) = \frac{1}{2}(t^n + t^{-n})$, $\sin(n\theta) = \frac{1}{2i}(t^n - t^{-n})$
  $\Rightarrow$ need Laurent polynomials not polynomials

This is what you should do but how?

Quadrature on an arc $[a, b]$ the unit circle $\mathbb{T}$

Find $\int_a^b f(t)d\mu(t)$, μ positive Borel measure on $[a, b]$,

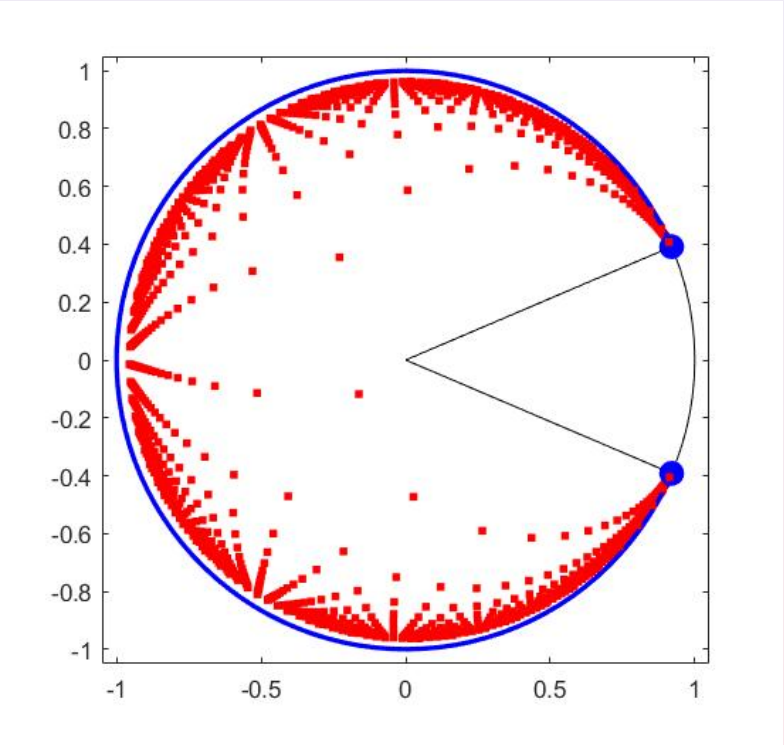
- Choose $\{t_{k,n}\}_{k=1}^n \subset \mathbb{T}$ & Interpolate $L_{n-1}(t) = \sum_{k=1}^n f(t_{k,n})\ell_{k,n}(t)$
- Integrate $\int_a^b f(t)d\mu \approx \sum_{k=1}^n f(t_{k,n})w_{k,n}, \quad w_{k,n} = \int_a^b \ell_{k,n}(t)d\mu > 0$
- exact $\forall f \in \mathcal{L}_{n-1} \rightarrow$ Szegő quadrature
  - Choose $\{t_{k,n}\} = \mathcal{Z}(P_n)$, POPUC P_n (see next slide)
  - then If $t_{k,n} \in (a, b)$ $k = 1, \ldots, n$ and
  - quadrature exact in $\mathcal{L}_{2n-1}$ of dimension $2n$
  - $\{t_{k,n}, w_{k,n}\}$ can be computed from eigenvalue decomposition of unitary Hessenberg instead of tridiagonal Jacobi

Para-orthogonal polynomials

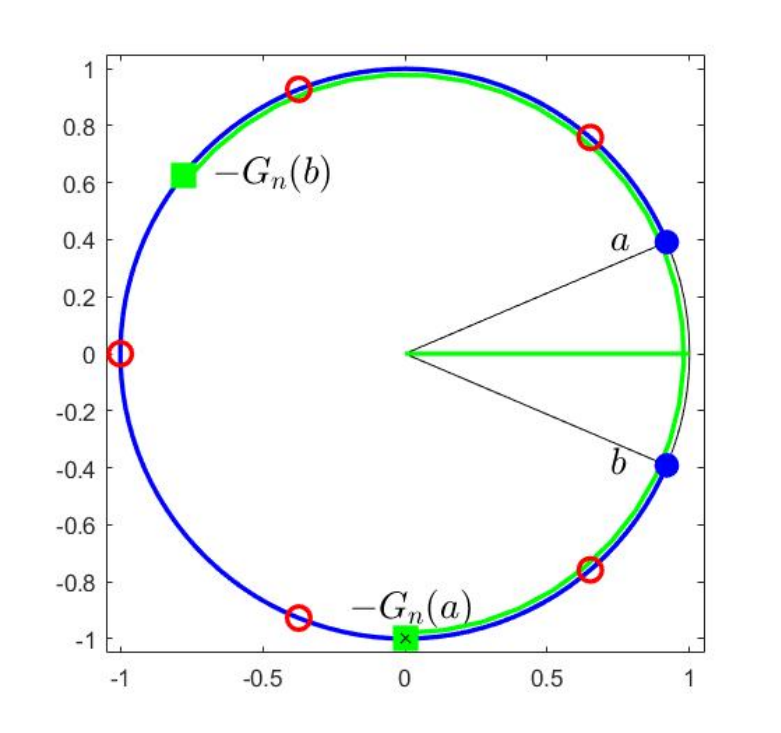
Theorem

- *monic OPUC:*
  $\rho_k(z) = z\rho_{k-1}(z) + \delta_k \rho_{k-1}^*(z), \quad \delta_k \in \mathbb{D}, \quad p_k^*(z) = z^k \overline{p_k(1/\overline{z})}$
  monic POPUC:
  $P_n(z) = P_n(z;\tau_n) = z\rho_{n-1}(z) + \tau_n \rho_{n-1}^*(z) = \nu_n^{-1}[\rho_n(z) + \hat{\tau}_n \rho_n^*(z)],$
  $\tau_n, \hat{\tau}_n \in \mathbb{T}$, $\nu_n = 1 + \overline{\delta_n}\hat{\tau}_n$
- $\varphi_\delta(z) = \dfrac{z+\delta}{1+\overline{\delta}z}$ *then* $\tau_n = \varphi_{\delta_n}(\hat{\tau}_n) \quad \Leftrightarrow \quad \hat{\tau}_n = \varphi_{-\delta_n}(\tau_n)$
- P_n *is* τ_n*-invariant:* $P_n(z) = \tau_n P_n^*(z)$. *(to get* $\mathcal{Z}(P_n) \subset \mathbb{T}$*)*
- *orthogonality* $P_n \perp \operatorname{span}\{z, \ldots, z^{n-1}, z^n - \hat{\tau}_n\}$
  (to get max degree exactness)
- $\mathcal{Z}(P_n)$ *are simple and at least* $n-1$ *are in* $\mathring{\Gamma}$, $\Gamma = \operatorname{supp}(\mu) =$ *arc on* $\mathbb{T}$.

OPUC and POPUC



$\mathcal{Z}(\rho_n), \quad n = 1, \ldots, 50$



$\mathcal{Z}(P_n), \quad n = 5 \quad \rightarrow$ mp4 mp4

- a and b ▬ points to τ_n
- ■ $G_n(a)$ and $G_n(b)$, $G_n = \rho_n / \rho_n^*$

$\mathcal{Z}(P_n) \subset \Gamma \Leftrightarrow \hat{\tau}_n \in -G_n(\Gamma)$

Szegő-Radau quadrature

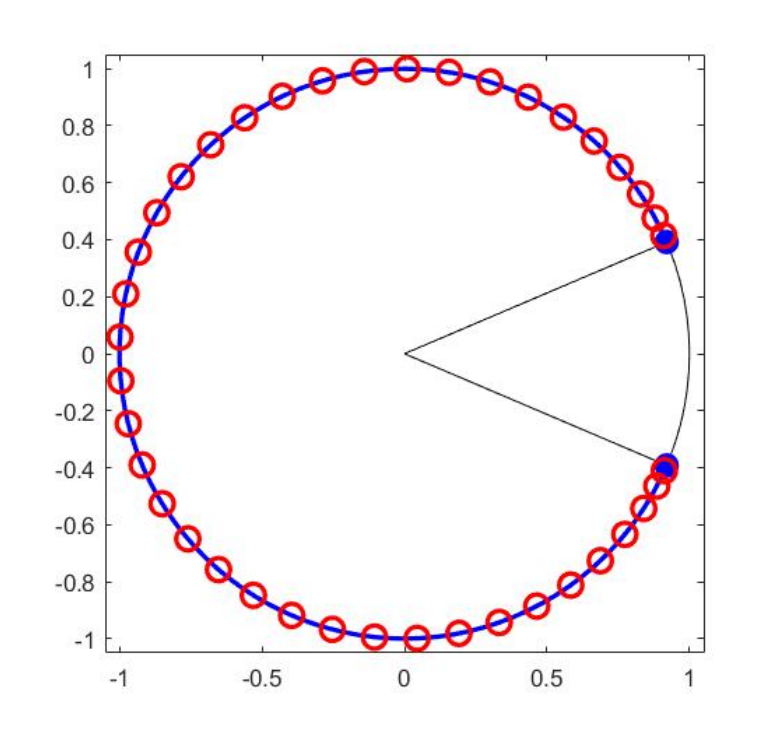
Choose τ_n or $\hat{\tau}_n$ to make a or b a node of the q.f. when $\operatorname{supp}(\mu) = [a, b]$.

$$P_n(z) = \nu_n^{-1}[\rho_n(z) + \hat{\tau}_n \rho_n^*(z)]_{z=c} = 0 \Leftrightarrow \hat{\tau}_n = -\frac{\rho_n(c)}{\rho_n^*(c)} = -G_n(c).$$

Theorem

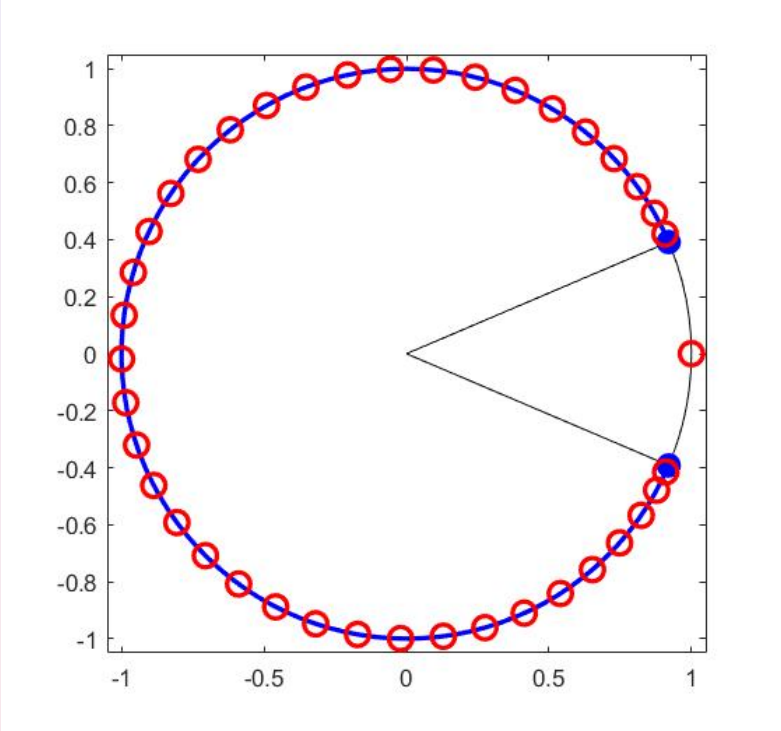
Let μ be a measure with $\operatorname{supp}(\mu) = [a, b] \subset \mathbb{T}$. Define the monic POPUC $P_n(z) = \nu_n^{-1}[\rho_n(z) + \hat{\tau}_n \rho_n^*(z)]$, $\hat{\tau}_n = -G_n(c)$, $c \in \{a, b\}$.
Then a Szegő-Radau quadrature exists as $I_n(f) := \sum_{n=1}^{n} \lambda_{k,n} f(z_{k,n})$ with $\lambda_{k,n} > 0$ and $z_{k,n} \in \mathcal{Z}(P_n)$, are simple and in Γ.
The quadrature is exact for all $f \in \operatorname{span}\{z^{-(n-1)}, \ldots, z^{n-1}, z^n - \frac{\hat{\tau}_n}{\overline{\tau}_n} z^{-n}\}$.
Moreover the sets $\mathcal{Z}(P_n) \setminus \{c\}$ and $\mathcal{Z}(P_{n+1}) \setminus \{c\}$ interlace (in Γ).
$\lim_{n\to\infty} I_n(f) = I(f)$ for all Riemann-Stieltjes integrable f.

POPUC $\mathcal{Z}(P_n(\cdot;\tau_n))$ depends on τ_n



$\mathcal{Z}(P_n), \quad n = 1, \ldots, 40$

$\tau_n = \frac{\delta_n}{|\delta_n|}$ mp4 mp4



$\mathcal{Z}(P_n), \quad n = 1, \ldots, 40$

$\tau_n = -\frac{\delta_n}{|\delta_n|}$ mp4 mp4

Theorem

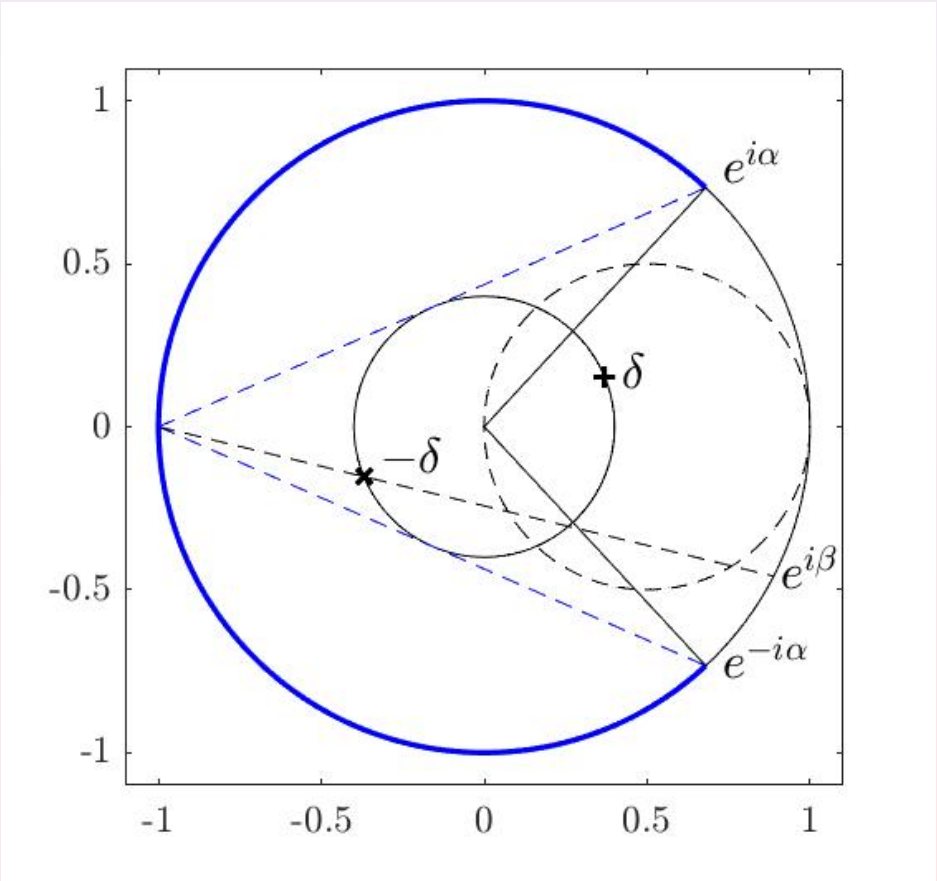
$-\hat{\tau}_n \in G_n(\Gamma) \Rightarrow \mathcal{Z}(P_n) \subset \Gamma = \mathrm{supp}(\mu)$

Interval vs arc

interval $[a, b] \subset \mathbb{R}$	arc $[a, b] \subset \mathbb{T}$
polynomial interpolation	Laurent polynomial interpolation
OPRL ϕ_n, $\mathcal{Z}(\phi_n) \subset [a, b]$	OPUC ρ_n, $\mathcal{Z}(\rho_n) \subset \mathbb{D}$
	POPUC depend on $\tau_n \in \mathbb{T}$
	$P_n(z) = z\rho_{n-1}(z) + \tau_n \rho^*_{n-1}(z)$,
Gauss nodes $\mathcal{Z}(\phi_n)$	Szegő nodes $\mathcal{Z}(P_n)$??
3Term Recurrence	Szegő recurrence
EVP Jacobi matrix	EVP unitary Hessenberg (CMV)
what if coefficients not known	what if coefficients not known 1
$\mathcal{Z}(\phi_n) \subset [a, b]$ is natural	$\mathcal{Z}(P_n) \subset \mathbb{T}$ depends on τ_n
	what is the natural choice for τ_n
	such that $\mathcal{Z}(P_n) \subset [a, b]$? 2
	Two problems 1 & 2

Geronimus class $\mathcal{G}$

Standard example of measure supported on an arc:
$\rho_k(z) = z\rho_{k-1}(z) + \delta\rho^*_{k-1}(z)$, i.e., $\delta_k = \delta \in \mathbb{D}$ for all $k \in \mathbb{N}$.



$$\mu' = \omega(\theta) = \frac{\sqrt{\sin(\frac{\theta+\alpha}{2})\sin(\frac{\theta-\alpha}{2})}}{\sin(\frac{\theta-\beta}{2})}$$

$$\sin(\tfrac{\alpha}{2}) = |\delta|, \quad \beta = 2\arg(1-\delta)$$

Mass point in $e^{i\beta} \in \Gamma^c$

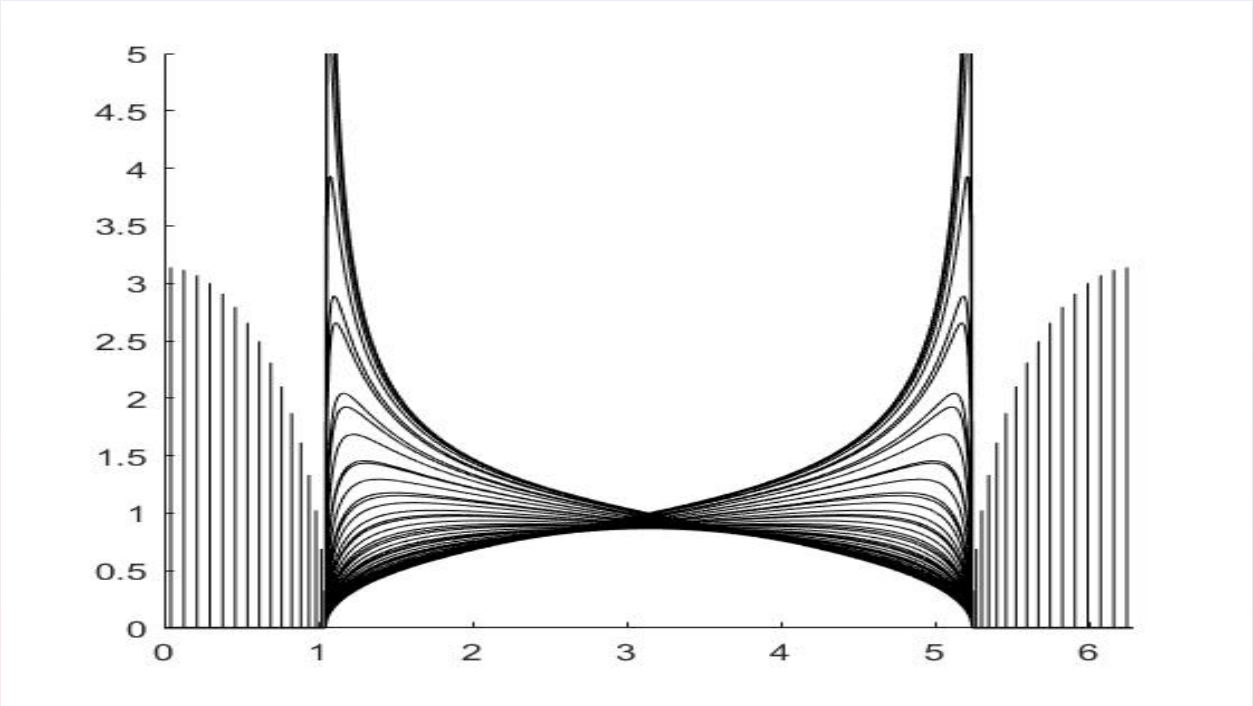
if $|1-2\delta| > 1$

We assume $|1-2\delta| \leq 1$.

$$\Gamma = \mathrm{supp}(\mu) = [a, \overline{a}]$$

Geronimus class $\mathcal{G}$

All possible Geronimus weights for $|\delta| = 0.5$ (defines support)



$\omega(\theta), \quad \delta = 0.5e^{i\gamma}, \gamma \in [0, 2\pi]$ mp4 mp4

Question 2: Solved for $\mathcal{G}$![1]

$\rightarrow$ Take $\tau_n = d := \delta/|\delta|$, then $\mathcal{Z}(P_n(z; d)) \subset \Gamma$

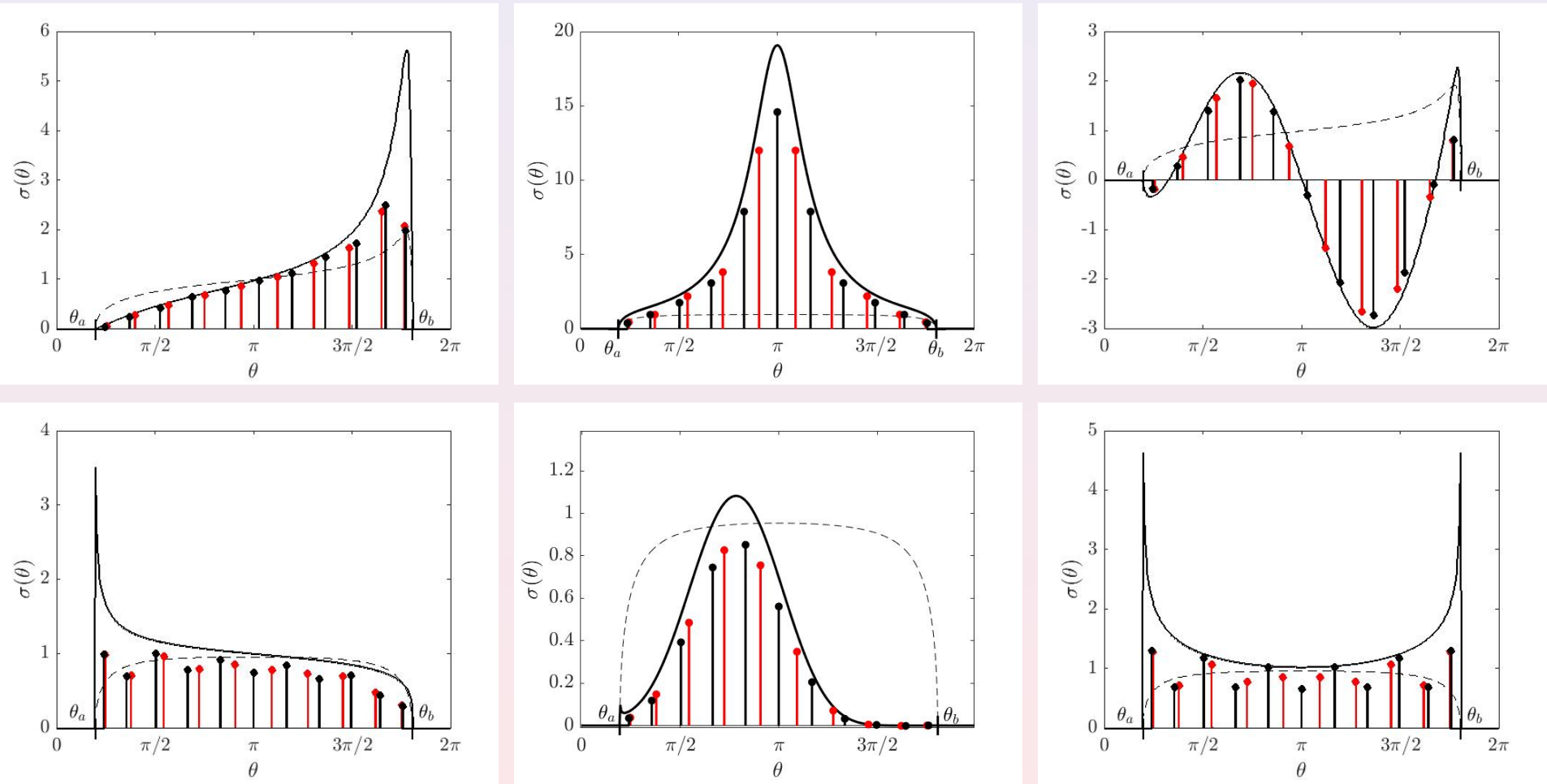
[1] *Positive quadrature formulas on an arc of the unit circle*, Numer. Alg. 100(2) 769-802,2025. (Marc's proceedings)

Question 1: interpolating quadrature

Question 1: Verblunsky coefficients not known for $d\mu = \sigma(\theta)d\theta$,

- rotate to make the arc symmetric
- take $\mathcal{Z}(P_n(z;d))$ POPUC for Geronimus weight ω on that arc,
- construct a Laurent interpolating polynomial in $\mathcal{L}_{n-1}$
  $\mathcal{L}_{2k} = \mathrm{span}\{z^{-k}, \ldots, z^k\}, \quad \mathcal{L}_{2k-1}(\lambda_k) = \mathcal{L}_{2k-2} \cup \mathrm{span}\{z^k + \frac{\lambda_k}{z^k}\}$
- integrate to obtain a quadrature $J_n^{\sigma,\omega}[f]$
- the weights for $J_n^{\sigma,\omega}[f]$ are modifications of $I_n^{\omega}[f]$
- quadrature $J_n^{\sigma,\omega}[f]$ exact $\forall f \in \mathcal{L}_{n-1}(-d)$
- $\lim_{n\to\infty} |J_n^{\sigma,\omega}[f] - I^{\sigma}[f]| \leq \frac{C_n}{\lfloor \frac{n-1}{2} \rfloor^{k+\gamma}}, \quad \forall f \in \mathcal{L}(k,\gamma)$ if $\frac{|\sigma|^2}{\omega} \in L_1(\Gamma)$.
- ω need not be the Geronimus weight
- If $\omega = \sigma$ then $J_n^{\sigma,\sigma}[f] = I_n^{\sigma}[f]$ is ordinary Szegő quadrature.

Question 1: interpolating quadrature



weights σ (full) and ω (dashed); $(z_{k,n}, w_{k,n})$, $n = 10$ (black), $n = 11$ (red)

Interpolatory quadrature formulas on an arc of the unit circle, Numer. Alg. (these proceedings)

Question 2 $\rightarrow$ 'natural' value of τ_n

- What is so special about OPRL ϕ_n and POPUC $B_n(z) := P_n(z; d)$ for Geronimus such that all their zeros are in the support?
- $\|\phi_n\|_2 = \min_{p_n \in \mathbb{R}_n^1} \|p_n\|_{2,\mu}$,
  $\mathbb{R}_n^1 =$ monic real polynomials of degree n
- $\|B_n\|_2 = \min_{p_n \in \mathbb{P}_n^1} \|p_n\|_{2,\mu}$, (if $\delta_n \neq 0$)
  $\mathbb{P}_n^1 =$ monic invariant polynomials of degree n

Hypothesis

In general: if $d_n = \frac{\delta_n}{|\delta_n|} \Rightarrow \mathcal{Z}(P_n(z; d_n)) \subset \mathrm{supp}(\mu)$

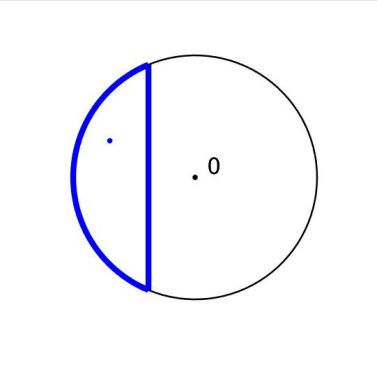
For Geronimus: Yes! $d = \frac{\delta}{|\delta|}$

Question 2 Minimum norm POPUC

Hypothesis

if $d_n = \frac{\delta_n}{|\delta_n|} \Rightarrow \mathcal{Z}(P_n(z; d_n)) \subset \mathrm{supp}(\mu)$

Partial Proof (If $|\Gamma| \leq \pi$)



- $\delta_n \neq 0$ by contradiction. If $\delta_n = 0$ then $\rho_n(z) = z\rho_{n-1}(z)$ and $\langle t\rho_{n-1}, \rho_{n-1}\rangle = 0$ but $\int_\Gamma t \underbrace{|\rho_{n-1}|^2 d\mu}_{>0} \in \mathrm{Co}(\Gamma)$ and $0 \notin \mathrm{Co}(\Gamma)$

- $B_n(z) = \prod_{j=1}^n (z - t_{j,n})$, $\quad Q_{n-1}(z) = \frac{B_n(z)}{z - t_{k,n}} = z^{n-1} + zP_{n-3}(z) + \overline{t}_{k,n} d_n$

$$\begin{aligned} \langle B_n, Q_{n-1}\rangle &= -t_{k,n}\overline{d}_n \langle B_n, 1\rangle = -t_{k,n}(1 - |\delta_n|)\|\rho_{n-1}\|^2 = -t_{k,n}A, \quad A > 0 \\ &= \int_\Gamma (z - t_{k,n}) \underbrace{|Q_{n-1}|^2 d\mu}_{d\tilde{\mu}>0} = \underbrace{\int_\Gamma z d\tilde{\mu}}_{\in\, \mathrm{Co}(\Gamma)} - t_{k,n} \underbrace{\int_\Gamma d\tilde{\mu}}_{B>0} \end{aligned}$$

Collect $t_{k,n}$ terms: $t_{k,n} \underbrace{(B - A)}_{>0} = \underbrace{\int_\Gamma z d\tilde{\mu}}_{\in \mathrm{Co}(\Gamma)} \Rightarrow t_{k,n} \in \Gamma.$ $\square$

Szegő-Lobatto quadrature

- fix a and b as nodes of the quadrature
- all the other nodes should be in (a, b).
- $P_n(z) = (z-a)(z-b)\hat{P}_{n-2}(z), \quad P_n(0) \in \mathbb{T}$
- $\hat{P}_{n-2} \in \mathbb{P}^1_{n-2}$ (invariant) and $\hat{P}_{n-2}(0) = \overline{ab} P_n(0)$
- maximize space of exactness $\Leftrightarrow$ maximize orthogonality
  $P_n \perp_\mu \mathcal{L} = \operatorname{span}\{z^2, \ldots, z^{n-2}, z^{n-1} + \lambda z\}, \ \lambda \in \mathbb{T}$ (*)
- 2 Questions:
  - is $\mathcal{Z}(\hat{P}_{n-2}) \subset (a, b)$?
  - Is $\hat{P}_{n-2}$ of minimal norm? Which is the proper norm?

Szegő-Lobatto quadrature

- maximize space of exactness:
  $P_n \perp_\mu \mathcal{L} = \operatorname{span}\{z^2, \ldots, z^{n-2}, z^{n-1} + \lambda z\},\ \lambda \in \mathbb{T}$ (*)
- 2 Observations: set $c = \frac{\sqrt{ab}}{4}$ then
  - $c\frac{(t-a)(t-b)}{t} = \sin\frac{\theta-\theta_a}{2}\sin\frac{\theta_b-\theta}{2} = |s(\theta)|^2 \geq 0, \quad t = e^{i\theta} \in \Gamma,$
  - $c\langle (t-a)(t-b)g(t), f(t)\rangle_\mu = \langle \underbrace{c\frac{(t-a)(t-b)}{t}}_{|s(\theta)|^2} g(t), \frac{f(t)}{t}\rangle_\mu = \langle g(t), \frac{f(t)}{t}\rangle_{\hat{\mu}},$
    $d\hat{\mu} = |s(\theta)|^2 d\mu$
- $\langle (t-a)(t-b)g(t), f(t)\rangle_\mu = 0 \ \Leftrightarrow\ \langle g(t), \frac{f(t)}{t}\rangle_{\hat{\mu}},$ (**)
- (*) & (**) $\hat{P}_{n-2} \perp_{\hat{\mu}} \operatorname{span}\{z, \ldots, z^{n-3}, z^{n-3} + \lambda\},$
  $\Leftrightarrow \hat{P}_{n-2}$ is a POPUC for $\hat{\mu}$
  $\Leftrightarrow \hat{P}_{n-2}(z) = z\hat{\rho}_{n-3}(z) + \hat{\tau}_{n-2}\hat{\rho}^*_{n-3}(z), \quad \hat{\tau}_{n-2} \in \mathbb{T}$

Szegő-Lobatto quadrature (recap)

- $P_n(z) = (z-a)(z-b)\hat{P}_{n-2}(z), \quad P_n(0) \in \mathbb{T}$
- $\hat{P}_{n-2} \in \mathbb{P}^1_{n-2}$ (invariant) and $\hat{P}_{n-2}(0) = \overline{ab} P_n(0)$
- $\hat{P}_{n-2}(z) = z\hat{\rho}_{n-3}(z) + \hat{\tau}_{n-2}\hat{\rho}^*_{n-3}(z), \quad \hat{\tau}_{n-2} \in \mathbb{T}$
- $\hat{P}_{n-2}$ has minimal $\hat{\mu}$-norm if $\hat{\tau}_{n-2} = \hat{d}_{n-2} = \frac{\hat{\rho}_{n-2}(0)}{|\hat{\rho}_{n-2}(0)|}$
- If $|\Gamma| \leq \pi$ then $\mathcal{Z}(\hat{P}_{n-2}) \subset (a,b)$.
- Szegő-Lobatto q.f. exact in a space of maximal dimension $2n-3$, i.e., $\forall f \in \operatorname{span}\{z^{-(n-2)}, \ldots, z^{n-2}, z^{n-1} - \lambda_n z^{-(n-1)}\}, \quad \lambda_n = ab\hat{d}^2_{n-2}$

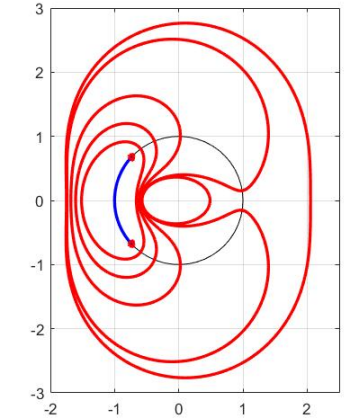
Convergence

Theorem (Szegő/Radau/Lobatto)

- Let $I_n[f]$ be an n-point quadrature formula for $I[f] = \int_\Gamma f(t)d\mu$
- "minimal norm" (Q)POPUC P_n and $\boxed{|\Gamma| \leq \pi}$
  Then $\mathcal{Z}(P_n) \subset \Gamma$) and

$$\limsup_{n\to\infty} |I_n[f] - I[f]|^{1/n} < 1/s$$

- s is the smallest r such that f is analytic in $\gamma(s)$
- Conformal map from $G : \mathbb{C}\backslash\Gamma \to \mathbb{C}\backslash\mathbb{D}$, $G(\infty) = \infty$,
  Set $H(z) = \overline{G(\frac{1}{\overline{z}})}G(z)$ then $\Gamma = \{z : |H(z)| = 1\}$,
  Define $\gamma(r) = \{z : |H(z)| \leq r\}, \quad r \in [1, \infty)$
- If f entire then $\lim_{n\to\infty} |I_n[f] - I[f]|^{1/n} = 0$.











$65 - \epsilon$









$$65 - \epsilon$$

$$\text{Relative error } \delta = \frac{5}{65 \times 12} \times 100 \approx 0.64\%$$