ERROR ANALYSIS IN PHYSICAL EXPERIMENTS USING MAPLE

ANTOON VAN DEN ABEELE, GUY DURINCK

Laboratorium voor Lichttechnologie, Katholieke Hogeschool Sint-Lieven, Associatie KULeuven, Gebroeders Desmetstraat 1, B-9000 Gent, Belgium
E-mail: toon.vandenabeele@kahosl.be, guy.durinck@kahosl.be

Error analysis of experimental results is not popular with engineering students and their teachers. Because of the often laborious calculations it is simply omitted in many laboratory sessions. The use of an advanced mathematical software package like Maple can make error analysis a lot less daunting. Error propagation can be easily calculated by linearization of the functions involved, the programming that is required is elementary. By using the Maple packages 'Tolerances' or 'ScientificErrorAnalysis' even this elementary programming can be avoided. Teachers and students no longer have an excuse for omitting a thorough error analysis in laboratory sessions. In this paper we report on the use of Maple as an error analysis tool by first year engineering students in physics laboratory sessions.

Keywords: Error analysis, Maple, Physics laboratory, Laboratory session

1   Introduction

First year students in science and engineering are usually not aware that experimental results have a limited accuracy. Being critical of experimental results is something that has to be learned. For this reason a basic training in error analysis is a very important part of the first year physics laboratory sessions. However, when error analysis is performed in the classical way, using rules of thumb and tables with results and errors, it becomes an extremely unpopular part of the laboratory sessions because of the laborious and uninspiring calculations.

At our institution the software tool Maple(1) is an integral part of the first year mathematics course so students participating in the physics laboratory sessions are already familiar with it. It is a logical next step to bring this software tool in the physics laboratory.

2   Method of error analysis

2.1   Basic expressions

When we have a number of measurements x_i of independent variables with errors Δx_i an error estimate of $y = f(x_1, x_2, .. x_n)$ can be found by using the linearization(2) of $f(x_1, x_2, .. x_n)$ and the inequality $|x_1 + x_2 + ... + x_n| \leq |x_1| + |x_2| + ... + |x_2|$. The error estimate is

$$\Delta y = \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} \right| \Delta x_i \quad [1]$$

It is assumed that the Δx_i are small, i.e. $\Delta x_i \ll x_i$.

In situations where standard deviations are used it is found from statistics(3) that the standard deviation of y is

$$\sigma_y = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 \sigma_{x_i}^{\ 2}} \quad [2]$$

2.2 Error analysis using tables

When mathematical software is not available it becomes impractical to systematically apply expression [1] to every case. Therefore expression [1] is applied to a number of standard functions (sum of two variables, product of two variables, etc…) which leads to a number of rules of thumb. When $y = f(x_1, x_2, ..x)$ is a more complicated function it is broken down in pieces to which the rules of thumb can be applied. Respectively for a sum and a product of two variables it is found that $\Delta(x_1 \pm x_2) = \Delta x_1 + \Delta x_2$ and $\dfrac{\Delta(x_1.x_2)}{x_1.x_2} = \dfrac{\Delta x_1}{x_1} + \dfrac{\Delta x_2}{x_2}$. During the calculation the errors are rounded to two significant digits while the final error is rounded to one significant digit. Errors are always rounded to a larger number. The final result $y = f(x_1, x_2, ..x)$ is rounded to the same digit as the final error.

This method of error analysis has a few drawbacks. It is often tedious and therefore unpopular with students and teachers. It is not uncommon that for this reason in more advanced engineering laboratory sessions error analysis is simply omitted. There is a risk that the critical attitude toward the accuracy of measured quantities that the students – hopefully- achieved in the first year physics laboratory is lost later in their education. Also, in some cases this method leads to incorrect results because, when the function $y = f(x_1, x_2, ..x)$ is broken down in pieces, it is very easy for students to incorrectly treat one of these pieces as an independent variable. We illustrate this by calculating the total resistance of two resistors in parallel: $R_1 = (250 \pm 5)\,\Omega$ and $R_1 = (500 \pm 10)\,\Omega$. The total resistance $R = (\dfrac{1}{R_1} + \dfrac{1}{R_2})^{-1}$ can be written as $R = \dfrac{R_1 R_2}{R_1 + R_2}$. The error analysis is shown in the next table:

	x	Δx		$\Delta x / x$
R_1	250 →	5	→	0,020↓
R_2	500 →	10	→	0,020↓
$R_1 + R_2$	750 →	15	→	0.020
$R_1.R_2$	125 000			0.040
$R = \dfrac{R_1 R_2}{R_1 + R_2}$	166.7 →	**10.00**		←0.060

However, when expression [1] is rigorously applied one finds $R = (167 \pm 4)\,\Omega$.

In the analysis above $R_1 R_2$ is implicitly treated as an independent variable which leads to an overestimate of the final error.

In the next table the analysis is carried out correctly:

	x	Δx		Δx / x
R_1	250 →	5	→	0,020
R_2	500 →	10	→	0,020
$1/R_1$	0.00400 →	0.000080		←0.020
$1/R_2$	0.00200 →	0.000040		←0.020
$1/R$	0.00600 →	0.000120	→	0.02
R	166.7 →	**3.4**		←0.02

Thus $R = (167 \pm 4)\ \Omega$.

2.3   Error analysis using Maple

Because students at our institution are already familiar with Maple from the calculus course and a number of computers is available in the physic laboratory it is a logical step to use Maple as a tool for error analysis. In Maple several packages are available for error analysis but in order to avoid confusion we introduce the options one at the time over a period of several months. In the laboratory sessions early in the academic year the students have to write a small program to calculate errors using expression [1]. In this way they are made aware that they are applying mathematics and not just punching buttons on a computer keyboard. Later they are introduced to the Maple package 'Tolerances' that is not based on expression [1] but calculates the maximum and minimum possible values of $y = f(x_1, x_2, ..x_n)$ using the errors Δx_i and takes the average of these values to be the final value of y. When expression [1] applies (i.e. when $\Delta x_i \ll x_i$) expression [1] and 'Tolerances' usually yield the same result. It has to be noted however that when $y = f(x_1, x_2, ..x)$ is complicated it is more straightforward to use expression [1]. Students are given the choice to use the package or keep programming expression [1] to do the error analysis. In the introductory physics laboratory propagation of standard deviation, expression [2], is not used. However, if it is required an equivalent approach can be used: first make students program expression [2] to make them aware of the statistics, and later introduce them to the package 'ScientificErrorAnalysis' which applies expression [2] behind the scenes.

In Appendix A the several methods of using Maple to do an error analysis are demonstrated in a number of simple examples.

3   Student response

In order to get an idea of how first year students respond to the introduction of error analysis using Maple we divided the student population in three groups:
- The first group was taught error analysis the traditional way, using rules of thumb and tables.
- The second group was taught error analysis using Maple only.
- The third group was first taught the rules of thumb and table method and was later on shown how error analysis can be done using Maple. They were given the choice whether to use Maple or keep using the table method.

The first group strongly disliked the calculations and when it was not made compulsory they performed no error analysis. At first the second group had some difficulties because they were not used to using Maple in this way, but after some time error analysis became a natural part

of a physics laboratory session for them. The third group was at first reluctant to use Maple but once they had tried it a couple of times all students started to use Maple. Like the second group performing an error analysis became just a natural part of the laboratory session.

4   Conclusion

Error analysis which traditionally is a laborious and often uninspiring part of physics laboratory sessions can be made a lot more attractive to students and teachers by using advanced mathematical software like Maple. By applying the real error propagation formulas, which is unpractical when one has to manually calculate, and not a set of rules of thumb erroneous results can be avoided. If the mathematical software is also used to support the teaching of mathematics the students are encouraged to look over the boundaries of individual courses.

Teachers of more advanced engineering laboratory sessions that come after the first year now no longer have an excuse to omit error analysis which in our opinion is an essential part of good engineering or science practice.

Appendix A: Examples

1. Applying expression [1]

$x = 100{,}5$ m, $y = 20{,}3$ m, $z = 41{,}0$ m.

Calculate: $v = \dfrac{\sqrt{x^2 - y^2}}{2z}$.

symbolic evaluation

```
> restart:
> v:=sqrt(x^2-y^2)/(2*z);
```

$$v := \frac{\sqrt{x^2 - y^2}}{2\,z}$$

```
> dv := abs(diff(v, x))*dx+abs(diff(v, y))*dy+abs(diff(v,
z))*dz;
```

$$dv := \frac{1}{2}\left|\frac{x}{\sqrt{x^2 - y^2}\,z}\right| dx + \frac{1}{2}\left|\frac{y}{\sqrt{x^2 - y^2}\,z}\right| dy + \frac{1}{2}\left|\frac{\sqrt{x^2 - y^2}}{z^2}\right| dz$$

Numerical evaluation

```
> subs(x=100.5,y=20.3,z=41.0,v);
```
$$1.20030$$

➢ `subs(x=100.5,y=20.3,z=41,dx=0.1,dy=0.1,dz=0.1,dv);ev`
   `alf(%);`
➢
$$0.05000\,|\,0.02491\,|+0.05000\,|\,0.00503\,|+0.05000\,|\,0.05855\,|$$
$$0.00442$$

Conclusion: $v = (1.200 \pm 0.005)$

2. Using 'Tolerances'

Calculate the total resistance of two parallel resistors.

```
> restart: with(Tolerances):
> R1:=500 &+-5;R2:=250 &+-5;
> R:=(1/R1+1/R2)^(-1);
```

$$R1 := 500.000+-(10.000)$$
$$R2 := 250.000 +-(5.000)$$
$$R := 166.663 +-(3.333)$$

3. Applying expression [2]

Calculate the total resistance of two parallel resistors.

$R_1 = 250\ \Omega\ \ \sigma_{R_1} = 5\ \Omega\ \ R_2 = 125\ \Omega\ \ \sigma_{R_1} = 3\ \Omega$

```
> restart;
> R:=1/(1/R1+1/R2);
> dR:=sqrt((diff(R,R1)*dR1)^2+(diff(R,R2)*dR2)^2);
> subs(R1=250.0,R2=125,R);
> subs(R1=250.0,R2=125,dR1=5,dR2=3,dR);
```

$$R := \cfrac{1}{\cfrac{1}{R1} + \cfrac{1}{R2}}$$

$$dR := \sqrt{\frac{dR1^2}{\left(\dfrac{1}{R1} + \dfrac{1}{R2}\right)^4 R1^4} + \frac{dR2^2}{\left(\dfrac{1}{R1} + \dfrac{1}{R2}\right)^4 R2^4}}$$

$$83.33300$$
$$1.44440$$

$R = 83.3\ \Omega\ \ \sigma_R = 1.5\ \Omega$

4. Using 'ScientificErrorAnalysis'

```
> restart;with(ScientificErrorAnalysis):
> R:=1/(1/R1+1/R2);
> R1:=Quantity(250.0,5);
> R2:=Quantity(125,3);
> combine( R, errors );
```

$$R := \cfrac{1}{\cfrac{1}{R1} + \cfrac{1}{R2}}$$

$$R1 := \text{Quantity}(250.00000, 5)$$

$$R2 := \text{Quantity}(125, 3)$$

$$\text{Quantity}(83.33300, 1.44440)$$

$R = 83.3\ \Omega \quad \sigma_R = 1.5\ \Omega$

References

1. http://www.maplesoft.com/

2. William G. McCallum, Deborah Hughes-Hallett, Andrew M. Gleason, Multivariate Calculus, John Wiley & Sons Inc., 1998

3. John P. Bentley, Principles of Measurement Systems, Longman Scientific & Technical, 1995