

1  
2  
3  
4  
5  
6  
7  
8  
9 A parameterized  $L^2$  metric between fuzzy numbers  
10 and its parameter interpretation <sup>1</sup>  
11  
12  
13

14 Beatriz Sinova<sup>a</sup>, María Ángeles Gil <sup>\*a</sup>, María Teresa López<sup>a</sup>,  
15 and Stefan Van Aelst<sup>b</sup>  
16

17 <sup>a</sup>*Departamento de Estadística e I.O. y D.M.,*  
18 *Universidad de Oviedo, 33071 Oviedo, Spain*

19 <sup>b</sup>*Department of applied Mathematics, Computer Science and Statistics,*  
20 *Ghent University, 9000 Gent, Belgium*  
21  
22  
23  
24

---

25 **Abstract**  
26

27 When handling fuzzy number data, it is common practice to make use of a  
28 metric to quantify distances between fuzzy numbers. Several metrics have  
29 been suggested in the literature for this purpose. When statistically analyzing  
30 fuzzy number-valued data,  $L^2$  metrics become especially useful. This paper  
31 introduces a new family of generalized  $L^2$  metrics which take into account  
32 key features of the involved fuzzy numbers, namely, a measure of central  
33 location and two measures associated with the shape of the fuzzy numbers  
34 are used. A crucial property related to these three measures is that necessary  
35 and sufficient conditions can be established for them to characterize fuzzy  
36 numbers. Furthermore, the family of generalized  $L^2$  metrics depends on  
37 one parameter. A discussion is provided regarding the interpretation of this  
38 parameter which can guide selection of its value in practice.  
39  
40  
41  
42  
43

---

44  
45  
46  
47  
48  
49  
50  
51  
52  
53  
54  
55  
56 <sup>\*</sup>corresponding author: magil@uniovi.es, Fax: (+34) 985103354.  
57

1  
2  
3  
4  
5  
6  
7  
8  
9 A parameterized  $L^2$  metric between fuzzy numbers  
10 and its parameter interpretation <sup>1</sup>  
11  
12  
13

14 Beatriz Sinova<sup>a</sup>, María Ángeles Gil <sup>\*a</sup>, María Teresa López<sup>a</sup>,  
15 and Stefan Van Aelst<sup>b</sup>  
16

17 <sup>a</sup>*Departamento de Estadística e I.O. y D.M.,*  
18 *Universidad de Oviedo, 33071 Oviedo, Spain*

19 <sup>b</sup>*Department of applied Mathematics, Computer Science and Statistics,*  
20 *Ghent University, 9000 Gent, Belgium*  
21  
22  
23  
24

---

25 **Abstract**

26  
27 When handling fuzzy number data, it is common practice to make use of a  
28 metric to quantify distances between fuzzy numbers. Several metrics have  
29 been suggested in the literature for this purpose. When statistically analyzing  
30 fuzzy number-valued data,  $L^2$  metrics become especially useful. This paper  
31 introduces a new family of generalized  $L^2$  metrics which take into account  
32 key features of the involved fuzzy numbers, namely, a measure of central  
33 location and two measures associated with the shape of the fuzzy numbers  
34 are used. A crucial property related to these three measures is that necessary  
35 and sufficient conditions can be established for them to characterize fuzzy  
36 numbers. Furthermore, the family of generalized  $L^2$  metrics depends on  
37 one parameter. A discussion is provided regarding the interpretation of this  
38 parameter which can guide selection of its value in practice.  
39  
40  
41  
42

43 *Keywords:* fuzzy number,  $L^2$ -metric, wabl/ldev/rdev representation of  
44 fuzzy numbers,  $L^2$  wabl/ldev/rdev metric, weighting parameter  
45  
46

---

47  
48 **1. Introduction**  
49

50 A metric is a numerical description of how far apart objects of a given  
51 space are. It is a powerful tool in many fields from Physics to Statistics, and  
52 the nature of objects a metric can be applied to is very diverse.  
53  
54  
55

---

56 <sup>\*</sup>corresponding author: magil@uniovi.es, Fax: (+34) 985103354.  
57  
58

1  
2  
3  
4  
5  
6  
7  
8  
9  
10  
11  
12  
13  
14  
15  
16  
17  
18  
19  
20  
21  
22  
23  
24  
25  
26  
27  
28  
29  
30  
31  
32  
33  
34  
35  
36  
37  
38  
39  
40  
41  
42  
43  
44  
45  
46  
47  
48  
49  
50  
51  
52  
53  
54  
55  
56  
57  
58  
59  
60  
61  
62  
63  
64  
65

In the last decades the notion of fuzzy number, also referred to as fuzzy interval in the literature, has become appealing to model real life objects that cannot be quantified precisely such as linguistic terms, experimental data associated with human ratings, perceptions, judgments, etc.

To analyze fuzzy number data, a metric between fuzzy numbers can be very helpful. Such a metric allows us to formalize problems involving fuzzy numbers and facilitates related developments and methods to solve them. For example, metrics have been used

- to classify fuzzy data (see, for instance, Coppi *et al.* [1], Ferraro and Giordani [2] and Guillaume *et al.* [3]),
- to obtain some limit and probabilistic results for random fuzzy numbers (see, for instance, Colubi *et al.* [4], Molchanov [5], Terán [6, 7], Quang and Thuan [8], Aletti and Bongiorno [9]),
- in optimization problems (see, for instance, Abbasbandy and Asady [10], Abbasbandy and Amirfakhrian [11], Prochelvi *et al.* [12], Báez-Sánchez *et al.* [13], Bana and Coroianu [14], Bera *et al.* [17], Coroianu [15], Coroianu *et al.* [16])
- and especially in performing many statistical analyses (see, for instance, Näther [18, 19], Körner and Näther [20], Körner [21], García *et al.* [22], Montenegro *et al.* [23, 24], Gil *et al.* [25], Coppi *et al.* [26], González-Rodríguez *et al.* [29, 30, 31], Ferraro *et al.* [27], Ferraro and Giordani [28], Ramos-Guajardo and Lubiano [32], Sinova *et al.* [39]).

In the literature on fuzzy numbers and more general fuzzy sets, several metrics have been suggested (see, for instance, Puri and Ralescu [33], Klement *et al.* [34]). One of the best known metrics is the  $L^2$  distance based on the support function studied by Diamond and Kloeden [35]. In the case of fuzzy numbers one of the strengths of this  $L^2$  distance is that it uses the inf/sup representation of fuzzy numbers. This is a representation for which necessary and sufficient conditions have been stated to characterize fuzzy numbers.

The metric of Diamond and Kloeden has been extended with the aim to take into account not only the extreme-points of the level sets of the fuzzy numbers but the complete level sets. One of these extensions has been given by Bertoluzza *et al.* [36] (see also Trutschnig *et al.* [37] for a definition for general fuzzy values and Grzegorzewski [38] for a more general metric), which can be characterized by using the mid/spread (i.e., center/radius) representation of fuzzy numbers. This family of metrics is quite versatile and has a rather intuitive interpretation. The family can be treated as a parameterized

1  
2  
3  
4  
5  
6  
7  
8  
9 distance weighting level-wise the deviation in location with respect to the  
10 deviation in vagueness of the level sets of the involved fuzzy numbers. Along  
11 this century, a statistical methodology has been developed on the basis of  
12 Bertoluzza *et al.*'s metrics and its Trutschnig *et al.*'s extension. However,  
13 a drawback of these metrics is that for the mid/spread representation there  
14 are no necessary and sufficient conditions that allow to characterize fuzzy  
15 numbers.  
16  
17

18 Recently (Sinova *et al.* [40]) introduced a new parametric family of  $L^1$   
19 metrics with the goal of defining a robust centrality measure for fuzzy data.  
20 These metrics are based on a new representation of fuzzy numbers that coin-  
21 cides with the mid/spread representation in case of symmetric fuzzy numbers.  
22 An advantage of this new representation is that it has been possible to state  
23 necessary and sufficient conditions to characterize fuzzy numbers. However,  
24 no discussion has been carried out regarding the interpretation of the in-  
25 volved parameter, which makes it difficult to make a suitable choice for this  
26 parameter in practice.  
27  
28

29 This paper introduces a new parametric family of  $L^2$  metrics between  
30 fuzzy numbers that is based on a modification of the fuzzy number repre-  
31 sentation in [40]. This new parametric family of metrics aims to allow an  
32 intuitive interpretation of the involved parameter which facilitates the selec-  
33 tion of its value in practice. Moreover, the modified representation of fuzzy  
34 numbers preserves the important property that necessary and sufficient condi-  
35 tions to characterize fuzzy numbers can be stated. This new representation  
36 and hence the associated  $L^2$  metrics take into account the (squared) deviation  
37 in central location and level-wise the (squared) deviations in shape between  
38 fuzzy numbers.  
39  
40  
41

42 The rest of this paper is organized as follows. Section 2 recalls some basic  
43 concepts on fuzzy numbers and introduces the proposed representation of  
44 fuzzy numbers. Necessary and sufficient conditions for this representation  
45 to characterize fuzzy numbers are derived as well. In Section 3, the new  
46 parametric family of  $L^2$  metrics is presented and their metric and topological  
47 properties are examined. Section 4 presents an equivalent expression for the  
48 new metric which facilitates the interpretation of the role of the parameter in  
49 this family of  $L^2$  metrics. Finally, Section 5 provides a discussion on potential  
50 applications of this new family of metrics.  
51  
52  
53  
54  
55  
56  
57  
58

1  
2  
3  
4  
5  
6  
7  
8  
9 **2. The wabl/ldev/rdev representation of fuzzy numbers**

10  
11 A **bounded fuzzy number** (also called fuzzy interval) is a fuzzy set of  
12  $\mathbb{R}$ , that is, a mapping  $\tilde{U} : \mathbb{R} \rightarrow [0, 1]$  such that for all  $\alpha \in [0, 1]$  the  $\alpha$ -level of  
13  $\tilde{U}$ , defined as  
14

$$15 \tilde{U}_\alpha = \begin{cases} \{x \in \mathbb{R} : \tilde{U}(x) \geq \alpha\} & \text{if } \alpha \in (0, 1] \\ \text{cl}\{x \in \mathbb{R} : \tilde{U}(x) > \alpha\} & \text{if } \alpha = 0, \end{cases}$$

16  
17  
18  
19  
20  
21 is a nonempty compact interval. The value  $\tilde{U}(x)$  is intuitively interpreted as  
22 the ‘degree of compatibility of  $x$  with  $\tilde{U}$ ’, or ‘degree of membership of  $x$  to  
23  $\tilde{U}$ ’, or ‘degree of truth of the assertion “ $x$  is  $\tilde{U}$ ” ’.  
24

25 Equivalently, a bounded fuzzy number can be defined as a normalized  
26 upper semicontinuous element of  $[0, 1]^{\mathbb{R}}$  with bounded 0-level (i.e., the closure  
27 of the support set of the fuzzy set). The class of the bounded fuzzy numbers  
28 is denoted by  $\mathcal{F}_c^*(\mathbb{R})$ .  
29

30 A representative value of the central location of a bounded fuzzy number  
31 has been introduced by Yager [41], and later extended by de Campos and  
32 González [42] as the *.5-average index*, and by Nasibov [43] as the *weighted*  
33 *averaging based on levels* (see also Nasibov *et al.* [44]). For any  $\tilde{U} \in \mathcal{F}_c^*(\mathbb{R})$ ,  
34 the .5-average index is defined as the real number in the interior set  $\text{int}(\tilde{U}_0)$   
35 such that  
36

$$37 \text{wabl}^\varphi(\tilde{U}) = \int_{[0,1]} \text{mid } \tilde{U}_\alpha d\varphi(\alpha),$$

38 where  $\text{mid } \tilde{U}_\alpha = (\inf \tilde{U}_\alpha + \sup \tilde{U}_\alpha)/2$ .  $\varphi$  is a weighting measure on the measur-  
39 able space  $([0, 1], \mathcal{B}_{[0,1]})$  which can be formalized by means of an absolutely  
40 continuous probability measure with positive mass function on  $(0, 1)$ . It  
41 should be pointed out that no stochastic meaning is actually associated with  
42  $\varphi$ , but it allows us to weight the ‘degrees of compatibility’ in the  $\alpha$ -levels. The  
43  $\text{wabl}^\varphi$  measure of central location is often used as a *defuzzification* function  
44 to rank fuzzy numbers.  
45

46 The  $\text{wabl}^\varphi$  measure satisfies several valuable properties. A relevant prop-  
47 erty for this paper is related to the usual arithmetic with fuzzy numbers.  
48 Namely,  
49

$$50 \text{wabl}^\varphi(\tilde{U} + \tilde{V}) = \text{wabl}^\varphi(\tilde{U}) + \text{wabl}^\varphi(\tilde{V}), \quad \text{wabl}^\varphi(\gamma \cdot \tilde{U}) = \gamma \cdot \text{wabl}^\varphi(\tilde{U}),$$

for  $\tilde{U}, \tilde{V} \in \mathcal{F}_c^*(\mathbb{R})$ ,  $\gamma \in \mathbb{R}$  where the fuzzy arithmetic is based on Zadeh's extension principle [45], which extends level-wise the usual interval arithmetic, i.e. for all  $\alpha \in [0, 1]$

$$(\tilde{U} + \tilde{V})_\alpha = \{x + y : x \in \tilde{U}_\alpha, y \in \tilde{V}_\alpha\}, \quad (\gamma \cdot \tilde{U})_\alpha = \{\gamma \cdot x : x \in \tilde{U}_\alpha\}.$$

Moreover, these properties can be joined and generalized to the so-called Aumann-type expected value of random elements taking on fuzzy number values (see López-Díaz and Gil [46]), for which the  $\text{wabl}^\varphi$  of the Aumann-type expected value equals the expected value of the  $\text{wabl}^\varphi$  of the random element.

We use the central location measure  $\text{wabl}^\varphi$  as one of the three components of our new representation of fuzzy numbers. The other two components are level-wise representative values of the shape of a bounded fuzzy number relative to this value of central location. They can be formalized as the two functions:

$$\text{ldev}_\varphi^\varphi : [0, 1] \rightarrow \mathbb{R}, \quad \alpha \mapsto \text{ldev}_\varphi^\varphi(\alpha) = \text{wabl}^\varphi(\tilde{U}) - \inf \tilde{U}_\alpha,$$

$$\text{rdev}_\varphi^\varphi : [0, 1] \rightarrow \mathbb{R}, \quad \alpha \mapsto \text{rdev}_\varphi^\varphi(\alpha) = \sup \tilde{U}_\alpha - \text{wabl}^\varphi(\tilde{U}).$$

On the basis of these three components we obtain the following representation of fuzzy numbers.

**Definition 2.1.** *Given an absolutely continuous probability measure  $\varphi$  on the measurable space  $([0, 1], \mathcal{B}_{[0,1]})$  with positive mass function on  $(0, 1)$ , the  $\varphi$ -**wabl/ldev/rdev representation** of  $\tilde{U} \in \mathcal{F}_c^*(\mathbb{R})$  is given by the real number  $\text{wabl}^\varphi(\tilde{U})$  and the two functions  $\text{ldev}_\varphi^\varphi$  and  $\text{rdev}_\varphi^\varphi$ . It follows that for each  $\alpha \in [0, 1]$*

$$\tilde{U}_\alpha = \left[ \text{wabl}^\varphi(\tilde{U}) - \text{ldev}_\varphi^\varphi(\alpha), \text{wabl}^\varphi(\tilde{U}) + \text{rdev}_\varphi^\varphi(\alpha) \right].$$

For symmetric fuzzy numbers this representation coincides with the mid/spread representation (whatever  $\varphi$  may be). Consequently, it also coincides with the mid/spread representation in the case that fuzzy numbers reduce to compact intervals.

The following result establishes sufficient and necessary conditions to characterize each fuzzy number by the  $\varphi$ -wabl/ldev/rdev representation.

**Proposition 2.1.** *Given a fuzzy number  $\tilde{U} \in \mathcal{F}_c^*(\mathbb{R})$  there exist a value  $m \in \mathbb{R}$  and two functions  $l^* : [0, 1] \rightarrow \mathbb{R}$ ,  $r^* : [0, 1] \rightarrow \mathbb{R}$  satisfying that*

- i)  $l^*$  and  $r^*$  are*
- left-continuous functions at any  $\alpha \in (0, 1]$ ,*
  - right-continuous at 0,*
  - and non-increasing on  $[0, 1]$ ,*

*with*

- ii)  $-l^*(1) \leq r^*(1)$ ,*

*and such that for all  $\alpha \in [0, 1]$ ,*

$$\tilde{U}_\alpha = [m - l^*(\alpha), m + r^*(\alpha)].$$

*Conversely, let  $m \in \mathbb{R}$  and let  $l^* : [0, 1] \rightarrow \mathbb{R}$ ,  $r^* : [0, 1] \rightarrow \mathbb{R}$  be functions satisfying conditions i) and ii). Then there exists a unique  $\tilde{U} \in \mathcal{F}_c^*(\mathbb{R})$  such that for all  $\alpha \in [0, 1]$*

$$\tilde{U}_\alpha = [m - l^*(\alpha), m + r^*(\alpha)].$$

*Furthermore, if there is an absolutely continuous probability measure  $\varphi$  on  $([0, 1], \mathcal{B}_{[0,1]})$  with positive mass function on  $(0, 1)$  and such that*

$$\text{iii) } \int_{[0,1]} l^*(\alpha) d\varphi(\alpha) = \int_{[0,1]} r^*(\alpha) d\varphi(\alpha),$$

*then,  $(m, l^*, r^*)$  is the  $\varphi$ -wabl/ldev/rdev representation of  $\tilde{U}$ .*

*Proof.* First, take  $\tilde{U} \in \mathcal{F}_c^*(\mathbb{R})$ . Because of the properties of the infimum and supremum of the level intervals as functions of  $\alpha$  (see, for example, Ming [47], and Negoita and Ralescu [48]), the functions  $\text{ldev}_{\tilde{U}}^\varphi$  and  $\text{rdev}_{\tilde{U}}^\varphi$  should be left-continuous functions of  $\alpha$  on  $(0, 1]$  and right-continuous at 0. They should also be non-increasing functions of  $\alpha$  on  $[0, 1]$ . Moreover,

$$\text{rdev}_{\tilde{U}}^\varphi(1) \geq -\text{ldev}_{\tilde{U}}^\varphi(1),$$

$$\int_{[0,1]} \text{ldev}_{\tilde{U}}^\varphi(\alpha) d\varphi(\alpha) = \int_{[0,1]} \text{rdev}_{\tilde{U}}^\varphi(\alpha) d\varphi(\alpha) \geq 0.$$

Since  $\text{wabl}^\varphi(\tilde{U}) \in \mathbb{R}$  and for all  $\alpha \in [0, 1]$  it holds that

$$\tilde{U}_\alpha = \left[ \text{wabl}^\varphi(\tilde{U}) - \text{ldev}_{\tilde{U}}^\varphi(\alpha), \text{wabl}^\varphi(\tilde{U}) + \text{rdev}_{\tilde{U}}^\varphi(\alpha) \right],$$

it follows that the real number  $\text{wabl}^\varphi(\tilde{U})$  and the two functions  $\text{ldev}_{\tilde{U}}^\varphi$  and  $\text{rdev}_{\tilde{U}}^\varphi$  indeed satisfy conditions (i)-(iii).

On the other hand, given  $m \in \mathbb{R}$ , if  $l^* : [0, 1] \rightarrow \mathbb{R}$ ,  $r^* : [0, 1] \rightarrow \mathbb{R}$  are mappings satisfying Conditions *i) – ii)*, then, the functions  $l : [0, 1] \rightarrow \mathbb{R}$ ,  $r : [0, 1] \rightarrow \mathbb{R}$  given by

$$l(\alpha) = m - l^*(\alpha), \quad r(\alpha) = m + r^*(\alpha)$$

satisfy that

- $l$  is a left-continuous nondecreasing function of  $\alpha \in (0, 1]$  and right-continuous at  $\alpha = 0$ ,
- $r$  is a left-continuous nonincreasing function of  $\alpha \in (0, 1]$  and right-continuous at  $\alpha = 0$ ,
- $l(1) = m - l^*(1) \leq m + r^*(1) = r(1)$ ,

then, it is well-known that there exists a unique  $\tilde{U} \in \mathcal{F}_c^*(\mathbb{R})$  such that

$$\tilde{U}_\alpha = [l(\alpha), r(\alpha)] = [m - l^*(\alpha), m + r^*(\alpha)]$$

for  $\alpha \in [0, 1]$  (see, for instance, Ming [47]).

Furthermore, if there exists an absolutely continuous probability measure  $\varphi$  on  $([0, 1], \mathcal{B}_{[0,1]})$ , with positive mass function on  $(0, 1)$  such that

$$\int_{[0,1]} l^*(\alpha) d\varphi(\alpha) = \int_{[0,1]} r^*(\alpha) d\varphi(\alpha), \text{ then}$$

$$\begin{aligned} \int_{[0,1]} l(\alpha) d\varphi(\alpha) &= m - \int_{[0,1]} l^*(\alpha) d\varphi(\alpha) \\ &= m - \int_{[0,1]} r^*(\alpha) d\varphi(\alpha) = 2m - \int_{[0,1]} r(\alpha) d\varphi(\alpha) \end{aligned}$$

. Hence,

$$m = \int_{[0,1]} \frac{l(\alpha) + r(\alpha)}{2} d\varphi(\alpha) = \int_{[0,1]} \text{mid } \tilde{U}_\alpha d\varphi(\alpha) = \text{wabl}^\varphi(\tilde{U}).$$

Moreover, for all  $\alpha \in [0, 1]$

$$\text{ldev}_{\tilde{U}}^\varphi(\alpha) = \text{wabl}^\varphi(\tilde{U}) - \inf \tilde{U}_\alpha = m - l(\alpha) = l^*(\alpha),$$

$$\text{rdev}_{\tilde{U}}^\varphi(\alpha) = \sup \tilde{U}_\alpha - \text{wabl}^\varphi(\tilde{U}) = r(\alpha) - m = r^*(\alpha). \quad \square$$

To illustrate this result, we now consider the following example.



**Example 2.1.** Let  $m = 8$ ,  $l^*(\alpha) = 5 - 3\alpha^2$  and  $r^*(\alpha) = 7 - 6\alpha$  for  $\alpha \in [0, 1]$ . Since functions  $l^*$  and  $r^*$  satisfy Conditions *i)* and *ii)* in Proposition 2.1, there exists a unique bounded fuzzy number  $\tilde{U}$  such that

$$\tilde{U}_\alpha = [m - l^*(\alpha), m + r^*(\alpha)] = [3 + 3\alpha^2, 15 - 6\alpha]$$

for every  $\alpha \in [0, 1]$ . This fuzzy number  $\tilde{U}$  is shown in Figure 2.1 and is given by

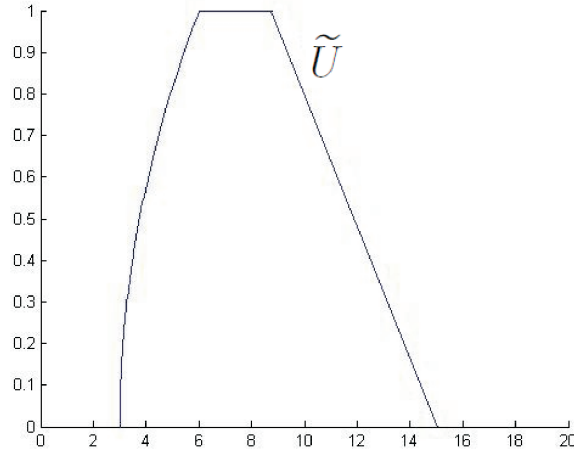


Figure 1: Fuzzy number for the given  $\ell$ -wabl/ldev/rdev representation in Example 2.1

$$\tilde{U}(x) = \begin{cases} \sqrt{(x-3)/3} & \text{if } x \in [3, 6) \\ 1 & \text{if } x \in [6, 9) \\ (15-x)/6 & \text{if } x \in [9, 15] \\ 0 & \text{otherwise} \end{cases}$$

On the other hand, the equality

$$\int_{[0,1]} (5 - 3\alpha^2) d\alpha = 4 = \int_{[0,1]} (7 - 6\alpha) d\alpha$$

implies that for  $\varphi \equiv \ell \equiv$  Lebesgue measure in  $[0, 1]$  we have that

$$\text{wabl}^\ell(\tilde{U}) = 8, \text{ ldev}_{\tilde{U}}^\ell(\alpha) = 5 - 3\alpha^2, \text{ rdev}_{\tilde{U}}^\ell(\alpha) = 7 - 6\alpha.$$

The representation above is not the only wabl/ldev/rdev representation for this fuzzy number. In fact, for this example it is possible to find such a

representation for every choice of  $\varphi$ . For instance, if one chooses  $\varphi \equiv \beta_{5,1}$ , i.e. the probability measure associated with the Beta(5, 1) distribution, then

$$\text{wabl}^{\beta_{5,1}}(\tilde{U}) = 53/7, \text{ldev}^{\beta_{5,1}}(\alpha) = (32 - 21\alpha^2)/7, \text{rdev}^{\beta_{5,1}}(\alpha) = (52 - 42\alpha)/7.$$

Note that for the Beta(5, 1) distribution it holds that the larger the  $\alpha$ -level of a set, i.e. the greater the degree of compatibility with  $\tilde{U}$ , the larger its weight is in the corresponding wabl/ldev/rdev representation.

**Remark 2.1.** It should be emphasized that on the basis of the Weighted Mean Value Theorem for integrals, whenever  $\text{mid } \tilde{U}_\alpha$  is a continuous function of  $\alpha$  in  $[0, 1]$ , then for each  $\varphi$  there exists at least one  $\beta_\varphi \in [0, 1]$  such that

$$\text{mid } \tilde{U}_{\beta_\varphi} = \int_{[0,1]} \text{mid } \tilde{U}_\alpha d\varphi(\alpha) = \text{wabl}^\varphi(\tilde{U}).$$

Hence, for each wabl/ldev/rdev representation of a fuzzy number in Definition 2.1 there is a corresponding representation as introduced in Sinova *et al.* [40].

### 3. The wabl/ldev/rdev-based $L^2$ metrics

$L^2$  metrics between fuzzy numbers have been shown to be very suitable in the development of statistical methodology for experimental fuzzy data. The recent review of Blanco-Fernández *et al.* [49] gathers most of these statistical methods. González-Rodríguez *et al.* [31] provides a detailed explanation of this approach when using the Bertoluzza *et al.*'s  $L^2$  metric. Moreover, they reveal a clear and interesting connection with the statistical analysis of functional data through the so-called support function of fuzzy sets.

Bertoluzza *et al.*'s  $L^2$  metric can be expressed in terms of the mid/spread representation for fuzzy numbers (see, for instance, Gil *et al.* [50]). In the particular case of interval-valued data one can easily establish necessary and sufficient conditions for this representation to characterize a compact interval (given simply by the non-negativeness of the spread). However, for the general case of fuzzy numbers this is not possible anymore. It is known that the spread function should be a nonnegative, left-continuous at  $(0, 1]$ , right-continuous at 0, and a nonincreasing function of  $\alpha$ , but nothing can be imposed on the mid function to characterize a fuzzy number. This concern

can be a setback for certain developments, especially for those related to certain minimization problems.

We now introduce a family of metrics based on the  $\varphi$ -wabl/ldev/rdev representation with the purpose of establishing  $L^2$  metrics between fuzzy numbers with the following properties.

- The metrics extend the mid/spread-based metric for interval-valued data;
- The metrics take into account the central location and shape of the involved fuzzy numbers;
- The metrics are based on a characterizing representation of fuzzy numbers.

The new metrics are also partially inspired by previous  $L^2$  distances (see Yang and Ko [51], Ferraro *et al.* [52]).

**Definition 3.1.** *Given an absolutely continuous probability measure  $\varphi$  on  $([0, 1], \mathcal{B}_{[0,1]})$  with positive mass function on  $(0, 1)$ , and a parameter  $\theta \in (0, 1]$ , the  $\varphi$ -wabl/ldev/rdev-based  $L^2$  metric is the mapping  $\mathfrak{D}_\theta^\varphi : \mathcal{F}_c^*(\mathbb{R}) \times \mathcal{F}_c^*(\mathbb{R}) \rightarrow [0, +\infty)$  such that for  $\tilde{U}, \tilde{V} \in \mathcal{F}_c^*(\mathbb{R})$*

$$\mathfrak{D}_\theta^\varphi(\tilde{U}, \tilde{V}) = \left[ \left( \text{wabl}^\varphi(\tilde{U}) - \text{wabl}^\varphi(\tilde{V}) \right)^2 + \theta \int_{[0,1]} \left( \frac{1}{2} \left[ \text{ldev}_{\tilde{U}}^\varphi(\alpha) - \text{ldev}_{\tilde{V}}^\varphi(\alpha) \right]^2 + \frac{1}{2} \left[ \text{rdev}_{\tilde{U}}^\varphi(\alpha) - \text{rdev}_{\tilde{V}}^\varphi(\alpha) \right]^2 \right) d\varphi(\alpha) \right]^{1/2}.$$

Note that  $\theta$  and  $\varphi$  do not have a stochastic meaning but have a weighting interpretation in the definition of this metric. It is clear that the role of the parameter  $\theta$  in the distance is to weigh the influence of the ‘deviation in shape’ between the fuzzy numbers (quantified through ldev and rdev) with respect to the influence of their ‘deviation in central location’ (quantified through wabl). A more detailed interpretation of the values for  $\theta$  that can be chosen follows in Section 4. Moreover, the choice of  $\varphi$  allows us to weight the influence of each  $\alpha$ -level (i.e., the different degrees of ‘compatibility’).

1  
2  
3  
4  
5  
6  
7  
8  
9 *3.1. Metric and topological properties of  $\mathfrak{D}_\theta^\varphi$*

10  
11 To examine some of the main properties of  $\mathfrak{D}_\theta^\varphi$ , it is convenient to point  
12 out that given the absolutely continuous probability measure  $\varphi$  on  $([0, 1], \mathcal{B}_{[0,1]})$   
13 with positive mass function on  $(0, 1)$ , an arbitrary fuzzy number  $\tilde{U} \in \mathcal{F}_c^*(\mathbb{R})$   
14 can be characterized by its  $v^\varphi$ -vector given by  
15  
16

$$17 \quad v_{\tilde{U}}^\varphi : [0, 1] \rightarrow \mathbb{R}^3, \quad \alpha \mapsto v_{\tilde{U}}^\varphi(\alpha) = (\text{wabl}^\varphi(\tilde{U}), \text{ldev}_{\tilde{U}}^\varphi(\alpha), \text{rdev}_{\tilde{U}}^\varphi(\alpha)).$$

18  
19  
20 The  $v^\varphi$ -vector function can be stated as the function

$$21 \quad v^\varphi : \mathcal{F}_c^*(\mathbb{R}) \rightarrow \mathbb{H}^2, \quad v^\varphi(\tilde{U}) \mapsto v_{\tilde{U}}^\varphi,$$

22  
23 where  $\mathbb{H}^2 = \{L^2 \text{ type 3-dimensional vector-valued functions defined on } [0, 1]\}$ .

24  
25 Moreover, this function allows us to induce a parametric family of  $L^2$   
26 metrics on  $\mathcal{F}_c^*(\mathbb{R})$  from a parametric family of  $L^2$  norms on  $\mathbb{R}^3$ . More pre-  
27 cisely, for an arbitrarily fixed parameter value  $\theta \in (0, 1]$  we can consider the  
28  $L^2$  norm on  $\mathbb{R}^3$  which is given by  
29  
30

$$31 \quad |\mathbf{x} - \mathbf{y}|_\theta = \sqrt{[x_1 - y_1]^2 + \frac{\theta}{2} \cdot [x_2 - y_2]^2 + \frac{\theta}{2} \cdot [x_3 - y_3]^2},$$

32  
33 for any  $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ . Actually, the  $v^\varphi$ -vector en-  
34 ables to isometrically embed the space  $\mathcal{F}_c(\mathbb{R})$  in a convex cone of  $\mathbb{H}^2$ . The  
35 corresponding  $L^2$  norm in this subset of  $\mathbb{H}^2$  would be:  
36  
37  
38

$$39 \quad \|f - g\|_\theta^\varphi = \sqrt{\int_{[0,1]} (|f(\alpha) - g(\alpha)|_\theta)^2 d\varphi(\alpha)},$$

40  
41 where  $f$  and  $g$  represent two elements in the cone. Using this relation and  
42 ideas similar to those in Trutschnig *et al.* [37], one can conclude that  
43  
44  
45  
46

47 **Proposition 3.1.** *Let  $\varphi$  be an arbitrarily fixed absolutely continuous proba-*  
48 *bility measure on  $([0, 1], \mathcal{B}_{[0,1]})$ , with positive mass function on  $(0, 1)$ , and  $\theta \in$*   
49  *$(0, 1]$  be a weight parameter. The mapping  $\mathfrak{D}_\theta^\varphi : \mathcal{F}_c^*(\mathbb{R}) \times \mathcal{F}_c^*(\mathbb{R}) \rightarrow [0, +\infty)$*   
50 *satisfies that*  
51  
52

- 53 *i)  $(\mathcal{F}_c^*(\mathbb{R}), \mathfrak{D}_\theta^\varphi)$  is a metric space.*
- 54 *ii)  $\mathfrak{D}_\theta^\varphi$  is an  $L^2$ -type metric, and it is translation and rotation invariant.*
- 55 *iii) For fixed  $\varphi$ , the  $v^\varphi$ -vector function satisfies that*  
56  
57  
58  
59  
60  
61  
62  
63  
64  
65

- $v^\varphi$  is an isometry
- $v^\varphi(\tilde{U} + \tilde{V}) = v^\varphi(\tilde{U}) + v^\varphi(\tilde{V})$  for all  $\tilde{U}, \tilde{V} \in \mathcal{F}_c^*(\mathbb{R})$
- $v^\varphi(\gamma \cdot \tilde{U}) = \gamma \cdot v^\varphi(\tilde{U})$  for all  $\tilde{U} \in \mathcal{F}_c^*(\mathbb{R})$  and  $\lambda > 0$ .

The proof of this proposition is based on the fact that the metric  $\mathfrak{D}_\theta^\varphi$  defined in Definition 3.1 coincides with the norm  $\|\cdot\|_\theta^\varphi$  of the difference of the corresponding  $v^\varphi$ -vector functions:

$$\mathfrak{D}_\theta^\varphi(\tilde{U}, \tilde{V}) = \sqrt{\int_{[0,1]} \left( |v_{\tilde{U}}^\varphi(\alpha) - v_{\tilde{V}}^\varphi(\alpha)|_\theta \right)^2 d\varphi(\alpha)} = \|v^\varphi(\tilde{U}) - v^\varphi(\tilde{V})\|_\theta^\varphi.$$

As a consequence, the  $v^\varphi$ -vector function preserves the semilinearity of  $\mathcal{F}_c^*(\mathbb{R})$  and relates the fuzzy arithmetic to the functional arithmetic, which implies that  $\mathcal{F}_c^*(\mathbb{R})$  can be isometrically embedded in a convex cone of  $\mathbb{H}^2$ .

For the special case  $\varphi \equiv l$  with  $l$  the Lebesgue measure as in Example 2.1, the corresponding metric  $\mathfrak{D}_\theta^\ell$  is topologically equivalent to some well-known metrics between fuzzy numbers. In particular, the next proposition shows that  $\mathfrak{D}_\theta^\ell$  is equivalent to the well-known  $L^2$  metric  $\rho_2$  based on the inf/sup representation of fuzzy numbers (Diamond and Kloeden [35]) and given by

$$\rho_2(\tilde{U}, \tilde{V}) = \sqrt{\frac{1}{2} \cdot \int_{[0,1]} \left[ \inf \tilde{U}_\alpha - \inf \tilde{V}_\alpha \right]^2 d\alpha + \frac{1}{2} \cdot \int_{[0,1]} \left[ \sup \tilde{U}_\alpha - \sup \tilde{V}_\alpha \right]^2 d\alpha}.$$

**Proposition 3.2.** *Let  $\ell$  be the Lebesgue measure on  $([0, 1], \mathcal{B}_{[0,1]})$ , and  $\theta \in (0, 1]$  be a weight parameter. Then, the metric  $\mathfrak{D}_\theta^\ell$  is topologically equivalent to the metric  $\rho_2$  on  $\mathcal{F}_c^*(\mathbb{R})$ . More precisely,*

$$\sqrt{\theta} \cdot \rho_2(\tilde{U}, \tilde{V}) \leq \mathfrak{D}_\theta^\ell(\tilde{U}, \tilde{V}) \leq \rho_2(\tilde{U}, \tilde{V})$$

for all  $\tilde{U}, \tilde{V} \in \mathcal{F}_c^*(\mathbb{R})$ .

*Proof.* Indeed,

$$\begin{aligned} (|v_{\tilde{U}}^\ell(\alpha) - v_{\tilde{V}}^\ell(\alpha)|_\theta)^2 &= \left[ \text{wabl}^\ell(\tilde{U}) - \text{wabl}^\ell(\tilde{V}) \right]^2 \\ &+ \frac{\theta}{2} \cdot \left[ \text{wabl}^\ell(\tilde{U}) - \inf \tilde{U}_\alpha - \text{wabl}^\ell(\tilde{V}) + \inf \tilde{V}_\alpha \right]^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{\theta}{2} \cdot \left[ \sup \tilde{U}_\alpha - \text{wabl}^\ell(\tilde{U}) - \sup \tilde{V}_\alpha + \text{wabl}^\ell(\tilde{V}) \right]^2 \\
& = (1 - \theta) \cdot \left[ \text{wabl}^\ell(\tilde{U}) - \text{wabl}^\ell(\tilde{V}) \right]^2 \\
& + \frac{\theta}{2} \cdot \left[ \inf \tilde{U}_\alpha - \inf \tilde{V}_\alpha \right]^2 + \frac{\theta}{2} \cdot \left[ \sup \tilde{U}_\alpha - \sup \tilde{V}_\alpha \right]^2,
\end{aligned}$$

whence

$$\begin{aligned}
\mathfrak{D}_\theta^\ell(\tilde{U}, \tilde{V}) & = \sqrt{(1 - \theta) \cdot \left[ \text{wabl}^\ell(\tilde{U}) - \text{wabl}^\ell(\tilde{V}) \right]^2 + \theta \cdot (\rho_2(\tilde{U}, \tilde{V}))^2} \\
& \geq \sqrt{\theta} \cdot \rho_2(\tilde{U}, \tilde{V}).
\end{aligned}$$

On the other hand,

$$\left[ \text{wabl}^\ell(\tilde{U}) - \text{wabl}^\ell(\tilde{V}) \right]^2 = \left[ \int_{[0,1]} \left( \text{mid } \tilde{U}_\alpha - \text{mid } \tilde{V}_\alpha \right) d\alpha \right]^2$$

and by applying twice Jensen's inequality

$$\begin{aligned}
& \left[ \text{wabl}^\ell(\tilde{U}) - \text{wabl}^\ell(\tilde{V}) \right]^2 \leq \int_{[0,1]} \left( \text{mid } \tilde{U}_\alpha - \text{mid } \tilde{V}_\alpha \right)^2 d\alpha \\
& = \int_{[0,1]} \left[ \frac{1}{2} \cdot (\inf \tilde{U}_\alpha - \inf \tilde{V}_\alpha) + \frac{1}{2} \cdot (\sup \tilde{U}_\alpha - \sup \tilde{V}_\alpha) \right]^2 d\alpha \\
& \leq \int_{[0,1]} \left[ \frac{1}{2} \cdot (\inf \tilde{U}_\alpha - \inf \tilde{V}_\alpha)^2 + \frac{1}{2} \cdot (\sup \tilde{U}_\alpha - \sup \tilde{V}_\alpha)^2 \right] d\alpha = (\rho_2(\tilde{U}, \tilde{V}))^2.
\end{aligned}$$

Therefore,

$$\mathfrak{D}_\theta^\ell(\tilde{U}, \tilde{V}) \leq \sqrt{(1 - \theta) \cdot (\rho_2(\tilde{U}, \tilde{V}))^2 + \theta \cdot (\rho_2(\tilde{U}, \tilde{V}))^2} = \rho_2(\tilde{U}, \tilde{V}). \quad \square$$

**Remark 3.1.** In an analogous way, if we extend  $\rho_2$  by incorporating the weighting measure  $\varphi$ , we can state that the metric  $\mathfrak{D}_\theta^\varphi$  is topologically equivalent on  $\mathcal{F}_c^*(\mathbb{R})$  to the metric  $\rho_2^\varphi$  given by

$$\rho_2^\varphi(\tilde{U}, \tilde{V}) = \sqrt{\frac{1}{2} \cdot \int_{[0,1]} \left( \left[ \inf \tilde{U}_\alpha - \inf \tilde{V}_\alpha \right]^2 + \left[ \sup \tilde{U}_\alpha - \sup \tilde{V}_\alpha \right]^2 \right) d\varphi(\alpha)}.$$

Thus,

$$\sqrt{\theta} \cdot \rho_2^\varphi(\tilde{U}, \tilde{V}) \leq \mathfrak{D}_\theta^\varphi(\tilde{U}, \tilde{V}) \leq \rho_2^\varphi(\tilde{U}, \tilde{V}).$$

Moreover, since  $\rho_2$  is also topologically equivalent to the metric  $d_2$  given by

$$d_2(\tilde{U}, \tilde{V}) = \sqrt{\int_{[0,1]} d_H(\tilde{U}_\alpha, \tilde{V}_\alpha)^2 d\alpha},$$

where  $d_H$  denotes the well-known Hausdorff metric on the space of nonempty closed and bounded intervals, and  $(\mathcal{F}_c^*(\mathbb{R}), d_2)$  is a separable metric space (see Klement *et al.* [34]), one can immediately derive that

**Proposition 3.3.**  $(\mathcal{F}_c^*(\mathbb{R}), \mathfrak{D}_\theta^\ell)$  is a separable metric space.

More generally, it is rather straightforward to extend the proof for the separability of  $(\mathcal{F}_c^*(\mathbb{R}), d_2)$  (see Klement *et al.* [34] and Diamond and Kloeden [35]) to the general metric  $d_2^\varphi$  given by

$$d_2^\varphi(\tilde{U}, \tilde{V}) = \sqrt{\int_{[0,1]} d_H(\tilde{U}_\alpha, \tilde{V}_\alpha)^2 d\varphi(\alpha)}.$$

Moreover, this metric  $d_2^\varphi$  is topologically equivalent to the metric  $\rho_2^\varphi$ , so that we can more generally conclude that  $(\mathcal{F}_c^*(\mathbb{R}), \mathfrak{D}_\theta^\varphi)$  is a separable metric space.

#### 4. Interpretation of the weight parameter $\theta$

A key question that arises when employing the metric  $\mathfrak{D}_\theta^\varphi$  in real-life problems involving fuzzy data is the selection of a particular element of the family of metrics. For this purpose, it is very valuable to interpret the roles played by  $\varphi$  and  $\theta$  in the metric.

The probability measure  $\varphi$  can be formally identified with a measure weighting the ‘importance’ given to the different  $\alpha$ -levels of the fuzzy numbers. For instance,

- the choice  $\varphi \equiv \ell$  indicates that in quantifying the distance between fuzzy numbers one gives the same relevance to all levels;
- choosing  $\varphi$  such that the greater the value of  $\alpha$  the greater its weight (e.g.  $\varphi \equiv \text{Beta}(p, 1)$  with  $p \gg 1$ ) indicates that in quantifying the distance between fuzzy numbers one gives higher relevance to high levels, that is, one mainly focuses on the levels with high degree of compatibility;

- choosing  $\varphi$  such that the greater the value of  $\alpha$  the lower its weight (e.g.  $\varphi \equiv \text{Beta}(1, p)$  with  $p \gg 1$ ) indicates that in quantifying the distance between fuzzy numbers one gives higher relevance to low levels, that is, one mainly focuses on the levels with low degree of compatibility.

We illustrate these assertions with the following example.

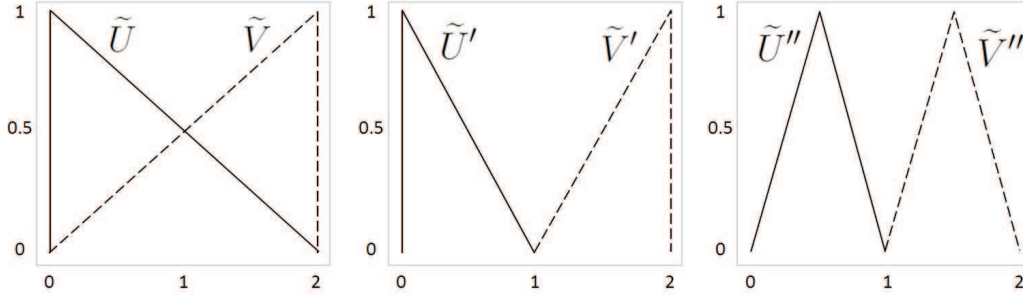


Figure 2: Three couples of fuzzy numbers

**Example 4.1.** Consider the three couples of triangular fuzzy numbers in Figure 2. Note that  $\tilde{U}$  and  $\tilde{V}$  share the 0-level (interval  $[0, 2]$ ) but differ strongly in the shape.  $\tilde{U}'$  and  $\tilde{V}'$  have 0-levels (intervals  $[0, 1]$  and  $[1, 2]$ ) which only overlap in a singleton and also show strongly different shapes. Finally,  $\tilde{U}''$  and  $\tilde{V}''$  have disjoint 0-levels (intervals  $[0, 1]$  and  $[1, 2]$ ) but show the same shape and are symmetric. Figure 3 graphically displays the distance  $\mathfrak{D}_{1/3}^\varphi$  between the fuzzy numbers  $\tilde{U}$  and  $\tilde{V}$ ,  $\tilde{U}'$  and  $\tilde{V}'$ , and  $\tilde{U}''$  and  $\tilde{V}''$  as a function of  $p \in (0, \infty)$  when  $\varphi$  is taken as  $\beta(p, 1)$  and  $\beta(1, p)$  respectively (note that  $p = 1$  corresponds to  $\varphi \equiv \ell$ ). The conclusions from Figure 3 are not unexpected given their location and shape in Figure 2. First note that due to the symmetry of these fuzzy numbers  $\mathfrak{D}_{1/3}^\varphi(\tilde{U}'', \tilde{V}'')$  is constantly equal to 1, irrespectively of the choice of  $\varphi$ . When  $\varphi$  mainly weights the high levels (i.e.,  $\varphi \equiv \beta(p, 1)$  with  $p \gg 1$ ), then  $\mathfrak{D}_{1/3}^\varphi(\tilde{U}, \tilde{V}) > \mathfrak{D}_{1/3}^\varphi(\tilde{U}', \tilde{V}') \gg \mathfrak{D}_{1/3}^\varphi(\tilde{U}'', \tilde{V}'')$ , whereas when  $\varphi$  mainly weights the low levels (i.e.,  $\varphi \equiv \beta(1, p)$  with  $p > 1$ ), then  $\mathfrak{D}_{1/3}^\varphi(\tilde{U}', \tilde{V}') > \mathfrak{D}_{1/3}^\varphi(\tilde{U}, \tilde{V}) > \mathfrak{D}_{1/3}^\varphi(\tilde{U}'', \tilde{V}'')$ , and for  $p \gg 1$ ,  $\mathfrak{D}_{1/3}^\varphi(\tilde{U}', \tilde{V}') > \mathfrak{D}_{1/3}^\varphi(\tilde{U}'', \tilde{V}'') \gg \mathfrak{D}_{1/3}^\varphi(\tilde{U}, \tilde{V})$ .

As explained in the previous section, the general role of the parameter  $\theta$  in the metric  $\mathfrak{D}_\theta^\varphi$  is to weight the influence of the squared deviation in



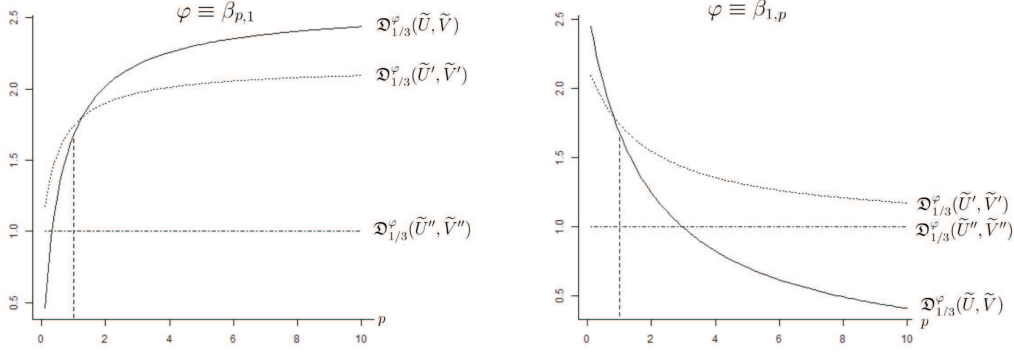


Figure 3: Distance  $\mathfrak{D}_{1/3}^\varphi$  between the couples of fuzzy numbers in Fig. 2 for  $\varphi \equiv \beta(p, 1)$  (left) and  $\beta(1, p)$  (right).

central location (measured by  $\text{wabl}^\varphi$ ) of the fuzzy numbers in contrast to the influence of their deviation in shape (measured level-wise by the  $\text{ldev}^\varphi$  and  $\text{rdev}^\varphi$  functions). However, the exact meaning of the value of  $\theta$  is not yet clear. Therefore, we now establish a result that allows us to interpret better the value of  $\theta$ . For this purpose, we take into account that if we consider any fuzzy number  $\tilde{U} \in \mathcal{F}_c^*(\mathbb{R})$ , and a level  $\alpha \in [0, 1]$ , then each

$$x \in \left[ \min\{\text{wabl}^\varphi(\tilde{U}), \inf \tilde{U}_\alpha\}, \max\{\text{wabl}^\varphi(\tilde{U}), \sup \tilde{U}_\alpha\} \right]$$

can be written as a particular linear combination of the components of  $v_{\tilde{U}}^\varphi(\alpha)$ . More precisely, for any of these  $x$ 's there exists a  $\lambda \in [-1, 1]$  such that

$$x = f_{\tilde{U}}^\varphi(\alpha, \lambda) = \text{wabl}^\varphi(\tilde{U}) - M_0(-\lambda) \cdot \text{ldev}_{\tilde{U}}^\varphi(\alpha) + M_0(\lambda) \cdot \text{rdev}_{\tilde{U}}^\varphi(\alpha),$$

with  $M_0(\lambda) = \max\{0, \lambda\}$ .

Note that  $[\min\{\text{wabl}^\varphi(\tilde{U}), \inf \tilde{U}_\alpha\}, \max\{\text{wabl}^\varphi(\tilde{U}), \sup \tilde{U}_\alpha\}]$  does not always represent the level  $\tilde{U}_\alpha$ , but in case that  $\text{wabl}^\varphi(\tilde{U}) \notin \tilde{U}_\alpha$  the  $\alpha$  level interval is 'enlarged' to include  $\text{wabl}^\varphi(\tilde{U})$ .

Based on the above expression, for any two fuzzy numbers  $\tilde{U}$  and  $\tilde{V}$ , a one-to-one correspondence between them can be stated by considering the functions  $f_{\tilde{U}}^\varphi(\alpha, \lambda)$  and  $f_{\tilde{V}}^\varphi(\alpha, \lambda)$ , so that it seems plausible to consider the distance between  $\tilde{U}$  and  $\tilde{V}$  as given by

$$\mathcal{D}_\eta^\varphi(\tilde{U}, \tilde{V}) = \sqrt{\int_{[0,1]} \int_{[-1,1]} \left[ f_{\tilde{U}}^\varphi(\alpha, \lambda) - f_{\tilde{V}}^\varphi(\alpha, \lambda) \right]^2 d\eta(\lambda) d\varphi(\alpha)},$$

where  $\eta$  is a measure which can formally be identified with a symmetric and non-degenerate probability measure on  $([-1, 1], \mathcal{B}_{[-1,1]})$ .

Since  $\eta$  is assumed to be symmetric on  $[-1, 1]$ , it can be expressed as a finite mixture  $\eta = .5 \cdot \zeta + .5 \cdot \xi$  with  $\xi$  a probability measure on  $[0, 1]$  which is non-degenerate at 0, and  $\zeta(\lambda) = \xi(-\lambda)$ . Therefore, the distance  $\mathcal{D}_\eta^\varphi(\tilde{U}, \tilde{V})$  can be rewritten by taking into account that

$$\begin{aligned} & \int_{[-1,1]} \left[ f_{\tilde{U}}^\varphi(\alpha, \lambda) - f_{\tilde{V}}^\varphi(\alpha, \lambda) \right]^2 d\eta(\lambda) \\ &= \frac{1}{2} \int_{[0,1]} \left[ \left( \text{wabl}^\varphi(\tilde{U}) - \lambda \cdot \text{ldev}_{\tilde{U}}^\varphi(\alpha) \right) - \left( \text{wabl}^\varphi(\tilde{V}) - \lambda \cdot \text{ldev}_{\tilde{V}}^\varphi(\alpha) \right) \right]^2 d\xi(\lambda) \\ &+ \frac{1}{2} \int_{[0,1]} \left[ \left( \text{wabl}^\varphi(\tilde{U}) + \lambda \cdot \text{rdev}_{\tilde{U}}^\varphi(\alpha) \right) - \left( \text{wabl}^\varphi(\tilde{V}) + \lambda \cdot \text{rdev}_{\tilde{V}}^\varphi(\alpha) \right) \right]^2 d\xi(\lambda). \end{aligned}$$

The next result shows that this distance is an equivalent definition for  $\mathfrak{D}_\theta^\varphi$ . Based on this equivalence the role of the parameter  $\theta$  becomes easier to interpret.

**Theorem 4.1.** *The family of metrics  $\mathfrak{D}_\theta^\varphi$  is equivalent to the family of metrics  $\mathcal{D}_\eta^\varphi$ .*

*Proof.* Indeed, for each  $\alpha \in [0, 1]$

$$\begin{aligned} & \frac{1}{2} \int_{[0,1]} \left[ \left( \text{wabl}^\varphi(\tilde{U}) - \lambda \cdot \text{ldev}_{\tilde{U}}^\varphi(\alpha) \right) - \left( \text{wabl}^\varphi(\tilde{V}) - \lambda \cdot \text{ldev}_{\tilde{V}}^\varphi(\alpha) \right) \right]^2 d\xi(\lambda) \\ &+ \frac{1}{2} \int_{[0,1]} \left[ \left( \text{wabl}^\varphi(\tilde{U}) + \lambda \cdot \text{rdev}_{\tilde{U}}^\varphi(\alpha) \right) - \left( \text{wabl}^\varphi(\tilde{V}) + \lambda \cdot \text{rdev}_{\tilde{V}}^\varphi(\alpha) \right) \right]^2 d\xi(\lambda) \\ &= \left[ \text{wabl}^\varphi(\tilde{U}) - \text{wabl}^\varphi(\tilde{V}) \right]^2 + \frac{1}{2} \left[ \text{ldev}_{\tilde{U}}^\varphi(\alpha) - \text{ldev}_{\tilde{V}}^\varphi(\alpha) \right]^2 \cdot \int_{[0,1]} \lambda^2 d\xi(\lambda) \\ &\quad + \frac{1}{2} \left[ \text{rdev}_{\tilde{U}}^\varphi(\alpha) - \text{ldev}_{\tilde{U}}^\varphi(\alpha) \right]^2 \cdot \int_{[0,1]} \lambda^2 d\xi(\lambda) \\ &\quad - \left[ \text{wabl}^\varphi(\tilde{U}) - \text{wabl}^\varphi(\tilde{V}) \right]^2 \cdot \int_{[0,1]} 2\lambda d\xi(\lambda) \\ &\quad + \left[ \text{wabl}^\varphi(\tilde{U}) - \text{wabl}^\varphi(\tilde{V}) \right] \cdot \left[ \text{mid } \tilde{U}_\alpha - \text{mid } \tilde{V}_\alpha \right] \cdot \int_{[0,1]} 2\lambda d\xi(\lambda), \end{aligned}$$

whence  $\mathcal{D}_\eta^\varphi(\tilde{U}, \tilde{V}) = \mathfrak{D}_{\theta_\xi}^\varphi(\tilde{U}, \tilde{V})$  with  $\theta_\xi = \int_{[0,1]} \lambda^2 d\xi(\lambda)$ .

Consequently, for any non-degenerate symmetric probability measure  $\eta$  on  $[-1, 1]$  there is a probability measure  $\xi$  on  $[0, 1]$  and non-degenerate at 0 such that  $\eta(\lambda) = .5 \cdot \xi(-\lambda) + .5 \cdot \xi(\lambda)$  and  $\mathcal{D}_\eta^\varphi(\tilde{U}, \tilde{V}) = \mathfrak{D}_{\theta_\xi}^\varphi(\tilde{U}, \tilde{V})$  with  $\theta_\xi = \int_{[0,1]} \lambda^2 d\xi(\lambda)$ . Conversely, for any parameter value  $\theta \in (0, 1]$  there are probability measures  $\xi_\theta$  on  $[0, 1]$  and non-degenerate at 0 such that  $\mathcal{D}_{\eta_\theta}^\varphi(\tilde{U}, \tilde{V}) = \mathfrak{D}_\theta^\varphi(\tilde{U}, \tilde{V})$  for  $\eta_\theta(\lambda) = .5 \cdot \xi_\theta(-\lambda) + .5 \cdot \xi_\theta(\lambda)$ . For instance, we can consider  $\xi_\theta$  to be the Bernoulli distribution with parameter  $\{(\sqrt{1+4\theta}-1)/2\}$  or the Beta( $(\sqrt{\theta^2+8\theta}+\theta)/2, 1$ ) distribution, etc.  $\square$

As an immediate implication from the preceding theorem we can interpret some choices of the value of  $\theta$  which will be useful for practical purposes. Among the most relevant metrics we can highlight

- the choice of  $\theta = 1/3$  (considered in Example 4.1) corresponds to choosing  $\xi$  as the Lebesgue measure  $\ell$  on  $[0, 1]$  (i.e. the points in each ‘enlarged’ level being uniformly weighted);
- the choice  $\theta = 1$  corresponds, among others, to choosing  $\xi$  as the indicator function of  $\{1\}$ , so that  $\mathfrak{D}_1^\varphi = \rho_2^\varphi$ .

## 5. Concluding remarks

A family of metrics  $\mathfrak{D}_\theta^\varphi$  has been introduced in this paper as  $L^2$  metrics based on a representation of fuzzy sets which takes into account the central locations and shape of the involved fuzzy numbers, and for which necessary and sufficient conditions can be stated to characterize a fuzzy number. In this representation the central location of a fuzzy number  $\tilde{U}$  is measured by  $\text{wabl}^\varphi(\tilde{U})$  while its shape is measured by the two functions  $\text{ldev}_U^\varphi$  and  $\text{rdev}_U^\varphi$ .

Since it can be shown that these metrics are coherent with the Fréchet principle in case of fuzzy number-valued random elements and the mean value is approached by the Aumann-type expectation (see, Puri and Ralescu [33]), a statistical methodology following the guidelines in [49] can be developed in the future. A comparative study can then be performed to compare the power of these metrics. The new metrics can also be used to extend the notion of  $L^2$  median to fuzzy number-valued random elements by extending the notion of Sinova et al. [53].

A much deeper discussion on the choice of  $\varphi$  would be desirable. However, this is not a point to be covered in this paper but an open problem. It should be pointed out that the influence of the selection of  $\varphi$  will strongly depend on

1  
2  
3  
4  
5  
6  
7  
8  
9 the particular problem being studied. Studies should be conducted to analyze  
10 the influence on the power of hypothesis tests, on the MSE of estimators, on  
11 the robustness of some measures, etc. Moreover, the discussion should be  
12 developed separately for each problem to which the metric is applied.  
13  
14

## 15 **Acknowledgements**

16  
17  
18 This research has been partially supported by/benefited from the Span-  
19 ish Ministry of Science and Innovation Grant MTM2009-09440-C02-01, the  
20 FPU Grant AP2009-1197 from the Spanish Ministry and an Ayuda de Inves-  
21 tigación 2011 from the Fundación Banco Hererro (the last two for Sinova).  
22 The research by Stefan Van Aelst was supported by a grant of the Fund for  
23 Scientific Research-Flanders (FWO-Vlaanderen) and by IAP research net-  
24 work grant nr. P7/06 of the Belgian government (Belgian Science Policy).  
25 Their financial support is gratefully acknowledged.  
26  
27  
28

## 29 **References**

- 30  
31 [1] R. Coppi, P. D'Urso, P. Giordani, Fuzzy and possibilistic clustering for fuzzy data,  
32 *Comput. Stat. Data An.* 56 (2012) 915–927.  
33  
34 [2] M.B. Ferraro, P. Giordani, On possibilistic clustering with repulsion constraints for  
35 imprecise data, *Inform. Sciences* (2013) doi:10.1016/j.ins.2013.04.008.  
36  
37 [3] S. Guillaume, B. Charnomordic, P. Loisel, Fuzzy partitions: a way to in-  
38 tegrate expert knowledge into distance calculations, *Inform. Sciences* (2013)  
39 doi:10.1016/j.ins.2012.07.045.  
40  
41 [4] A. Colubi, M. López-Díaz, J.S. Domínguez-Menchero, M.A. Gil, A generalized Strong  
42 Law of Large Numbers, *Prob. Theor. Rel. Fields* 114 (1999) 401–417.  
43  
44 [5] I.S. Molchanov, On Strong Laws of Large Numbers for random upper semicontinuous  
45 functions, *Journal of Mathematical Analysis and Applications* 235 (1999) 349–355.  
46  
47 [6] P. Terán, Strong law of large numbers for  $t$ -normed arithmetics, *Fuzzy Set Syst.* 159  
48 (2008) 343–360.  
49  
50 [7] P. Terán, Algebraic, metric and probabilistic properties of convex combinations based  
51 on the  $t$ -normed extension principle: The strong law of large numbers, *Fuzzy Set Syst.*  
52 223 (2013) 1–25.  
53  
54 [8] N.V. Quang, N.T. Thuan, Strong laws of large numbers for adapted arrays of set-  
55 valued and fuzzy-valued random variables in Banach space, *Fuzzy Set Syst.* 209  
56 (2012) 14–32.  
57  
58

- 1  
2  
3  
4  
5  
6  
7  
8  
9 [9] G. Aletti, E.N. Bongiorno, A decomposition theorem for fuzzy set-valued random  
10 variables, *Fuzzy Set Syst.* 219 (2013) 98-112.  
11  
12 [10] S. Abbasbandy, B. Asady, The nearest trapezoidal fuzzy number to a fuzzy quantity,  
13 *Appl. Math. Comput.* 156 (2004) 381-386.  
14  
15 [11] S. Abbasbandy, M. Amirfakhrian, The nearest approximation of a fuzzy quantity in  
16 parametric form, *Appl. Math. Comput.* 172 (2006) 624-632.  
17  
18 [12] R.S. Porchelvi, A. Nagoorgani, R.I. Hepzibah, An algorithmic approach to multi  
19 objective fuzzy linear programming problem. *Int. J. Algor., Comput. Math.* 3 (2010)  
20 61-66.  
21  
22 [13] A.D. Báez-Sánchez, A.C. Moretti, M.A. Rojas-Medar, On polygonal fuzzy sets and  
23 numbers, *Fuzzy Set Syst.* 209 (2012) 54-65.  
24  
25 [14] A.I. Bana, L. Coroianu, Nearest interval, triangular and trapezoidal approximation  
26 of a fuzzy number preserving ambiguity, *Int. J. Approx. Reason.* 53 (2012) 805-836.  
27  
28 [15] L. Coroianu, Lipschitz functions and fuzzy number approximations, *Fuzzy Set Syst.*  
29 200 (2012) 116-135.  
30  
31 [16] L. Coroianu, S.G. Gal, B. Bede, Approximation of fuzzy numbers by  
32 max-product Bernstein operators, *Fuzzy Set Syst.* (accepted, in press,  
33 doi:10.1016/j.fss.2013.04.010)  
34  
35 [17] U.K. Bera, M.K. Maiti, M. Maiti, Inventory model with fuzzy lead-time and dynamic  
36 demand over finite time horizon using a multi-objective genetic algorithm, *Comput.*  
37 *Math. Appl.* 64 (2012) 1822-1838.  
38  
39 [18] W. Näther, Linear statistical inference for random fuzzy data, *Statistics* 29 (1997)  
40 221-240.  
41  
42 [19] W. Näther, Regression with fuzzy random data, *Comput. Stat. Data An.* 51 (2006)  
43 915-927.  
44  
45 [20] R. Körner, W. Näther, Linear regression with random fuzzy variables: extended  
46 classical estimates, best linear estimates, least squares estimates, *Inform. Sciences*  
47 109 (1998) 95-118.  
48  
49 [21] R. Körner, An asymptotic  $\alpha$ -test for the expectation of random fuzzy variables, *J.*  
50 *Stat. Plann. Infer.* 83 (2000) 331-346.  
51  
52 [22] D. García, M.A. Lubiano, M.C. Alonso, Estimating the expected value of fuzzy  
53 random variables in the stratified random sampling from finite populations, *Inform.*  
54 *Sciences* 138 (2001) 165-184.  
55  
56  
57  
58

- 1  
2  
3  
4  
5  
6  
7  
8  
9 [23] M. Montenegro, M.R. Casals, M.A. Lubiano, M.A. Gil, Two-sample hypothesis tests  
10 of means of a fuzzy random variable, *Inform. Sciences* 133 (2001) 89–100.  
11  
12 [24] M. Montenegro, A. Colubi, M.R. Casals, M.A. Gil, Asymptotic and Bootstrap tech-  
13 niques for testing the expected value of a fuzzy random variable, *Metrika* 59 (2004)  
14 31–49.  
15  
16 [25] M.A. Gil, M. Montenegro, G. González-Rodríguez, A. Colubi, M.R. Casals, Boot-  
17 strap approach to the multi-sample test of means with imprecise data, *Comput. Stat.*  
18 *Data An.* 51 (2006) 148–162.  
19  
20 [26] R. Coppi, P. Giordani, P. D’Urso, Component models for fuzzy data, *Psychometrika*  
21 71 (2006) 733–761.  
22  
23 [27] M.B. Ferraro, A. Colubi, G. González-Rodríguez, R. Coppi, A determination coeffi-  
24 cient for a linear regression model with imprecise response, *Environmetrics* 22 (2011)  
25 516–529.  
26  
27 [28] M.B. Ferraro, P. Giordani, A multiple linear regression model for imprecise informa-  
28 tion, *Metrika* 75 (2012) 1049–1068.  
29  
30 [29] G. González-Rodríguez, A. Blanco, A. Colubi, M.A. Lubiano, Estimation of a simple  
31 linear regression model for fuzzy random variables, *Fuzzy Set Syst.* 160 (2009) 357–  
32 370.  
33  
34 [30] G. González-Rodríguez, M. Montenegro, A. Colubi, M.A. Gil, Bootstrap techniques  
35 and fuzzy random variables: Synergy in hypothesis testing with fuzzy data, *Fuzzy*  
36 *Set Syst.* 157 (2006) 2608–2613.  
37  
38 [31] G. González-Rodríguez, A. Colubi, M.A. Gil, Fuzzy data treated as functional data:  
39 A one-way ANOVA test approach, *Comput. Stat. Data An.* 56 (2012) 943–955.  
40  
41 [32] A.B. Ramos-Guajardo, M.A. Lubiano,  $K$ -Sample tests for equality of variances of  
42 random fuzzy sets, *Comput. Stat. Data An.* 56 (2012) 956–966.  
43  
44 [33] M.L. Puri, D.A. Ralescu, Fuzzy random variables, *J. Math. Anal. Appl.* 114 (1986)  
45 409–422.  
46  
47 [34] E.P. Klement, M.L. Puri, D.A. Ralescu, Limit theorems for fuzzy random variables,  
48 *Proc. R. Soc. Lond. A* 407 (1986) 171–182.  
49  
50 [35] P. Diamond, P. Kloeden, Metric spaces of fuzzy sets, *Fuzzy Set Syst.* 100 (1999)  
51 63–71.  
52  
53 [36] C. Bertoluzza, N. Corral, A. Salas, On a new class of distances between fuzzy num-  
54 bers, *Math. Soft. Comput.* 2 (1995) 71–84.  
55  
56  
57  
58  
59  
60  
61  
62  
63  
64  
65

- 1  
2  
3  
4  
5  
6  
7  
8  
9 [37] W. Trutschnig, G. González-Rodríguez, A. Colubi, M.A. Gil, A new family of metrics  
10 for compact, convex (fuzzy) sets based on a generalized concept of mid and spread.  
11 Inform. Sciences 179 (2009) 3964–3972.  
12
- 13 [38] P. Grzegorzewski, Metrics and orders in space of fuzzy numbers, Fuzzy Set Syst. 97  
14 (1998) 83–94.  
15
- 16 [39] B. Sinova, M.A. Gil, A. Colubi, S. Van Aelst, The median of a random fuzzy number.  
17 The 1-norm distance approach. Fuzzy Set Syst. 200 (2012) 99–115.  
18
- 19 [40] B. Sinova, S. de la Rosa de Sáa, M.A. Gil, A generalized  $L^1$ -type metric between  
20 fuzzy numbers for an approach to central tendency of fuzzy data, Inform. Sciences  
21 242 (2013) 22–34.  
22
- 23 [41] R.R. Yager, A procedure for ordering fuzzy subsets of the unit interval, Inform.  
24 Sciences 24 (1981) 143–161.  
25
- 26 [42] L. de Campos, A. González, A subjective approach for ranking fuzzy numbers, Fuzzy  
27 Set Syst., 29 (1989) 145–153.  
28
- 29 [43] E.N. Nasibov, To linear transformations with fuzzy arguments, Trans. Acad. Sciences  
30 Azerbaijan, Ser. Phys-tecn. and Math. Sciences, 6 (1989) 164–169.  
31
- 32 [44] E.N. Nasibov, O. Baskan, A. Mert, A learning algorithm for level sets weights in  
33 weighted level-based averaging method. Fuzzy Optim. Dec. Making 4 (2005) 279–  
34 291.  
35
- 36 [45] L.A. Zadeh. The concept of a linguistic variable and its application to approximate  
37 reasoning, Part 1. Inform. Sciences 8 (1975) 199–249; Part 2. Inform. Sciences 8  
38 (1975) 301–353; Part 3. Inform. Sciences 8 (1975) 43–80.  
39
- 40 [46] M. López-Díaz, M.A. Gil, The  $\lambda$ -average value and the fuzzy expectation of a fuzzy  
41 random variable. Fuzzy Set Syst. 99 (1998) 347–352.  
42
- 43 [47] M. Ming, On embedding problems of fuzzy number spaces: Part 4. Fuzzy Set Syst.  
44 58 (1993) 185–193.  
45
- 46 [48] C.V. Negoita, D.A. Ralescu, Representation theorems for fuzzy concepts. Kybernetes  
47 4 (1975) 169–174.  
48
- 49 [49] A. Blanco-Fernández, M.R. Casals, A. Colubi, N. Corral, M. García-Bárcana, M.A.  
50 Gil, G. González-Rodríguez, M.T. López, M.A. Lubiano, M. Montenegro, A.B.  
51 Ramos-Guajardo, S. de la Rosa de Sáa, B. Sinova, Random fuzzy sets: a mathe-  
52 matical tool to develop statistical fuzzy data analysis. Iranian J. Fuzzy Syst. 10, 2  
53 (2013), 1–28.  
54
- 55 [50] M.A. Gil, M.A. Lubiano, M. Montenegro, M.T. López-García, Least squares fitting  
56 of an affine function and strength of association for interval-valued data. Metrika 56  
57 (2002) 97–111.  
58  
59  
60  
61  
62  
63  
64  
65

1  
2  
3  
4  
5  
6  
7  
8  
9  
10  
11  
12  
13  
14  
15  
16  
17  
18  
19  
20  
21  
22  
23  
24  
25  
26  
27  
28  
29  
30  
31  
32  
33  
34  
35  
36  
37  
38  
39  
40  
41  
42  
43  
44  
45  
46  
47  
48  
49  
50  
51  
52  
53  
54  
55  
56  
57  
58  
59  
60  
61  
62  
63  
64  
65

[51] M.-S. Yang, C.H. Ko. On a class of fuzzy  $c$ -numbers clustering procedures for fuzzy data. *Fuzzy Sets Syst.* 84 (1996) 49–60.

[52] M.B. Ferraro, R. Coppi, G. González-Rodríguez, A. Colubi. A linear regression model for imprecise response. *Int. J. Approx. Reas.* 51 (2010) 759-770.

[53] B. Sinova, G. González-Rodríguez, S. Van Aelst, An alternative approach to the median of a random interval using an  $L^2$  metric, in: R. Kruse, M.R. Berthold, C. Moewes, M.A. Gil, P. Grzegorzewski, O. Hryniewicz (Eds.), *Sinergies of Soft Computing and Statistics for Intelligent Data Analysis*, Springer, Heidelberg, 2013, pp. 273–281.