A FAVARD THEOREM FOR RATIONAL FUNCTIONS WITH POLES ON THE UNIT CIRCLE

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ABSTRACT

The purpose of the present paper is to establish a Favard theorem for rational functions with poles on the unit circle in the complex plane. We show that under some mild conditions, a sequence of rational functions which satisfies a certain three term recurrence relation, is orthogonal with respect to a Hermitian positive-definite inner product.

KEYWORDS: Positive-definite, Hermitian inner product, orthogonal rational functions, recurrence relation.

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1. Introduction. In [2] a Favard theorem for Laurent polynomials was given. After that several Favard theorems for classes of rational functions determined by restrictions on the poles were given. We mention [1], [3] and [4]. In none of the Favard theorems we know, the poles of the functions may vary freely on the unit circle or in the real line. In the present paper we give a Favard theorem for rational functions with poles on the unit circle in the complex plane. The case of poles in the real line may be treated in a similar way.

Let $T = \{z \in \mathbb{C} : |z| = 1\}$ and let α_n, $n = 1, 2, 3, \ldots$ be given points in T. We define the functions ω_n by

$$\omega_0(z) = 1 \quad \text{and} \quad \omega_n(z) = \prod_{k=1}^{n}(z - \alpha_k), \quad n = 1, 2, 3, \ldots,$$

and the linear spaces $\mathcal{L}_n$ and $\mathcal{L}$ by

$$\mathcal{L}_n = \text{span}\{\frac{1}{\omega_0}, \frac{1}{\omega_1}, \ldots, \frac{1}{\omega_n}\}, \quad n = 0, 1, 2, \ldots, \quad \text{and} \quad \mathcal{L} = \bigcup_{n=0}^{\infty} \mathcal{L}_n.$$

Then $\mathcal{L}_n$ consists of the functions f which may be written as

$$f(z) = \frac{p(z)}{\omega_n(z)}$$

where p belongs to the set Π_n of all polynomials with degree at most n. The substar conjugate f_* of a function f is defined as

$$f_*(z) = \overline{f(1/\overline{z})}.$$

Then

$$\omega_{n*}(z) = \frac{(-1)^n}{\alpha_1\alpha_2\ldots\alpha_n} z^{-n}\omega_n(z). \tag{1.1}$$

This implies that the spaces $\mathcal{L}_n$ and $\mathcal{L}$ are invariant under substar conjugation.

Let us assume now that M is a linear functional on $\mathcal{L}\cdot\mathcal{L}$ such that

$$M(R_*) = \overline{M(R)} \quad \text{for} \quad R \in \mathcal{L}\cdot\mathcal{L}$$

and

$$M(PP_*) > 0 \quad \text{for} \quad P \in \mathcal{L}, \quad P \neq 0.$$

Notice that in general $\mathcal{L}\cdot\mathcal{L} \supset \mathcal{L} + \mathcal{L}$ but $\mathcal{L}\cdot\mathcal{L} \neq \mathcal{L} + \mathcal{L}$. Then the functional M induces an inner product $\langle\cdot,\cdot\rangle$ in $\mathcal{L}$ by $\langle P, Q\rangle = M(PQ_*)$ for $P, Q \in \mathcal{L}$. As

$$\overline{\langle Q, P\rangle} = \overline{M(QP_*)} = M(PQ_*) = \langle P, Q\rangle \quad \text{for} \quad P, Q \in \mathcal{L} \quad \text{and} \quad \langle P, P\rangle > 0 \quad \text{for} \quad P \in \mathcal{L}, \quad P \neq 0,$$

the inner product is Hermitian and positive-definite on $\mathcal{L}$.

Orthogonalization of the sequence $\{\frac{1}{\omega_n}\}_{n=0}^{\infty}$ leads to an orthogonal sequence $\{\phi_n\}_{n=0}^{\infty}$ of functions of the form

$$\phi_n = \sum_{m=0}^{n} b_m^{(n)} \frac{1}{\omega_m}$$

where $b_0^{(n)}, \ldots, b_n^{(n)}$ are constants with $b_n^{(n)} \neq 0$. Of course, in the representation

$$\phi_n = \frac{p_n}{\omega_n} \quad \text{with} \quad p_n \in \Pi_n \tag{1.2}$$

we then have $p_n(\alpha_n) \neq 0$. When in the representation (1.2) also $p_n(\alpha_{n-1}) \neq 0$, for $n = 2, 3, \ldots$, the sequence $\{\phi_n\}_{n=0}^{\infty}$ is called **regular**. As in [5] it can be shown that if the sequence $\{\phi_n\}_{n=0}^{\infty}$ is regular, the ϕ_n satisfy the following recurrence relations

$$\begin{aligned}
\phi_1(z) &= \frac{A_1 + B_1(z - \alpha_1)}{z - \alpha_1}\phi_0(z), \\
\phi_2(z) &= \frac{A_2 + B_2(z - \alpha_1)}{z - \alpha_2}\phi_1(z) + \frac{C_2(z - \alpha_1)}{z - \alpha_2}\phi_0(z), \\
\phi_n(z) &= \frac{A_n + B_n(z - \alpha_{n-1})}{z - \alpha_n}\phi_{n-1}(z) + \frac{C_n(z - \alpha_{n-2})}{z - \alpha_n}\phi_{n-2}(z), \quad n \geq 3,
\end{aligned} \tag{1.3}$$

where A_n, B_n and C_n are constants such that

$$A_n \neq 0 \quad \text{for} \quad n = 1, 2, 3, \ldots \quad \text{and} \quad C_n \neq 0 \quad \text{for} \quad n = 2, 3, \ldots. \tag{1.4}$$

Next to the recurrence relations, which will be the basis of our Favard theorem, we give some additional consequences of the orthogonality. At first we show that

$$\phi_{n*}(z) = \kappa_n \ \phi_n(z) \quad \text{with} \quad \kappa_n \in T \quad \text{for all} \quad n \tag{1.5}$$

and next we express the positive-definite character of the inner product in terms of the constants A_n and C_n.

It follows from (1.1) and (1.2) that

$$\phi_{n*}(z) = \delta_n \ \frac{z^n p_{n*}(z)}{\omega_n(z)} \quad \text{where} \quad \delta_n = (-1)^n \alpha_1 \alpha_2 \ldots \alpha_n \in T.$$

If $v_n(z) = \delta_n \ z^n p_{n*}(z)$ then $v_n \in \Pi_n$ and

$$\phi_{n*} = \frac{v_n}{\omega_n}$$

while

$$v_n(z) = \delta_n \ z^n \overline{p_n(z)} \quad \text{for} \quad z \in T,$$

so in particular $v_n(\alpha_n) \neq 0$ and $v_n(\alpha_{n-1}) \neq 0$. Hence $\{\phi_{n*}\}_{n=0}^{\infty}$ is also a basis for $\mathcal{L}$, $\phi_{n*} \in \mathcal{L} \setminus \mathcal{L}_{n-1}$, $n = 1, 2, \ldots$, ϕ_{0*} is a constant different from zero and $\phi_{0*}, \ldots, \phi_{n*}$ is a basis for $\mathcal{L}_n$, $n = 0, 1, 2, \ldots$. Thus $M(\phi_n \phi_k) = 0$ for $k < n$, and by symmetry, $M(\phi_n \phi_k) = 0$ for $n \neq k$, while also $M(\phi_n^2) \neq 0$ for all n. As $\phi_{n*} = \kappa_0 \phi_0 + \ldots + \kappa_n \phi_n$ for some $\kappa_j \in \mathbb{C}$, it follows now that $0 = M(\phi_k \phi_{n*}) = \kappa_k M(\phi_k^2)$ for $k = 0, 1, \ldots, n-1$, thus $\kappa_k = 0$ for $k = 0, 1, \ldots, n-1$. So $\phi_{n*} = \kappa_n \phi_n$, and by substar conjugation $\phi_n = \overline{\kappa_n} \phi_{n*}$. This implies that $|\kappa_n| = 1$. As also $v_n = \kappa_n p_n$, it follows that $\delta_n \ z^n p_{n*}(z) = \kappa_n p_n(z)$, and

$$\delta_n \ z^n \overline{p_n(z)} = \kappa_n p_n(z) \quad \text{for} \quad z \in T.$$

So in particular

$$\kappa_n = \delta_n \alpha_n^n \frac{\overline{p_n(\alpha_n)}}{p_n(\alpha_n)}, \quad n = 1, 2, 3, \ldots. \tag{1.6}$$

If we take the ϕ_n **monic** with respect to the basis functions $\frac{1}{\omega_n}$, meaning that

$$\phi_n = \sum_{m=0}^{n} b_m^{(n)} \frac{1}{\omega_n} \quad \text{wth} \quad b_n^{(n)} = 1,$$

then $p_n(\alpha_n) = 1$, since

$$\phi_n(z) = \frac{p_n(z)}{\omega_n(z)} = \frac{p_n(z) - p_n(\alpha_n)}{\omega_n(z)} + \frac{p_n(\alpha_n)}{\omega_n(z)} = \psi(z) + \frac{p_n(\alpha_n)}{\omega_n(z)}$$

where $\psi \in \mathcal{L}_{n-1}$. So in this case (1.6) reduces to

$$\kappa_n = \delta_n \ \alpha_n^n. \tag{1.7}$$

We remark that this normalization is different from the one in [5]. There the ϕ_n are monic with respect to the functions σ_n given by

$$\sigma_0(z) = 1 \quad \text{and} \quad \sigma_n(z) = \mathrm{i}^n \left[\prod_{k=1}^{n} (1 + \alpha_k) \right] \frac{(1+z)^n}{\omega_n(z)}, \quad n = 1, 2, \ldots,$$

which are real valued on T.

Taking substar conjugates of the recurrence relations (1.3) it follows easily that $\phi_{n*} = \kappa_n \phi_n$ if and only if

$$\begin{aligned} B_1 &= \frac{\kappa_0}{\kappa_1}(\overline{B_1} - \alpha_1 \overline{A_1}), \\ B_2 &= \frac{\kappa_1}{\kappa_2}\frac{\alpha_2}{\alpha_1}(\overline{B_2} - \alpha_1 \overline{A_2}), \quad C_2 = \frac{\kappa_0}{\kappa_2}\frac{\alpha_2}{\alpha_1}\overline{C_2}, \\ B_n &= \frac{\kappa_{n-1}}{\kappa_n}\frac{\alpha_n}{\alpha_{n-1}}(\overline{B_n} - \alpha_{n-1}\overline{A_n}), \quad C_n = \frac{\kappa_{n-2}}{\kappa_n}\frac{\alpha_n}{\alpha_{n-2}}\overline{C_n}, \quad n \geq 3 \end{aligned}$$

are satified.

Now we consider the positivity of the inner product. It follows from the recurrence relations and (1.2) that

$$p_{n+1}(\alpha_n) = A_{n+1} p_n(\alpha_n), \quad n = 1, 2, \ldots. \tag{1.8}$$

It follows also from (1.2) that

$$\phi_n(z) = \frac{p_n(z) - p_n(\alpha_n)}{z - \alpha_n}\frac{1}{\omega_{n-1}(z)} + \frac{p_n(\alpha_n)}{\omega_n(z)},$$

so

$$\frac{p_n(\alpha_n)}{\omega_n(z)} = \phi_n(z) - \frac{q_{n-1}(z)}{\omega_{n-1}(z)} \quad \text{with} \quad q_{n-1} \in \Pi_{n-1}. \tag{1.9}$$

In the same way we get

$$\begin{aligned} \frac{p_n(\alpha_n)}{\omega_n(z)} &= \phi_n(z) - \frac{q_{n-1}(z) - q_{n-1}(\alpha_{n-1})}{z - \alpha_{n-1}}\frac{1}{\omega_{n-2}(z)} - \frac{q_{n-1}(\alpha_{n-1})}{\omega_{n-1}(z)} \\ &= \phi_n(z) - \frac{q_{n-1}(\alpha_{n-1})}{\omega_{n-1}(z)} - \frac{r_{n-2}(z)}{\omega_{n-2}(z)} \end{aligned} \tag{1.10}$$

where $r_{n-2} \in \Pi_{n-2}$. Using (1.9) with n replaced by $n-1$ we obtain from (1.10)

$$\begin{aligned} \frac{p_n(\alpha_n)}{\omega_n(z)} &= \phi_n(z) - \frac{q_{n-1}(\alpha_{n-1})}{p_{n-1}(\alpha_{n-1})}\frac{p_{n-1}(\alpha_{n-1})}{\omega_{n-1}(z)} - \frac{r_{n-2}(z)}{\omega_{n-2}(z)} \\ &= \phi_n(z) - \frac{q_{n-1}(\alpha_{n-1})}{p_{n-1}(\alpha_{n-1})}\left[\phi_{n-1}(z) - \frac{q_{n-2}(z)}{\omega_{n-2}(z)}\right] - \frac{r_{n-2}(z)}{\omega_{n-2}(z)} \\ &= \phi_n(z) - \frac{q_{n-1}(\alpha_{n-1})}{p_{n-1}(\alpha_{n-1})}\phi_{n-1}(z) + \psi(z), \end{aligned}$$

where $\psi \in \mathcal{L}_{n-2}$. When $\alpha_n \neq \alpha_{n-1}$, then

$$q_{n-1}(\alpha_{n-1}) = \frac{p_n(\alpha_{n-1}) - p_n(\alpha_n)}{\alpha_{n-1} - \alpha_n},$$

and

$$q_{n-1}(\alpha_{n-1}) = p_n'(\alpha_n)$$

if $\alpha_n = \alpha_{n-1}$. If $\alpha_n \neq \alpha_{n-1}$, then there is a $\psi \in \mathcal{L}_{n-2}$ such that

$$\frac{p_n(\alpha_n)}{\omega_n(z)} = \phi_n(z) - \frac{p_n(\alpha_{n-1}) - p_n(\alpha_n)}{(\alpha_{n-1} - \alpha_n)p_{n-1}(\alpha_{n-1})}\phi_{n-1}(z) + \psi(z). \tag{1.11}$$

In order to calculate $M(\phi_n^2)$ for $n \geq 2$ we write (1.3) in the form

$$\frac{z - \alpha_{n+1}}{\omega_n(z)}\phi_{n+1}(z) = \frac{A_{n+1}}{\omega_n(z)}\phi_n(z) + \frac{B_{n+1}}{\omega_{n-1}(z)}\phi_n(z) + \frac{C_{n+1}(z - \alpha_{n-1})}{\omega_n(z)}\phi_{n-1}(z). \tag{1.12}$$

Applying M on (1.12) we get

$$A_{n+1} M\left(\frac{1}{\omega_n(z)}\phi_n(z)\right) = -C_{n+1} M\left(\frac{z - \alpha_{n-1}}{\omega_n(z)}\phi_{n-1}(z)\right). \tag{1.13}$$

Now we consider two cases.
A. If $\alpha_n = \alpha_{n-1}$, then

$$A_{n+1} M\left(\frac{1}{\omega_n(z)}\phi_n(z)\right) = -C_{n+1} M\left(\frac{1}{\omega_{n-1}(z)}\phi_{n-1}(z)\right),$$

and it follows from (1.8) and (1.9) that

$$M(\phi_n^2) = -\frac{C_{n+1}A_n}{A_{n+1}} M(\phi_{n-1}^2).$$

B. If $\alpha_n \neq \alpha_{n-1}$, then

$$\frac{z-\alpha_{n-1}}{\omega_n(z)} = \frac{1}{\omega_{n-1}(z)} + \frac{\alpha_n - \alpha_{n-1}}{\omega_n(z)}. \tag{1.14}$$

Here (1.11) gives

$$\frac{\alpha_n - \alpha_{n-1}}{\omega_n(z)} = \frac{\alpha_n - \alpha_{n-1}}{p_n(\alpha_n)}\phi_n(z) + \frac{p_n(\alpha_{n-1}) - p_n(\alpha_n)}{p_n(\alpha_n)p_{n-1}(\alpha_{n-1})}\phi_{n-1}(z) + \psi(z)$$

with $\psi \in \mathcal{L}_{n-2}$. Hence

$$M\left(\frac{\alpha_n - \alpha_{n-1}}{\omega_n(z)}\phi_{n-1}(z)\right) = \frac{p_n(\alpha_{n-1}) - p_n(\alpha_n)}{p_n(\alpha_n)p_{n-1}(\alpha_{n-1})} M(\phi_{n-1}^2(z)).$$

As also

$$M\left(\frac{1}{\omega_{n-1}(z)}\phi_{n-1}(z)\right) = \frac{1}{p_{n-1}(\alpha_{n-1})} M(\phi_{n-1}^2(z))$$

we obtain from (1.14) that

$$\begin{aligned} M\left(\frac{z-\alpha_{n-1}}{\omega_n(z)}\phi_{n-1}(z)\right) &= \left[\frac{1}{p_{n-1}(\alpha_{n-1})} + \frac{p_n(\alpha_{n-1}) - p_n(\alpha_n)}{p_n(\alpha_n)p_{n-1}(\alpha_{n-1})}\right] M(\phi_{n-1}^2(z)) \\ &= \frac{p_n(\alpha_{n-1})}{p_n(\alpha_n)p_{n-1}(\alpha_{n-1})} M(\phi_{n-1}^2(z)) \end{aligned}.$$

Thus (1.13) becomes

$$M(\phi_n^2) = -\frac{C_{n+1}}{A_{n+1}} \frac{p_n(\alpha_n)p_n(\alpha_{n-1})}{p_n(\alpha_n)p_{n-1}(\alpha_{n-1})} M(\phi_{n-1}^2).$$

By (1.8) this gives again

$$M(\phi_n^2) = -\frac{C_{n+1}A_n}{A_{n+1}} M(\phi_{n-1}^2), \quad n \geq 2. \tag{1.15}$$

So (1.15) holds in both cases (A) and (B). In a similar way we obtain

$$M(\phi_1^2) = -\frac{C_2 A_1}{A_2} M(\phi_0^2).$$

Hence

$$M(\phi_n^2) = -\frac{C_{n+1}A_n}{A_{n+1}} M(\phi_{n-1}^2), \quad \text{for} \quad n = 1, 2, 3, \ldots. \tag{1.16}$$

By (1.5) it follows from (1.16) that

$$M(\phi_n \phi_{n*}) = -\frac{\kappa_n}{\kappa_{n-1}} \frac{C_{n+1}A_n}{A_{n+1}} M(\phi_{n-1}\phi_{(n-1)*}), \quad \text{for} \quad n = 1, 2, 3, \ldots. \tag{1.17}$$

Now the positivity of the inner product implies

$$-\frac{\kappa_n}{\kappa_{n-1}} \frac{C_{n+1}A_n}{A_{n+1}} > 0, \quad n = 1, 2, 3, \ldots. \tag{1.18}$$

The purpose of the present paper is to give a Favard theorem based on the recurrence (1.3) together with (1.5) and (1.18).

2. Extended recurrence. In the following two lemmas we assume that $\{\phi_n\}_{n=0}^{\infty}$ is a sequence of monic functions which satisfy the recurrence relations (1.3) and (1.4). This implies that $\{\phi_n\}_{n=0}^{\infty}$ is regular.

LEMMA 2.1. Let m and j be be integers with $m \geq j+2$ and $j \geq 1$ and suppose that α_m is different from $\alpha_{m-1}, \alpha_{m-2}, \ldots, \alpha_{j+1}$. Then there are constants $\lambda_1, \lambda_2, \ldots, \lambda_{m-j-1}, \mu_{m-j-1}, \mu_{m-j-2}$, depending on m, such that

$$\begin{aligned}\phi_m(z) &= \lambda_1\phi_{m-1}(z) + \ldots + \lambda_{m-j-1}\phi_{j+1}(z) \\ &\quad + \mu_{m-j-1}\frac{z-\alpha_{j+1}}{z-\alpha_m}\phi_{j+1}(z) + \mu_{m-j-2}C_{j+2}\frac{z-\alpha_j}{z-\alpha_m}\phi_j(z).\end{aligned} \tag{2.1}$$

Proof. For $j = m-2$ formula (2.1) reduces to (1.3), case $n \geq 3$. We proceed by induction. Assume that

$$\begin{aligned}\phi_m(z) &= \lambda_1\phi_{m-1}(z) + \ldots + \lambda_{m-k}\phi_k(z) \\ &\quad + \mu_{m-k}\frac{z-\alpha_k}{z-\alpha_m}\phi_k(z) + \mu_{m-k-1}C_{k+1}\frac{z-\alpha_{k-1}}{z-\alpha_m}\phi_{k-1}(z)\end{aligned}$$

for some k with $m-2 \geq k \geq j+2$. Then by (1.3), case $n \geq 3$, applied to ϕ_k in the right hand side, we have

$$\begin{aligned}&\phi_m(z) - [\lambda_1\phi_{m-1}(z) + \ldots + \lambda_{m-k}\phi_k(z)] \\ &\quad = \mu_{m-k}\frac{z-\alpha_k}{z-\alpha_m}\left[\frac{A_k + B_k(z-\alpha_{k-1})}{z-\alpha_k}\phi_{k-1}(z) + \frac{C_k(z-\alpha_{k-2})}{z-\alpha_k}\phi_{k-2}(z)\right] \\ &\quad + \mu_{m-k-1}C_{k+1}\frac{z-\alpha_{k-1}}{z-\alpha_m}\phi_{k-1}(z) \\ &\quad = \frac{\mu_{m-k}[A_k + B_k(z-\alpha_{k-1})] + \mu_{m-k-1}C_{k+1}(z-\alpha_{k-1})}{z-\alpha_m}\phi_{k-1}(z) + \mu_{m-k}C_k\frac{z-\alpha_{k-2}}{z-\alpha_m}\phi_{k-2}(z).\end{aligned}$$

Since $\alpha_m \neq \alpha_{k-1}$ there are constants λ_{m-k+1} and μ_{m-k+1} such that

$$\mu_{m-k}[A_k + B_k(z-\alpha_{k-1})] + \mu_{m-k-1}C_{k+1}(z-\alpha_{k-1}) = \lambda_{m-k+1}(z-\alpha_m) + \mu_{m-k+1}(z-\alpha_{k-1}),$$

so

$$\begin{aligned}&\phi_m(z) - [\lambda_1\phi_{m-1}(z) + \ldots + \lambda_{m-k}\phi_k(z)] \\ &\quad = \lambda_{m-k+1}\phi_{k-1}(z) + \mu_{m-k+1}\frac{z-\alpha_{k-1}}{z-\alpha_m}\phi_{k-1}(z) + \mu_{m-k}C_k\frac{z-\alpha_{k-2}}{z-\alpha_m}\phi_{k-2}(z).\end{aligned}$$

□

When $j \geq 1$ we may perform an extra step in formula (2.1). For $j \geq 2$ this gives

LEMMA 2.2. When $m \geq j+2$, $j \geq 2$ and $\alpha_m \neq \alpha_{m-1}, \alpha_{m-2}, \ldots, \alpha_{j+1}$, Then there are constants $\lambda_1, \lambda_2, \ldots, \lambda_{m-j-1}, \lambda'_{m-j}, \nu_{m-j}, \mu_{m-j-1}$ such that

$$\begin{aligned}\phi_m(z) &= \lambda_1\phi_{m-1}(z) + \ldots + \lambda_{m-j-1}\phi_{j+1}(z) + \lambda'_{m-j}\phi_j(z) \\ &\quad + \frac{\nu_{m-j}}{z-\alpha_m}\phi_j(z) + \mu_{m-j-1}C_{j+1}\frac{z-\alpha_{j-1}}{z-\alpha_m}\phi_{j-1}(z).\end{aligned} \tag{2.2}$$

Here λ'_{m-j} and ν_{m-j} are given by

$$\lambda'_{m-j} = \mu_{m-j-1}B_{j+1} + \mu_{m-j-2}C_{j+2} \tag{2.3}$$

and

$$\nu_{m-j} = \mu_{m-j-1}[A_{j+1} + B_{j+1}(\alpha_m - \alpha_j)] + \mu_{m-j-2}C_{j+2}(\alpha_m - \alpha_j) \tag{2.4}$$

where μ_{m-j-2} is as in (2.1)
Proof. Lemma 2.1 and (1.3). □

3. The Favard theorem. In this section we assume that $\{\phi_n\}_{n=0}^{\infty}$ is a sequence of monic functions ϕ_n in $\mathcal{L}$ which satisfy the recurrence relations (1.3) with the "positivity conditions" (1.18) and which satisfy (1.5). Then $A_n \neq 0$ for $n = 1, 2, \ldots$ and $C_n \neq 0$ for $n = 2, 3, \ldots$ and the ϕ_n form a regular sequence.

As $\{\phi_n\}_{n=0}^{\infty}$ is a basis for $\mathcal{L}$ we define the functional M on $\mathcal{L}$ by

$$M(\phi_0) > 0 \quad \text{and} \quad M(\phi_n) = 0 \quad \text{for} \quad n = 1, 2, \ldots .$$

We have to extend M to $\mathcal{L} \cdot \mathcal{L}$, which is in general larger then $\mathcal{L}$, such that $M(\phi_n \phi_k) = 0$ for $n \neq k$. Therefore we consider the products $\phi_n \phi_j$ with $n = 1, 2, \ldots$ and $j = n+1, n+2, \ldots$ in the following order. At first we make sure that $M(\phi_1 \phi_2) = 0$. Thereafter

$$M(\phi_1 \phi_3), M(\phi_1 \phi_4), \ldots, M(\phi_1 \phi_n), \ldots$$

may be treated successively with mathematical induction. Next we deal with $M(\phi_n \phi_{n+1})$ for $n \geq 2$, provided that $M(\phi_k \phi_m) = 0$ for $k = 0, \ldots, n-1$ and $m = k+1, k+2, \ldots$. Finally we consider $M(\phi_n \phi_j)$ for $n \geq 2$ and $j \geq n+2$, given that $M(\phi_k \phi_m) = 0$ for $k = 0, \ldots, n-1$ and $m = k+1, k+2, \ldots$ and $M(\phi_n \phi_k) = 0$ for $k = n+1, \ldots, j-1$.

Thus we obtain $M(\phi_n \phi_k) = 0$ for $n \neq k$. This enables us to determine $M(\phi_n^2)$ for $n = 1, 2, \ldots$ as in the introduction. Then the functional M is defined on $\mathcal{L} \cdot \mathcal{L}$ by linearity, and it follows easily from (1.5) and (1.18) that M has the desired properties and that $\{\phi_n\}_{n=0}^{\infty}$ is orthogonal with respect to the induced inner product.

We take the following steps.

a) $M(\phi_1 \phi_2)$.
When $\phi_1 \phi_2 \notin \mathcal{L}$ we may define $M(\phi_1 \phi_2) = 0$. Thus we may assume that $\phi_1 \phi_2 \in \mathcal{L}$. Then there are $k \geq 0$ and $q_k \in \Pi_k$ such that

$$\phi_1 \phi_2 = \frac{q_k}{\omega_k}, \quad \text{so} \quad \frac{p_1 p_2}{\omega_1 \omega_2} = \frac{q_k}{\omega_k}.$$

Then $k \geq 3$. For if $k \leq 2$, $z = \alpha_1$ in

$$p_1 p_2 = \omega_1 \frac{\omega_2}{\omega_k} q_k$$

gives $0 \neq p_1(\alpha_1) p_2(\alpha_1) = 0$, a contradiction. As now

$$\frac{\omega_k}{\omega_2} p_1 p_2 = \omega_1 q_k$$

ω_k / ω_2 must be zero for $z = \alpha_1$, so there is an $m \geq 3$ such that $\alpha_m = \alpha_1$. Let m be the smallest one.

If $m = 3$, then (1.3) gives

$$\phi_3 = \frac{A_3 + B_3(\alpha_1 - \alpha_2)}{z - \alpha_1} \phi_2 + B_3 \phi_2 + C_3 \phi_1$$

where $A_3 + B_3(\alpha_1 - \alpha_2) \neq 0$ by the linear independence of the ϕ_n. This implies

$$M(\phi_1 \phi_2) = M\left(\frac{1}{\omega_1} \phi_2\right) = 0$$

since ϕ_1 is monic.

If $m \geq 4$, then $\alpha_m \neq \alpha_{m-1}, \ldots, \alpha_3$. Hence by Lemma 2.2 with $j = 2$ and $\alpha_m = \alpha_1$ we get

$$\phi_m = \lambda_1 \phi_{m-1} + \ldots + \lambda_{m-3} \phi_3 + \lambda'_{m-2} \phi_2 + \frac{\nu_{m-2}}{z - \alpha_1} \phi_2 + \mu_{m-3} C_3 \phi_1$$

where $\nu_{m-2} \neq 0$. Thus again

$$M(\phi_1 \phi_2) = M\left(\frac{1}{\omega_1} \phi_2\right) = 0.$$

b) The values $M(\phi_1 \phi_n)$, given that $M(\phi_1 \phi_k) = 0$ for $k = 2, \ldots, n-1$, can be handled as in d) below. Thus we may assume that $M(\phi_1 \phi_n) = 0$ for $n \geq 2$.

c) $M(\phi_n \phi_{n+1})$ for $n \geq 2$, assumed that $M(\phi_k \phi_m) = 0$ for $k = 0, 1, \ldots, n-1$ and $m \geq k+1$.

When

$$\phi_n\phi_{n+1} \notin \text{span}\{\mathcal{L} \cup \{\phi_k\phi_m : 1 \le k \le n-1, m \ge k+1\}\}$$

we define $M(\phi_n\phi_{n+1}) = 0$. Thus we have to consider the case that

$$\phi_n\phi_{n+1} \in \text{span}\{\mathcal{L} \cup \{\phi_k\phi_m : 1 \le k \le n-1, m \ge k+1\}\}.$$

In this case there is a $k \ge 0$ and there are $q_{k,0}, \ldots, q_{k,n-1} \in \Pi_k$ such that

$$\phi_n\phi_{n+1} + \phi_1\frac{q_{k,1}}{\omega_k} + \ldots + \phi_{n-1}\frac{q_{k,n-1}}{\omega_k} = \frac{q_{k,0}}{\omega_k}$$

which may be written as

$$\frac{p_n}{\omega_n}\frac{p_{n+1}}{\omega_{n+1}} + \frac{p_1}{\omega_1}\frac{q_{k,1}}{\omega_k} + \ldots + \frac{p_{n-1}}{\omega_{n-1}}\frac{q_{k,n-1}}{\omega_k} = \frac{q_{k,0}}{\omega_k}$$

and as

$$p_np_{n+1} + \frac{\omega_n}{\omega_1}\frac{\omega_{n+1}}{\omega_k}p_1q_{k,1} + \ldots + \frac{\omega_n}{\omega_{n-1}}\frac{\omega_{n+1}}{\omega_k}p_{n-1}q_{k,n-1} = \omega_n\frac{\omega_{n+1}}{\omega_k}q_{k,0}.$$

If $k \le n+1$, then for $z = \alpha_n$ this gives $0 \ne p_n(\alpha_n)p_{n+1}(\alpha_n) = 0$. A contradiction. Hence $k \ge n+2$. But then

$$\frac{\omega_k}{\omega_{n+1}}p_np_{n+1} + \frac{\omega_n}{\omega_1}p_1q_{k,1} + \ldots + \frac{\omega_n}{\omega_{n-1}}p_{n-1}q_{k,n-1} = \omega_nq_{k,0}$$

shows that ω_k/ω_{n+1} has a zero at α_n, so there exists an $m \ge n+2$ such that $\alpha_m = \alpha_n$. Let m be the smallest one. We consider two cases.

(i) $m = n+2$,
(i) $m \ge n+3$.

In case (i) it follows from (1.3) with $\alpha_{n+2} = \alpha_m$ that

$$\frac{1}{\omega_{n-1}}\phi_{n+2} = \frac{A_{n+2} + B_{n+2}(\alpha_n - \alpha_{n+1})}{\omega_n}\phi_{n+1} + \frac{B_{n+2}}{\omega_{n-1}}\phi_{n+1} + \frac{C_{n+2}}{\omega_{n-1}}\phi_n,$$

where $A_{n+2} + B_{n+2}(\alpha_n - \alpha_{n+1}) \ne 0$ by the independence of the ϕ_j. Application of M gives

$$M(\phi_n\phi_{n+1}) = M\left(\frac{1}{\omega_n}\phi_{n+1}\right) = 0.$$

In case (ii) we have $\alpha_m = \alpha_n$ and $\alpha_m \ne \alpha_{m-1}, \ldots, \alpha_{n+2}$, so we may use the extended recursion formula (2.2) with $j = n+1$. This gives

$$\phi_m = \lambda_1\phi_{m-1} + \ldots + \lambda_{m-n-2}\phi_{n+2} + \lambda'_{m-n-1}\phi_{n+1} + \frac{\nu_{m-n-1}}{z-\alpha_n}\phi_{n+1} + \mu_{m-n-2}C_{n+2}\phi_n, \tag{3.1}$$

where $\nu_{m-n-1} \ne 0$. After division by ω_{n-1} followed by application of M, (3.1) gives

$$M(\phi_n\phi_{n+1}) = M\left(\frac{1}{\omega_n}\phi_{n+1}\right) = 0.$$

d) $M(\phi_n\phi_j)$ for $n \ge 2$ and $j \ge n+2$, given that $M(\phi_k\phi_m) = 0$ for $k = 0, \ldots, n-1$ and $m \ge k+1$ and $M(\phi_n\phi_k) = 0$ for $k = n+1, \ldots, j-1$.
From (1.3) we get

$$\frac{z-\alpha_j}{\omega_n}\phi_j = \frac{A_j + B_j(z-\alpha_{j-1})}{\omega_n}\phi_{j-1} + \frac{C_j(z-\alpha_{j-2})}{\omega_n}\phi_{j-2}.$$

Here the second term in the right hand side becomes

$$\frac{C_j}{\omega_{n-1}}\phi_n$$

when $j = n+2$. Hence application of M gives

$$M\left(\frac{z-\alpha_j}{\omega_n}\phi_j\right) = 0.$$

If $\alpha_j \neq \alpha_n$ this implies

$$M(\phi_n\phi_j) = M\left(\frac{1}{\omega_n}\phi_j\right) = 0.$$

Assume now that $\alpha_j = \alpha_n$, and let $\mathcal{S}$ be the subspace of $\mathcal{L}\cdot\mathcal{L}$ on which M already has been defined. Then

$$\mathcal{S} = \text{span}\{\mathcal{L} \cup \{\phi_k\phi_m : 1 \leq k \leq n-1, m \geq k+1\} \cup \{\phi_n\phi_k : n+1 \leq k \leq j-1\}\}.$$

When $\phi_n\phi_j \notin \mathcal{S}$, we define $M(\phi_n\phi_j) = 0$. Therefore we may assume that $\phi_n\phi_j \in \mathcal{S}$. Then there is a $k \geq 0$ and there are polynomials $q_{k,i} \in \Pi_k$ and there are $\gamma_i \in \mathbb{C}$ such that

$$\begin{aligned}\frac{p_n}{\omega_n}\frac{p_j}{\omega_j} + \gamma_{j-1}\frac{p_n}{\omega_n}\frac{p_{j-1}}{\omega_{j-1}} + \ldots + \gamma_{n+1}\frac{p_n}{\omega_n}\frac{p_{n+1}}{\omega_{n+1}} \\ + \frac{p_{n-1}}{\omega_{n-1}}\frac{q_{k,n-1}}{\omega_k} + \ldots + \frac{p_1}{\omega_1}\frac{q_{k,1}}{\omega_k} = \frac{q_{k,0}}{\omega_k}.\end{aligned}$$

(Recall that $\phi_n = p_n/\omega_n$ for all n). This gives

$$\begin{aligned}p_np_j + \gamma_{j-1}\frac{\omega_j}{\omega_{j-1}}p_np_{j-1} + \ldots + \gamma_{n+1}\frac{\omega_j}{\omega_{n+1}}p_np_{n+1} \\ + \frac{\omega_n}{\omega_{n-1}}\frac{\omega_j}{\omega_k}p_{n-1}q_{k,n-1} + \ldots + \frac{\omega_n}{\omega_1}\frac{\omega_j}{\omega_k}p_1q_{k,1} = \omega_n\frac{\omega_j}{\omega_k}q_{k,0},\end{aligned}$$

and as $0 \neq p_n(\alpha_n)p_j(\alpha_j) = p_n(\alpha_j)p_j(\alpha_j)$, it follows that $k \geq j+1$. Now $z = \alpha_n$ in

$$\begin{aligned}\frac{\omega_k}{\omega_j}p_np_j + \gamma_{j-1}\frac{\omega_k}{\omega_{j-1}}p_np_{j-1} + \ldots + \gamma_{n+1}\frac{\omega_k}{\omega_{n+1}}p_np_{n+1} \\ + \frac{\omega_n}{\omega_{n-1}}p_{n-1}q_{k,n-1} + \ldots + \frac{\omega_n}{\omega_1}p_1q_{k,1} = \omega_nq_{k,0}\end{aligned}$$

implies that ω_k/ω_j is zero for $z = \alpha_n$. Hence there is an $m \geq j+1$ such that $\alpha_m = \alpha_n$. Again, let m be the smallest one. Again we consider two cases.

(i) $m = j+1$,
(ii) $m \geq j+2$.

In case (i) the recurrence relations with $\alpha_n = \alpha_{j+1}$ give

$$\frac{1}{\omega_{n-1}}\phi_{j+1} = \frac{A_{j+1}}{\omega_n}\phi_j + \frac{B_{j+1}}{\omega_{n-1}}\phi_j + \frac{C_{j+1}(z-\alpha_{j-1})}{\omega_n}\phi_{j-1}.$$

As $A_{j+1} \neq 0$ this yields

$$M(\phi_n\phi_j) = M\left(\frac{1}{\omega_n}\phi_j\right) = 0.$$

In case (ii) we have $\alpha_m = \alpha_j = \alpha_n$ and $\alpha_m \neq \alpha_{m-1}, \ldots, \alpha_{j+1}$. Thus by Lemma 2.1 we have

$$\phi_m = \lambda_1\phi_{m-1} + \ldots + \lambda_{m-j-1}\phi_{j+1} + \mu_{m-j-1}\frac{z-\alpha_{j+1}}{z-\alpha_n}\phi_{j+1} + \mu_{m-j-2}C_{j+2}\phi_j,$$

which shows that $\mu_{m-j-1} \neq 0$. Under the same conditions Lemma 2.2 gives

$$\begin{aligned}\phi_m =& \lambda_1\phi_{m-1} + \ldots + \lambda_{m-j-1}\phi_{j+1} + \lambda'_{m-j}\phi_j \\ &+ \frac{\nu_{m-j}}{z-\alpha_n}\phi_j + \mu_{m-j-1}C_{j+1}\frac{z-\alpha_{j-1}}{z-\alpha_n}\phi_{j-1}\end{aligned} \tag{3.2}$$

with $\nu_{m-j} = \mu_{m-j-1}A_{j+1} \neq 0$. In (3.2) we devide by ω_{n-1} and we apply M. Then we get

$$M(\phi_n\phi_j) = M\left(\frac{1}{\omega_n}\phi_j\right) = 0.$$

So far we have shown that there exists a linear functional M on the subspace

$$\mathcal{V} = \text{span}\{\{\phi_0\} \cup \{\phi_n\phi_k : n,k \geq 0, n \neq k\}\}$$

of $\mathcal{L}\cdot\mathcal{L}$ such that

$$M(\phi_0) > 0 \quad \text{and} \quad M(\phi_n\phi_k) = 0 \quad \text{if} \quad n \neq k.$$

However, writing (1.3), case $n = 2$, in the form

$$\frac{z-\alpha_2}{z-\alpha_1}\phi_2 = \frac{A_2}{z-\alpha_1}\phi_1 + B_2\phi_1 + C_2\phi_0$$

we see that $\phi_1/\omega_1 \in \mathcal{V}$, and thus that $\phi_1^2 \in \mathcal{V}$. Moreover it follows from (1.3), case $n \geq 3$, written in the form (1.12) that $\phi_n^2 \in \mathcal{V}$ if $\phi_{n-1}^2 \in \mathcal{V}$, for $n = 2,3,\ldots$. Hence $\mathcal{V} = \mathcal{L}\cdot\mathcal{L}$. By the reasoning given in the introduction we get (1.16). Because $M(\phi_0^2) = M(\phi_0) > 0$ it follows that $M(\phi_n^2) \neq 0$ for all n.

As we have assumed that (1.5) holds, also (1.17) must be valid, with κ_n as in (1.7). Since $M(\phi_0\phi_{0*}) = M(\phi_0^2) > 0$, the assumption (1.18) implies that

$$M(\phi_n\phi_{n*}) > 0 \quad \text{for} \quad n = 0,1,\ldots.$$

Of course now we have

$$M(R_*) = \overline{M(R)} \quad \text{for all} \quad R \in \mathcal{L}\cdot\mathcal{L}. \tag{3.3}$$

Indeed, every $R \in \mathcal{L}\cdot\mathcal{L}$ is of the form $R = PQ_*$ with $P,Q \in \mathcal{L}$, so if

$$P = \sum_{k=0}^{n}\beta_k\phi_k \quad \text{and} \quad Q = \sum_{j=0}^{n}\gamma_j\phi_j,$$

then

$$\begin{aligned} M(R_*) = M(P_*Q) &= M(\sum_{k=0}^{n}\overline{\beta_k}\phi_{k*}\sum_{j=0}^{n}\gamma_j\phi_j) \\ &= \sum_{k=0}^{n}\overline{\beta_k}\gamma_k M(\phi_{k*}\phi_k) = \overline{\sum_{k=0}^{n}\beta_k\overline{\gamma_k}M(\phi_k\phi_{k*})} \\ &= \overline{M(\sum_{k=0}^{n}\beta_k\phi_k\sum_{j=0}^{n}\overline{\gamma_j}\phi_{j*})} = \overline{M(PQ_*)} = \overline{M(R)} \end{aligned}$$

.

In a similar way we see that

$$M(PP_*) > 0 \quad \text{for} \quad P \in \mathcal{L}, \quad P \neq 0. \tag{3.4}$$

If

$$P = \sum_{k=0}^{n}\beta_k\phi_k \neq 0,$$

then

$$M(PP_*) = M(\sum_{k=0}^{n}\beta_k\phi_k\sum_{j=0}^{n}\overline{\beta_j}\phi_{j*}) = \sum_{k=0}^{n}|\beta_k|^2M(\phi_k\phi_{k*}) > 0.$$

Hence the induced inner product $\langle\cdot,\cdot\rangle$ is Hermitian and positive-definite on $\mathcal{L}$.

We sum up the results in the following Favard theorem.

THEOREM 3.1. Let $\{\phi_n\}_{n=0}^{\infty}$ be a sequence of monic functions in $\mathcal{L}$ which satisfy the recurrence relations (1.3) together with (1.5) and (1.18). Then the sequence $\{\phi_n\}_{n=0}^{\infty}$ is orthogonal with respect to a Hermitian positive-definite inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{L}$. The inner product is given by $\langle P, Q \rangle = M(PQ_*)$ for $P, Q \in \mathcal{L}$, where M is a linear functional on $\mathcal{L} \cdot \mathcal{L}$ which satisfies (3.3) and (3.4).

We conclude the paper with the following remark.

REMARK 3.1. Let M be a linear functional on $\mathcal{L} \cdot \mathcal{L}$ such that (3.3) and (3.4) hold. Then M is positive on T in the following sence. If $P \in \mathcal{L}_n$ for some n and $P(z) \geq 0$ for $z \in T \setminus \{\alpha_1, \ldots, \alpha_n\}$, and $P \neq 0$, then $M(P) > 0$.
Proof. Let $\{\phi_n\}_{n=0}^{\infty}$ be obtained by orthogonalization of $\{\frac{1}{\omega_n}\}_{n=0}^{\infty}$ with respect to the inner product induced by M. Let $P \in \mathcal{L}_n$, $P(z) \geq 0$ for $z \in T \setminus \{\alpha_1, \ldots, \alpha_n\}$, and $P \neq 0$. Assume that $P = q_n/\omega_n$ with $q_n \in \Pi_n$. Then

$$f(z) = \omega_{n*}(z) q_n(z) = \omega_{n*}(z)\omega_n(z)P(z) \geq 0 \quad \text{for} \quad z \in T.$$

¿From (1.1) it follows that

$$q_{n*}(z) = \omega_{n*}(z)P(z) = \frac{(-1)^n z^{-n}}{\alpha_1 \ldots \alpha_n} q_n(z)$$

so

$$q_n(z) = (-1)^n \alpha_1 \ldots \alpha_n z^n q_{n*}(z).$$

As

$$\omega_{n*}(z) = \prod_{k=1}^{n} (z^{-1} - \overline{\alpha_k})$$

is a Laurent polynomial, $f(z) = \omega_{n*}(z) q_n(z)$ is a nonnegative Laurent polynomial on T, so there exists a polynomial $s \in \Pi_n$ such that

$$f(z) = |s(z)|^2 \quad \text{for} \quad z \in T,$$

(see for instance [6]). But then $f(z) = s_*(z)s(z)$ and

$$P(z) = \frac{f(z)}{\omega_{n*}(z)\omega_n(z)} = \frac{s_*(z)s(z)}{\omega_{n*}(z)\omega_n(z)},$$

where $s/\omega_n \neq 0$ since $P \neq 0$. Hence

$$M(P) = M\left(\left(\frac{s}{\omega_n}\right)_* \frac{s}{\omega_n}\right) > 0.$$

□

References

1. E. Hendriksen and O. Njåstad, "A Favard theorem for rational functions", *J. Math. Anal. and Applic.* **142**(2) (1989), 508-520.
2. E. Hendriksen and H. van Rossum, "Orthogonal Laurent polynomials", *Proc. Kon. Ned. Akad. v. Wet.*, Amsterdam, ser. A, **89**(1) = *Indag. Math.* **48**(1) (1986), 17-36.
3. A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, "Orthogonal rational functions similar to Szegő polynomials", *Orthogonal polynomials and their applications, IMACS annals on computing and applied mathematics* **9** (1991), 195-204.
4. A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, "A Favard theorem for orthogonal rational functions on the unit circle", *Numerical Algorithms* **3** (1992), 81-90.
5. A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, "Orthogonal rational functions with poles on the unit circle", *J. Math. Anal. Appl.* **120** (1) (1994), 221-243.
6. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ., vol.23, 4th ed. Amer. Math. Soc., Providence, RI,1975.