Can we solve nonlinear eigenvalue problems?

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Abstract

Let $A(\lambda)$ be an $n \times n$ matrix whose entries are functions in λ. The nonlinear eigenvalue problem is defined as finding pairs $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$, with $x \neq 0$ so that

$$A(\lambda)x = 0.$$

The last decade the number of applications is increasing. In physics, the Schroedinger equation for determining the bound states in a semiconductor device, introduces terms with square roots of different shifts of the eigenvalue:

$$(A_0 + \lambda I + E_L e^{i\Delta x\sqrt{\lambda-\mu_L}} + E_R e^{i\Delta x\sqrt{\lambda-\mu_R}})x = 0.$$

In mechanical and civil engineering, new materials often have nonlinear damping properties. For the vibration analysis of such materials, this leads to nasty functions of λ in the system matrix. One particular example is the sandwhich beam problem, where a layer of damping material is sandwhiched between two layers of steel:

$$\left(K_e + \frac{G_0 + G_\infty(i\lambda\tau)^\alpha}{1 + (i\lambda\tau)^\alpha} K_v - \lambda^2 M\right) x = 0,$$

where K_e and M are the stiffness and mass matrices, respectively, G_0, G_∞, τ and α are material properties that are obtained from measurements. A last example is the stability analysis of the noise produced by burners in a combustion chamber. The burners lead to a boundary condition with delay terms (exponentials of the eigenvalue):

$$Kp - \lambda^2 Mp = e^{i\lambda\tau} C(\lambda)p.$$

We often receive the question: "How can we solve a nonlinear eigenvalue problem?" Therefore we decided to write an abstract and explain the different steps, which have to be taken. This talk reviews research results with Stefan Güttel, Roel Van Beeumen, Wim Michiels and Elias Jarlebring. We first give an overview of the different methods that we have considered. Newton's method and variations such as *residual inverse iteration* have been proposed a long time ago [7], but their convergence is not easy to understand. In addition, these methods can only be used for computing a single eigenvalue. Several eigenvalues can be computed by embedding the methods in a subspace method [2], but a small scale nonlinear eigenvalue still needs to be solved on each iteration.

Other methods map the nonlinear eigenvalue problem to a linear one. Contour integration methods use the Keldysh theorem and 'moments' to construct a linear eigenvalue problem whose eigenvalues correspond to the eigenvalues of $A(\lambda)x = 0$ lying inside a closed contour [1]. The moments are defined as

$$A_p = \int_\Gamma z^p A(z)^{-1} V dz \quad , \quad V \in \mathbb{C}^{n\times k} \quad , \quad p = 0, 1,$$

where the columns of V form the basis of a subspace. The moments A_0 and A_1 are used to compute k eigenvalues inside the contour by solving the order k linear eigenvalue problem

$$V_0^* A_1 z = \lambda(V_0^* A_0)z$$

where $V_0 \in \mathbb{C}^{n\times k}$ is chosen so that $V_0^* A_0$ is nonsingular. The integral is computed by the trapezoid rule and requires the computation of N terms of the form $A(z)^{-1}V$.

This talk is devoted to explaining how to use polynomial approximation and linearization in Krylov methods for solving the nonlinear eigenvalue problem. The solution is performed in three steps:

i) Approximate $A(\lambda)$ by a (rational) polynomial $P(\lambda) = \sum_{j=0}^{d} P_j \phi_j(\lambda)$ where $\phi_0, \ldots, \phi_d$ forms a basis of polynomials;

ii) Transform the polynomial $P(\lambda)$ to a matrix pencil, which we call a linearization;

iii) Solve the linear eigenvalue problem using a (rational) Krylov method.

We now discuss the different steps.

Most problems we encountered during our research on solving nonlinear eigenvalue problems have the form as shown in the examples above, i.e.,

$$A(\lambda) = \sum_{j=1}^{m} A_j f_j(\lambda)$$

where A_j is an $n \times n$ matrix and f_j a scalar (nonlinear) function. Usually, m is much smaller than n. The first step consists of approximating each of these scalar functions by a (rational) polynomial of degree d. The coefficients and the degree are determined by an approximation criterion. The simplest approach is to use a Taylor series or sequentially picking interpolation points near computed Ritz values, but the best approach is to use global approximation criteria for the region of interest in the complex plane. We have used results from potential theory for building rational approximations of the f_js. The combination of the approximations for all f_j lead to the polynomial $P(\lambda)$. The choice of interpolation points and the degree can be fixed in advance following a global criterion, but they can also be chosen dynamically, using, e.g., a Leja or Leja-Bagby approach [4]. Other global approximation criteria rely on a spectral discretization [5] or a Chebyshev least squares approximation [3]. The choice of polynomial basis relies on the chosen criterion. A small approximation error on f_j guarantees a small residual norm for the nonlinear eigenvalue problem.

The second step is writing the polynomial eigenvalue problem as a linear one. All matrix polynomials first need to be written in the form

$$P(\lambda) = \sum_{j=0}^{d-1} (A_j - \lambda B_j)\phi_j(\lambda).$$

With the polynomial basis, a $(d-1) \times d$ linear pencil $M - \lambda N$ is associated, so that $(M - \lambda N)[\phi_0, \ldots, \phi_{d-1}]^T = 0$. This leads to the following linearization of the matrix polynomial, which we call the CORK-linearization:

$$\mathbb{A} - \lambda\mathbb{B} = \begin{bmatrix} (A_0 - \lambda B_0) & \cdots & (A_{d-1} - \lambda B_{d-1}) \\ & (M - \lambda N) \otimes I & \end{bmatrix}$$

For a strong linearization the pencil $\mathbb{A} - \lambda\mathbb{B}$ has the same eigenvalues as $P(\lambda)$, which are approximations to eigenvalues of $A(\lambda)$.

The third step is the solution of the linear matrix pencil. Since the dimension is $nd \times nd$, standard Krylov methods may not be suitable: the storage cost of the iteration vectors may be unacceptably

high. The Compact Rational Krylov algorithm (CORK) [6][8] represents the iteration vectors in a compact form, using the fact that there is a large amount of redundancy in the iteration vectors. In fact, k iteration vectors of length nd can be represented by $d + k$ vectors of length n. We will show several numerical examples illustrating the effectiveness of the method, including the use of implicit restarting and locking. The computational cost of CORK is mainly dominated by the number of sparse LU factorizations of $A(\lambda)$. We will show how the strong features of CORK and contour integration methods can be combined.

Finally, we will discuss open problems for solving nonlinear and polynomial eigenvalue problems. First, scaling of the system matrices may influence the accuracy of the construction of Krylov spaces. We are not aware of any results for polynomial eigenvalue problems. Second, rational polynomials are key for good approximations of nonlinear functions. Yet, the rational eigenvalue problem is still not well understood, as well as their linearizations.

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