K.U.Leuven Optimization in Engineering Center

Opening new horizons

Efficient numerical methods for robust periodic optimal control with Lyapunov equations

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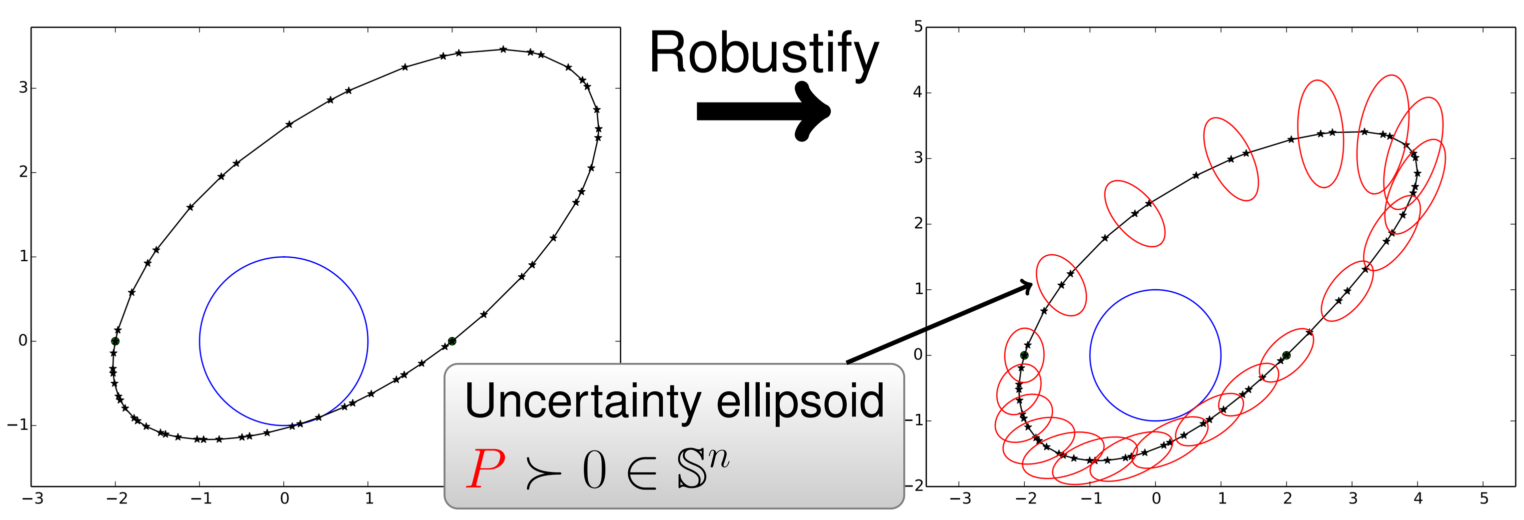
Abstract

This work compares various numerical methods to robustify periodic optimal control problems using the paradigm of Lyapunov differential equations. In this paradigm, estimates for state-covariance are obtained by solving the periodic Lyapunov equations for the a system linearised along a to-be-optimized trajectory, and are added to objective or constraints of the original optimal control problem. For non-trivial dynamical systems, method details were found to be critical to obtain algorithms with reasonable time complexity. An application for time-optimal quadcopter flight is worked out numerically with the optimal-control tool CasADi, which was extended by the author to solve discrete periodic Lyapunov equations using the SLICOT library.

Problem statement



Goal: Robustify the path constraints in a periodic optimal control problem (OCP), using the method of Lyapunov differential equations[2]

$$
\begin{array}{lll}
\underset{x(t),u(t),P(t)}{\text{minimize}} & J(x(\bullet),u(\bullet)) & \text{Objective} \\
\text{s.t. } \dot{x}(t) = f(x(t),u(t)) & & \text{System dynamics} \\
x(0) = x(T) & & \text{Periodic state} \\
0 \leq h(x(t)) + \underbrace{\gamma}_{\text{tuning knob}} \sqrt{\underbrace{\frac{\partial h}{\partial x} P(t) \frac{\partial h}{\partial x}^T}_{\text{Variance of } h(x)}} & & \text{Scalar path constraint} \\
\dot{P}(t) = \underbrace{A(t)P(t) + P(t)A^T(t)}_{\text{sink}} + \underbrace{Q(t)}_{\text{source}} & & \text{Covariance propagation} \\
P(0) = P(T) & & \text{Periodic covariance}
\end{array}
$$

Classic method: Augment state $[x; \mathrm{vec}(P)]$ and feed to your favourite OCP solver (multiple shooting, direct collocation, ...)

- 👍 pendulum example ($n = 2 \ldots 4$ states)
- 👎 real applications ($n = 10 \ldots 20$)

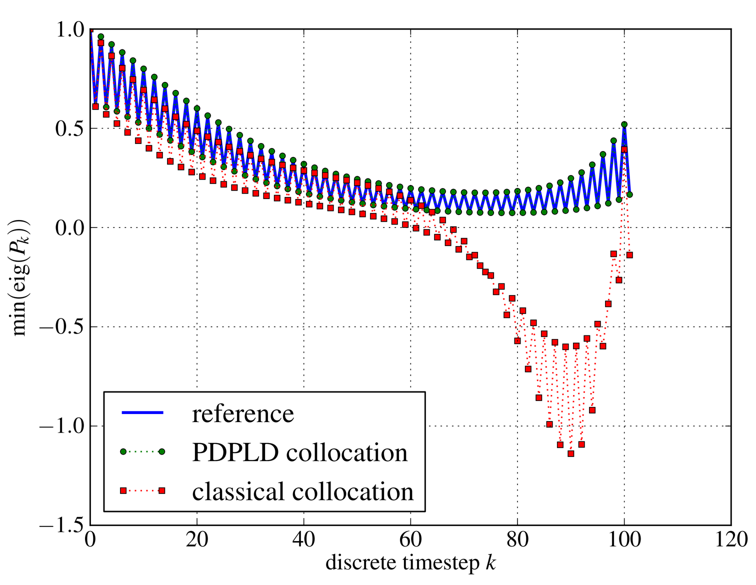
Smarter discretization[1]

Problem: Property of positive-definiteness not preserved during integration, e.g. forward Euler:

$$
\begin{aligned}
P_{k+1} &= P_k + \Delta t (A_k P_k + P_k A_k^T) \\
&= (\mathbf{1} + A_k \Delta t) P_k (\mathbf{1} + A_k \Delta t)^T - (\Delta t)^2 A_k P_k A_k^T
\end{aligned}
$$

Solution: Work directly in discrete time, using integrator sensitivities $\frac{\partial I}{\partial x}(x_k, u_k) \equiv \tilde{A}_k$ (automatic differentiation – AD):

$$P_{k+1} = \tilde{A}_k P_k \tilde{A}_k^T$$



Better complexity $O(n^6) \to O(n^3)$

Observation: The (discrete) robustified OCP has $n^2 N$ extra decision variables ($P_\bullet$) and $n^2 N$ extra (linear) constraints:

$$P_{(k+1) \mod N} = \tilde{A}_k P_k \tilde{A}_k^T + \tilde{Q}_k, \quad k = 0 \ldots (N-1)$$

→ Non-linear problem (NLP) with dense $n^2 - \text{by} - n^2$ blocks in constraint Jacobian, $O(n^6)$ runtime

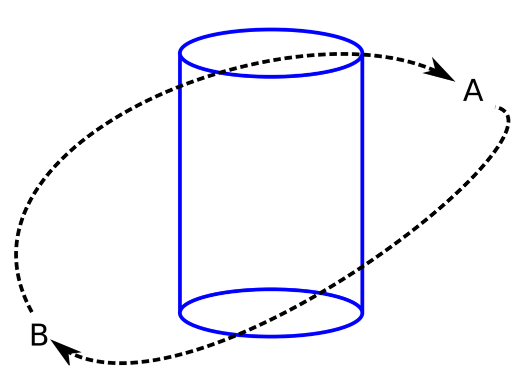
Improvement: Eliminate $P_\bullet$ and its constraints from the NLP:

$$P_\bullet = \mathtt{LyapSolver}\left(\tilde{A}_\bullet(x_\bullet, u_\bullet), \tilde{Q}_\bullet(x_\bullet, u_\bullet)\right)$$

→ Implemented periodic Schur decomposition solver[3], using SLICOT, $O(n^3)$ runtime

→ Embedded in a CasADi expression graph, implemented forward and adjoint mode AD

Quadcopter application



→ Fly periodically $A \to B$ around obstacle as fast as possible

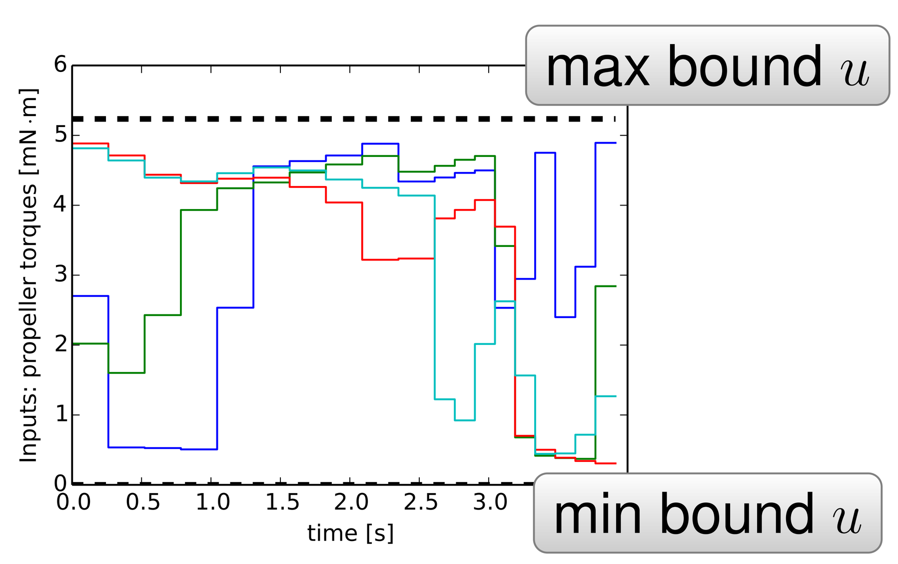
→ Nonlinear model ($n = 17$) with quaternions for orientation

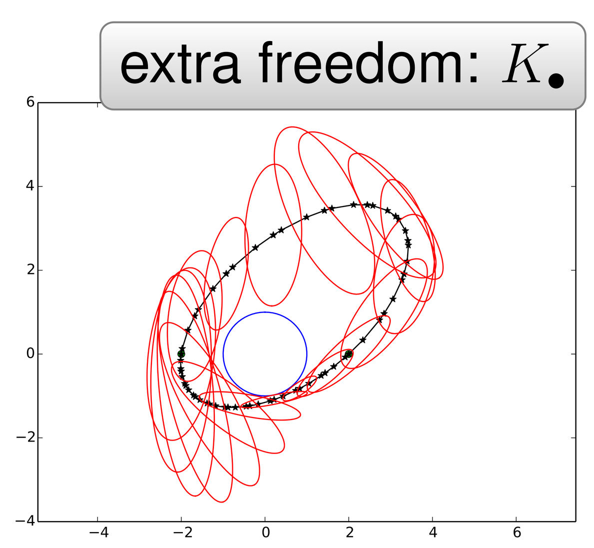
→ Linear feedback controller $K_\bullet$ to stabilize the system

Numerical: $N = 20$, 3rd-order Radau, IPOPT with BFGS

→ 1444 variables, 1548 constraints, 183739 nonzeros in Jacobian

→ 41 iterations to convergence, 2.4s for a Jacobian evaluation





References

[1] J. Gillis and M. Diehl, *A Positive Definiteness Preserving Discretization Method for nonlinear Lyapunov Differential Equations*, in Proceedings of the 52nd IEEE Conference on Decision and Control, 2013.

[2] B. Houska, *Robustness and Stability Optimization of Open-Loop Controlled Power Generating Kites*, Master's thesis, University of Heidelberg, 2007.

[3] A. Varga, *Periodic Lyapunov equations: some applications and new algorithms*, International Journal of Control, 67 (1997), pp. 69–88.