

Basket option pricing and implied correlation in a Lévy copula model

Daniël Linders, Wim Schoutens



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Abstract

In this paper we employ the Lévy copula model to determine basket option prices. More precisely, basket option prices are determined by replacing the real basket with an appropriate approximation. For the approximate basket we determine the underlying characteristic function and hence we can derive the related basket option prices by using the Carr-Madan formula. Two approaches are considered. In the first approach, we replace the arithmetic sum by an appropriate geometric sum, whereas the second approach can be considered as a three-moments-matching method. Numerical examples illustrate the accuracy of our approximations; several Lévy models are calibrated to market data and basket option prices are determined.

In a last part we show how our newly designed basket option pricing formula can be used to define implied Lévy correlation by matching model and market prices for basket options. Our main finding is that the implied Lévy correlation smile is flatter than its Gaussian counterpart. Furthermore, if (near) at-the-money option prices are used, the corresponding implied Gaussian correlation estimate is a good proxy for the implied Lévy correlation.

Keywords: basket options, characteristic function, implied correlation, Lévy market, Variance-Gamma.

1 Introduction

Nowadays, an increased volume of multi-asset derivatives is traded. An example of such a derivative is a *basket option*. The basic version of such a multivariate product has the same characteristics as a vanilla option, but now the underlying is a basket of stocks instead of a single stock. The pricing of these derivatives is not a trivial task because it requires a model that jointly describes the stock prices involved.

Stock price models based on the lognormal model proposed in Black and Scholes (1973) are popular choices from a computational point of view, however, they are not capable of capturing

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the skewness and kurtosis observed for log returns of stocks and indices. The class of Lévy processes provides a much better fit of the observed log returns and, consequently, the pricing of options and other derivatives in a Lévy setting is much more reliable. In this paper we consider the problem of pricing multi-asset derivatives in a multivariate Lévy model.

The most straightforward extension of the univariate Black & Scholes model is the *Gaussian copula model*, also called the multivariate Black & Scholes model. In the Gaussian copula model, the stocks composing the basket are assumed to be lognormal distributed and a Gaussian copula connects the marginals. Even in this simple setting, the price of a basket option is not given in closed form and has to be approximated; see e.g. Hull and White (1993), Brooks et al. (1994), Milevsky and Posner (1998), Rubinstein (1994), Deelstra et al. (2004), Carmona and Durrleman (2006) and Linders (2013), among others. However, the normality assumption for the marginals used in this pricing framework is too restrictive. Indeed, in Linders and Schoutens (2014) it is shown that calibrating the Gaussian copula model to market data can lead to non-meaningful parameter values. This dysfunctioning of the Gaussian copula model is typical observed in distressed periods. In this paper we extend the classical Gaussian pricing framework in order to overcome this problem.

Several extensions of the multivariate Black & Scholes model are proposed in order to model the joint dynamics of a number of stocks in a more realistic way. For example, Luciano and Schoutens (2006) introduce a multivariate Variance Gamma model where the dependence is modeled through a common jump component. This model was generalized in Semeraro (2008), Luciano and Semeraro (2010) and Guillaume (2013). A framework for modeling dependence in finance using copulas was described in Cherubini et al. (2004). However, the pricing of basket options in these advanced multivariate stock price models is not a straightforward task. There are several attempts to derive closed form approximations for the price of a basket option in a non-Gaussian world. In Linders and Stassen (2014), approximate basket option prices in a multivariate Variance Gamma model are derived, whereas Xu and Zheng (2010, 2014) consider a local volatility jump diffusion model. McWilliams (2011) derives approximations for the basket option price in a stochastic delay model.

In this paper we start from the one-factor Lévy model introduced in Albrecher et al. (2007) to build a multivariate stock price model with correlated Lévy marginals. Stock prices are assumed to be driven by an idiosyncratic and a systematic factor. Conditional on the common (or market) factor, the stock prices are independent. We show that our model generalizes the Gaussian model (with single correlation). Indeed, the idiosyncratic and systematic component are constructed from a Lévy process. Employing in that construction a Brownian motion delivers the Gaussian copula model, but other Lévy copulas arise by employing different Lévy processes like VG, NIG, Meixner, ... As a result, this new *Lévy copula model* is more flexible and can capture other types of dependence. From a tractability point of view the copula is still on the basis of a single correlation number.

In a first part of this paper, we consider the problem of finding accurate approximations for the price of a basket option in the Lévy copula model. In order to value a basket option, the distribution of this basket has to be determined. However, the basket is a weighted sum of dependent stock prices and its distribution function is unknown or too complex to work with. Therefore, we replace the random variable describing the basket price at maturity by a random variable with a more simple structure. Moreover, the characteristic function of the log transfor-

mation of this approximate random variable is given in closed form, such that the Carr-Madan formula can be used to determine approximate basket option prices. We propose two different approximations. Both methods are already applied for deriving approximate basket option prices in the multivariate Black & Scholes model. In this paper we show how to generalize the methodologies to the Lévy case.

In this paper, two methodologies for pricing basket options in a Gaussian copula model, proposed in Korn and Zeytun (2013) and Brigo et al. (2004), are generalized to our Lévy copula model. A basket is an arithmetic sum of dependent random variables. However, stock prices are modeled as exponentials of stochastic processes and therefore, a geometric average has a lot of computational advantages. The first methodology, proposed in Korn and Zeytun (2013), consists of constructing an approximate basket by replacing the arithmetic sum by a geometric sum. The second valuation formula is based on a moment-matching approximation. To be more precise, the distribution of the basket is replaced by a shifted random variable having the same first three moments than the original basket. This idea was first proposed in Brigo et al. (2004). Note that the approximations proposed in Korn and Zeytun (2013) and Brigo et al. (2004) were only worked out in the Gaussian copula model, whereas the approximations introduced in this paper allow for Lévy marginals and a Lévy copula. Furthermore, we determine the approximate basket option price using the Carr-Madan formula, whereas a closed form expression is available in the special situation considered in Korn and Zeytun (2013) and Brigo et al. (2004). Numerical examples illustrating the accuracy of the approximations are provided.

In a second part of the paper we show how the well-established notions of implied volatility and implied correlation can be generalized in our Lévy copula model; see also Corcuera et al. (2009). We assume that a finite number of options, written on the basket and the components, are traded. The prices of these derivatives are observable and will be used to calibrate the parameters of our stock price model. One main advantage of our Lévy copula model is that each stock is described by a volatility parameter and that the marginal parameters can be calibrated separately from the correlation parameter. We give numerical examples to show how to use the vanilla option curves to determine an implied Lévy volatility for each stock based on a Normal, VG, NIG and Meixner process and determine basket option prices for different choices of the correlation parameter. However, the available market prices for basket options together with our newly designed basket option pricing formula enables us to determine *implied Lévy correlation* estimates. Indeed, once we have calibrated the model to the vanilla option curves, the only unspecified parameter in our (approximate) basket option pricing formula is the correlation and an implied Lévy correlation estimate arises when we match the market and the model price of a basket option with given strike. We observe that implied correlation depends on the strike and the so-called *implied Lévy correlation smile* is flatter than its Gaussian (i.e. Black & Scholes) counterpart. The standard technique to price non-traded basket options (or other multi-asset derivatives), is by interpolating on the implied correlation curve. It is shown in Linders and Schoutens (2014) that in the Gaussian copula model, this can sometimes lead to non-meaningful correlation values. We show that the Lévy version of the implied correlation solves (at least to some extent) this problem.

This paper is organized as follows. In Section 2 we introduce the Lévy copula model as an extension of the classical Gaussian copula model. In Section 3 and Section 4 we propose two different approximating random variables and show how the Carr-Madan formula, discussed

in Section 5, can be used to determine approximate basket option prices. We give numerical illustrations in Section 6. Implied Lévy volatility and correlation are defined and investigated in Section 7.

2 Lévy copula model

We consider a market where n stocks are traded. The price level of stock j at some future time t , $0 \leq t \leq T$ is denoted by $S_j(t)$ ¹. Dividends are assumed to be paid continuously and the dividend yield of stock j is constant and deterministic over time. We denote this dividend yield by q_j . The price level at time t of a basket of stocks is denoted by $S(t)$ and given by

$$S(t) = \sum_{j=1}^n w_j S_j(t),$$

where $w_j > 0$ are weights which are fixed upfront. The pay-off of a basket option with strike K and maturity T is given by $(S(T) - K)_+$, where $(x)_+ = \max(x, 0)$. The price of this basket option is denoted by $C[K, T]$. We assume that the market is arbitrage-free and that there exists a risk-neutral pricing measure \mathbb{Q} such that the basket option price $C[K, T]$ can be expressed as the discounted risk-neutral expected value. In this pricing formula, discounting is performed using the risk-free interest rate r , which is, for simplicity, assumed to be deterministic and constant over time. Throughout the paper, we always assume that all expectations we encounter are well-defined and finite.

Since the seminal papers of Bachelier (1900) and Black and Scholes (1973), various models are proposed to capture the dynamics of the stocks and their dependence relations. We first revisit the Gaussian copula model. Next, we generalize this popular multidimensional model by allowing more flexibility when fitting the marginals.

2.1 Gaussian copula model

We show how the Gaussian copula model can be constructed using standard Brownian motions. Consider the independent standard Brownian motions $W = \{W(t) | t \geq 0\}$ and $W_j = \{W_j(t) | t \geq 0\}$, for $j = 1, 2, \dots, n$. Let $\rho \in [0, 1]$. The log returns of each stock j are driven by the r.v. Z_j . We assume that these log returns consist of a systematic component and a stock-specific, idiosyncratic, component. Therefore, the r.v. Z_j is given by:

$$Z_j = W(\rho) + W_j(1 - \rho), \quad j = 1, 2, \dots, n. \quad (1)$$

Because the Brownian motions W and W_j are independent, we find that $Z_j \stackrel{d}{=} N(0, 1)$. Furthermore, for $i \neq j$, the correlation between Z_i and Z_j is equal to ρ . Indeed, we have that

$$\begin{aligned} \text{Corr}[Z_i, Z_j] &= \text{Var}[W(\rho)] \\ &= \rho. \end{aligned}$$

¹We use the common approach to describe the financial market via a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. Furthermore, $(\mathcal{F}_t)_{0 \leq t \leq T}$ is the natural filtration and we assume that $\mathcal{F}_T = \mathcal{F}$.

The log returns of the different stocks are assumed to be described by the correlated r.v.'s Z_j , $j = 1, 2, \dots, n$. Each r.v. Z_j is standard normal distributed. In order to adjust the mean and the variance of the time- T stock price $S_j(T)$, we add a stock specific drift parameter $\mu_j \in \mathbb{R}$ and a volatility parameter $\sigma_j > 0$. The stock prices $S_j(T)$, $j = 1, 2, \dots, n$ at time T are then given by

$$S_j(T) = S_j(0)e^{\mu_j T + \sigma_j \sqrt{T} Z_j}, \quad j = 1, 2, \dots, n. \quad (2)$$

The stock price model (2) is also called the multivariate Black & Scholes model or Gaussian copula model, because the log returns are modeled by Normal marginals and a Gaussian copula connects these marginals. For a detailed description of the Black & Scholes model and its extensions, we refer to Black and Scholes (1973), Björk (1998), Carmona and Durrleman (2006) and Dhaene et al. (2013).

2.2 Generalization: the Lévy copula model

A crucial (and simplifying) assumption in the Gaussian copula model (2) is the normality assumption for the risk factors Z_j . Indeed, the r.v.'s Z_j are driven by a systematic factor $W(\rho)$ and a stock specific (idiosyncratic) factor $W_j(1 - \rho)$, where W and W_j are standard Brownian motions. However, it is well-known that log returns do not pass the test for normality. Indeed, log returns exhibit a skewed and leptokurtic distribution which cannot be captured by a normal distribution; see e.g. Schoutens (2003).

We generalize the Gaussian copula model outlined in the previous section, by allowing the risk factors to be distributed according to any infinitely divisible distribution with known characteristic function. This larger class of distributions increases the flexibility to find a more realistic distribution for the log returns. In Albrecher et al. (2007), a similar framework was considered for pricing CDO tranches. The Variance Gamma case was considered in Moosbrucker (2006), whereas Guillaume et al. (2009) consider the pricing of CDO-squared tranches in this one-factor Lévy model.

Consider an infinitely divisible distribution for which the characteristic function is denoted by ϕ . A stochastic process X can be build using this distribution. Such a process is called a Lévy process with mother distribution having characteristic function ϕ . The Lévy process $X = \{X(t) | t \geq 0\}$ based on this infinitely distribution starts at zero and has independent and stationary increments. Furthermore, for $t, s \geq 0$ the characteristic function of the increment $X(t + s) - X(t)$ is ϕ^s . For more details on Lévy processes, we refer to Sato (1999) and Schoutens (2003).

Assume that the random variable L has an infinitely divisible distribution and denote its characteristic function by ϕ_L . Consider the Lévy process $X = \{X(t) | t \in [0, 1]\}$ based on the distribution L . We assume that the process is standardized, i.e. $\mathbb{E}[X(1)] = 0$ and $\text{Var}[X(1)] = 1$. One can then show that $\text{Var}[X(t)] = t$, for $t \geq 0$. Define also a series of independent and standardized processes $X_j = \{X_j(t) | t \in [0, 1]\}$, for $j = 1, 2, \dots, n$ which are all build on the same distribution L and also independent from X . Take $\rho \in [0, 1]$. The r.v. A_j is defined as

$$A_j = X(\rho) + X_j(1 - \rho), \quad j = 1, 2, \dots, n. \quad (3)$$

In this construction, $X(\rho)$ and $X_j(1 - \rho)$ are random variables having characteristic function ϕ_L^ρ and $\phi_L^{1-\rho}$, respectively. Because the processes X and X_j are independent and standardized, we immediately find that

$$\mathbb{E}[A_j] = 0, \quad \text{Var}[A_j] = 1 \quad \text{and} \quad A_j \stackrel{d}{=} L, \quad \text{for } j = 1, 2, \dots, n. \quad (4)$$

The assumption that X and X_j are both Lévy processes based on the same mother distribution L is crucial for obtaining the equality $A_j \stackrel{d}{=} L$. However, this assumption can be relaxed and we can take X_j a Lévy process based on the mother infinitely distribution L_j . In this case, A_j has not the same distribution as L , but its characteristic function is given by $\phi_L^\rho \phi_{L_j}^{1-\rho}$. For notational convenience, we always consider the situation where all processes are based on the same distribution L in the rest of the paper.

The parameter ρ describes the correlation between A_i and A_j , if $i \neq j$. Indeed, it was proven in Albrecher et al. (2007) that in case A_j , $j = 1, 2, \dots, n$ is defined by (3), we have that

$$\text{Corr}[A_i, A_j] = \rho. \quad (5)$$

We model the stock price levels $S_j(T)$ at time T for $j = 1, 2, \dots, n$ as follows

$$S_j(T) = S_j(0)e^{\mu_j T + \sigma_j \sqrt{T} A_j}, \quad j = 1, 2, \dots, n, \quad (6)$$

where $\mu_j \in \mathbb{R}$ and $\sigma_j > 0$. Note that in this setting, each time- T stock price is modeled as the exponential of a Lévy process. Furthermore, a drift μ_j and a volatility parameter σ_j are added to match the characteristics of stock j . Our model, which we will call the *Lévy copula model*, can be considered as a generalization of the Gaussian copula model (2). Indeed, instead of a normal distribution, we allow for a Lévy distribution, while the Gaussian copula is generalized to a so-called Lévy copula. This Lévy copula model can also, at least to some extent, be considered as a generalization to the multidimensional case of the model proposed in Corcuera et al. (2009) and the parameter σ_j in (6) can then be interpreted as the Lévy space (implied) volatility of stock j . Another related model was proposed in Kawai (2009), where the dynamics of each stock price are modeled by a linear combination of independent Lévy processes. The idea of building a multivariate asset model by taking a linear combination of a systematic and an idiosyncratic process can also be found in Ballotta and Bonfiglioli (2014) and Ballotta et al. (2014).

2.3 The risk-neutral stock price processes

In order to obtain a martingale for the stock price processes, we change the drift μ_j of stock j such that the following relation holds

$$\mathbb{E}[S_j(T)] = e^{(r-q_j)T} S_j(0).$$

Plugging expression (6) in this equation results in

$$\mathbb{E} \left[S_j(0)e^{\mu_j T + \sigma_j \sqrt{T} A_j} \right] = e^{(r-q_j)T} S_j(0), \quad j = 1, 2, \dots, n,$$

from which we find that

$$\mu_j = (r - q_j) - \frac{1}{T} \log \phi_L \left(-i\sigma_j \sqrt{T} \right). \quad (7)$$

From expression (7) we conclude that the risk-neutral dynamics of the stocks in the Lévy copula model are given by

$$S_j(T) = S_j(0) e^{(r - q_j - \omega_j)T + \sigma_j \sqrt{T} A_j}, \quad j = 1, 2, \dots, n, \quad (8)$$

where $\omega_j = \frac{1}{T} \log \phi_L \left(-i\sigma_j \sqrt{T} \right)$ is a mean-correction which puts us directly in a risk-neutral world. We always assume that ω_j is finite.

The first three moments of $S_j(T)$ can be expressed in function of the characteristic function ϕ_L . By the martingale property, we have that $\mathbb{E}[S_j(T)] = S_j(0) e^{(r - q_j)T}$. The risk-neutral variance $\text{Var}[S_j(T)]$ can be written as follows

$$\text{Var}[S_j(T)] = S_j(0)^2 e^{2(r - q_j - \omega_j)T} \mathbb{E} \left[e^{2\sigma_j \sqrt{T} A_j} \right] - S_j(0)^2 e^{2(r - q_j)T}.$$

Because $A_j \stackrel{d}{=} L$ and the characteristic function of L is ϕ_L , we find

$$\text{Var}[S_j(T)] = S_j(0)^2 e^{2(r - q_j)T} \left(e^{-2\omega_j T} \phi_L \left(-i2\sigma_j \sqrt{T} \right) - 1 \right).$$

In a similar way, the second and third moment of $S_j(T)$ can be expressed in terms of the characteristic function ϕ_L :

$$\begin{aligned} \mathbb{E}[S_j(T)^2] &= \mathbb{E}[S_j(T)]^2 \frac{\phi_L \left(-i2\sigma_j \sqrt{T} \right)}{\phi_L \left(-i\sigma_j \sqrt{T} \right)^2}, \\ \mathbb{E}[S_j(T)^3] &= \mathbb{E}[S_j(T)]^3 \frac{\phi_L \left(-i3\sigma_j \sqrt{T} \right)}{\phi_L \left(-i\sigma_j \sqrt{T} \right)^3}. \end{aligned}$$

If the process X_j does not have the same mother distribution L than X , we can still determine the moments of $S_j(T)$. Indeed, assume that X_j has mother infinitely distribution L_j . Then we have to replace ϕ_L by $\phi_L^\rho \phi_{L_j}^{1-\rho}$ in expression (7) and the formulas for $\mathbb{E}[S_j(T)^2]$ and $\mathbb{E}[S_j(T)^3]$.

In the following sections, we propose two methodologies to approximate the risk-neutral distribution of the sum $S(T)$. In Section 3 we replace the arithmetic sum $S(T)$ by a geometric sum, whereas a three-moments matching approach is considered in Section 4. In both situations, the r.v. $S(T)$ is replaced by an approximate r.v. S^* for which the characteristic function $\phi_{\log S^*}$ is known. Approximate basket option prices can then be derived by using the Carr-Madan formula.

3 Approximating the arithmetic sum by a geometric sum

3.1 The geometric sum

The random variable $S(T)$ is a weighted sum of the dependent random variables $S_j(T)$, $j = 1, 2, \dots, n$. Its distribution function cannot be determined in an analytical form, which makes the calculation of $C[K, T]$ a difficult task. In this section we approximate the r.v. $S(T)$ by a more tractable r.v. S^* such that the characteristic function $\phi_{\log S^*}$ is known. A suitable approximating random variable is based on the following lemma by Korn and Zeytun (2013).

Lemma 1 *Let s_1, s_2, \dots, s_n be a set of non-negative numbers. Then the following asymptotic relations hold:*

$$\lim_{\kappa \rightarrow \infty} \frac{\left(\frac{1}{n} \sum_{j=1}^n (s_j + \kappa)\right)^n}{\prod_{j=1}^n (s_j + \kappa)} = 1, \quad (9)$$

$$\frac{\left(\frac{1}{n} \sum_{j=1}^n (s_j + \kappa)\right)^n}{\prod_{j=1}^n (s_j + \kappa)} = 1 + \frac{1}{\kappa} \sum_{j=1}^n s_j + \mathcal{O}\left(\frac{1}{\kappa^2}\right). \quad (10)$$

A geometric sum of stock prices is more tractable than an arithmetic sum, because stock prices are modeled as exponentials of correlated Lévy processes. The lemma stated above provides a way to move from an arithmetic sum to a geometric sum. Indeed, Lemma 1 shows that the geometric average $\prod_{j=1}^n (s_j + \kappa)^{1/n}$ is a good approximation for the arithmetic average $\frac{1}{n} \sum_{j=1}^n (s_j + \kappa)$, provided κ is sufficiently large.

The pay-off of the basket option $C[K, T]$ can be rewritten as follows

$$\left(\sum_{j=1}^n w_j S_j(T) - K\right)_+ = \left(\frac{1}{n} \sum_{i=1}^n (nw_j S_j(T) + \kappa) - (K + \kappa)\right)_+,$$

where $\kappa \in \mathbb{R}$. If κ is large, we can use Lemma 1 and derive that the pay-off can be approximated as follows

$$\left(\sum_{j=1}^n w_j S_j(T) - K\right)_+ \approx \left(\prod_{j=1}^n (nw_j S_j(T) + \kappa)^{1/n} - (K + \kappa)\right)_+ \quad (11)$$

$$= \left(\tilde{S}(T) - \tilde{K}\right)_+, \quad (12)$$

where

$$\tilde{S}(T) = \prod_{j=1}^n (nw_j S_j(T) + \kappa)^{1/n}, \quad (13)$$

and $\tilde{K} = K + \kappa$. By adding a sufficiently large number κ to the basket, one can approximate the arithmetic sum by a geometric sum of r.v.'s, where each r.v. is now shifted by the constant κ . However, we would like to decompose $\log \tilde{S}(T)$ in terms of the logarithms $\log S_j(T)$, but the shift κ throws a spanner in the works. In the next section, we show how to deal with this problem.

3.2 Approximating the shifted log infinitely divisible distribution

In order to determine the basket option price $C[K, T]$, we replace the arithmetic sum $S(T)$ by the geometric sum $\tilde{S}(T)$, defined in (13). The terms of the geometric sum $\tilde{S}(T)$ are denoted by Y_j , $j = 1, 2, \dots, n$ and we have that $Y_j = nw_j S_j(T) + \kappa$. The key ingredient for applying the Carr-Madan option pricing formula is an analytical expression for the characteristic function of $\log \tilde{S}(T)$. Unfortunately, the characteristic function of $\log Y_j$, and hence also the characteristic function of $\log \tilde{S}(T)$, is unknown. Therefore, we replace the r.v. Y_j by Y_j^* which is defined as

$$Y_j^* = e^{\mu_j^* + \sigma_j^* A_j}. \quad (14)$$

Remember that the characteristic function of A_j is denoted by ϕ_L . If the parameters μ^* , σ^* and the characteristic function ϕ_L are known, also the characteristic function of $\log Y_j^*$ is given in an analytical form.

We can now show how to determine the parameters μ^* and σ^* . It can easily be obtained that

$$\mathbb{E}[Y_j^*] = e^{\mu_j^*} \phi_L(-i\sigma_j^*), \quad \text{Var}[Y_j^*] = e^{2\mu_j^*} \left(\phi_L(-i2\sigma_j^*) - \phi_L(-i\sigma_j^*)^2 \right). \quad (15)$$

The parameters μ_j^* and σ_j^* are determined such that the first and second moment of Y_j and Y_j^* coincide. This results in:

$$\mathbb{E}[Y_j^*] = \mathbb{E}[Y_j], \quad (16)$$

$$\text{Var}[Y_j^*] = \text{Var}[Y_j]. \quad (17)$$

Using the expression (15) for the mean of Y_j^* , we find that equation (16) can be rewritten as follows:

$$e^{\mu_j^*} \phi_L(-i\sigma_j^*) = \mathbb{E}[Y_j],$$

from which we derive that

$$\mu_j^* = \log \left(\frac{\mathbb{E}[Y_j]}{\phi_L(-i\sigma_j^*)} \right). \quad (18)$$

Plugging expression (18) for μ_j^* in the equation for the variance $\text{Var}[Y_j^*]$ yields

$$\begin{aligned} \text{Var}[Y_j^*] &= e^{2\log \left(\frac{\mathbb{E}[Y_j]}{\phi_L(-i\sigma_j^*)} \right)} \left(\phi_L(-i2\sigma_j^*) - \phi_L(-i\sigma_j^*)^2 \right) \\ &= \mathbb{E}[Y_j]^2 \left(\frac{\phi_L(-i2\sigma_j^*)}{\phi_L(-i\sigma_j^*)^2} - 1 \right). \end{aligned}$$

The equation (17) then becomes

$$\mathbb{E}[Y_j]^2 \left(\frac{\phi_L(-i2\sigma_j^*)}{\phi_L(-i\sigma_j^*)^2} - 1 \right) = \text{Var}[Y_j].$$

We find that σ_j^* can be determined from the following implicit relation

$$\frac{\phi_L(-i2\sigma_j^*)}{\phi_L(-i\sigma_j^*)^2} = \frac{\text{Var}[Y_j]}{\mathbb{E}[Y_j]^2} + 1. \quad (19)$$

We always silently assume that $\Pr[L > 0] > 0$, which implies that $\lim_{a \rightarrow +\infty} \phi_L(-ia) = +\infty$. Define the function f as $f(a) = \frac{\phi_L(-i2a)}{\phi_L(-ia)^2}$. Then f is a continuous function and $\lim_{a \rightarrow 0} f(a) = 1$. By the Cauchy-Schwarz inequality, we also find that $\lim_{a \rightarrow +\infty} f(a) = +\infty$. Moreover, the right-hand-side of equation (19) is always larger than one. As a result we find that (19) always has a solution σ_j^* , for $j = 1, 2, \dots, n$. We have that the mean $\mathbb{E}[Y_j]$ is equal to $nw_j\mathbb{E}[S_j(T)] + \kappa$ and the variance $\text{Var}[Y_j]$ is equal to $n^2w_j^2\text{Var}[S_j(T)]$. These moments can be determined using the expressions derived in Section 2.3 for $\mathbb{E}[S_j(T)]$ and $\text{Var}[S_j(T)]$.

3.3 Approximating the basket option price

The basket option price $C[K, T]$ is approximated by replacing the arithmetic sum $S(T)$ by an appropriate geometric sum and by changing the strike K ; see (11). For each j , the random variable $nw_jS_j(T) + \kappa$ is then approximated by Y_j^* , defined in (14), where σ_j^* has to be determined using the relation (19) and μ_j^* then follows from (18). This results in the approximation $C^{GA}[K, T]$ for $C[K, T]$:

$$C^{GA}[K, T] = e^{-rT} \mathbb{E} \left[\left(\prod_{j=1}^n (Y_j^*)^{1/n} - (K + \kappa) \right)_+ \right]. \quad (20)$$

We define the random variable S^* as follows

$$S^* = \prod_{j=1}^n (Y_j^*)^{1/n}. \quad (21)$$

In the following theorem, we give an expression for the characteristic function of the random variable $\log S^*$ in terms of the characteristic function ϕ_L of the mother infinitely distribution L .

Theorem 1 *Consider the Lévy copula model (8) with mother infinitely divisible distribution L . The characteristic function $\phi_{\log S^*}$ of the random variable $\log S^*$ is given by*

$$\phi_{\log S^*}(u) = \mathbb{E} [e^{iu \log S^*}] = e^{\frac{iu}{n} \sum_{j=1}^n \mu_j^*} \phi_L \left(u \sum_{j=1}^n \frac{\sigma_j^*}{n} \right)^\rho \prod_{j=1}^n \phi_L \left(\frac{u\sigma_j^*}{n} \right)^{1-\rho}. \quad (22)$$

Proof. Combining expressions (21) and (14), we find that

$$\log S^* = \frac{1}{n} \left(\sum_{j=1}^n \mu_j^* + \sum_{j=1}^n \sigma_j^* A_j \right).$$

Then we can express $\mathbb{E} [e^{iu \log S^*}]$ as follows

$$\mathbb{E} [e^{iu \log S^*}] = e^{\frac{iu}{n} \sum_{j=1}^n \mu_j^*} \mathbb{E} \left[\exp \left\{ \frac{iu}{n} \sum_{j=1}^n \sigma_j^* (X(\rho) + X_j(1 - \rho)) \right\} \right].$$

We can now rewrite $\mathbb{E} [e^{iu \log S^*}]$ using the independence between $X(\rho)$ and $\sum_{j=1}^n X_j(1 - \rho)$

$$\mathbb{E} [e^{iu \log S^*}] = e^{\frac{iu}{n} \sum_{j=1}^n \mu_j^*} \mathbb{E} \left[\exp \left\{ \frac{iu}{n} \sum_{j=1}^n \sigma_j^* X(\rho) \right\} \right] \mathbb{E} \left[\exp \left\{ \frac{iu}{n} \sum_{j=1}^n \sigma_j^* X_j(1 - \rho) \right\} \right].$$

Note also that X_j , $j = 1, 2, \dots, n$ are independent processes, which results in the following expression:

$$\mathbb{E} [e^{iu \log S^*}] = e^{\frac{iu}{n} \sum_{j=1}^n \mu_j^*} \mathbb{E} \left[\exp \left\{ \frac{iu}{n} \sum_{j=1}^n \sigma_j^* X(\rho) \right\} \right] \prod_{j=1}^n \mathbb{E} \left[\exp \left\{ \frac{iu}{n} \sigma_j^* X_j(1 - \rho) \right\} \right].$$

The processes X and X_j , $j = 1, 2, \dots, n$ have all the same mother infinitely divisible distribution L . This implies that

$$\mathbb{E} [e^{iuX(\rho)}] = \phi_L(u)^\rho, \quad (23)$$

$$\mathbb{E} [e^{iuX_j(1-\rho)}] = \phi_L(u)^{1-\rho}, \quad \text{for } j = 1, 2, \dots, n, \quad (24)$$

from which we find that (22) holds. ■

In Section 5 we show that if $\phi_{\log S^*}$ is given in closed form, the approximate basket option prices $C^{GA}[K, T]$ can be determined using a robust and fast algorithm. Note also that it is not hard to adapt the proof in order to find the characteristic function $\phi_{\log S^*}$ in case the processes X and X_j , $j = 1, 2, \dots, n$ do not have the same mother distribution L . Indeed, in this situation, the right-hand-side of expression (24) has to be replaced by $\phi_{L_j}^{1-\rho}$.

4 A three-moments-matching approximation

In this section we introduce a second approach for approximating $C[K, T]$ by replacing the sum $S(T)$ with an appropriate random variable $\tilde{S}(T)$ which has a simpler structure, but for which the first three moments coincide with the first three moments of the original basket $S(T)$. This moment-matching approach was also considered in Brigo et al. (2004) for the multivariate Black & Scholes model. Consider the Lévy process $Y = \{Y(t) \mid 0 \leq t \leq 1\}$ with infinitely divisible distribution L . Furthermore, we define the random variable A as

$$A = Y(1).$$

In this case, the characteristic function of A is given by ϕ_L . The sum $S(T)$ is a weighted sum of dependent random variables and its cdf is unknown. We approximate the sum $S(T)$ by $\tilde{S}(T)$, defined as

$$\tilde{S}(T) = \bar{S}(T) + \lambda, \quad (25)$$

where $\lambda \in \mathbb{R}$ and

$$\bar{S}(T) = S(0) \exp \left\{ (\bar{\mu} - \bar{\omega})T + \bar{\sigma}\sqrt{T}A \right\}. \quad (26)$$

The parameter $\bar{\mu} \in \mathbb{R}$ determines the drift and $\bar{\sigma} > 0$ is the volatility parameter. These parameters, as well as the shifting parameter λ are determined such that the first three moments of $\bar{S}(T)$ are coinciding with the corresponding moments of $S(T)$. The parameter $\bar{\omega}$, defined as follows

$$\bar{\omega} = \frac{1}{T} \log \phi_L \left(-i\bar{\sigma}\sqrt{T} \right),$$

is a mean-correcting parameter and is assumed to be finite.

4.1 Matching the first three moments

The first three moments of the basket $S(T)$ are denoted by m_1, m_2 and m_3 respectively. In the following lemma, we express the moments m_1, m_2 and m_3 in terms of the characteristic function ϕ_L and the marginal parameters. A proof of this lemma is provided in the appendix.

Lemma 2 *Consider the Lévy copula model (8) with mother infinitely divisible distribution L . The first two moments m_1 and m_2 of the basket $S(T)$ can be expressed as follows*

$$m_1 = \sum_{j=1}^n w_j \mathbb{E} [S_j(T)], \quad (27)$$

$$m_2 = \sum_{j=1}^n \sum_{k=1}^n w_j w_k \mathbb{E} [S_j(T)] \mathbb{E} [S_k(T)] \left(\frac{\phi_L \left(-i(\sigma_j + \sigma_k)\sqrt{T} \right)}{\phi_L \left(-i\sigma_j\sqrt{T} \right) \phi_L \left(-i\sigma_k\sqrt{T} \right)} \right)^{\rho_{j,k}}, \quad (28)$$

where

$$\rho_{j,k} = \begin{cases} \rho, & \text{if } j \neq k; \\ 1, & \text{if } j = k. \end{cases}$$

The third moment m_3 of the basket $S(T)$ is given by

$$m_3 = \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n w_j w_k w_l \mathbb{E} [S_j(T)] \mathbb{E} [S_k(T)] \mathbb{E} [S_l(T)] \times \frac{\phi_L \left(-i(\sigma_j + \sigma_k + \sigma_l)\sqrt{T} \right)^\rho}{\phi_L \left(-i\sigma_j\sqrt{T} \right) \phi_L \left(-i\sigma_k\sqrt{T} \right) \phi_L \left(-i\sigma_l\sqrt{T} \right)} A_{j,k,l}, \quad (29)$$

where

$$A_{j,k,l} = \begin{cases} \left(\phi_L \left(-i\sigma_j\sqrt{T} \right) \phi_L \left(-i\sigma_k\sqrt{T} \right) \phi_L \left(-i\sigma_l\sqrt{T} \right) \right)^{1-\rho}, & \text{if } j \neq k, k \neq l \text{ and } j \neq l; \\ \left(\phi_L \left(-i(\sigma_j + \sigma_k)\sqrt{T} \right) \phi_L \left(-i\sigma_l\sqrt{T} \right) \right)^{1-\rho}, & \text{if } j = k, k \neq l. \\ \left(\phi_L \left(-i(\sigma_k + \sigma_l)\sqrt{T} \right) \phi_L \left(-i\sigma_j\sqrt{T} \right) \right)^{1-\rho}, & \text{if } j \neq k, k = l. \\ \left(\phi_L \left(-i(\sigma_j + \sigma_l)\sqrt{T} \right) \phi_L \left(-i\sigma_k\sqrt{T} \right) \right)^{1-\rho}, & \text{if } j = l, k \neq l. \\ \phi_L \left(-i(\sigma_j + \sigma_k + \sigma_l)\sqrt{T} \right)^{1-\rho}, & \text{if } j = k = l. \end{cases}$$

In Section 2.3 we derived the first three moments for each stock j , $j = 1, 2, \dots, n$. Taking into account the similarity between the price $S_j(T)$ defined in (8) and the approximate r.v. $\bar{S}(T)$, defined in (26), we can determine the first three moments of $\bar{S}(T)$:

$$\begin{aligned}\mathbb{E} [\bar{S}(T)] &= S(0)e^{\bar{\mu}T} =: \xi, \\ \mathbb{E} [\bar{S}(T)^2] &= \mathbb{E} [\bar{S}(T)]^2 \frac{\phi_L(-i2\bar{\sigma}\sqrt{T})}{\phi_L(-i\bar{\sigma}\sqrt{T})^2} =: \xi^2\alpha, \\ \mathbb{E} [\bar{S}(T)^3] &= \mathbb{E} [\bar{S}(T)]^3 \frac{\phi_L(-i3\bar{\sigma}\sqrt{T})}{\phi_L(-i\bar{\sigma}\sqrt{T})^3} =: \xi^3\beta.\end{aligned}$$

These expressions can now be used to determine the first three moments of the approximate r.v. $\tilde{S}(T)$:

$$\begin{aligned}\mathbb{E} [\tilde{S}(T)] &= \mathbb{E} [\bar{S}(T)] + \lambda, \\ \mathbb{E} [\tilde{S}(T)^2] &= \mathbb{E} [\bar{S}(T)^2] + \lambda^2 + 2\lambda\mathbb{E} [\bar{S}(T)], \\ \mathbb{E} [\tilde{S}(T)^3] &= \mathbb{E} [\bar{S}(T)^3] + \lambda^3 + 3\lambda^2\mathbb{E} [\bar{S}(T)] + 3\lambda\mathbb{E} [\bar{S}(T)^2].\end{aligned}$$

Determining the parameters $\bar{\mu}$, $\bar{\sigma}$ and the shifting parameter λ by matching the first three moments results in the following set of equations

$$\begin{aligned}m_1 &= \xi + \lambda, \\ m_2 &= \xi^2\alpha + \lambda^2 + 2\lambda\xi, \\ m_3 &= \xi^3\beta + \lambda^3 + 3\lambda^2\xi + 3\lambda\xi^2\alpha.\end{aligned}$$

These equations can be recast in the following set of equations

$$\begin{aligned}\lambda &= m_1 - \xi, \\ \xi^2 &= \frac{m_2 - m_1^2}{\alpha - 1}, \\ 0 &= \left(\frac{m_2 - m_1^2}{\alpha - 1}\right)^{3/2} (\beta + 2 - 3\alpha) + 3m_1m_2 - 2m_1^3 - m_3.\end{aligned}$$

Remember that α and β are defined by

$$\alpha = \frac{\phi_L(-i2\bar{\sigma}\sqrt{T})}{\phi_L(-i\bar{\sigma}\sqrt{T})^2} \quad \text{and} \quad \beta = \frac{\phi_L(-i3\bar{\sigma}\sqrt{T})}{\phi_L(-i\bar{\sigma}\sqrt{T})^3}.$$

Solving the third equation results in the parameter $\bar{\sigma}$. Note that this equation does not always have a solution. This issue was also discussed in Brigo et al. (2004) for the Gaussian copula

case. However, in our numerical studies we did not encounter any numerical problems. If we know $\bar{\sigma}$, we can also determine ξ and λ from the first two equations. Next, the drift $\bar{\mu}$ can be determined from

$$\bar{\mu} = \frac{1}{T} \log \frac{\xi}{S(0)}.$$

Finally, the mean-correcting parameter $\bar{\omega}$ is given by $\frac{1}{T} \log \phi_L \left(-i\bar{\sigma}\sqrt{T} \right)$.

4.2 Approximate basket option pricing

The price of a basket option with strike K and maturity T is denoted by $C[K, T]$. This unknown price is in this section approximated by $C^{MM}[K, T]$, which is defined as

$$C^{MM}[K, T] = e^{-rT} \mathbb{E} \left[\left(\tilde{S}(T) - K \right)_+ \right].$$

Using expression (25) for $\tilde{S}(T)$, the price $C^{MM}[K, T]$ can be expressed as

$$C^{MM}[K, T] = e^{-rT} \mathbb{E} \left[\left(\bar{S}(T) - (K - \lambda) \right)_+ \right].$$

Note that $\bar{S}(T)$ is also depending on the choice of λ . In order to determine the price $C^{MM}[K, T]$, we should be able to price an option written on $\bar{S}(T)$, with a shifted strike $K - \lambda$. Determining the approximation $C^{MM}[K, T]$ using the Carr-Madan formula requires knowledge about the characteristic function $\phi_{\log \bar{S}(T)}$ of $\log \bar{S}(T)$:

$$\phi_{\log \bar{S}(T)}(u) = \mathbb{E} \left[e^{iu \log \bar{S}(T)} \right].$$

Using expression (26) we find

$$\phi_{\log \bar{S}(T)}(u) = \mathbb{E} \left[\exp \left\{ iu \left(\log S(0) + (\bar{\mu} - \bar{\omega})T + \bar{\sigma}\sqrt{T}A \right) \right\} \right].$$

The characteristic function of A is ϕ_L , from which we find

$$\phi_{\log \bar{S}(T)}(u) = \exp \{ iu (\log S(0) + (\bar{\mu} - \bar{\omega})T) \} \phi_L \left(u\bar{\sigma}\sqrt{T} \right).$$

Note that nowhere in this section, we used the assumption that the basket weights w_j are strictly positive. Therefore, the three-moments-matching approach proposed in this section can also be used to price e.g. spread options.

5 The FFT method and basket option pricing

Our two methodologies for pricing basket options consist both in approximating the basket $S(T)$ by a random variable with a simpler structure and for which the characteristic function

is known in closed form. Denote this approximate random variable by X . In this section we show that in case the characteristic function $\phi_{\log X}$ of a random variable X is known, one can approximate

$$e^{-rT} \mathbb{E} [(X - K)_+],$$

for any $K > 0$.

Let $\alpha > 0$ and assume that $\mathbb{E} [X^{\alpha+1}]$ exists and is finite. It was proven in Carr et al. (1999) that the price $e^{-rT} \mathbb{E} [(X - K)_+]$ can be expressed as follows

$$e^{-rT} \mathbb{E} [(X - K)_+] = \frac{e^{-\alpha \log(K)}}{\pi} \int_0^{+\infty} \exp \{-iv \log(K)\} g(v) dv, \quad (30)$$

where

$$g(v) = \frac{e^{-rT} \phi_{\log X}(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}. \quad (31)$$

In order to determine the approximation $C^{GA}[K, T]$ proposed in Section 3, the random variable X has to be understood as the geometric sum S^* . Furthermore, $C[K, T]$ is approximated by determining option prices written on S^* but where the strike price K is shifted by a constant κ :

$$C^{GA}[K, T] = e^{-rT} \mathbb{E} [(S^* - (K + \kappa))_+].$$

Note that the random variable S^* also depends on κ . According to Lemma 1, the shift κ has to be chosen sufficiently large; typical values are in the range $[10^4, 10^7]$. The approximation $C^{MM}[K, T]$ was introduced in Section 4 and the random variable X now denotes the moment-matching approximation $\tilde{S}(T) = \bar{S}(T) + \lambda$. The approximation $C^{MM}[K, T]$ can then be determined as the option price written on $\bar{S}(T)$ and with shifted strike price $K - \lambda$.

6 Examples and numerical illustrations

The Gaussian copula model is a member of our class of Lévy copula models and is already described in Section 2. In this section we discuss how to build the Variance Gamma, Normal Inverse Gaussian and Meixner copula models. However, the reader is invited to construct Lévy copula models based on other Lévy based distributions; e.g. CGMY, Generalized hyperbolic, ... distributions. In each situation, the methodology for pricing basket options consists of replacing the original basket by an appropriate approximate basket. This idea was also considered in Dhaene et al. (2002a,b) for the Gaussian copula case and extended in Valdez et al. (2009) to the elliptical copula case.

Table 1 summarizes the Variance Gamma, Normal Inverse Gaussian and the Meixner distributions, which are all infinitely divisible. In the last row, it is shown how to construct for each of these distributions a standardized version. We assume that L is distributed according to one of these standardized distributions. Hence, L has zero mean and unit variance. Furthermore, the characteristic function ϕ_L of L is given in closed form. We can then define the Lévy processes X and X_j , $j = 1, 2, \dots, n$ based on the mother distribution L . The random variables A_j , $j = 1, 2, \dots, n$, are modeled as follows

$$A_j = X(\rho) + X_j(1 - \rho), \quad j = 1, 2, \dots, n,$$

where X and $X_j, j = 1, 2, \dots, n$ are independent Lévy processes with mother infinitely divisible distribution L . More details can be found in Albrecher et al. (2007).

6.1 Variance Gamma

Although pricing basket options under a normality assumption is tractable from a computational point of view, it introduces a high degree of model risk; see e.g. Leoni and Schoutens (2008). The Variance Gamma distribution has already been proposed as a more flexible alternative to the Brownian setting; see e.g. Madan and Seneta (1990) and Madan et al. (1998).

We consider two numerical examples where L has a Variance Gamma distribution with parameters $\sigma = 0.5695, \nu = 0.75, \theta = -0.9492, \mu = 0.9492$. Table 2 contains numerical values for a four-basket option paying $\left(\frac{1}{4} \sum_{j=1}^4 S_j(T) - K\right)_+$ at time T . We use the following parameter values: $r = 6\%, T = 0.5, \rho = 0$ and $S_1(0) = 40, S_2(0) = 50, S_3(0) = 60, S_4(0) = 70$. These parameter values are also used in Section 5 of Korn and Zeytun (2013). We denote by $C^{mc}[K, T]$ the corresponding Monte Carlo estimate for the price $C[K, T]$. Here, 10^6 number of simulations are used. One can approximate the price $C[K, T]$ by replacing the arithmetic sum by a geometric sum as explained in Section 3. This approximation is denoted by $C^{GA}[K, T]$. In order to determine $C^{GA}[K, T]$, we use $C = 10^4$. Alternatively, the price $C[K, T]$ is approximated by using the moment-matching approach outlined in Section 4. This approximation is denoted by $C^{MM}[K, T]$.

In a second example, we consider the basket $S(T) = w_1 X_1(T) + w_2 X_2(T)$. The interest rate r is set to 5%. We determine option prices for the maturities $T = 1$ and $T = 3$. Note that strike prices are expressed in terms of forward moneyness. A basket strike price K has forward moneyness equal to $\frac{K}{\mathbb{E}[S]}$. We assume that the current prices of the non-dividend paying stocks are given by $X_1(0) = X_2(0) = 100$ and the weights are equal, $w_1 = w_2 = 0.5$. These parameter values are also used in Section 7 of Deelstra et al. (2004). Table 3 gives numerical values for these basket options. We can conclude that the three-moments-matching approximation gives more accurate results than the geometric average approximation. Especially for far out-of-the money call options, the approximation based on the geometric average is not able to closely approximate the real basket option price, whereas the accuracy of the three-moments-matching approximation is better.

6.2 Pricing basket options

In this subsection we explain how to determine the price of a basket option in a realistic situation where option prices of the components of the basket are available and used to calibrate the marginal parameters. In our example, the basket under consideration consists of 2 major stock market indices, the S&P500 and the Nasdaq. The pricing date is February 19, 2009 and we determine prices for the Normal, VG, NIG and Meixner case. The details of the basket are listed in Table 4. The weights w_1 and w_2 are chosen such that the initial price $S(0)$ of the basket is equal to 100. The maturity of the basket option is equal to 30 days. The S&P 500 and Nasdaq option curves are denoted by C_1 and C_2 respectively, and are shown in Figure 1. These

Table 1: Overview of infinitely divisible distributions.

	Gaussian	Variance Gamma
Parameters	$\mu \in \mathbb{R}, \sigma > 0$	$\mu, \theta \in \mathbb{R}, \sigma, \nu > 0$
Notation	$N(\mu, \sigma^2)$	$VG(\sigma, \nu, \theta, \mu)$
$\phi(u)$	$e^{iu\mu + \frac{1}{2}\sigma^2 u^2}$	$e^{iu\mu} (1 - iu\theta\nu + u^2\sigma^2\nu/2)^{-1/\nu}$
Mean	μ	$\mu + \theta$
Variance	σ^2	$\sigma^2 + \nu\theta^2$
Standardized version	$N(0, 1)$	$VG(\kappa\sigma, \nu, \kappa\theta, -\kappa\theta)$ where $\kappa = \frac{1}{\sqrt{\sigma^2 + \theta^2\nu}}$
	Normal Inverse Gaussian	Meixner
Parameters	$\alpha, \delta > 0, \beta \in (-\alpha, \alpha), \mu \in \mathbb{R}$	$\alpha, \delta > 0, \beta \in (-\pi, \pi), \mu \in \mathbb{R}$
Notation	$NIG(\alpha, \beta, \delta, \mu)$	$MX(\alpha, \beta, \delta, \mu)$
$\phi(u)$	$e^{iu\mu - \delta(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2})}$	$e^{iu\mu} \left(\frac{\cos(\beta/2)}{\cosh((\alpha u - i\beta)/2)} \right)^{2\delta}$
Mean	$\mu + \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}}$	$\mu + \alpha\delta \tan(\beta/2)$
Variance	$\alpha^2\delta (\alpha^2 - \beta^2)^{-3/2}$	$\cos^{-2}(\beta/2)\alpha^2\delta/2$
Standardized version	$NIG\left(\alpha, \beta, (\alpha^2 - \beta^2)^{3/2}, \frac{-(\alpha^2 - \beta^2)\beta}{\alpha^2}\right)$	$MX\left(\alpha, \beta, \frac{2\cos^2(\frac{\beta}{2})}{\alpha^2}, \frac{-\sin(\beta)}{\alpha}\right)$

Table 2: Basket option prices in the VG copula model with $S_1(0) = 40$, $S_2(0) = 50$, $S_3(0) = 60$, $S_4(0) = 70$ and $\rho = 0$.

K	$C^{mc}[K, T]$	$C^{GA}[K, T]$	$C^{MM}[K, T]$
$\sigma_1 = 0.2; \sigma_2 = 0.2; \sigma_3 = 0.2; \sigma_4 = 0.2$			
50	6.5770	6.5601	6.5676
55	2.4373	2.4357	2.4781
60	0.2658	0.2675	0.2280
$\sigma_1 = 0.5; \sigma_2 = 0.5; \sigma_3 = 0.5; \sigma_4 = 0.5$			
55	4.1034	4.0780	4.2089
60	1.7768	1.5862	1.7976
65	0.5478	0.3428	0.4637
$\sigma_1 = 0.8; \sigma_2 = 0.8; \sigma_3 = 0.8; \sigma_4 = 0.8$			
60	3.2459	2.9522	3.3371
65	1.6826	1.2463	1.6429
70	0.7589	0.3612	0.6375
$\sigma_1 = 0.6; \sigma_2 = 1.2; \sigma_3 = 0.3; \sigma_4 = 0.9$			
55	5.5243	5.4535	5.6766
60	3.2397	2.8540	3.1933
65	1.7050	1.1421	1.4524
70	0.7923	0.2933	0.4763

Table 3: Basket option prices in the VG copula model with $r = 0.05$, $w_1 = w_2 = 0.5$, $X_1(0) = X_2(0) = 100$.

	T	ρ	σ_1	$C^{mc}[K, T]$	$C^{GA}[K, T]$	$C^{MM}[K, T]$	
K=115.64	1	0.3	0.2	1.3995	1.0474	1.3113	
			0.4	5.5724	4.4345	5.6267	
			0.7	0.2	1.8963	1.4532	1.8706
				0.4	6.9451	5.5577	7.0095
K=127.80	3	0.3	0.2	4.4427	3.498	4.4565	
			0.4	11.3138	9.371	11.5920	
			0.7	0.2	5.6002	4.4588	5.6368
				0.4	13.7444	11.2444	13.9336
K=105.13	1	0.3	0.2	5.5312	5.2622	5.5965	
			0.4	10.1471	9.5181	10.3515	
			0.7	0.2	6.327	5.958	6.3731
				0.4	11.7163	10.7703	11.8379
K=116.18	3	0.3	0.2	8.9833	8.4505	9.1489	
			0.4	15.8784	14.8124	16.2498	
			0.7	0.2	10.3513	9.5619	10.4528
				0.4	18.4042	16.7648	18.6214
K=94.61	1	0.3	0.2	12.3514	12.4384	12.4371	
			0.4	16.213	16.247	16.4493	
			0.7	0.2	13.0696	13.1218	13.1269
				0.4	17.7431	17.5251	17.8690
K=104.57	3	0.3	0.2	15.1888	15.2468	15.3869	
			0.4	21.3994	21.2958	21.7592	
			0.7	0.2	16.5069	16.3782	16.6232
				0.4	23.8489	23.2868	24.0507

option curves can be used to determine the parameters describing the distribution of L and the volatility parameters σ_1 and σ_2 . An approximate basket option price can then be determined by choosing a suitable correlation parameter ρ .

In the Lévy copula model (8), the characteristic function $\phi_{\log S_j(T)}$, $j = 1, 2$ can be determined if the volatilities σ_1 and σ_2 and the characteristic function ϕ_L of the mother distribution L are known. Assume that the vector containing the model parameters of L is denoted by Θ . The Carr-Madan formula can then be used to determine the model prices of the vanilla options, notation $C_j^{model}[K, T; \Theta, \sigma_j]$, $j = 1, 2$. We use the observed option curves C_1 and C_2 to determine the calibrated model parameters by minimizing the relative error. Notice that the calibrated parameters σ_1 and σ_2 can be interpreted as implied Lévy volatility parameters; see also Corcuera et al. (2009). These calibrated parameters together with the calibration error are listed in Table 5. Note that the relative error in the VG, Meixner and NIG case is significantly smaller than in the normal case.

Using the calibrated parameters for the mother distribution L together with the volatility parameters σ_1 and σ_2 , we can determine basket option prices in the different model settings. Note that here and in the sequel of the paper, we always use the three-moments-matching approximation for determining basket option prices. We put $T = 30$ days and consider the cases where the correlation parameter ρ is given by 0.1, 0.5 and 0.8. The corresponding basket option prices are listed in Table 6. One can observe from the table that each model generates a different basket option price, i.e. there is model risk. However, the difference between the Gaussian and the non-Gaussian models is much more pronounced than the difference within the non-Gaussian models. We also find that using normally distributed log returns, one underestimates the in-the-money basket option prices. Indeed, the basket option price $C^{VG}[K, T]$, $C^{Meixner}[K, T]$ and $C^{NIG}[K, T]$ are larger than $C^{BLS}[K, T]$ when K is below 105. Note, however, that for the strike $K = 110$, the price $C^{BLS}[K, T]$ is much closer to the other Lévy basket values. The reason for this behavior is that marginal log returns in the non-Gaussian situations are negatively skewed, whereas these distributions are symmetric in the Gaussian case. This skewness results in a lower probability of ending in the money for options with a sufficiently large strike. In the next section, we encounter situations where the Gaussian basket option price is larger than the corresponding VG price for out-of-the-money options.

Table 4: Input data for the basket option.

Date	Feb 19, 2009	
Maturity	March 21, 2009	
	S&P 500	Nasdaq
Forward	777.76	116.72
Weights	0.06419	0.0428

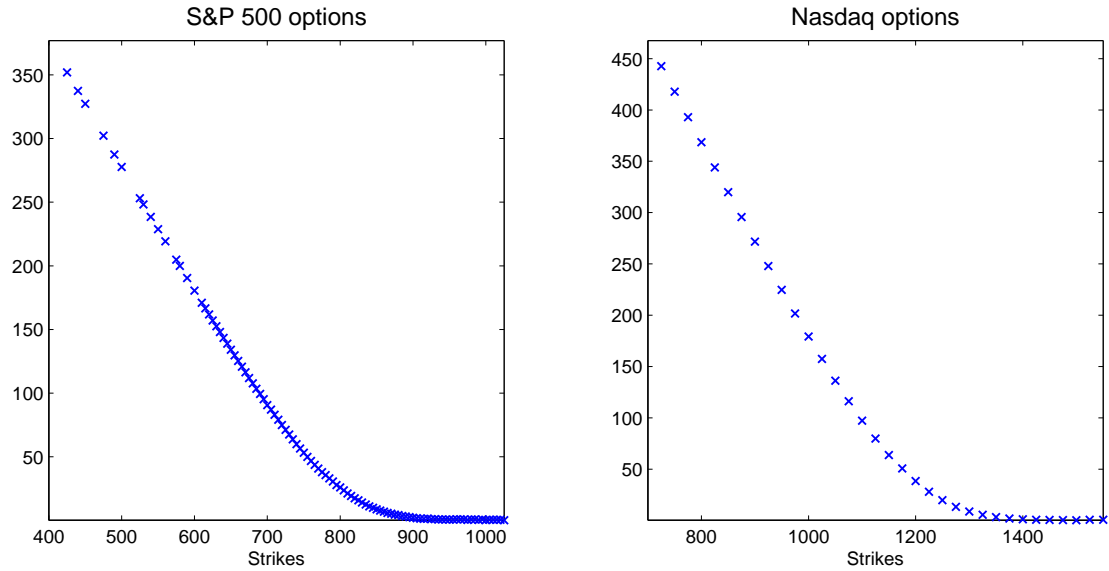


Figure 1: Available option data on February 19, 2009 for the S&P 500 (left) and the Nasdaq (right), with time to maturity 30 days.

Table 5: Lévy copula models: Calibrated model parameters

Model	Calibration error	Model Parameters			Volatilities	
Normal	15.91%	μ_{normal}	σ_{normal}		σ_1	σ_2
		0	1		0.2863	0.2762
VG	9.39 %	σ_{VG}	ν_{VG}	θ_{VG}		
		0.3640	0.7492	-0.3123	0.3876	0.3729
Meixner	9.33%	$\alpha_{Meixner}$	$\beta_{Meixner}$			
		1.5794	-1.6235		0.4015	0.3833
NIG	9.51 %	α_{NIG}	β_{NIG}			
		1.5651	-1.0063		0.4130	0.3941

Table 6: Basket option prices for the basket given in Table 4.

ρ	K	$C^{BLS}[K, T]$	$C^{VG}[K, T]$	$C^{Meixner}[K, T]$	$C^{NIG}[K, T]$
0.1	90	10.1906	10.8285	10.8957	10.9126
	95	5.9737	6.7858	6.8183	6.795
	100	2.8761	3.4893	3.4658	3.4237
	105	1.1009	1.2942	1.2802	1.2714
	110	0.3307	0.3721	0.3723	0.3764
0.5	90	10.3741	11.2496	11.3205	11.3242
	95	6.35	7.316	7.3414	7.3026
	100	3.3556	4.0513	4.017	3.9607
	105	1.5057	1.736	1.7024	1.68p01
	110	0.5706	0.5768	0.5715	0.5737
0.8	90	10.5226	11.5343	11.6064	11.6023
	95	6.6127	7.6707	7.6923	7.6457
	100	3.6748	4.4373	4.3983	4.3352
	105	1.7868	2.0708	2.0276	1.9973
	110	0.758	0.7669	0.7586	0.7582

7 Implied Lévy correlation

In Section 6.2 we showed how the basket option formulas can be used to obtain basket option prices in the Lévy copula model. The parameter vector Θ describing the mother distribution L and the implied Lévy volatility parameters σ_j can be calibrated using the observed vanilla option curves $C_j[K, T]$ of the stocks composing the basket $S(T)$. In this section we show how an implied Lévy correlation estimate ρ can be obtained if on top of the vanilla options, also market prices for the basket option are available for some strikes.

We assume that $S(T)$ represents the time- T price of a stock market index. Examples of such stock market indices are the Dow Jones, S&P 500, EUROSTOXX 50, Furthermore, options on $S(T)$ are traded and their prices are observable for a finite number of strikes. In this situation, pricing these index options is not a real issue; we denote the market price of an index option with maturity T and strike K by $C[K, T]$. Assume now that the stocks composing the index can be described by the Lévy copula model (8). If the parameter vector Θ and the marginal volatility vector $\underline{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_n)$ are determined, the model price $C^{model}[K, T; \underline{\sigma}, \Theta, \rho]$ for the basket option only depends on the correlation ρ . An *implied correlation* estimate for ρ arises when we match the model price with the observed index option price.

Definition 1 (implied Lévy correlation) *Consider the Lévy copula model defined in (8). The implied Lévy correlation of the index $S(T)$ with moneyness $\pi = \frac{S(T)}{S(0)}$, denoted by $\rho[\pi]$, is defined by the following equation:*

$$C^{model}[K, T; \underline{\sigma}, \Theta, \rho[\pi]] = C[K, T], \quad (32)$$

where $\underline{\sigma}$ contains the marginal implied volatilities and Θ is the parameter vector of L .

Determining an implied correlation estimate $\rho\left[\frac{K}{S(0)}\right]$ requires an inversion of the pricing formula $\rho \rightarrow C^{model}[K, T; \underline{\sigma}, \Theta, \rho]$. However, the basket option price is not given in closed form and determining this price using Monte Carlo simulation is not an option, because the calibration would be too slow. Therefore, from here on, $C^{model}[K, T; \underline{\sigma}, \Theta, \rho]$ has to be interpreted as the three-moments-matching approximation. In this case, implied correlations can be determined in a fast and efficient way. The idea of determining implied correlation estimates based on an approximate basket option pricing formula was already proposed in Linders and Schoutens (2014).

Note that in case we take L to be the normal distribution, $\rho[\pi]$ is the classical Black & Scholes implied correlation; see e.g. Chicago Board Options Exchange (2009) and Skintzi and Refenes (2005). Equation (32) can be considered as a generalization of the Black & Scholes implied correlation. Indeed, instead of determining the single correlation parameter in a multivariate model with normal log returns and a Gaussian copula, we can now extend the model to the situation where the log returns follow a Lévy distribution and a Lévy copula connects the marginals. A similar idea was proposed in Garcia et al. (2009) and further studied in Masol and Schoutens (2011). In these papers, Lévy base correlation is defined using CDS and CDO prices.

Note that when we use the three-moments-matching approximation to determine the model prices $C^{model}[K, T; \underline{\sigma}, \Theta, \rho[\pi]]$, we can also derive implied correlation estimates from traded spread options in a Lévy copula model. This approach was also considered in Tavin (2013).

7.1 Variance Gamma

In order to illustrate the proposed methodology for determining implied Lévy correlation estimates, we use the Dow Jones Industrial Average (DJ). The DJ is composed of 30 underlying stocks and for each underlying we have a finite number of option prices to which we can calibrate the parameter vector Θ and the Lévy volatility parameters σ_j . Using the available vanilla option data for June 20, 2008 we will work out the normal and the Variance Gamma case. Note that options on components of the Dow Jones are of American type. In the sequel, we assume that the American option price is a good proxy for the corresponding European option price. This assumption is justified because we use short term and out-of-the-money options.

Assuming a normal distribution for L , the implied Black & Scholes volatility of stock j , notation σ_j^{BLS} , can be determined using solely the option curve of stock j . In the Variance Gamma case, the parameter vector Θ and the implied VG volatilities, notation σ_j^{VG} , $j = 1, 2, \dots, n$, have to be calibrated simultaneously. The parameter vector Θ is given in Table 8. The implied volatilities are listed in Table 7. In the Gaussian copula model, each stock j is captured by the single implied volatility parameter σ_j^{BLS} . In case we consider the Variance Gamma copula model, we have three additional common parameters which can be calibrated to the vanilla option curves, resulting in an improved fit. Figure 2 shows the model (Gaussian and Variance Gamma) and market prices for General Electric and IBM, both members of the Dow Jones, based on the implied volatility parameters listed in Table 7. We observe that the Variance Gamma copula model is more suitable in capturing the dynamics of the components of the Dow Jones than the Gaussian copula model.

Given the volatility parameters for the Variance Gamma case and the normal case, listed in Table 7, the implied correlation defined by equation (32), can be determined based on the available Dow Jones index options on June 20, 2008. For a given index strike K , the moneyness π is defined as $\pi = \frac{K}{S(0)}$. The Black & Scholes implied correlation is denoted by $\rho^{BLS}[\pi]$ and the corresponding implied Lévy correlation, based on a VG distribution, is denoted by $\rho^{VG}[\pi]$. However, in order to more closely match the vanilla option curves, we use a volatility parameter with moneyness π for each stock j , which we denote by $\sigma_j[\pi]$. For a detailed and step-by-step plan for the calculation of these volatility parameters, we refer to Linders and Schoutens (2014).

Figure 3 shows that both the implied Black & Scholes and implied Lévy correlation depend on the moneyness π . However, for low strikes, we observe that $\rho^{VG}[\pi] < \rho^{BLS}[\pi]$, whereas the opposite inequality holds for large strikes, making the implied Lévy correlation curve less steep than its Black & Scholes counterpart. In Linders and Schoutens (2014), the authors discussed the shortcomings of the implied Black & Scholes correlation and showed that implied Black & Scholes correlations can become larger than one for low strike prices. Considering our more general approach and using the implied Lévy correlation solves, at least to some extent, this problem. Indeed, the region where the implied correlation stays below 1 is much larger for the flatter implied Lévy correlation curve than for its Black & Scholes counterpart. We also observe that near the at-the-money strikes, VG and Black & Scholes correlation estimates are comparable, which may be a sign that in this region, the use of implied Black & Scholes correlation (as defined in Linders and Schoutens (2014)) is justified. Figure 4 shows implied correlation curves for March, April, July and August, 2008. In all these situations, the time to maturity is close to 30 days. The calibrated parameters for each trading day are listed in 8.

Table 7: Implied Variance Gamma volatilities σ_j^{VG} and implied Black & Scholes volatilities σ_j^{BLS} for June 20, 2008.

Stock	σ_j^{VG}	σ_j^{BLS}
Alcoa Incorporated	0.5805	0.5458
American Express Company	0.5104	0.4774
American International group	0.577	0.5282
Bank of America	0.653	0.5962
Boeing Corporation	0.3209	0.3047
Caterpillar	0.2916	0.2782
JP Morgan	0.5517	0.5147
Chevron	0.281	0.2688
Citigroup	0.6723	0.6232
Coca Cola Company	0.2533	0.2417
Walt Disney Company	0.2757	0.2626
DuPont	0.2655	0.2531
Exxon Mobile	0.2866	0.2743
General Electric	0.3728	0.3521
General Motors	0.9943	0.9096
Hewlet-Packard	0.2938	0.28
Home Depot	0.3546	0.3362
Intel	0.4268	0.4051
IBM	0.2932	0.2789
Johnson & Johnson	0.175	0.1645
McDonald's	0.2488	0.2388
Merck & Company	0.2987	0.2762
Microsoft	0.328	0.3102
3M	0.2354	0.2236
Pfizer	0.279	0.2595
Practer & Gamble	0.1946	0.1872
AT&T	0.294	0.2797
United Technologies	0.2819	0.2668
Verizon	0.3109	0.2943
Wal-Mart Stores	0.2632	0.2517

Table 8: Calibrated VG parameters for different trading days.

	VG Parameters				
	$S(0)$	T	σ	ν	θ
March 25, 2008	125.33	25 days	0.2981	0.5741	-0.1827
April 18, 2008	128.49	29 days	0.3606	0.5247	-0.2102
June 20, 2008	118.43	29 days	0.3587	0.4683	-0.1879
July 18, 2008	114.97	29 days	0.2639	0.5222	-0.1641
August 20, 2008	114.17	31 days	0.2467	0.3770	-0.1887

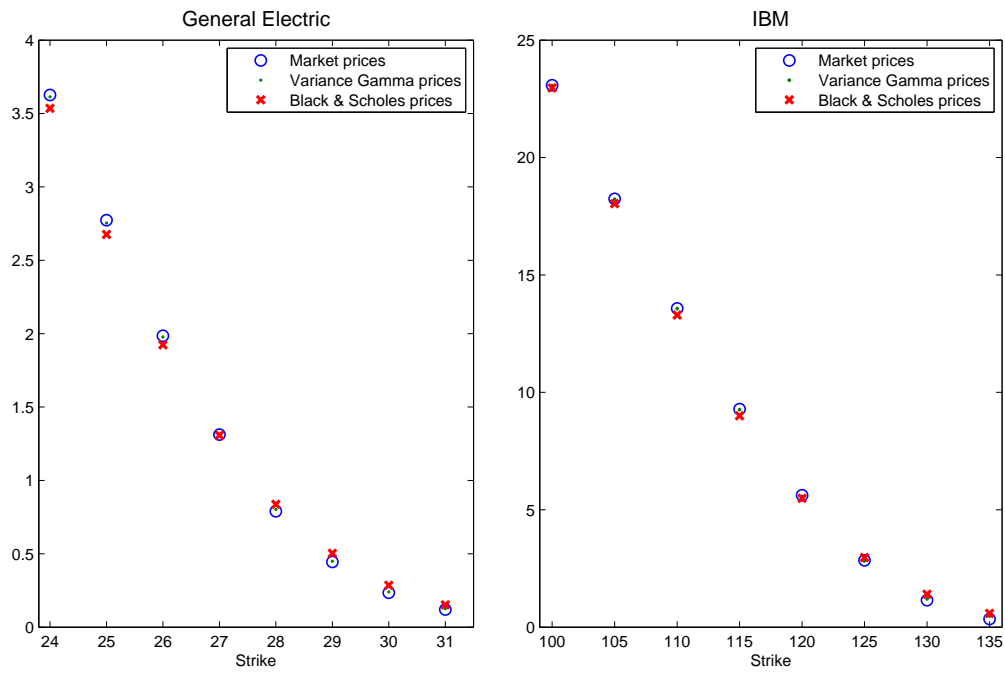


Figure 2: Option prices (model and market) for General Electric and IBM on June 20, 2008 based on the volatility parameters listed in Table 7. The time to maturity is 30 days.

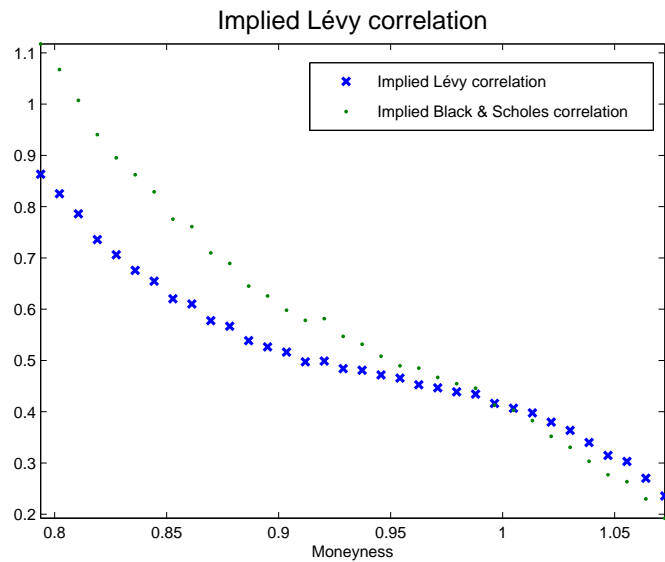


Figure 3: Implied correlation smile for the Dow Jones, based on a Gaussian (dots) and a Variance Gamma copula model (crosses) for June 20, 2008.

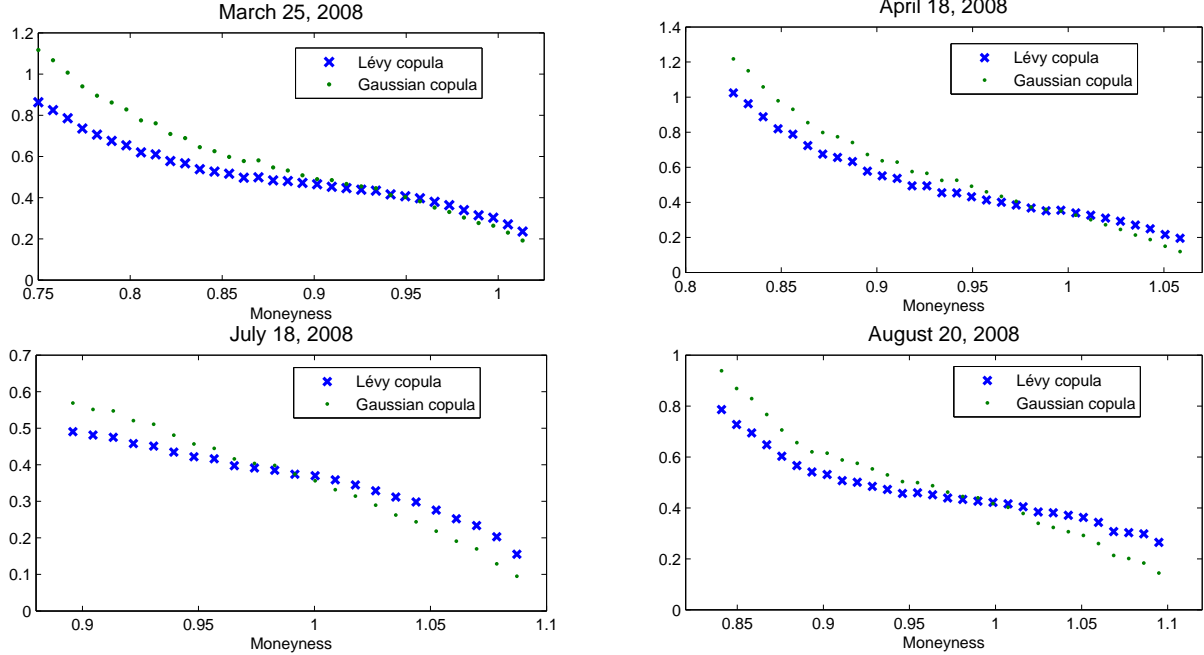


Figure 4: Implied correlation smile for the Dow Jones, based on a Gaussian (dots) and a Variance Gamma copula model (crosses) for different trading days.

7.2 Double Exponential

In the previous subsection, we showed that the Lévy copula model allows for determining more robust implied correlation estimates. However, calibrating this model can be a computational challenging task. Indeed, in case we deal with the Dow Jones Industrial Average, there are 30 underlying stocks and each stock has approximately 5 traded option prices. Calibrating the parameter vector Θ and the volatility parameters σ_j has to be done simultaneously. This contrasts sharply with the Gaussian copula model, where the calibration can be done stock per stock.

In this subsection we consider a model with the computational attractive calibration property of the Gaussian copula model, but without imposing any normality assumption on the marginal log returns. To be more precise, given the convincing arguments exposed in Figure 4 we would like to keep L a $VG(\sigma, \nu, \theta, \mu)$ distribution. However, we do not calibrate the parameter vector $\Theta = (\sigma, \nu, \theta, \mu)$ to the vanilla option curves, but we fix these parameters upfront as follows

$$\begin{aligned}\mu &= 0, \\ \theta &= 0, \\ \nu &= 1, \\ \sigma &= 1.\end{aligned}$$

Then L is a standardized distribution and its characteristic function ϕ_L is given by

$$\phi_L(u) = \frac{1}{1 + \frac{u^2}{2}}, \quad u \in \mathbb{R}.$$

From its characteristic function, we see that L has a *Standard Double Exponential distribution*, also called Laplace distribution, and its pdf f_L is given by

$$f_L(u) = \frac{1}{2\sqrt{2}} e^{-\frac{|u|}{\sqrt{2}}}, \quad u \in \mathbb{R}.$$

The Standard Double Exponential distribution is symmetric and centered around zero, while it has variance 1. Note, however, that it is straightforward to generalize this distribution such that it has center μ and variance σ^2 . Moreover, the kurtosis of this Double Exponential is 6.

By using the Double Exponential distribution instead of the, more general, Variance Gamma distribution, a little bit flexibility is lost for modeling the marginals. However, the Double Exponential distribution is still a much better distribution for modeling the stock returns than the normal distribution. Moreover, in this simplified setting, the only parameters to be calibrated are the marginal volatility parameters, which we denote by σ_j^{DE} , and the correlation parameter ρ^{DE} . Similar to the Gaussian copula model, calibrating the volatility parameter σ_j^{DE} only requires the option curve of stock j . As a result, the time to calibrate the Double Exponential copula model is comparable to its Gaussian counterpart and much shorter than the general Variance Gamma copula model.

Consider the DJ on March 25, 2008. The time to maturity is 25 days. We determine the implied marginal volatility parameter for each stock in a Variance Gamma copula model and a Double Exponential framework. The results are listed in Table 9. One can observe that the implied volatility in each of these models is more or less the same. Given this information, we can determine the prices $C^{VG}[K, T]$ and $C^{DE}[K, T]$ for a basket option in a Variance-Gamma and a Double Exponential copula model, respectively. We have put $\rho = 0.5$. The option prices $C^{VG}[K, T]$ and $C^{DE}[K, T]$ are shown in Figure 5 for different choices of K and a time to maturity of 25 days. For out-of-the-money strikes K , we find that $C^{DE}[K, T]$ is significantly bigger than $C^{VG}[K, T]$, whereas the prices are more or less comparable for in-the-money strikes. This difference between the Variance Gamma and the Double Exponential copula model for out-of-the-money strikes has its impact when determining implied correlation estimates. Indeed, it is shown in Figure 6 that the implied Variance Gamma correlation is larger than its Double Exponential counterpart for a moneyness bigger than one, whereas both implied correlation estimates are relatively close to each other in the other situation.

8 Conclusion

In this paper we extended the classical Gaussian copula model to a Lévy copula model. We proposed two methods for approximating the price of a basket option. Both approximations consist of replacing the original r.v. describing the basket at maturity by a r.v. which has a more simple structure. Furthermore, we showed that the Carr-Madan formula can be used to

Table 9: Volatility parameters of the Variance Gamma and the Double Exponential copula model

Stocks	σ_j^{VG}	σ_j^{DE}
Alcoa Incorporated	0.5255	0.5187
American Express Company	0.4636	0.4498
Kraft Foods	0.5027	0.4957
Bank of America	0.4270	0.4232
Boeing Corporation	0.3053	0.3013
Caterpillar	0.3652	0.3648
JP Morgan	0.4968	0.4765
Chevron	0.3145	0.3139
Citigroup	0.6862	0.6440
Coca Cola Company	0.2146	0.2158
Walt Disney Company	0.2755	0.2842
DuPont	0.3152	0.3190
Exxon Mobile	0.3087	0.3054
General Electric	0.2849	0.2865
General Motors	0.8372	0.7914
Hewlet-Packard	0.3110	0.3077
Home Depot	0.4093	0.4001
Intel	0.4325	0.4259
IBM	0.2985	0.2962
Johnson & Johnson	0.1937	0.1958
McDonald's	0.2635	0.2642
Merck & Company	0.3524	0.3472
Microsoft	0.3093	0.3049
3M	0.2489	0.2490
Pfizer	0.2748	0.2763
Practer & Gamble	0.1780	0.1780
AT&T	0.3155	0.3105
United Technologies	0.2821	0.2830
Verizon	0.3208	0.3164
Wal-Mart Stores	0.2609	0.2674

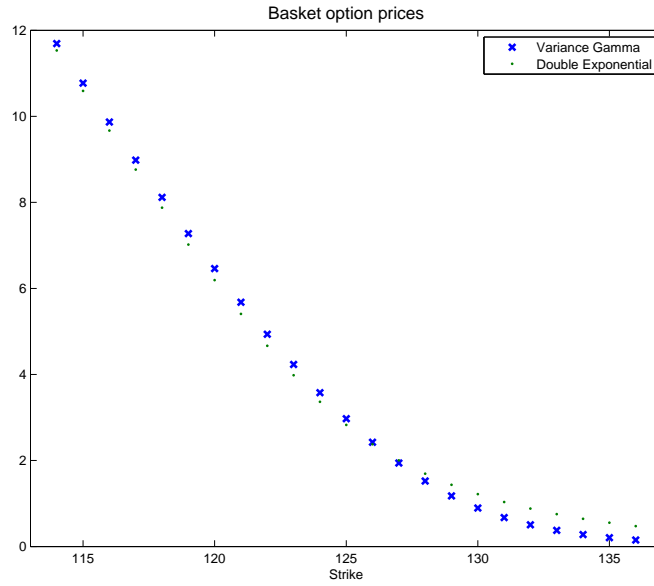


Figure 5: Dow Jones option prices in the Variance Gamma and the Double Exponential copula model for March 25, 2008 and a time to maturity of 25 days.

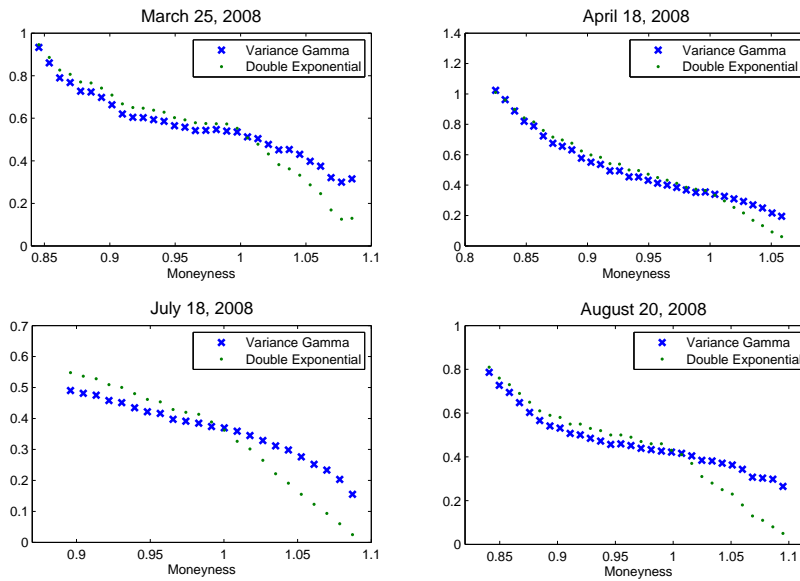


Figure 6: Implied correlation smiles in the Variance Gamma and the Double Exponential copula model.

determine the approximate basket option prices. Well-known distributions like the Normal, Variance Gamma, NIG, Meixner, . . . can be used in the Lévy copula model. We calibrate these different models to market data and determine basket option prices for the different model settings. Our newly designed (approximate) basket option pricing formula can be used to define implied Lévy correlation. We showed that implied Lévy correlation improves the classical Gaussian implied correlation.

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References

- Albrecher, H., Ladoucette, S. and Schoutens, W. (2007), A generic one-factor lvy model for pricing synthetic CDOs, in M. Fu, R. Jarrow, J.-Y. Yen and R. Elliott, eds, 'Advances in Mathematical Finance', Applied and Numerical Harmonic Analysis, Birkhäuser Boston, pp. 259–277.
- Bachelier, L. (1900), *Theorie de la spéculation*, PhD thesis, Faculté des sciences de Paris.
- Ballotta, L. and Bonfiglioli, E. (2014), 'Multivariate asset models using lévy processes and applications', *The European Journal of Finance* **0**(0), 1–31.
- Ballotta, L., Deelstra, G. and Rayée, G. (2014), Pricing derivatives written on more than one underlying asset in a multivariate lévy framework, Eighth World Congress of the Bachelier Finance Society in Brussels, Belgium .
- Björk, T. (1998), *Arbitrage theory in continuous time*, Oxford University Press. pp. 311.
- Black, F. and Scholes, M. (1973), 'The pricing of options and corporate liabilities', *The Journal of Political Economy* **81**(3), pp. 637–654.
- Brigo, D., Mercurio, F., Rapisarda, F. and Scotti, R. (2004), 'Approximated moment-matching dynamics for basket-options pricing', *Quantitative Finance* **4**(1), 1–16.
- Brooks, R., Corson, J. and Wales, J. D. (1994), 'The pricing of index options when the underlying assets all follow a lognormal diffusion', *Advances in Futures and Options Research* **7**.
- Carmona, R. and Durrleman, V. (2006), 'Generalizing the black-scholes formula to multivariate contingent claims', *Journal of Computational Finance* **9**, 43–67.
- Carr, P., Madan, D. B. and Smith, R. H. (1999), 'Option valuation using the fast fourier transform', *Journal of Computational Finance* **2**, 61–73.
- Cherubini, U., Luciano, E. and Vecchiato, W. (2004), *Copula Methods in Finance*, The Wiley Finance Series, Wiley.
- Chicago Board Options Exchange (2009), CBOE S&P 500 implied correlation index, *Working Paper*.
- Corcuera, J. M., Guillaume, F., Leoni, P. and Schoutens, W. (2009), 'Implied Lévy volatility', *Quantitative Finance* **9**(4), 383–393.
- Deelstra, G., Liinev, J. and Vanmaele, M. (2004), 'Pricing of arithmetic basket options by conditioning', *Insurance: Mathematics and Economics* **34**(1), 55 – 77.
- Dhaene, J., Denuit, M., Goovaerts, M., Kaas, R. and Vyncke, D. (2002a), 'The concept of comonotonicity in actuarial science and finance: theory', *Insurance: Mathematics & Economics* **31**(1), 3–33.

- Dhaene, J., Denuit, M., Goovaerts, M., Kaas, R. and Vyncke, D. (2002b), ‘The concept of comonotonicity in actuarial science and finance: applications’, *Insurance: Mathematics & Economics* **31**(2), 133–161.
- Dhaene, J., Kukush, A. and Linders, D. (2013), ‘The multivariate black & scholes market: conditions for completeness and no-arbitrage’, *Theory of Probability and Mathematical Statistics* (88), 1–14.
- Garcia, J., Goossens, S., Masol, V. and Schoutens, W. (2009), ‘Levy base correlation’, *Wilmott Journal* **1**, 95–100.
- Guillaume, F. (2013), ‘The α VG model for multivariate asset pricing: calibration and extension’, *Review of Derivatives Research* **16**(1), 25–52.
- Guillaume, F., Jacobs, P. and Schoutens, W. (2009), ‘Pricing and hedging of CDO-squared tranches by using a one factor Lévy model’, *International Journal of Theoretical and Applied Finance* **12**(05), 663–685.
- Hull, J. and White, S. (1993), ‘Efficient procedures for valuing european and american path-dependent options’, *The Journal of Derivatives* **1**(1), 21–31.
- Kawai, R. (2009), ‘A multivariate Lévy process model with linear correlation’, *Quantitative Finance* **9**(5), 597–606.
- Korn, R. and Zeytun, S. (2013), ‘Efficient basket monte carlo option pricing via a simple analytical approximation’, *Journal of Computational and Applied Mathematics* **243**(1), 48 – 59.
- Leoni, P. and Schoutens, W. (2008), ‘Multivariate smiling’, *Wilmott Magazin* **march 2008**.
- Linders, D. (2013), Pricing index options in a multivariate Black & Scholes model, Research report AFI-1383 FEB, Leuven: KU Leuven - Faculty of Business and Economics.
- Linders, D. and Schoutens, W. (2014), ‘A framework for robust measurement of implied correlation’, *Journal of Computational and Applied Mathematics* **271**, 39–52.
- Linders, D. and Stassen, B. (2014), Pricing basket options using a moment-matching approximation in a multivariate variance gamma model, Research report AFI, FEB, KU Leuven.
- Luciano, E. and Schoutens, W. (2006), ‘A multivariate jump-driven financial asset model’, *Quantitative Finance* **6**(5), 385–402.
- Luciano, E. and Semeraro, P. (2010), ‘Multivariate time changes for Lévy asset models: Characterization and calibration’, *J. Comput. Appl. Math.* **233**, 1937–1953.
- Madan, D. B., Carr, P. and Chang, E. C. (1998), ‘The variance gamma process and option pricing.’, *European Finance Review* **2**, 79–105.
- Madan, D. B. and Seneta, E. (1990), ‘The variance gamma (v.g.) model for share market returns’, *The Journal of Business* **63**(4), pp. 511–524.

- Masol, V. and Schoutens, W. (2011), ‘Comparing alternative lévy base correlation models for pricing and hedging cdo tranches’, *Quantitative Finance* **11**(5), 763–773.
- McWilliams, N. (2011), Option pricing techniques understochastic delay models, PhD thesis, University of Edinburgh.
- Milevsky, M. and Posner, S. (1998), ‘A closed-form approximation for valuing basket options’, *The Journal of Derivatives* **5**(4), 54–61.
- Moosbrucker, T. (2006), ‘Explaining the correlation smile using Variance Gamma distributions’, *The Journal of Fixed Income* **16**(1), 71–87.
- Rubinstein, M. (1994), ‘Implied binomial trees’, *The Journal of Finance* **49**(3), 771–818.
- Sato, K. I. (1999), *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Studies in Advanced Mathematics, Cambridge University Press.
- Schoutens, W. (2003), *Lévy processes in Finance: Pricing Financial Derivatives*, Wiley.
- Semeraro, P. (2008), ‘A multivariate Variance Gamma model for financial applications’, *International Journal of Theoretical and Applied Finance (IJTAF)* **11**(01), 1–18.
- Skintzi, V. D. and Refenes, A. N. (2005), ‘Implied correlation index: A new measure of diversification’, *Journal of Futures Markets* **25**, 171–197. doi: 10.1002/fut.20137.
- Tavin, B. (2013), Hedging dependence risk with spread options via the power frank and power student t copulas, Technical report, Université Paris I Panthéon-Sorbonne. Available at SSRN: <http://ssrn.com/abstract=2192430>.
- Valdez, E. A., Dhaene, J., Maj, M. and Vanduffel, S. (2009), ‘Bounds and approximations for sums of dependent log-elliptical random variables’, *Insurance: Mathematics and Economics* **44**(3), 385 – 397.
- Xu, G. and Zheng, H. (2010), ‘Basket options valuation for a local volatility jump diffusion model with the asymptotic expansion method’, *Insurance: Mathematics and Economics* **47**(3), 415 – 422.
- Xu, G. and Zheng, H. (2014), ‘Lower bound approximation to basket option values for local volatility jump-diffusion models’, *International Journal of Theoretical and Applied Finance* **17**(01), 1450007.

A Proof of Lemma 2

The proof for expression (27) is straightforward.

The second moment m_2 can be rewritten as follows

$$\begin{aligned}
m_2 &= \sum_{j=1}^n \sum_{k=1}^n w_j w_k \mathbb{E} [S_j(T) S_k(T)] \\
&= \sum_{j=1}^n \sum_{k=1}^n w_j w_k S_j(0) S_k(0) \\
&\quad \times \mathbb{E} \left[\exp \left\{ (2r - q_j - q_k - \omega_j - \omega_k)T + (\sigma_j A_j + \sigma_k A_k) \sqrt{T} \right\} \right] \\
&= \sum_{j=1}^n \sum_{k=1}^n w_j w_k \frac{\mathbb{E} [S_j(T)] \mathbb{E} [S_k(T)]}{\phi_L(-i\sigma_j \sqrt{T}) \phi_L(-i\sigma_k \sqrt{T})} \mathbb{E} \left[\exp \left\{ (\sigma_j A_j + \sigma_k A_k) \sqrt{T} \right\} \right].
\end{aligned}$$

In the last step, we used the expression $\omega_j = \frac{1}{T} \log \phi_L(i\sigma_j \sqrt{T})$. If we use expression (3) to decompose A_j and A_k in the common component $X(\rho)$ and the independent components $X_j(1-\rho)$ and $X_k(1-\rho)$, we find the following expression for m_2

$$m_2 = \sum_{j=1}^n \sum_{k=1}^n w_j w_k \frac{\mathbb{E} [S_j(T)] \mathbb{E} [S_k(T)]}{\phi_L(-i\sigma_j \sqrt{T}) \phi_L(-i\sigma_k \sqrt{T})} \mathbb{E} \left[e^{(\sigma_j + \sigma_k)X(\rho)} e^{\sigma_j \sqrt{T} X_j(1-\rho)} e^{\sigma_k \sqrt{T} X_k(1-\rho)} \right].$$

The r.v. $X(\rho)$ is independent from $X_j(1-\rho)$ and $X_k(1-\rho)$. Furthermore, the characteristic function of $X(\rho)$ is ϕ_L^ρ , which results in

$$\begin{aligned}
m_2 &= \sum_{j=1}^n \sum_{k=1}^n w_j w_k \frac{\mathbb{E} [S_j(T)] \mathbb{E} [S_k(T)]}{\phi_L(-i\sigma_j \sqrt{T}) \phi_L(-i\sigma_k \sqrt{T})} \phi_L(-i(\sigma_j + \sigma_k) \sqrt{T})^\rho \\
&\quad \times \mathbb{E} \left[e^{\sigma_j \sqrt{T} X_j(1-\rho)} e^{\sigma_k \sqrt{T} X_k(1-\rho)} \right]
\end{aligned}$$

If $j \neq k$, $X_j(1-\rho)$ and $X_k(1-\rho)$ are i.i.d. with characteristic function $\phi_L^{1-\rho}$, which gives the following expression for m_2 :

$$m_2 = \sum_{j=1}^n \sum_{k=1}^n w_j w_k \mathbb{E} [S_j(T)] \mathbb{E} [S_k(T)] \left(\frac{\phi_L(-i(\sigma_j + \sigma_k) \sqrt{T})}{\phi_L(-i\sigma_j \sqrt{T}) \phi_L(-i\sigma_k \sqrt{T})} \right)^\rho.$$

If $j = k$, we find that

$$\mathbb{E} \left[e^{\sigma_j \sqrt{T} X_j(1-\rho)} e^{\sigma_k \sqrt{T} X_k(1-\rho)} \right] = \phi_L(-i(\sigma_j + \sigma_k) \sqrt{T}),$$

which gives

$$m_2 = \sum_{j=1}^n \sum_{k=1}^n w_j w_k \mathbb{E} [S_j(T)] \mathbb{E} [S_k(T)] \frac{\phi_L(-i(\sigma_j + \sigma_k) \sqrt{T})}{\phi_L(-i\sigma_j \sqrt{T}) \phi_L(-i\sigma_k \sqrt{T})}.$$

This proves expression (28) for m_2 .

The third moment m_3 can be written as

$$m_3 = \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n w_j w_k w_l \mathbb{E} [S_j(T) S_k(T) S_l(T)].$$

Similarly calculations as in the case m_2 result in

$$m_3 = \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n w_j w_k w_l \mathbb{E} [S_j(T)] \mathbb{E} [S_k(T)] \mathbb{E} [S_l(T)] \\ \times \frac{\phi_L \left(-i(\sigma_j + \sigma_k + \sigma_l) \sqrt{T} \right)^p}{\phi_L \left(-i\sigma_j \sqrt{T} \right) \phi_L \left(-i\sigma_k \sqrt{T} \right) \phi_L \left(-i\sigma_l \sqrt{T} \right)} A_{j,k,l},$$

where

$$A_{j,k,l} = \mathbb{E} \left[e^{\sigma_j \sqrt{T} X_j(1-\rho)} e^{\sigma_k \sqrt{T} X_k(1-\rho)} e^{\sigma_l \sqrt{T} X_l(1-\rho)} \right].$$

Differentiating between the situations $(j = k = l)$, $(j = k, k \neq l)$, $(j \neq k, k = l)$, $(j \neq k, k \neq l, j = l)$ and $(j \neq k \neq l, j \neq l)$, we find expression (29).

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