A FAMILY OF NONLINEAR DIFFERENCE EQUATIONS: EXISTENCE, UNIQUENESS, AND ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS

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Dedicated to Dick Askey

Abstract. We study solutions $(x_n)_{n\in\mathbb{N}}$ of nonhomogeneous nonlinear second order difference equations of the type

$$\ell_n = x_n\left(\sigma_{n,1}\,x_{n+1} + \sigma_{n,0}\,x_n + \sigma_{n,-1}\,x_{n-1}\right) + \kappa_n\,x_n, \quad n \in \mathbb{N},$$
$$\text{with given initial data } \{x_0 \in \mathbb{R} \ \ \& \ \ x_1 \in \mathbb{R}^+\}$$

where

$$(\ell_n)_{n\in\mathbb{N}} \in \mathbb{R}^+ \ \& \ (\sigma_{n,0})_{n\in\mathbb{N}} \in \mathbb{R}^+ \ \& \ (\kappa_n)_{n\in\mathbb{N}} \in \mathbb{R},$$

and the left and right σ-coefficients satisfy either

$$(\sigma_{n,1})_{n\in\mathbb{N}} \in \mathbb{R}^+ \ \& \ (\sigma_{n,-1})_{n\in\mathbb{N}} \in \mathbb{R}^+$$

or

$$(\sigma_{n,1})_{n\in\mathbb{N}} \in \mathbb{R}^+_0 \ \& \ (\sigma_{n,-1})_{n\in\mathbb{N}} \in \mathbb{R}^+_0.$$

Depending on one's standpoint, such equations originate either from orthogonal polynomials associated with certain Shohat-Freud-type exponential weight functions or from Painlevé's discrete equation #1, that is, d-$\mathrm{P_I}$.

1. Preliminaries

Since the authors come from different cultures using different mathematical notation, we need to fix some of it right now in order to avoid subsequent misunderstanding.

Date: March 25, 2014.

2010 *Mathematics Subject Classification.* 39A22, 65Q10, 65Q30.

Key words and phrases. nonhomogeneous nonlinear second order difference equations, Shohat-Freud-type exponential weight functions, Painlevé's discrete equation #1, existence of solutions, unicity of solutions, asymptotic behavior.

The research of Saud M. Alsulami and Paul Nevai was supported by KAU grant No. 20-130/1433 HiCi. The research of Walter Van Assche was supported by KU Leuven research grant OT/12/073 and FWO research grant G.0934.13.

The set of natural numbers $\mathbb{N}$ consists of all strictly positive integers. Furthermore, $\mathbb{R}^+ \stackrel{\text{def}}{=} \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_0^+ \stackrel{\text{def}}{=} \{x \in \mathbb{R} : x \geqslant 0\}$.

2. Introduction

This section will explain how the unlikely pair of JSZ and WVA became involved in this research via PN's manipulations. We justify its unusual length compared to the rest of the paper by the necessity of giving a proper historical perspective that will also serve as introduction for the subsequent papers that we plan to publish on nonlinear difference equations.

It was Géza Freud who brought the attention of the approximation theory and orthogonal polynomial communities to exponential weight functions with his extensive body of work in the 1970s that was suddenly interrupted by his untimely death in 1979 at the youthful age of 57 years.[1] In particular, Freud solved two special and, to some extent, simple cases of his *Freud conjectures* that, even today, are of extraordinary interest despite having been overshadowed by the incomparably deeper pathbreaking achievements by so many of us such as Alphonse Magnus, Evguenii A. Rakhmanov, Andrei A. Gonchar, Hrushikesh N. Mhaskar, Edward B. Saff, Doron S. Lubinsky, Vilmos Totik, and Guillermo López Lagomasino, in some kind of a chronological order.

The two special cases above refer to the asymptotic behavior of the recurrence coefficients in the three-term recurrence for the orthogonal polynomials associated with the weight functions $|x|^\rho \exp(-x^4)$ and $|x|^\rho \exp(-x^6)$ on $\mathbb{R}$ with $\rho > -1$, see [5]. In particular, [5, (23, p. 5] is the almost-birthplace of the equation

$$n + \frac{1-(-1)^n}{2}\rho = 4a_n^2\left(a_{n+1}^2 + a_n^2 + a_{n-1}^2\right), \qquad a_0 = 0, \quad n \in \mathbb{N}, \tag{2.1}$$

where $(a_n)_{n\in\mathbb{N}}$ are the recurrence coefficients for the orthogonal polynomials associated with $|x|^\rho \exp(-x^4)$. We wrote "almost-birthplace", because, as it was discovered in 1983 by Dick Askey, see [9, p. 285], Shohat in 1939 could have found (2.1) except that he was only interested in the weight function $\exp(-x^4)$, that is, when $\rho = 0$, see [12, (39), p. 407]. Even if Shohat found or could have found the equation, he did nothing with it and neither did Freud except that Freud used a clever lim inf–lim sup argument, we call it the *Freud Kunstgriff*, to find the asymptotic behavior of (a_n)'s in (2.1), see [5, part (b), p. 5].[2] Let us emphasize that for both Shohat and Freud the (a_n)'s were recurrence coefficients for the orthogonal polynomials although the equation itself could have been viewed independently of orthogonal polynomials with the stipulation that the (a_n)'s are positive.[3]

[1] This statement is not entirely accurate; e.g., Mkhitar Djrbashian (aka Dzhrbashjan & Jerbashian) has a large body of work that is not that different from some of Freud's work but its international impact was negligible. In addition, exponential weights have long been of definite interest in areas such as the moment problem.

[2] It was subsequently successfully adapted by several authors, see, [9, Theorem 1, p. 266], [6, Theorem 2.2(b), pp. 210–211], and [13, p. 695].

[3] The (a_n)'s appear squared in (2.1) so one could also simply require that they be real and nonzero.

PN's 1983 paper [9] was the first one to subject

$$n = 4a_n^2 \left(a_{n+1}^2 + a_n^2 + a_{n-1}^2\right), \qquad a_0 = 0, \quad n \in \mathbb{N}, \tag{2.2}$$

to a systematic analysis and it was the almost-birthplace of the theorem that is the starting point of the current paper, see [9, Theorem 3, p. 268].

Theorem 2.1. *The equation*

$$n = x_n \left(x_{n+1} + x_n + x_{n-1}\right), \qquad x_0 = 0, \quad n \in \mathbb{N}, \tag{2.3}$$

has a unique positive[4] *solution* $(x_n)_{n\in\mathbb{N}}$*, and this solution is obtained by setting*

$$x_1 = \frac{\int_{\mathbb{R}} x^2 \exp\left(-x^4\right)}{\int_{\mathbb{R}} \exp\left(-x^4\right)}\, dx = 2\,\frac{\Gamma\left(3/4\right)}{\Gamma\left(1/4\right)}\,.$$

We wrote "almost-birthplace", because while PN was working on [9], he visited the IBM Research Center in Yorktown Heights, New York, in December, 1981, where he discussed orthogonal polynomials with Freud-type exponential weights and mentioned a conjecture that was the essence of Theorem 2.1. John S. Lew was in the audience and one thing led to another. In the end, Lew, together with Donald A. Quarles, wrote a magnificent paper that, as far as we know, was the first study of generalizations of (2.3) where orthogonal polynomials no longer occupied a central place and the primary object of interest was existence and uniqueness of positive solutions. Lew–Quarles's equation is

$$\ell_n = x_n \left(x_{n+1} + x_n + x_{n-1}\right), \qquad x_0 \in \mathbb{R}, \quad \ell_n > 0, \quad n \in \mathbb{N}, \tag{2.4}$$

and they proved a very general theorem that contains, as a special case, [9, Theorem 3, p. 268], see [7, Theorem 6.3, p. 369].[5]

The year 1984 produced two more papers [10, 11][6] where (2.2) is discussed, see [10, middle of p. 420] and [11, (2), p. 1177]. However, nothing is done with the equation outside of the scope of orthogonal polynomials.

Real progress came in 1984 with [3][7] where Theorem 2.1 was extended to the following, see [3, (iii), p. 142].

Theorem 2.2. *Given* $c > 0$ *and* $K \in \mathbb{R}$*, the equation*

$$n = c\, x_n \left(x_{n+1} + x_n + x_{n-1}\right) + K\, x_n, \qquad x_0 = 0, \quad n \in \mathbb{N}, \tag{2.5}$$

has a unique positive solution $(x_n)_{n\in\mathbb{N}}$*, and this solution is obtained by setting*

$$x_1 = \frac{\int_{\mathbb{R}} x^2 \exp\left(-\frac{c}{4}x^4 - \frac{K}{2}x^2\right)}{\int_{\mathbb{R}} \exp\left(-\frac{c}{4}x^4 - \frac{K}{2}x^2\right)}\, dx\,. \tag{2.6}$$

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[4] In [9, Theorem 3, p. 268] the word "nonnegative" is used erroneously, PN's maxima culpa.

[5] Interestingly, PN and Lew–Quarles mutually cross-credit each other for the result; the reason being that (i) they corresponded while working on their papers, and (ii) PN happened to be the editor of Lew–Quarles's paper that was published in J. Approximation Theory.

[6] [11] was received by SIMA on April 5, 1983.

[7] [3] was received by JAT on March 28, 1983.

Of course, the c parameter adds nothing new and in all proofs it can be assumed, without loss of generality, to be equal to 1 or 4 or whatever one finds more convenient. However, the additional parameter K represents real progress. It comes from orthogonal polynomials associated with weights $\exp\left(-\frac{c}{4}x^4 - \frac{K}{2}x^2\right)$ on $\mathbb{R}$. As a matter of fact, [3] is the "almost-birthplace" of orthogonal polynomials associated explicitly with such weight functions. We wrote "almost-birthplace", because they also appear in Daniel Bessis' 1979 paper [1, (III.1), p. 151] where the weight function is $\exp\left(-\beta x^4 - \frac{1}{2}x^2\right)$ on $\mathbb{R}$. As long as $K > 0$, these two weights are equivalent to each other. However, as soon as $K < 0$, the rules of the game change drastically. We will return to this in a moment. For some reason unknown to us, [1] doesn't treat the case $K = 0$ even though in 1979 that would have been opening up new vistas as well. The equivalent of (2.5) is lurking in [1, (IV.18), p. 151] a telescoping summation leads to the equivalent of (2.5). In the 1980 Bessis–Itzykson–Zuber paper [2], the $K > 0$ equivalent also pops up although it's a little harder to recognize it, see, e.g., [2, (4.32), p. 126] where it is referred to as the "quartic case", and then [2, (4.33), p. 126] is the equation corresponding to (2.5). Since none of us is capable of understanding either of these papers, we won't comment on them except for emphasizing that in both papers $K > 0$. If the reader is interested, he can check out [2, §6, p. 128–131], especially the last sentence that refers to "$N \to \infty$ *selects out a unique initial condition, in the sense of asymptotic series, which is precisely (6.18)*" where the latter formula is essentially (2.6).

For the sake of fairness, let us point out that in the 1980s neither Lew–Quarles nor PN were familiar with [1, 2]. Had they been aware of these papers, it might have been a game changer.

The reason that we mentioned these two papers is that they subsequently became the standard reference as the birthplace of Painlevé's discrete equation #1, that is, d-$\mathrm{P_I}$ even though the case $K < 0$ was not even considered in them. On the other hand, [3] was fully ignored by practically all Painlevé experts. If the reader wants to find out what Painlevé d-$\mathrm{P_I}$ is, he can turn to Google or, even better, read one of Alphonse Magnus' excellent survey papers such as [8] who is also well familiar with the work done by PN and his collaborators in the 1980s.

In 1984 PN mentioned his papers and those of Stan Bonan and Lew-Quarles to Dan Hajela who at the time was a student in his introductory real analysis class, and told him how interesting it would be to find new approaches to studying difference equations of the type mentioned above. Hajela turned his attention to a combination of (2.4) and (2.5), and in [6] he came up with[8]

$$\ell_n = x_n\left(x_{n+1} + x_n + x_{n-1}\right) + \kappa_n\, x_n, \qquad x_0 \in \mathbb{R},\ \ell_n > 0,\ \kappa_n \in \mathbb{R},\ n \in \mathbb{N}. \tag{2.7}$$

Among others, he found a new proof of Theorem 2.2 but, very unfortunately, only for the case when $K \in \mathbb{R}_+$ where $\mathbb{R}_+$ is, again very unfortunately, undefined, although clearly it is either the set of positive or nonnegative real numbers, most likely the latter, see [6, Theorem 2.2, p. 210].

The proof of Theorem 2.1 is elementary whereas the proof of Theorem 2.2 is anything but elementary. Although PN was the editor of [6], he somehow missed or

[8]Hajela in [6, Theorem 2.2, p. 210] writes $\ell_n \geqslant 0$ but that appears to be a typo.

forgot that for uniqueness of positive solutions in (2.7) the parameter κ_n had to be nonnegative. Hence, for 25 years, PN was under the false impression that there is a proof of Theorem 2.2 that is not based on orthogonal polynomials, Fourier integrals, and the moment problem, but, instead, relies on some rather elementary fixed point arguments. Therefore, he no longer sought an elementary solution although the equation (2.5) was always on his mind. As a matter of fact, PN mentioned (2.5), and its special case (2.3) to Vilmos Totik March of 2003 who thought it would be a good problem for a Schweitzer competition,[9] and the uniqueness of positive solutions of (2.3) was indeed included as Problem #6 in 2003.[10] Thanks to the participants and Vilmos Totik, we were given access to some of the ingenious proofs by Rezső László Lovas, András Máthé, Tamás Terpai, and Péter Varjú. Varjú even included a proof for the existence of positive solutions that we borrowed and adopted in this paper, see Theorem 4.1.[11] We thank all of them for sharing their solutions with us.

Fast forward to February of 2013. PN and JSZ spent two weeks with SA at KAU in Jeddah chock-full of heated discussions of equations of the type described above and lamenting that there is a lack of any new developments in the area of existence and uniqueness of positive solutions. At the end of their visits they flew to Riyadh to attend a workshop on special functions where they met WVA who overheard them talking about the above equations and casually mentioned that his talk next day will be about discrete Painlevé equations which is just a fancy term describing the same object. The rest is history and this is the first installment of what is expected to be a long term research project.

3. Notation

For $a \in \mathbb{R}$, the negative and positive parts of a are denoted by a^- and a^+, respectively; they are defined the usual way, for instance, $a^- \stackrel{\text{def}}{=} (|a| - a)/2$.

We call a sequence, say, $\mathbf{Z} \stackrel{\text{def}}{=} (z_n)$ positive, if $(z_n) \in \mathbb{R}^+$, that is, $z_n > 0$ for each n in the domain of $\mathbf{Z}$. A sequence $\mathbf{Z} \stackrel{\text{def}}{=} (z_n)$ is nonnegative if $(z_n) \in \mathbb{R}_0^+$, that is, $z_n \geqslant 0$ for each n in the domain of $\mathbf{Z}$.

We will study solutions $\mathbf{X} \stackrel{\text{def}}{=} (x_n)_{n\in\mathbb{N}}$ of nonhomogeneous nonlinear second order difference equations (recurrence or recursive formulas) of the type

$$\begin{gathered} \ell_n = x_n\,(\sigma_{n,1}\,x_{n+1} + \sigma_{n,0}\,x_n + \sigma_{n,-1}\,x_{n-1}) + \kappa_n\,x_n, \quad n \in \mathbb{N}, \\ \text{with given initial data } \{x_0 \in \mathbb{R} \;\;\&\;\; x_1 \in \mathbb{R}^+\} \end{gathered} \tag{3.1}$$

where

$$(\ell_n)_{n\in\mathbb{N}} \in \mathbb{R}^+ \;\&\; (\sigma_{n,0})_{n\in\mathbb{N}} \in \mathbb{R}^+ \;\&\; (\kappa_n)_{n\in\mathbb{N}} \in \mathbb{R}, \tag{3.2}$$

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[9] see en.wikipedia.org/wiki/Miklós_Schweitzer_Competition

[10] Go to the website versenyvizsga.hu/external/vvszuro/vvszuro.php, first click on "*Böngészés*", then on "*Schweitzer Miklós Emlékverseny*', and scroll down to "*2003 1. kateg. 1. ford. 13. évfolyam*".

[11] As one of the referees pointed it out, [4] is a good source for fixed point theorems and monotonically decomposable iterative processes, see especially [4, §21].

and the left and right σ-coefficients satisfy either

$$(\sigma_{n,1})_{n\in\mathbb{N}} \in \mathbb{R}^+ \ \& \ (\sigma_{n,-1})_{n\in\mathbb{N}} \in \mathbb{R}^+ \tag{3.3}$$

or

$$(\sigma_{n,1})_{n\in\mathbb{N}} \in \mathbb{R}_0^+ \ \& \ (\sigma_{n,-1})_{n\in\mathbb{N}} \in \mathbb{R}_0^+. \tag{3.4}$$

Note that in (3.2) & (3.3) all σ-coefficients must be positive whereas in (3.2) & (3.4) the left and right σ-coefficients may vanish. However, the biggest semantic difference between (3.3) and (3.4) is that in the latter case, because the coefficient of x_{n+1} may vanish, pedantically speaking, the terms *recurrence* or *recursive formula* are no longer appropriate although the term *difference equation* is still valid.

4. Existence

In this section, we prove the following theorem about existence of positive solutions of (3.1).

Theorem 4.1. *Let the conditions in* (3.2) *and* (3.3) *be satisfied. Then, for every* $x_0 \in \mathbb{R}$*, there exists at least one* $x_1 \in \mathbb{R}^+$ *such that the equation* (3.1) *has a positive solution* $\mathbf{X} = (x_n)_{n\in\mathbb{N}}$.

Proof. Introducing $t \stackrel{\text{def}}{=} x_1$, we may view $x_n \equiv x_n(t)$, for each $n \in \mathbb{N}$, as a function of t on $\mathbb{R}_0^+$ with the exception of those points t where the equation (3.1) can't be solved for the senior term because either the previous one vanishes or when some of the earlier terms are undefined. For instance, $x_1(t) \equiv t$.

First, we will construct two strictly monotone sequences $\mathbf{A} \stackrel{\text{def}}{=} (\alpha_n)_{n\in\mathbb{N}}$ and $\mathbf{B} \stackrel{\text{def}}{=} (\beta_n)_{n\in\mathbb{N}}$ such that

$$-\infty < \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \beta_3 < \beta_2 < \beta_1 < \infty \tag{4.1}$$

with the property that, for each $n \in \mathbb{N}$, we have $x_n(t) > 0$ for $t \in (\alpha_n, \beta_n)$ and then (3.1) can be solved for $x_{n+1}(t)$ both in (α_n, β_n) and at those endpoints of (α_n, β_n) where $x_n(t) \neq 0$.

We will generate the sequences $\mathbf{A}$ and $\mathbf{B}$ in such a way that for each $n \in \mathbb{N}$ the function x_n is continuous in $[\alpha_n, \beta_n]$,

$$x_n(\alpha_n) = \begin{cases} 0 & \text{if } n \in \mathbb{N} \text{ is odd,} \\ \frac{-\kappa_n + \sqrt{\kappa_n^2 + 4\sigma_{n,0}\,\ell_n}}{2\sigma_{n,0}} & \text{if } n \in \mathbb{N} \text{ is even,} \end{cases} \tag{4.2}$$

and

$$x_n(\beta_n) = \begin{cases} 1 + \frac{-(\sigma_{1,-1}\, x_0 + \kappa_1) + \sqrt{(\sigma_{1,-1}\, x_0 + \kappa_1)^2 + 4\sigma_{1,0}\, \ell_1}}{2\sigma_{1,0}} & \text{if } n = 1, \\ \frac{-\kappa_n + \sqrt{\kappa_n^2 + 4\sigma_{n,0}\, \ell_n}}{2\sigma_{n,0}} & \text{if } n \in \mathbb{N}\backslash\{1\} \text{ is odd}, \\ 0 & \text{if } n \in \mathbb{N} \text{ is even}. \end{cases} \tag{4.3}$$

The construction will be made by semi-complete induction; the reason for treating $n = 1$ separately in (4.3) will be explained shortly.

We define the first term $\alpha_1 \overset{\text{def}}{=} 0$ and then pick $\beta_1 > 0$ in such a way that

$$\frac{\ell_1}{\beta_1} - \sigma_{1,0}\, \beta_1 - \sigma_{1,-1}\, x_0 - \kappa_1 < 0,$$

that is, β_1 is greater than the positive zero of the quadratic polynomial

$$\sigma_{1,0}\, t^2 + (\sigma_{1,-1}\, x_0 + \kappa_1)\, t - \ell_1,$$

and one possible choice for β_1 is

$$\beta_1 \overset{\text{def}}{=} 1 + \frac{-(\sigma_{1,-1}\, x_0 + \kappa_1) + \sqrt{(\sigma_{1,-1}\, x_0 + \kappa_1)^2 + 4\sigma_{1,0}\, \ell_1}}{2\sigma_{1,0}}\,.$$

We have $x_1(t) \equiv t$ so that x_1 is continuous in $[\alpha_1, \beta_1]$, is positive in (α_1, β_1), and the first relations in (4.2) and (4.3) are also satisfied with $n = 1$.

Now let $n = 1$. Then, by (3.1) we have

$$\sigma_{1,1}\, x_2(t) = \frac{\ell_1}{x_1(t)} - \sigma_{1,0}\, x_1(t) - \sigma_{1,-1}\, x_0 - \kappa_1,$$

so that x_2 is continuous on (α_1, β_1),

$$\lim_{t \to \alpha_1 + 0} x_2(t) = +\infty \tag{4.4}$$

and, by the choice of β_1 in (4.3),

$$\sigma_{1,1}\, x_2(\beta_1) = \frac{\ell_1}{x_1(\beta_1)} - \sigma_{1,0}\, x_1(\beta_1) - \sigma_{1,-1}\, x_0 - \kappa_1 < 0.$$

Therefore, x_2 has at least one zero in (α_1, β_1). Let

$$\beta_2 \overset{\text{def}}{=} \inf\{t : t > \alpha_1,\ x_2(t) = 0\},$$

and then $\alpha_1 < \beta_2 < \beta_1$ and, by continuity, $x_2(\beta_2) = 0$ as well. Using (4.4) and the intermediate value theorem, we can find $\alpha_2 \in (\alpha_1, \beta_2)$ such that

$$x_2(\alpha_2) = \frac{\sqrt{\kappa_2^2 + 4\sigma_{2,0}\, \ell_2} - \kappa_2}{2\sigma_{2,0}}\,.$$

Hence, α_2, β_2, and x_2 all have the prescribed properties.

The next step involves a semi-complete induction in the sense that passing to $n+1$ from n will also use the inductive assumption for $n-1$. The inductive step is almost but not exactly the same as we went to x_2 from x_1 and x_0.

Suppose the construction with the properties mentioned above has been done up to n, and now we proceed with it for $n+1$. We assume that n is odd; the other case is similar. Writing (3.1) in the form

$$\sigma_{n,1}\, x_{n+1}(t) = \frac{\ell_n}{x_n(t)} - \sigma_{n,0}\, x_n(t) - \sigma_{n,-1}\, x_{n-1}(t) - \kappa_n,$$

we can see that x_{n+1} is continuous on (α_n, β_n) because, by the inductive hypotheses, (i) x_n is positive and continuous on (α_n, β_n), (ii) x_{n-1} is positive and continuous on $(\alpha_{n-1}, \beta_{n-1})$, and (iii) $[\alpha_n, \beta_n] \subset (\alpha_{n-1}, \beta_{n-1})$. Furthermore, by the first relation in (4.2), the second in (4.3), and by simple algebra, we have

$$\lim_{t\to\alpha_n+0} x_{n+1}(t) = +\infty \quad \& \quad x_{n+1}(\beta_n) = -\frac{\sigma_{n,-1}}{\sigma_{n,1}} x_{n-1}(\beta_n) < 0 \tag{4.5}$$

because $\sigma_{n,1}$ and $\sigma_{n,-1}$ are both positive. Hence, x_{n+1} has at least one zero in (α_n, β_n). Let

$$\beta_{n+1} \stackrel{\text{def}}{=} \inf\{t : t > \alpha_n,\ x_{n+1}(t) = 0\}.$$

Then, by continuity, $x_{n+1}(\beta_{n+1}) = 0$ and, by the first limit in (4.5), $\beta_{n+1} \in (\alpha_n, \beta_n)$. By the construction, $x_{n+1} > 0$ in (α_n, β_{n+1}). Again by the first limit in (4.5), the intermediate value theorem guarantees the existence of $\alpha_{n+1} \in (\alpha_n, \beta_{n+1})$ such that

$$x_{n+1}(\alpha_{n+1}) = \frac{\sqrt{\kappa_{n+1}^2 + 4\sigma_{n+1,0}\,\ell_{n+1}} - \kappa_{n+1}}{2\sigma_{n+1,0}}.$$

This proves that x_{n+1} has all the required properties.

Now the theorem follows immediately, since, by the nested interval theorem,[12] we can pick a number t^*, that is, an initial value x_1 such that

$$\lim_{n\to\infty} \alpha_n \leqslant t^* \leqslant \lim_{n\to\infty} \beta_n,$$

and then the sequence $\{x_n(t^*)\}_{n\in\mathbb{N}}$ is a positive solution of (3.1). □

Note 4.2. Of course, if

$$\lim_{n\to\infty} \alpha_n < \lim_{n\to\infty} \beta_n$$

above, then every t^* between those two limits would yield a positive solution. However, for all practical purposes this observation is useless since we have no actual information about those limits. A similarly "useless" observation was made in [7, Theorem 4.3, p. 365], see $x^{\pm}$ there.

[12]We should rather say that by a version of the nested interval theorem since our intervals are open and we also need the fact that the closure of each interval lies inside the interior of its parent interval.

5. Uniqueness

In this section, we study uniqueness of positive solutions of (3.1). We will need the following lemma that is no doubt well known and is straightforward anyway.

Lemma 5.1. *If* $\mathbf{\Omega} \stackrel{\text{def}}{=} (\omega_n)_{n=0}^{\infty}$ *is a convex sequence of real numbers that grows slower than linear, that is,*

$$2\omega_n \leqslant \omega_{n+1} + \omega_{n-1}, \qquad \forall n \in \mathbb{N}, \tag{5.1}$$

and

$$\liminf_{n\to\infty} \frac{\omega_n}{n} \leqslant 0\,, \tag{5.2}$$

then $\mathbf{\Omega}$ *is a nonincreasing sequence. In particular, if* $\mathbf{\Omega}$ *is a nonnegative sequence with* $\omega_0 = 0$*, then* $\omega_n = 0$ *for* $n \in \mathbb{N}$*.*

Proof. Rewriting (5.1) as

$$\omega_n - \omega_{n-1} \leqslant \omega_{n+1} - \omega_n, \qquad n \in \mathbb{N},$$

shows that $(\omega_n - \omega_{n-1})_{n\in\mathbb{N}}$ is a nondecreasing sequence so that

$$\omega_n - \omega_{n-1} \leqslant \omega_p - \omega_{p-1}, \qquad n, p \in \mathbb{N} \quad \& \quad n \leqslant p,$$

from which

$$(q-n)\left(\omega_n - \omega_{n-1}\right) = \sum_{p=n+1}^{q} \left(\omega_n - \omega_{n-1}\right) \leqslant \sum_{p=n+1}^{q} \left(\omega_p - \omega_{p-1}\right) = \omega_q - \omega_n,$$
$$n, q \in \mathbb{N} \quad \& \quad n < q,$$

that is,

$$\omega_n - \omega_{n-1} \leqslant \frac{\omega_q - \omega_n}{q-n}\,, \qquad n, q \in \mathbb{N} \quad \& \quad n < q,$$

and now, fixing $n \in \mathbb{N}$, letting $q \to \infty$, and taking (5.2) into consideration, we finally see that $\omega_n - \omega_{n-1} \leqslant 0$ for $n \in \mathbb{N}$, that is, $\mathbf{\Omega}$ is a nonincreasing sequence. ☐

We define $(\sigma_n)_{n\in\mathbb{N}}$ by

$$\sigma_n \stackrel{\text{def}}{=} \max\left(\sigma_{n,-1}, \sigma_{n,1}\right), \qquad n \in \mathbb{N}, \tag{5.3}$$

see (3.4), so that $\sigma_n \geqslant 0$ for $n \in \mathbb{N}$.

Theorem 5.2. *With the conditions in* (3.2) & (3.4) *and with the notation* (5.3)*, assume*

$$\liminf_{n\to\infty} \frac{1}{n^2} \left(\frac{\ell_n}{\sigma_{n,0}} + \frac{\left(\kappa_n^-\right)^2}{\sigma_{n,0}^2} \right) = 0\,. \tag{5.4}$$

Let $\mathbb{N}$ *be representable as the disjoint union* $\mathbb{N} = \mathbb{N}_{✡} \cup \mathbb{N}_{✣}$ *such that, for each* $n \in \mathbb{N}$*, one of the following two displayed conditions*

$$2\,\sigma_n \leqslant \sigma_{n,0}\,, \qquad \text{if } n \in \mathbb{N}_{✡}, \tag{✡}$$

or

$$\sigma_n \leqslant \sigma_{n,0} < 2\sigma_n \;\; \& \;\; -2\,(\sigma_{n,0} - \sigma_n)\,\sqrt{\ell_n} \leqslant \kappa_n \sqrt{2\sigma_n - \sigma_{n,0}}\,, \quad \text{if } n \in \mathbb{N}_{✣} \qquad (✣)$$

is satisfied.
In addition, if $1 \in \mathbb{N}_{✡}$*, then simply let* x_0 *in* (3.1) *be an arbitrary real number, whereas if* $1 \in \mathbb{N}_{✣}$*, then we also assume that* x_0 *satisfies*

$$-2\,(\sigma_{1,0} - \sigma_1)\,\sqrt{\ell_1} \leqslant (\sigma_{1,-1}\,x_0 + \kappa_1)\,\sqrt{2\sigma_1 - \sigma_{1,0}}\,. \tag{5.5}$$

Then there exists a unique $x^* > 0$ *such that if a sequence* $\mathbf{X} = (x_n)_{n\in\mathbb{N}}$ *satisfying equation* (3.1) *is positive then* $x_1 = x^*$*, and, therefore,* (3.1) *can't have more than one positive solution.*

Before we prove Theorem 5.2, let us discuss a few special cases.

Example 5.3. *In the extreme case when the left and right* σ*-coefficients in* (3.4) *all vanish, we have* $\mathbb{N}_{✡} = \mathbb{N}$ & $\mathbb{N}_{✣} = \varnothing$*, and* (3.1) *takes the form*

$$\ell_n = \sigma_{n,0}\,x_n^2 + \kappa_n\,x_n, \quad n \in \mathbb{N},$$

so that, clearly, for each $n \in \mathbb{N}$*, the quadratic equation has a unique positive solution* x_n*.*

Example 5.4. *If we allowed the middle* σ*-coefficients in* (3.1) *to vanish too, then we could end up with*

$$\ell_n = \kappa_n\,x_n, \quad n \in \mathbb{N},$$

that would not have a positive solution $\mathbf{X}$ *unless* $(\kappa_n)_{n\in\mathbb{N}}$ *is positive.*

Example 5.5. *Other examples showing the significance of the middle* σ*-coefficients in* (3.1) *are the following. If*

$$1 = x_n(x_{n+1} + x_{n-1}), \quad n \in \mathbb{N},$$

then the substitution $y_n \stackrel{\text{def}}{=} x_{n-1}x_n$ *transforms it to*

$$1 = y_{n+1} + y_n, \quad n \in \mathbb{N},$$

so that $y_n = 1/2 + (-1)^n\,\text{const}$*, and if*

$$n = x_n(x_{n+1} + x_{n-1}), \quad n \in \mathbb{N},$$

then the same substitution leads to $y_n = n/2 - 1/4 + (-1)^n\,\text{const}$*.*[13]

[13] These examples were suggested by one of the referees.

Example 5.6. *One can take* $\sigma_{1,1} = \sqrt{2}$ & $\sigma_{1,0} = 1$ & $\sigma_{1,-1} = 0$, *and* $\sigma_{n,1} = \sqrt{\frac{n}{n+1}}$ & $\sigma_{n,0} = 1$ & $\sigma_{n,-1} = \sqrt{\frac{n}{n-1}}$ *for* $n \geqslant 2$. *Then* $x_n = \sqrt{n}$ *is a solution of*

$$3n = x_n(\sigma_{n,1}\, x_{n+1} + \sigma_{n,0}\, x_n + \sigma_{n,-1}\, x_{n-1}), \quad n \in \mathbb{N},$$

with $x_0 = 0$ *and* $x_1 = 1$. *The solution for* $x_0 = 0$ *and* $x_1 = -1$ *is* $x_n = -\sqrt{n}$. *These are the only "nice" solutions, that is, monotone, and either positive or negative. The expression* $\sqrt{n}$ *corresponds to the asymptotic behavior whenever* $\sigma_{n,1}$ *and* $\sigma_{n,-1}$ *converge to* 1 *as* $n \to \infty$.

Combining Theorems 4.1 & 5.2 and simplifying the conditions in the latter we get the following corollary.

Corollary 5.7. *Let* $\sigma_n \leqslant \sigma_{n,0}$ & $\kappa_n \geqslant 0$ *for* $n \in \mathbb{N}$, *let*

$$\liminf_{n\to\infty} \frac{\ell_n}{n^2\, \sigma_{n,0}} = 0\,, \tag{5.6}$$

and assume that either $x_0 = 0$ *or, at least,* x_0 *satisfies* $\sigma_{1,-1}x_0 \geqslant -\kappa_1$, *see* (3.4) *and* (5.3). *Then there exists a unique* $x_1 > 0$ *such that the sequence* $\mathbf{X}$ *satisfying equation* (3.1) *is positive.*

Proof of Theorem 5.2. Suppose (3.1) has at least one positive solution. Pick two, not necessarily distinct, positive solutions of (3.1), say, $\mathbf{U} \stackrel{\text{def}}{=} (u_n)_{n\in\mathbb{N}}$ and $\mathbf{V} \stackrel{\text{def}}{=} (v_n)_{n\in\mathbb{N}}$. Denoting $\varepsilon_n \stackrel{\text{def}}{=} u_n - v_n$ for $n \in \mathbb{N} \cup \{0\}$ and taking the difference of the corresponding equations, we obtain

$$\left(\sigma_{n,0} + \frac{\ell_n}{u_n v_n}\right) \varepsilon_n = -\sigma_{n,1}\varepsilon_{n+1} - \sigma_{n,-1}\varepsilon_{n-1}, \qquad n \in \mathbb{N}, \tag{5.7}$$

so that, by (5.3),

$$\left(\sigma_{n,0} + \frac{\ell_n}{u_n v_n}\right) |\varepsilon_n| \leqslant \sigma_n \left(|\varepsilon_{n+1}| + |\varepsilon_{n-1}|\right), \qquad n \in \mathbb{N}. \tag{5.8}$$

Step 1. Our first goal is to show that

$$2\,|\varepsilon_n| \leqslant |\varepsilon_{n+1}| + |\varepsilon_{n-1}|\,, \qquad \forall n \in \mathbb{N}. \tag{5.9}$$

If $n \in \mathbb{N}_{\text{✡}}$, then (5.9) holds trivially by the theorem's assumption (✡) since all terms in (5.8) are nonnegative.

If $n \in \mathbb{N}_{\text{✣}}$, then we have to consider separately when $n = 1$ and $n > 1$; the reason being that for $n = 1$ equation (3.1) includes the term x_0 that is not necessarily nonnegative, and, therefore, x_0 can't be thrown away when estimating x_1.

When $n \in \mathbb{N}_{\text{✣}}$ and $n = 1$, we obtain from (3.1) the inequality

$$\ell_1 \geqslant x_1\,(\sigma_{1,0}\, x_1 + \sigma_{1,-1}\, x_0) + \kappa_1\, x_1 = \sigma_{1,0}\, x_1^2 + (\sigma_{1,-1}\, x_0 + \kappa_1)\; x_1,$$

so that by (5.5)

$$\ell_1 \geqslant \sigma_{1,0}\, x_1^2 - \frac{2\,(\sigma_{1,0}-\sigma_1)\,\sqrt{\ell_1}}{\sqrt{2\sigma_1-\sigma_{1,0}}}\, x_1 .$$

Therefore, x_1 must lie between the roots of

$$\sigma_{1,0}\, x^2 - \frac{2\,(\sigma_{1,0}-\sigma_1)\,\sqrt{\ell_1}}{\sqrt{2\sigma_1-\sigma_{1,0}}}\, x - \ell_1 = 0\,.$$

Solving this quadratic equation, we obtain

$$x_1 \leqslant \frac{\sqrt{\ell_1}}{\sqrt{2\sigma_1-\sigma_{1,0}}}\,, \qquad 1 \in \mathbb{N}_{✣}. \tag{5.10}$$

Therefore,

$$2\sigma_1 - \sigma_{1,0} \leqslant \frac{\ell_1}{u_1 v_1}\,,$$

which, together with (5.8), implies (5.9) for $n = 1 \in \mathbb{N}_{✣}$.

When $n \in \mathbb{N}_{✣}$ and $n > 1$, we proceed the same way with a minor modification. Namely, we obtain from (3.1) the inequality

$$\ell_n \geqslant x_n\,(\sigma_{n,0}\, x_n) + \kappa_n\, x_n = \sigma_{n,0}\, x_n^2 + \kappa_n\, x_n,$$

so that by the second inequality in (✣)

$$\ell_n \geqslant \sigma_{1,0}\, x_n^2 - \frac{2\,(\sigma_{n,0}-\sigma_n)\,\sqrt{\ell_n}}{\sqrt{2\sigma_n-\sigma_{n,0}}}\, x_n,$$

and then the same "largest root of the quadratic equation" argument we find that

$$x_n \leqslant \frac{\sqrt{\ell_n}}{\sqrt{2\sigma_n-\sigma_{n,0}}}\,, \qquad 1 < n \in \mathbb{N}_{✣},$$

that is,

$$2\sigma_n - \sigma_{n,0} \leqslant \frac{\ell_n}{u_n v_n},$$

which, together with (5.8), implies (5.9) for $n \in \mathbb{N}_{✣} \backslash \{1\}$ as well.

Step 2. Our next goal is to show that

$$\liminf_{n\to\infty} \frac{|\varepsilon_n|}{n} = 0. \tag{5.11}$$

We will estimate $|\varepsilon_n|$ for all $n \in \mathbb{N}\backslash\{1\}$ in one fell swoop. Throwing away all nonnegative terms in (3.1), we obtain

$$\ell_n \geqslant \sigma_{n,0}\, x_n^2 + \kappa_n\, x_n, \qquad n > 1,$$

so that x_n must lie between the roots of $\sigma_{n,0}\, x^2 + \kappa_n x - \ell_n = 0$, and, therefore,

$$|\varepsilon_n| \leqslant u_n + v_n \leqslant 2 \times \sqrt{\frac{\ell_n}{\sigma_{n,0}}} \qquad \text{if } n > 1 \;\&\; \kappa_n \geqslant 0,$$

and

$$|\varepsilon_n| \leqslant u_n + v_n \leqslant \frac{\sqrt{\kappa_n^2 + 4\ell_n\, \sigma_{n,0}} - \kappa_n}{\sigma_{n,0}} \leqslant 2 \times \left(\sqrt{\frac{\ell_n}{\sigma_{n,0}}} - \frac{\kappa_n}{\sigma_{n,0}} \right), \qquad \text{if } n > 1 \;\&\; \kappa_n < 0,$$

so that

$$\varepsilon_n^2 \leqslant 8 \times \left(\frac{\ell_n}{\sigma_{n,0}} + \frac{\left(\kappa_n^-\right)^2}{\sigma_{n,0}^2} \right), \qquad \forall n > 1 \;\&\; \forall \kappa_n \in \mathbb{R},$$

and, in view of (5.4), the limit relationship in (5.11) follows.

Combining what was proved in steps 1 & 2, that is, (5.9) and (5.11), we can use now Lemma 5.1, applied with $\omega_n \stackrel{\text{def}}{=} |\varepsilon_n|$, to obtain immediately that the two solutions $\mathbf{U}$ and $\mathbf{V}$ are, in fact, identical. □

Note 5.8. It would be interesting to see either examples or conditions when one can produce precisely Q different initial data $x_1 > 0$ yielding positive solutions where $1 < Q \in \mathbb{N}$ is prescribed.

Note 5.9. Although Lew-Quarles' [7, Theorem 4.3, p. 365] uses an entirely different approach to uniqueness, it also assumes that $\liminf_{n\to\infty} \ell_n/n^2 = 0$ that is essentially the same as (5.6). On the other hand, Lew-Quarles' [7, Theorem 6.3, p. 369] and Hajela's [6, Theorem 2.2, p. 210] impose the condition $\lim_{n\to\infty} \ell_{n+1}/\ell_n > 0$ that is of a totally different nature.

6. Limits

In this section, we investigate asymptotic behavior of (not necessarily positive or negative) solutions of (3.1). As before, we always assume that $(\ell_n)_{n\in\mathbb{N}} \in \mathbb{R}^+$. For convenience, we rewrite (3.1), that is,

$$\ell_n = x_n \left(\sigma_{n,1}\, x_{n+1} + \sigma_{n,0}\, x_n + \sigma_{n,-1}\, x_{n-1}\right) + \kappa_n\, x_n$$

as

$$1 = \frac{x_n}{\sqrt{\ell_n}} \left(\sigma_{n,1} \sqrt{\frac{\ell_{n+1}}{\ell_n}}\, \frac{x_{n+1}}{\sqrt{\ell_{n+1}}} + \sigma_{n,0}\, \frac{x_n}{\sqrt{\ell_n}} + \sigma_{n,-1} \sqrt{\frac{\ell_{n-1}}{\ell_n}}\, \frac{x_{n-1}}{\sqrt{\ell_{n-1}}} \right) + \frac{\kappa_n}{\sqrt{\ell_n}}\, \frac{x_n}{\sqrt{\ell_n}}$$

or, introducing

$$t_n \stackrel{\text{def}}{=} \frac{x_n}{\sqrt{\ell_n}}\,, \tag{6.1}$$

as

$$1 = t_n \left(\sigma_{n,1} \sqrt{\frac{\ell_{n+1}}{\ell_n}}\, t_{n+1} + \sigma_{n,0}\, t_n + \sigma_{n,-1} \sqrt{\frac{\ell_{n-1}}{\ell_n}}\, t_{n-1} \right) + \frac{\kappa_n}{\sqrt{\ell_n}}\, t_n\,. \tag{6.2}$$

We start with the following observation.

Theorem 6.1. *Let the coefficients of* (3.1) *satisfy*

$$\liminf_{n\to\infty} \sigma_{n,0} > 0 \quad \& \quad \limsup_{n\to\infty} \sigma_{n,\pm 1}\sqrt{\frac{\ell_{n\pm 1}}{\ell_n}} < \infty \quad \& \quad \limsup_{n\to\infty} \frac{|\kappa_n|}{\sqrt{\ell_n}} < \infty\,.$$

If $\mathbf{X} = (x_n)_{n\in\mathbb{N}}$ *satisfies* (3.1)*, then*

$$\liminf_{n\to\infty} \frac{x_n}{\sqrt{\ell_n}} > -\infty \quad \Longleftrightarrow \quad \limsup_{n\to\infty} \frac{x_n}{\sqrt{\ell_n}} < \infty\,. \tag{6.3}$$

Corollary 6.2. *Let the coefficients of* (3.1) *satisfy*

$$\limsup_{n\to\infty} \sigma_{n,1} < \infty \quad \& \quad \liminf_{n\to\infty} \sigma_{n,0} > 0 \quad \& \quad \limsup_{n\to\infty} \sigma_{n,-1} < \infty$$

and

$$0 < \liminf_{n\to\infty} \sqrt{\frac{\ell_{n+1}}{\ell_n}} \leqslant \limsup_{n\to\infty} \sqrt{\frac{\ell_{n+1}}{\ell_n}} < \infty \quad \& \quad \limsup_{n\to\infty} \frac{|\kappa_n|}{\sqrt{\ell_n}} < \infty.$$

If $\mathbf{X} = (x_n)_{n\in\mathbb{N}}$ *satisfies* (3.1)*, then* (6.3) *holds.*

Note 6.3. Of course, we could have stated (6.3) as

$$\liminf_{n\to\infty} \frac{x_n}{\sqrt{\ell_n}} > -\infty \quad \Longleftrightarrow \quad \limsup_{n\to\infty} \frac{x_n}{\sqrt{\ell_n}} < \infty \quad \Longleftrightarrow \quad \limsup_{n\to\infty} \frac{|x_n|}{\sqrt{\ell_n}} < \infty\,.$$

Example 6.4. *If, for instance,* $\ell_n \stackrel{\text{def}}{=} 1$ *&* $\sigma_{n,1} \stackrel{\text{def}}{=} 1/\sqrt{n}$ *when* n *is even and* $\ell_n \stackrel{\text{def}}{=} n$ *&* $\sigma_{n,1} \stackrel{\text{def}}{=} \sqrt{n}$ *when* n *is odd,* $\sigma_{n,-1} \stackrel{\text{def}}{=} 1/\sigma_{n,1}$*, and* $\sigma_{n,0} \stackrel{\text{def}}{=} 1$ *&* $\kappa_n \stackrel{\text{def}}{=} 1$*, then the coefficient conditions in Theorem 6.1 are satisfied whereas those in Corollary 6.2 are not.*

Proof of Theorem 6.1. First, we will prove $\Longrightarrow$ in (6.3). Let $\mathbb{N}_* \stackrel{\text{def}}{=} \{n \in \mathbb{N} : x_n \geqslant 0\}$. Clearly, it is sufficient to estimate $x_n/\sqrt{\ell_n}$ from above only when $n \in \mathbb{N}_*$. Let $K \in \mathbb{R}$ be such that

$$t_n = \frac{x_n}{\sqrt{\ell_n}} > K, \qquad n \in \mathbb{N}.$$

Then, keeping in mind that $t_n \geqslant 0$ for $n \in \mathbb{N}_*$, equation (6.2) implies

$$1 \geqslant K\,\sigma_{n,1}\sqrt{\frac{\ell_{n+1}}{\ell_n}}\,t_n + \sigma_{n,0}\,t_n^2 + K\sigma_{n,-1}\sqrt{\frac{\ell_{n-1}}{\ell_n}}\,t_n + \frac{\kappa_n}{\sqrt{\ell_n}}\,t_n\,, \qquad n \in \mathbb{N}_*,$$

that is, the quadratic polynomial

$$\sigma_{n,0}\,t^2 + \left(K\,\sigma_{n,1}\sqrt{\frac{\ell_{n+1}}{\ell_n}} + K\sigma_{n,-1}\sqrt{\frac{\ell_{n-1}}{\ell_n}} + \frac{\kappa_n}{\sqrt{\ell_n}}\right)t - 1\,, \qquad n \in \mathbb{N}_*,$$

with positive leading coefficient is nonpositive at $t = t_n$ so that t_n is at most as big as its largest zero is, that is,

$$t_n \leqslant \frac{-B_n + \sqrt{B_n^2 + 4\,A_n}}{2\,A_n}\,, \qquad n \in \mathbb{N}_*,$$

where

$$A_n \stackrel{\text{def}}{=} \sigma_{n,0} \quad \& \quad B_n \stackrel{\text{def}}{=} K\,\sigma_{n,1}\sqrt{\frac{\ell_{n+1}}{\ell_n}} + K\sigma_{n,-1}\sqrt{\frac{\ell_{n-1}}{\ell_n}} + \frac{\kappa_n}{\sqrt{\ell_n}}\,.$$

Hence, by the conditions imposed on the coefficients of (3.1), $\limsup_{\substack{n\to\infty\\ n\in\mathbb{N}_*}} t_n < \infty$ that proves $\Longrightarrow$ in (6.3).

The reverse implication $\Longleftarrow$ in (6.3) follows from the $\Longrightarrow$ case by replacing $(x_n)_{n\in\mathbb{N}}$ by $(-x_n)_{n\in\mathbb{N}}$ in (3.1) that leads to a sign change for κ_n. $\square$

If we expect solutions of (3.1) to behave well as $n\to\infty$ then it is natural to assume that so do the coefficients. This is expressed in the following statement.

Theorem 6.5. *Let the coefficients of* (3.1) *be such that the following four limits satisfy*

$$\sigma_0 \stackrel{\text{def}}{=} \lim_{n\to\infty} \sigma_{n,0} > 0 \quad \& \quad p_{\pm 1} \stackrel{\text{def}}{=} \lim_{n\to\infty} \sigma_{n,\pm 1}\sqrt{\frac{\ell_{n\pm 1}}{\ell_n}} \in \mathbb{R} \quad \& \quad q \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{\kappa_n}{\sqrt{\ell_n}} \in \mathbb{R}\,. \tag{6.4}$$

If (the not necessarily positive) $\mathbf{X} = (x_n)_{n\in\mathbb{N}}$ *satisfies* (3.1) *and*

$$\liminf_{n\to\infty} \frac{x_n}{\sqrt{\ell_n}} \geqslant 0, \tag{6.5}$$

then $\lim_{n\to\infty} x_n/\sqrt{\ell_n}$ *exists and*

$$\lim_{n\to\infty} \frac{x_n}{\sqrt{\ell_n}} = \frac{-q + \sqrt{q^2 + 4\,(p_1 + \sigma_0 + p_{-1})}}{2\,(p_1 + \sigma_0 + p_{-1})}\,, \tag{6.6}$$

and if (the not necessarily negative) $\mathbf{X} = (x_n)_{n\in\mathbb{N}}$ *satisfies* (3.1) *and*

$$\limsup_{n\to\infty} \frac{x_n}{\sqrt{\ell_n}} \leqslant 0, \tag{6.7}$$

then again $\lim_{n\to\infty} x_n/\sqrt{\ell_n}$ *exists and*

$$\lim_{n\to\infty} \frac{x_n}{\sqrt{\ell_n}} = -\frac{q + \sqrt{q^2 + 4\,(p_1 + \sigma_0 + p_{-1})}}{2\,(p_1 + \sigma_0 + p_{-1})}\,. \tag{6.8}$$

Corollary 6.6. *Let the coefficients of* (3.1) *be such that the following five limits satisfy*

$$\sigma_1 \stackrel{\text{def}}{=} \lim_{n\to\infty} \sigma_{n,1} \geqslant 0 \quad \& \quad \sigma_0 \stackrel{\text{def}}{=} \lim_{n\to\infty} \sigma_{n,0} > 0 \quad \& \quad \sigma_{-1} \stackrel{\text{def}}{=} \lim_{n\to\infty} \sigma_{n,-1} \geqslant 0$$

and

$$p \stackrel{\text{def}}{=} \lim_{n\to\infty} \sqrt{\frac{\ell_{n+1}}{\ell_n}} > 0 \quad \& \quad q \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{\kappa_n}{\sqrt{\ell_n}} \in \mathbb{R}\,,$$

where all five limits are finite. If (the not necessarily positive) $\mathbf{X} = (x_n)_{n\in\mathbb{N}}$ *satisfies* (3.1) *and* (6.5), *then* $\lim_{n\to\infty} x_n/\sqrt{\ell_n}$ *exists and*

$$\lim_{n\to\infty} \frac{x_n}{\sqrt{\ell_n}} = \frac{-q + \sqrt{q^2 + 4\,(\sigma_1\, p + \sigma_0 + \sigma_{-1}\, p^{-1})}}{2\,(\sigma_1\, p + \sigma_0 + \sigma_{-1}\, p^{-1})}\,,$$

and if (the not necessarily negative) $\mathbf{X} = (x_n)_{n\in\mathbb{N}}$ *satisfies* (3.1) *and* (6.5), *then again* $\lim_{n\to\infty} x_n/\sqrt{\ell_n}$ *exists and*

$$\lim_{n\to\infty} \frac{x_n}{\sqrt{\ell_n}} = -\frac{q + \sqrt{q^2 + 4\,(\sigma_1\, p + \sigma_0 + \sigma_{-1}\, p^{-1})}}{2\,(\sigma_1\, p + \sigma_0 + \sigma_{-1}\, p^{-1})}\,.$$

Proof of Theorem 6.5. First we deal with the case (6.5). We will use the "t_n" notation, see (6.1). By Theorem 6.1,

$$0 \leqslant \ell \stackrel{\text{def}}{=} \liminf_{n\to\infty} t_n \leqslant L \stackrel{\text{def}}{=} \limsup_{n\to\infty} t_n < \infty\,,$$

and our goal is to show that $L \leqslant \ell$. Once this is done, then evaluating $\lim_{n\to\infty} t_n$ is straightforward. Our approach is based on the *Freud Kunstgriff*.

Use (6.5) to find a nonincreasing sequence of nonnegative numbers $(\varepsilon_n)_{n\in\mathbb{N}}$ such that[14]

$$\lim_{n\to\infty} \varepsilon_n = 0 \quad \& \quad t_n + \varepsilon_n \geqslant 0\,, \quad \forall n \in \mathbb{N}.$$

We rewrite equation (6.2) as

$$\begin{gathered} 1 = (t_n + \varepsilon_n) \times \\ \left(\sigma_{n,1}\sqrt{\frac{\ell_{n+1}}{\ell_n}}\,(t_{n+1}+\varepsilon_{n+1}) + \sigma_{n,0}\,(t_n+\varepsilon_n) + \sigma_{n,-1}\sqrt{\frac{\ell_{n-1}}{\ell_n}}\,(t_{n-1}+\varepsilon_{n-1})\right) + \\ \frac{\kappa_n}{\sqrt{\ell_n}}\,(t_n+\varepsilon_n) + R_n\,, \end{gathered} \tag{6.9}$$

where

$$\begin{gathered} R_n \stackrel{\text{def}}{=} -t_n\left(\sigma_{n,1}\sqrt{\frac{\ell_{n+1}}{\ell_n}}\,\varepsilon_{n+1} + \sigma_{n,0}\,\varepsilon_n + \sigma_{n,-1}\sqrt{\frac{\ell_{n-1}}{\ell_n}}\,\varepsilon_{n-1}\right) - \\ \varepsilon_n\left(\sigma_{n,1}\sqrt{\frac{\ell_{n+1}}{\ell_n}}\,(t_{n+1}+\varepsilon_{n+1}) + \sigma_{n,0}\,(t_n+\varepsilon_n) + \sigma_{n,-1}\sqrt{\frac{\ell_{n-1}}{\ell_n}}\,(t_{n-1}+\varepsilon_{n-1})\right) + \\ -\frac{\kappa_n}{\sqrt{\ell_n}}\,\varepsilon_n\,. \end{gathered}$$

Clearly,

$$\lim_{n\to\infty} R_n = 0$$

because each ε-term in it goes to 0 as $n \to \infty$ and every coefficient of every ε-term in it is $\mathcal{O}(1)$. The advantage of (6.9) is that, except for R_n, every $t+\varepsilon$-term is nonnegative in it and that every coefficient of every such $t+\varepsilon$-term is both nonnegative and convergent.

Now we are in the position to use the *Freud Kunstgriff*. Pick $\mathbb{N}_\ell \subset \mathbb{N}$ and $\mathbb{N}_L \subset \mathbb{N}$ such that

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}_\ell}} t_n = \ell \qquad \& \qquad \lim_{\substack{n\to\infty\\ n\in\mathbb{N}_L}} t_n = L,$$

and let $n \to \infty$ first over $\mathbb{N}_\ell$ and then over $\mathbb{N}_L$ in (6.9). We get

$$1 \leqslant \ell\,(p_1\,L + \sigma_0\,\ell + p_{-1}\,L) + q\,\ell$$

[14]If $\mathbf{X}$ is nonnegative, then just set $\varepsilon_n = 0$ for $n \in \mathbb{N}$.

and

$$1 \geqslant L\,(p_1\,\ell + \sigma_0\,L + p_{-1}\,\ell) + q\,L,$$

respectively, from which

$$L\,(p_1\,\ell + \sigma_0\,L + p_{-1}\,\ell) + q\,L \leqslant \ell\,(p_1\,L + \sigma_0\,\ell + p_{-1}\,L) + q\,\ell\,,$$

that is, $L \leqslant \ell$ so that $\ell = L$. Once we know that $T \stackrel{\text{def}}{=} \lim_{n\to\infty} t_n$ exists, we just let $n \to \infty$ either in (6.9) or in (6.2) to obtain

$$1 = T\,(p_1\,T + \sigma_0\,T + p_{-1}\,T) + q\,T$$

and the positive solution of

$$(p_1 + \sigma_0 + p_{-1}\,)\,T^2 + q\,T - 1 = 0$$

yields (6.6).

If, instead of (6.5), condition (6.7) holds, then, as observed in the proof of Theorem 6.1, replacing $(x_n)_{n\in\mathbb{N}}$ by $(-x_n)_{n\in\mathbb{N}}$ in (3.1) leads to a sign change for κ_n and that results in a sign change of q in (6.4) so that (6.8) follows from (6.6). □

Note 6.7. It remains to be seen if conditions (6.5) and (6.7) in Theorem 6.5 can be replaced by a one-sided $\mathcal{O}(1)$ condition similarly as it is done in Theorem 6.1.

7. Acknowledgments

We thank Vilmos Totik and those participants of the 2003 Schweitzer competition whose solutions of Problem #6 we had the privilege to study and to adopt, especially Péter Varjú, see the details at the end of §2. We also thank the referees whose suggestions helped to improve the presentation.

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