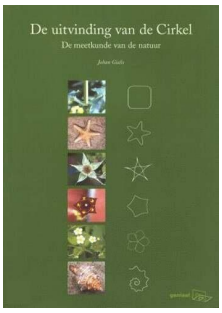
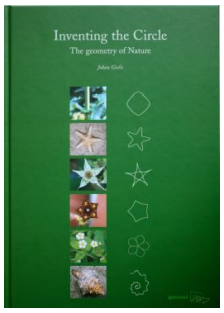
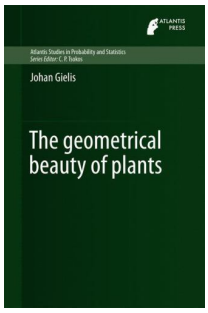
Inventing the circle: the geometry of nature Geniaal, 2003, isbn 9062167920 | **De uitvinding van de cirkel: de meetkunde van de natuur** Geniaal 2001, isbn 9789062157921, 188 p.
The geometrical beauty of plants, Atlantis Press, 2017, isbn 9789462391505, xxv+229 p.







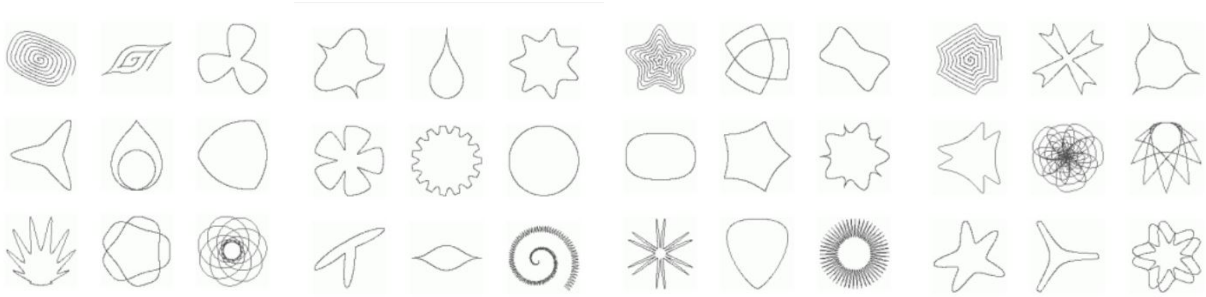


Johan Gielis

Johan Gielis is a botanical engineer from the University of Antwerp. He started using supercircles and superellipses in the 1990's while studying bamboo whose cross section had this kind of shape. These curves are described by $x^n + y^n = R^n$, or in polar coordinates $\rho(\theta) = [\cos(\theta)^n + \sin(\theta)^n]^{-1/n} R$. For $n = 2$, this is just $\rho(\theta) = R$, the usual Euclidean circle with radius R but it can of course easily be generalised to ellipses. These shapes were known and used by the Danish mathematician Piet Hein. Later, he generalised his formula to describe many other shapes of stems, leaves, flowers, etc. In several steps he arrived eventually at a formula that he called a superformula of the form

$$\rho(\theta) = \left[\frac{1}{a}\left|\cos\left(\frac{m_1\theta}{4}\right)\right|^{n_2} + \frac{1}{b}\left|\sin\left(\frac{m_2\theta}{4}\right)\right|^{n_3}\right]^{-1/n_1} R(\theta). \tag{1}$$

The parameters a and b give the "elliptic touch", m_1, m_2 somehow defines how many "peaks" or corners the result shall have, and the n_i parameters govern the size and the convexity of the result. The parametric function $R(\theta)$ was originally a constant, but if for example $R(\theta) = \theta$, this results in spirals. Some examples:



These were taken form `www.procato.com/superformula/` where you can experiment with the parameters interactively. For more interactive sites and octave code see the superformula on wikipedia. Of course it is not so difficult to generalise this to 3D which gives the most bizarre and diverse shapes: see this link for a YouTube video. Gielis took a patent and founded Genicap, a software company to exploit the possibilities of the formula as a subsidiary of his company *Geniaal bvba*. His first book on the topic was published in Dutch (2001) and later in English (2003) by Geniaal.

That book, *Inventing the circle*, is richly illustrated and introduces step by step the superformula starting from the circle. Many illustrations are from biology and especially from plants. In particular the bamboo example gets much attention, but also many shapes of leaves and flowers from other plants. Also eggs, seeds, cells, algae, pollen, fruits, sea shells, starfish, phyllotaxis and morphogenesis, you name it.

His conclusion is that everything is just a circle if you are willing to use a flexible anisotropic measure of distance. This is indeed what this formula is doing. If we use an isotropic fixed distance measure, then all points at a rigid distance R give the usual Euclidean circle with radius R. If we let the measure vary with the angle θ, then we can obtain all the shapes generated by the superformula.

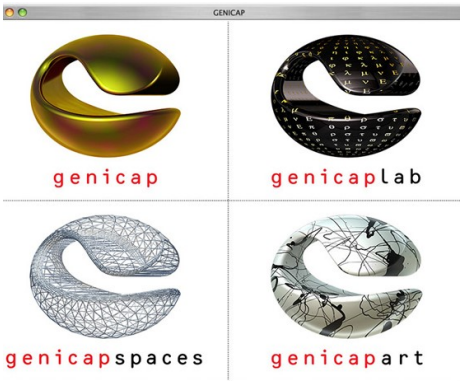
In the first book we learn to know Gielis as the bio-engineer who discovers this mathematical formula and becomes euphoric about its flexible applicability and its beauty in all its simplicity. In his second book, he has obviously grown into the subject, and we meet more of a mathematical engineer. His formula, which is now called the Gielis transform, is still applied to plants, but he has now discovered the connection with

many other applications and branches of mathematics, so that some chapters are more mathematical than biological. Although the title of the first book is more mathematical and the second title refers explicitly to plants, for the contents it is rather the other way around. His recent book is some kind of summary of all the mathematical and applied papers that he (co-)authored in the 15 years after his first book.



Not only is there more mathematics, there is also more historical background and even philosophy. He has subdivided the book into 6 parts. According to the commentaries of Proclus on Euclid's elements, this is how a theorem is proved in the Platonic tradition. One has to start with the *Propositio*, which is the announcement and exploration of an idea. It is followed by the *Expositio*, the particular announcement of the claim. To give the demonstration one has to collect the predicates in the *Determinatio*, then apply the postulates and prepare in the *Constructio* for the actual proof, which is given in the *Demonstratio*. The *Conclusio* is the summary of what has been proved.

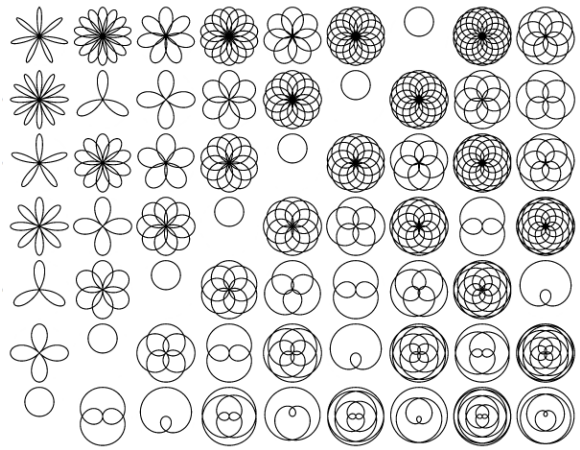
In his *Propositio* Gielis reflects on the difference between a pure mathematical object (which never exists in nature) and what is observed and how, in the mind's eye, mathematical models are used to describe reality, neglecting the "imperfections". Mathematical elements are explored like arithmetic and geometric mean, and the idea of a flexible elastic measure that depends on the direction in which one is measuring. The *Expositio* introduces the ideas on how to generalise the classical definitions of arithmetic and geometric mean by giving weights to the components that are averaged like $\mathrm{AM}_{i/n} = ((n-i)a + ib)/n$ and $\mathrm{GM}_{i/n} = \sqrt[n]{a^{n-i}b^i}$ and the harmonic mean $\mathrm{HM}_{i/n} = nab/(ia + (n-i)b)$. He links this to Pascal's triangle, Newton's fluxions, the monomiality principle (all polynomials are transformed monomials using derivative (P) and multiplication (M) operators) and the Heisenberg-Weyl algebra. Here Gielis seems to be dragged away a bit in his enthusiasm because at some points he states the commutation relation $[P, M] = 1$ without explaining what the notation means, one has to look up Dattoli's papers for that, and on page 37 he writes "Sine and cosine are examples of simple polynomials,...". Humph? The definitions of the means lead to Lamé curves (described by generalisations $x^n + y^n = R^n$ of the Pythagoras theorem) and the 3D analogs. And finally the superformula which is 'transformed' (pun intended) into the Gielis transform. On page 5 it is announced the Gielis is an acronym, but one has to wait till page 118 where it is explained as 'Generalised Intrinsic and Extrinsic Lengths In Submanifolds').



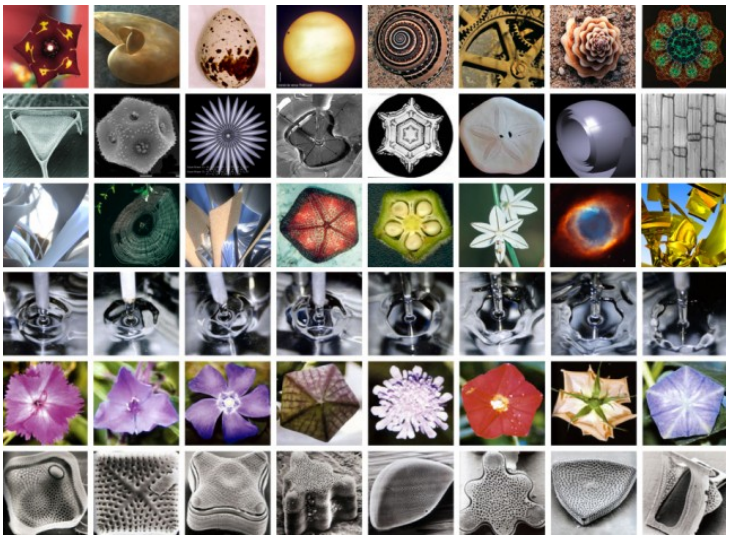
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In the *Determinatio* even more general versions are considered when the θ in the arguments of sine and cosine in the formula (1) are replaced by functions $f_i(\theta)$ and $f_2(\theta)$. Also the exponents could be replaced by functions, or a set of parameters is chosen for $i = 1, 2, \ldots, N$ and then these formulas are used in a linear combination of the corresponding $\rho_i(\theta)$. The latter is then some general form of a polynomial or even a Fourier expansion of a curve. Of course again 3D generalizations are feasible. One may generate m dimensional manifolds in an n dimensional space and Generalised Möbius-Listing (GML) bodies combining a degree of symmetry and a number of twists. The possibilities are quasi infinite. The advantage is that in this way the most complex objects are completely characterised by

a simple set of values for the parameters. Gielis calls these oligomials (oligo = few). This can be used in a definition for geometric or topological complexity. All Gielis curves are equally simple. Self-intersecting curves and surfaces are also possible which can represent for example complex dynamical systems. The mathematical aspects involve Lindenmayer L-systems and fractals. R-functions (named after Rvachev) are real functions whose sign can only change if the sign of one of its arguments changes. The sign can be interpreted as true or false and thus there is a connection with logic and set theory. The flexible measure of distance is linked to the Lorentz transform and the special relativity theory in a space-time continuum. The AM and GM means return in a discussion of Finsler-Riemann-Lamé-Minkowski (FRLM) geometry and the cosmological metrics à la Friedman-Lemaître-Robertsonn-Walker (FLRW). The Rhodonea curves of Guido Grandi (1722) are spirograph-like curves described by $\rho(\theta) = \cos(m\theta)$. Using Chebyshev polynomials ($T_m(x) = \cos(m\theta)$ if $x = \cos\theta$) allows to write the Gielis curves in terms of Chebyshev polynomials. With this duality between x and θ Gielis places himself in the sequence Pythagoras-Lamé-Gielis (he had only the polar description of Gielis curves before, but now he also has the Cartesian description).

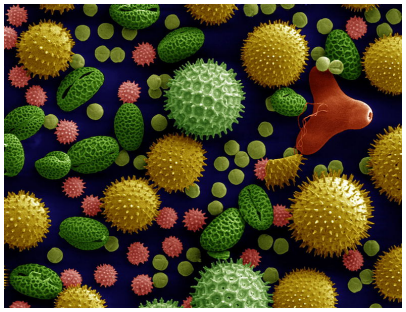


Rhodonea or Grandi curves



Time for the *Constructio* part. In order to define optimal values of the parameters in the equations, now partial differential equations (PDE) are introduced, more precisely boundary value problems (BVP). Here we have to cope with actual theorems and proofs. Environment stress can explain colour patterns in flowers and animal skins solving reaction-diffusion equations. Next curvature is brought to the front stage. Again the reader has to go through a set of mathematical definitions from differential geometry for curvatures of surfaces. Minimal surfaces have zero mean curvature and again this can be generalised to surfaces with constant *anisotropic* mean curvature (CAMC). With the Gielis transform everything can be reduced to only two basic shapes: the circle and the (logarithmic) spiral. Applying the Gielis transform to these basic shapes can generate nearly everything.

In the *Demonstratio* plant shapes are tested against the mathematical formulas: plant leaves and stems, tree rings, cell structures, plant pollen,... and their morphogenesis. All these shapes are the result of nature satisfying some optimality condition, possibly using an anisotropic measure.



plant pollen

The *Conclusio* summarises that all is just a generalisation of the Pythagoras theorem and reflects on world views that place plants and flowers, or physics, or humans at the centre.

This is a richly illustrated book, based on a simple, but general formula. The applications are found sometimes in an unexpected domain. Gielis was able to link his formula to many very diverse issues in mathematics. It can be seen as an invitation to read his papers that give many more mathematical details. On page 114, the proofreader had a local slump: $M(\sqrt{2}), 1)$ has an extra bracket and in $n''' = \sqrt{m''n'}$ the second n is missing a prime.

Adhemar Bultheel