Vanishing of the first continuous L^2-cohomology for II$_1$ factors

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Abstract

We prove that the continuous version of the Connes-Shlyakhtenko first L^2-cohomology for II$_1$ factors, as proposed by A. Thom in [Th06], always vanishes.

In [CS03], A. Connes and D. Shlyakhtenko developed an L^2-cohomology theory for finite von Neumann algebras M, and more generally for weakly dense $*$-subalgebras $A \subset M$ of such von Neumann algebras. Then in [Th06], A. Thom provided an alternative, Hochschild-type characterization of the first such L^2-cohomology of M as the quotient of the space of derivations $\delta : M \to \text{Aff}(M \overline{\otimes} M^{\text{op}})$ by the space of inner derivations, where $\text{Aff}(M \overline{\otimes} M^{\text{op}})$ denotes the $*$-algebra of operators affiliated with $M \overline{\otimes} M^{\text{op}}$. Thom also proposed in [Th06] a continuous version of the first L^2-cohomology, by considering the (smaller) space of derivations δ that are continuous from M with the operator norm to $\text{Aff}(M \overline{\otimes} M^{\text{op}})$ with the topology of convergence in measure. He noted that in many cases (e.g., when M has a Cartan subalgebra, or when M is not prime), this cohomology vanishes, i.e. any continuous derivation of M into $\text{Aff}(M \overline{\otimes} M^{\text{op}})$ is inner.

Following up on this work, V. Alekseev and D. Kyed have shown in [AK11] that the first continuous L^2-cohomology also vanishes when M has property (T), when M is finitely generated with nontrivial fundamental group, or when M has property Gamma. Recently, V. Alekseev proved in [Al13] that this is also the case for the free group factors $L(\mathbb{F}_n)$.

In this article, we prove that in fact the first continuous L^2-cohomology vanishes for all finite von Neumann algebras. The starting point of our proof is a key calculation in the proof of [Al13, Proposition 3.1], which provides a concrete sequence of elements y_n in the II$_1$ factor $M = L(\mathbb{F}_3)$ of the free group $\mathbb{F}_3$ with generators a, b, c, that tends to 0 in operator norm, but has the property that if a derivation $\delta : M \to \text{Aff}(M \overline{\otimes} M^{\text{op}})$ satisfies $\delta(u_a) = u_a \otimes 1$ and $\delta(u_b) = \delta(u_c) = 0$, then $\delta(y_n)$ does not tend to 0 in measure. More precisely, the y_n's in [Al13] are scalar multiples of words w_n in a, b, c with the property that $\delta(w_n)$ is a larger and larger sum of free independent Haar unitaries. In the case of an arbitrary II$_1$ factor M, we fix a hyperfinite II$_1$ factor $R \subset M$ with trivial relative commutant, and then use [Po92] to "simulate" (in distribution) $L(\mathbb{F}_3)$ inside M, with a any fixed unitary in M and b_m, c_m Haar unitaries in R such that a, b_m, c_m are asymptotically free. If now δ is a continuous derivation on M, then by subtracting an inner derivation, we may assume δ vanishes on R, thus on b_m, c_m. If $\delta(a) \neq 0$, and if we formally define y_n's via the same formula as Alekseev's, with a, b_m, c_m in lieu of a, b, c, then a careful estimation of norms of y_n and $\delta(y_n)$, which uses results in [HL99], shows that one still has $\|y_n\| \to 0$, while $\delta(y_n) \not\to 0$ in measure.

Let M be a finite von Neumann algebra. We denote by M^{op} the opposite von Neumann algebra and by $\mathrm{Aff}(M\,\overline{\otimes}\,M^{\mathrm{op}})$ the $*$-algebra of operators affiliated with $M\,\overline{\otimes}\,M^{\mathrm{op}}$. A *derivation* $\delta : M \to \mathrm{Aff}(M\,\overline{\otimes}\,M^{\mathrm{op}})$ is a linear map satisfying

$$\delta(ab) = (a \otimes 1)\delta(b) + (1 \otimes b^{\mathrm{op}})\delta(a) \quad \text{for all } a, b \in M .$$

For every $\xi \in \mathrm{Aff}(M\,\overline{\otimes}\,M^{\mathrm{op}})$, denote by $\partial\xi$ the *inner* derivation defined as

$$(\partial\xi)(a) = (a \otimes 1 - 1 \otimes a^{\mathrm{op}})\xi \quad \text{for all } a \in M .$$

We equip $\mathrm{Aff}(M\,\overline{\otimes}\,M^{\mathrm{op}})$ with the *measure topology,* i.e. the unique vector space topology with basic neighborhoods of 0 given by

$$B(\tau_1, \varepsilon) = \big\{ \xi \in \mathrm{Aff}(M\,\overline{\otimes}\,M^{\mathrm{op}}) \;\big|\; \exists \text{ projection } p \in M\,\overline{\otimes}\,M^{\mathrm{op}} \text{ with } \tau_1(p) > 1 - \varepsilon, \|\xi p\| < \varepsilon \big\}$$

for all normal tracial states $\tau_1 : M \to \mathbb{C}$ and all $\varepsilon > 0$. If $\tau : M \to \mathbb{C}$ is a normal *faithful* tracial state, then $\{B(\tau, \varepsilon) \mid \varepsilon > 0\}$ is a family of basic neighborhoods of 0 and there is no need to vary the trace.

Theorem 1. *Let M be a finite von Neumann algebra. Every derivation $\delta : M \to \mathrm{Aff}(M\,\overline{\otimes}\,M^{op})$ that is continuous from the norm topology on M to the measure topology on $\mathrm{Aff}(M\,\overline{\otimes}\,M^{op})$, is inner.*

The following lemma is quite standard, but we include a detailed proof for completeness.

Lemma 2. *It suffices to prove Theorem 1 for II_1 factors M with separable predual.*

Proof. We prove the lemma in different steps.

Step 1. It suffices to prove Theorem 1 for diffuse, countably decomposable M. Take a set $I = I_1 \sqcup I_2$ and an orthogonal family of projections $p_i \in \mathcal{Z}(M)$ with $\sum_{i \in I} p_i = 1$ and such that for all $i \in I_1$, we have that Mp_i is countably decomposable and diffuse, and such that for all $i \in I_2$, we have that Mp_i is a matrix algebra. Since the projections p_i are orthogonal, the element

$$\xi = \sum_{i \in I}(p_i \otimes 1)\delta(p_i)$$

is well defined in $\mathrm{Aff}(M\,\overline{\otimes}\,M^{\mathrm{op}})$. The following direct computation shows that

$$\delta(p_k) = (\partial\xi)(p_k) \quad \text{for all } k \in I . \tag{1}$$

Indeed,

$$(\partial\xi)(p_k) = (p_k \otimes 1)\delta(p_k) - \sum_{i \in I}(p_i \otimes p_k^{\mathrm{op}})\delta(p_i) . \tag{2}$$

But, for all i and k, we have

$$(1 \otimes p_k^{\mathrm{op}})\delta(p_i) = \delta(p_i p_k) - (p_i \otimes 1)\delta(p_k) .$$

Multiplying with $p_i \otimes 1$ and summing over i, we find that

$$\sum_{i \in I}(p_i \otimes p_k^{\mathrm{op}})\delta(p_i) = (p_k \otimes 1)\delta(p_k) - \delta(p_k) .$$

In combination with (2), we find that (1) holds.

Replacing δ by $\delta - \partial\xi$, we may assume that $\delta(p_i) = 0$ for all $i \in I$. It follows that $\delta(Mp_i) \subset \mathrm{Aff}(Mp_i \overline{\otimes} (Mp_i)^{\mathrm{op}})$ for every $i \in I$. We denote by δ_i the restriction of δ to Mp_i. By the assumption of step 1, δ_i is inner when $i \in I_1$. So for $i \in I_1$, we have that $\delta_i = \partial\xi_i$ for some $\xi_i \in \mathrm{Aff}(Mp_i \overline{\otimes} (Mp_i)^{\mathrm{op}})$. When $i \in I_2$, we have that Mp_i is a matrix algebra and we can take a complete system of matrix units e^i_{jk} for Mp_i. We then get that $\delta_i = \partial\xi_i$ for

$$\xi_i = \sum_k (e^i_{k1} \otimes 1)\delta_i(e^i_{1k}) \ .$$

The vector $\xi = \sum_{i \in I} \xi_i$ is well defined in $\mathrm{Aff}(M \overline{\otimes} M^{\mathrm{op}})$ and $\delta = \partial\xi$.

Step 2. It suffices to prove Theorem 1 when M is diffuse and has separable predual. Using step 1, we may already assume that M is diffuse and admits a faithful normal tracial state τ that we keep fixed. We start by proving the following three statements, using that M admits the faithful trace τ.

If $\xi \in \mathrm{Aff}(M \overline{\otimes} M^{op})$, there exists a von Neumann subalgebra $N \subset M$ with separable predual such that $\xi \in \mathrm{Aff}(N \overline{\otimes} N^{op})$. Take an increasing sequence of projections $p_n \in M \overline{\otimes} M^{\mathrm{op}}$ such that $\tau(1 - p_n) \to 0$ and $\xi p_n \in M \overline{\otimes} M^{\mathrm{op}}$ for all n. In particular, $\xi p_n \in L^2(M \overline{\otimes} M^{\mathrm{op}}) = L^2(M) \otimes L^2(M^{\mathrm{op}})$ and we can take separable Hilbert subspaces $H_n \subset L^2(M)$, $K_n \subset L^2(M^{\mathrm{op}})$ such that $\xi p_n \in H_n \otimes K_n$. We then find countable subsets $V_n \subset M$ such that for every n, the vector ξp_n belongs to the $\|\cdot\|_2$-closed linear span of $\{a \otimes b^{\mathrm{op}} \mid a, b \in V_n\}$. Defining N as the von Neumann subalgebra of M generated by all the sets V_n, our statement is proven.

Let $N_1 \subset M$ be a von Neumann subalgebra with separable predual. Then there exists a von Neumann subalgebra $N_2 \subset M$ with separable predual such that $N_1 \subset N_2$ and $\delta(N_1) \subset \mathrm{Aff}(N_2 \overline{\otimes} N_2^{op})$. Take a separable and weakly dense C^*-subalgebra $B_1 \subset N_1$. By the previous paragraph and because δ is norm-measure continuous, we can take $N_2 \subset M$ with separable predual such that $\delta(B_1) \subset \mathrm{Aff}(N_2 \overline{\otimes} N_2^{\mathrm{op}})$. Replacing N_2 by the von Neumann algebra generated by N_1 and N_2, we may assume that $N_1 \subset N_2$. Since $\delta(B_1) \subset \mathrm{Aff}(N_2 \overline{\otimes} N_2^{\mathrm{op}})$, it follows from [Th06, Lemma 4.2 and Theorem 4.3] that $\delta(N_1) \subset \mathrm{Aff}(N_2 \overline{\otimes} N_2^{\mathrm{op}})$.

Let $N_1 \subset M$ be a von Neumann subalgebra with separable predual. Then there exists a von Neumann subalgebra $N \subset M$ with separable predual such that $N_1 \subset N$ and $\delta(N) \subset \mathrm{Aff}(N \overline{\otimes} N^{op})$. Using the previous paragraph, we inductively find an increasing sequence of von Neumann subalgebras $N_1 \subset N_2 \subset \cdots$ with separable predual such that $\delta(N_k) \subset \mathrm{Aff}(N_{k+1} \overline{\otimes} N_{k+1}^{\mathrm{op}})$ for all k. We define N as the von Neumann algebra generated by all the N_k. By construction, N has separable predual and $\delta(N_k) \subset \mathrm{Aff}(N \overline{\otimes} N^{\mathrm{op}})$ for all k. Again using [Th06, Lemma 4.2 and Theorem 4.3], it follows that $\delta(N) \subset \mathrm{Aff}(N \overline{\otimes} N^{\mathrm{op}})$.

We can now conclude the proof of step 2. Since M is diffuse, we can fix a diffuse abelian von Neumann subalgebra $A \subset M$ with separable predual. By [Th06, Theorem 6.4], we can replace δ by $\delta - \partial\xi$ for some $\xi \in \mathrm{Aff}(M \overline{\otimes} M^{\mathrm{op}})$ and assume that $\delta(a) = 0$ for all $a \in A$. We prove that $\delta(x) = 0$ for all $x \in M$. Fix an arbitrary $x \in M$. Define N_1 as the von Neumann algebra generated by A and x. Note that N_1 has a separable predual. By the previous paragraph, we can take a von Neumann subalgebra $N \subset M$ with separable predual such that $N_1 \subset N$ and $\delta(N) \subset \mathrm{Aff}(N \overline{\otimes} N^{\mathrm{op}})$. By the initial assumption of step 2, the restriction of δ to N is inner. So we can take a $\xi \in \mathrm{Aff}(N \overline{\otimes} N^{\mathrm{op}})$ such that $\delta(y) = (\partial\xi)(y)$ for all $y \in N$. Since $A \subset N$ and $\delta(a) = 0$ for all $a \in A$, it follows that $(a \otimes 1)\xi = (1 \otimes a^{\mathrm{op}})\xi$ for all $a \in A$. Since A is diffuse, this implies that $\xi = 0$. Since $x \in N$, it then follows that $\delta(x) = 0$. This concludes the proof of step 2.

Step 3. Proof of the lemma : it suffices to prove Theorem 1 when M is a II_1 factor with separable predual. Using step 2, we may already assume that M is diffuse and has separable predual.

Let $p_0 \in \mathcal{Z}(M)$ be the maximal projection such that $\mathcal{Z}(M)p_0$ is diffuse (possibly, $p_0 = 0$). Let $p_1, p_2, \ldots$ be the minimal projections in $\mathcal{Z}(M)(1-p_0)$. Note that $\sum_{n=0}^{\infty} p_n = 1$. As in the proof of step 1, we may assume that $\delta(Mp_n) \subset \mathrm{Aff}(Mp_n \,\overline{\otimes}\, (Mp_n)^{\mathrm{op}})$ for all n. Denote by δ_n the restriction of δ to Mp_n. Since Mp_0 has a diffuse center, it follows from [Th06, Theorem 6.4] that δ_0 is inner. For all $n \geq 1$, we have that Mp_n is a II_1 factor with separable predual. So by assumption, all δ_n, $n \geq 1$, are inner. But then also δ is inner. $\qquad\square$

Proof of Theorem 1. Using Lemma 2, it suffices to take a II_1 factor M with separable predual and a derivation $\delta : M \to \mathrm{Aff}(M \,\overline{\otimes}\, M^{\mathrm{op}})$ that is continuous from the norm topology on M to the measure topology on $\mathrm{Aff}(M \,\overline{\otimes}\, M^{\mathrm{op}})$. Denote by τ the unique tracial state on M. By [Po81, Corollary 4.1], we can fix a copy of the hyperfinite II_1 factor $R \subset M$ such that $R' \cap M = \mathbb{C}1$. By [Th06, Theorem 6.4], we can replace δ by $\delta - \partial\xi$ and assume that $\delta(x) = 0$ for all $x \in R$. We prove that $\delta = 0$. Fix a unitary $u \in \mathcal{U}(M)$ with $\tau(u) = 0$. It suffices to prove that $\delta(u) = 0$.

Fix a free ultrafilter ω on $\mathbb{N}$ and consider the ultrapower M^ω. By [Po92, Corollary on p. 187], choose a unitary $v \in R^\omega$ such that the subalgebras $v^k M v^{-k} \subset M^\omega$, $k \in \mathbb{Z}$, are free. Fix a Haar unitary $a \in \mathcal{U}(R)$, i.e. a unitary satisfying $\tau(a^m) = 0$ for all $m \in \mathbb{Z} \setminus \{0\}$. Define, for $k \geq 1$, $a_k = v^k a v^{-k}$. It follows that $a_k \in \mathcal{U}(R^\omega)$ are $*$-free Haar unitaries that are moreover free w.r.t. M. Write $a_k = (a_{k,n})$ with $a_{k,n} \in \mathcal{U}(R)$.

Similar to the definition of x_n in the proof of [Al13, Proposition 3.1], we consider for all large n and m, the unitary

$$w_{m,n} = a_{1,n} u\, a_{2,n} u\, \cdots\, a_{m,n} u$$

and prove that either $\delta(u) = 0$ or $\delta(w_{m,n})$ is "very large almost everywhere", contradicting the continuity of δ.

Since $\delta(x) = 0$ for all $x \in R$, we get that

$$\delta(w_{m,n}) = \Big(\sum_{k=1}^{m} a_{1,n} u\, a_{2,n} \cdots a_{k-1,n} u\, a_{k,n} \otimes \big(a_{k+1,n} u \cdots a_{m,n} u\big)^{\mathrm{op}} \Big) \delta(u) \,, \qquad (3)$$

where, by convention, the first term in the sum is $a_{1,n} \otimes (a_{2,n} u \cdots a_{m,n} u)^{\mathrm{op}}$ and the last term is $a_{1,n} u \cdots a_{m-1,n} u\, a_{m,n} \otimes 1$.

Consider, in the ultrapower $(M \,\overline{\otimes}\, M^{\mathrm{op}})^\omega$, the element

$$T_m = \sum_{k=1}^{m} a_1 u\, a_2 u\, \cdots a_{k-1} u\, a_k \otimes \big(a_{k+1} u \cdots a_m u\big)^{\mathrm{op}} \,.$$

We claim that T_m is the sum of m $*$-free Haar unitaries. To prove this, it suffices to show that the first tensor factors $a_1, a_1 u a_2, a_1 u a_2 u a_3, \ldots$ form a $*$-free family of Haar unitaries. Since $a_1, a_2, a_3, \ldots$ is a $*$-free family of Haar unitaries that are $*$-free w.r.t. u, also the Haar unitaries $a_1, u a_2, u a_3, u a_4, \ldots$ are $*$-free. But then the conclusion follows by taking the product of the first k unitaries in this last sequence, again producing a $*$-free family of Haar unitaries.

Since T_m is the sum of m $*$-free Haar unitaries, we get from [HL99, Example 5.5] an explicit formula for the spectral distribution of $|T_m|$. It follows that $|T_m|$ has the same distribution as $2\sqrt{m-1}\, S_m$, where S_m is a sequence of random variables satisfying $0 \leq S_m \leq 1$ and converging in distribution to the normalized quarter circle law. Therefore, the spectral projections

$$q_m = \mathbf{1}_{[\sqrt{m}, +\infty)}(T_m^* T_m) = \mathbf{1}_{[m^{1/4}, +\infty)}\big(|T_m|\big) = \mathbf{1}_{\left[m^{1/4}(4(m-1))^{-1/2}, +\infty\right)}(S_m)$$

satisfy $\lim_m \tau(q_m) = 1$. Write $q_m = (q_{m,n})$ where $q_{m,n}$ are projections in $M \,\overline{\otimes}\, M^{\mathrm{op}}$.

Fix an arbitrary $\varepsilon > 0$. Since δ is continuous, fix $\rho > 0$ such that $\delta(z) \in B(\tau, \varepsilon/2)$ whenever $z \in M$ and $\|z\| < \rho$. Take m large enough such that $m^{-1/4} < \rho$ and $\tau(q_m) > 1 - \varepsilon$. For every n, the element $m^{-1/4} w_{m,n}$ has norm less than ρ. Therefore, $\delta(m^{-1/4} w_{m,n})$ belongs to $B(\tau, \varepsilon/2)$ and we find a projection $p_n \in M \overline{\otimes} M^{\mathrm{op}}$ with

$$\tau(p_n) > 1 - \varepsilon/2 \quad \text{and} \quad \left\| \delta\big(m^{-1/4} w_{m,n} \big) \, p_n \right\| < \varepsilon/2 \ .$$

We also fix a projection $e_0 \in M \overline{\otimes} M^{\mathrm{op}}$ with $\tau(e_0) > 1 - \varepsilon/2$ and such that $\delta(u)e_0 \in M \overline{\otimes} M^{\mathrm{op}}$. We write $e_n = e_0 \wedge p_n$ and view $e = (e_n)$ as a projection in $(M \overline{\otimes} M^{\mathrm{op}})^\omega$. By (3), we have in $(M \overline{\otimes} M^{\mathrm{op}})^\omega$ the equality $(\delta(w_{m,n})e_0)_n = T_m (\delta(u)e_0)$, and therefore also that $(\delta(w_{m,n})e_n)_n = T_m (\delta(u)e_0) e$. We then find that

$$
\begin{aligned}
\varepsilon^2 &> \lim_{n \to \omega} \left\| \delta\big(m^{-1/4} w_{m,n} \big) \, e_n \right\|^2 \\
&\geq \lim_{n \to \omega} \left\| \delta\big(m^{-1/4} w_{m,n} \big) \, e_n \right\|_2^2 \\
&= \tau\big(e \, (\delta(u)e_0)^* \, m^{-1/2} T_m^* T_m \, (\delta(u)e_0) \, e \big) \\
&\geq \tau\big(e \, (\delta(u)e_0)^* \, q_m \, (\delta(u)e_0) \, e \big) \ .
\end{aligned}
$$

Since $\tau(q_m) > 1 - \varepsilon$, we can fix n such that

$$\| q_{m,n} \, \delta(u) \, e_n \|_2 < \varepsilon \quad \text{and} \quad \tau(q_{m,n}) > 1 - \varepsilon \ .$$

Since $\tau(e_n) > 1 - \varepsilon$, we have proven that for every $\varepsilon > 0$, there exist projections $p, q \in M \overline{\otimes} M^{\mathrm{op}}$ such that $\tau(p) > 1 - \varepsilon$, $\tau(q) > 1 - \varepsilon$ and $\| q \delta(u) p \|_2 < \varepsilon$. This means that $\delta(u) = 0$. $\qquad\square$

The proof of Theorem 1 gives no indication as to whether or not the Connes-Shlyakhtenko first L^2-cohomology vanishes as well. Note however that in order for a first L^2-cohomology theory to "work well" for II$_1$ factors M, the corresponding derivations should be uniquely determined by their values on a set of elements generating M as a von Neumann algebra. In order for this to be the case, the derivations should normally satisfy some continuity property, even if that continuity is "very weak". However, by combining Theorem 1 with the closed graph theorem, it follows that any derivation from M into $\mathrm{Aff}(M \overline{\otimes} M^{\mathrm{op}})$ that satisfies some "reasonable" weak continuity property, must in fact be inner (see also Remark 4 hereafter):

Corollary 3. *Let M be a finite von Neumann algebra and write $\mathcal{E} = \mathrm{Aff}(M \overline{\otimes} M^{op})$. Assume that $\delta : M \to \mathcal{E}$ is a derivation. If δ has a closed graph for the norm topology on M and the measure topology on $\mathcal{E}$, then δ is inner.*

This is in particular the case if δ is norm-$\mathcal{T}$-continuous w.r.t. any vector space topology $\mathcal{T}$ on $\mathcal{E}$ satisfying the following two properties:

- *the inclusion $M \overline{\otimes} M^{op} \to \mathcal{E}$ is norm-$\mathcal{T}$-continuous;*

- *for every fixed $a \in M \overline{\otimes} M^{op}$, the map $\mathcal{E} \to \mathcal{E} : \xi \mapsto \xi a$ is $\mathcal{T}$-$\mathcal{T}$-continuous.*

Proof. Take an orthogonal family of projections $p_i \in \mathcal{Z}(M)$ such that $\sum_{i \in I} p_i = 1$ and every Mp_i is countably decomposable. As in step 1 of Lemma 2, we may assume that $\delta(Mp_i) \subset \mathrm{Aff}(Mp_i \overline{\otimes} (Mp_i)^{\mathrm{op}})$. If δ has closed graph, the restrictions δ_i of δ to Mp_i still have closed graph. If all these restrictions δ_i are inner, also δ is inner. So to prove the first part of the corollary, we may assume that M admits a faithful normal tracial state τ that we keep fixed. But then, the formula

$$d(\xi, \eta) = \inf\{ \varepsilon > 0 \mid \exists \text{ projection } p \in M \overline{\otimes} M^{\mathrm{op}} : \tau(1-p) < \varepsilon \text{ and } \|(\xi - \eta)p\| < \varepsilon \}$$

defines a translation invariant, complete metric on $\mathrm{Aff}(M \,\overline{\otimes}\, M^{\mathrm{op}})$ that induces the measure topology. So by [Ru91, 2.15], the closed graph theorem is valid and we find that δ is continuous, and hence inner by Theorem 1.

Assume now in general that δ is norm-$\mathcal{T}$-continuous. To prove that δ has closed graph, assume that $\|x_n\| \to 0$ and $\delta(x_n) \to \xi$ in the measure topology. We have to show that $\xi = 0$. Fix a normal tracial state τ on M. Choose projections $p_n \in M \,\overline{\otimes}\, M^{\mathrm{op}}$ such that $\tau(p_n) > 1 - 2^{-n}$ and $\|(\delta(x_n) - \xi)p_n\| < 1/n$. Define the projections $q_k = \bigwedge_{n \geq k} p_n$ and note that q_k is increasing and satisfies $\tau(q_k) \to 1$. For every fixed k, we get that $(\delta(x_n) - \xi)q_k$ converges to 0 in norm, as $n \to \infty$. By the first assumption on $\mathcal{T}$, this convergence also holds in $\mathcal{T}$. But also $\delta(x_n) \to 0$ in $\mathcal{T}$ so that, by our second assumption on $\mathcal{T}$, the sequence $\delta(x_n)q_k$ converges to 0 in $\mathcal{T}$ as $n \to \infty$. We conclude that $\xi q_k = 0$ for all k. This implies that $\xi z_\tau = 0$ where $z_\tau \in \mathcal{Z}(M)$ is the support projection of τ. Since τ was arbitrary, it follows that $\xi = 0$. $\qquad\square$

Remark 4. We should point out that we have no concrete examples of vector topologies on $\mathrm{Aff}(M \,\overline{\otimes}\, M^{\mathrm{op}})$ satisfying the conditions in the second part of Corollary 3 and that are strictly weaker than the measure topology (in fact, it is not even clear whether such a topology exists!). Let us also point out that there are other weak continuity properties of δ implying that δ has closed graph, thus following inner by the first part of Corollary 3. For instance, by using a similar argument as above, one can easily prove that this is the case when δ satisfies the following weak continuity property: whenever x_n is a sequence in M such that $\|x_n\| \to 0$, there exists a sequence of projections $p_n \in M \,\overline{\otimes}\, M^{\mathrm{op}}$ such that $p_n \to 1$ strongly, $\delta(x_n)p_n \in M \,\overline{\otimes}\, M^{\mathrm{op}}$ and $\delta(x_n)p_n \to 0$ σ-weakly.

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