SOME COMMENTS ON THE INEXACT MINIMAL PARTIAL REALIZATION PROBLEM

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! $H_i \in \mathbb{R}^{m \times n}$, $i = 1, 2, \ldots, N$ MARKOV PARS

? $C(z) \in \mathbb{R}^{m \times n}[z]$

$A(z) \in \mathbb{R}^{n \times n}[z]$

s.t.

1. $C(z) A(z)^{-1} = H_1 z^{-1} + H_2 z^{-2} + \cdots + H_N z^{-N} + \mathcal{O}_-(z^{-N-1})$
2. $C(z), A(z)$ coprime
3. $\deg \det A(z)$ minimal

MARKOV PARAMETERS GIVEN WITH SOME RELATIVE PRECISION

$A(z) = [a_1(z) \mid a_2(z) \mid \cdots \mid a_n(z)]$ SOLUTION VECTS

$a_j(z) = a_{0j} + a_{1j} z + \cdots + a_{d_j, j} z^{d_j}$ $a_{ij} \in \mathbb{R}^n$

$HD_A = [a_{d_1,1} \mid a_{d_2,2} \mid \cdots \mid a_{d_n,n}]$

HIGHEST DEGREE COEF. MATRIX

$A(z)$ COLUMN REDUCED $\Leftrightarrow$ HD_A NONSINGULAR

$\Rightarrow \deg \det A(z) = \sum_{j=1}^{n} d_j$

Thm $\forall A(z) \in \mathbb{R}^{n\times n}[z]$

$\exists\, U(z)$ UNIMODULAR $\quad (\det U(z) \in \mathbb{R}\setminus\{0\})$

s.t.

$A'(z) = A(z)\,U(z)$ COLUMN REDUCED

$\Rightarrow$ WE LOOK FOR COLUMN REDUCED SOLUTIONS

SOLUTION VECTOR OF $\begin{cases}\text{DEGREE } d\\ \text{FITTING } N \text{ MARKOV PARS}\end{cases}$

SATISFY:

$$\underbrace{\begin{bmatrix} H_1 & H_2 & \cdots & H_{d+1} \\ H_2 & \iddots & & \\ \vdots & & & \\ H_{N-d} & & & H_N \end{bmatrix}}_{H_{N,d}} \underbrace{\begin{bmatrix} a_0 \\ \vdots \\ a_d \end{bmatrix}}_{a(z)} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}_{=0} \quad \Leftrightarrow \quad a(z) \in \ker H_{N,d} \qquad (*)$$

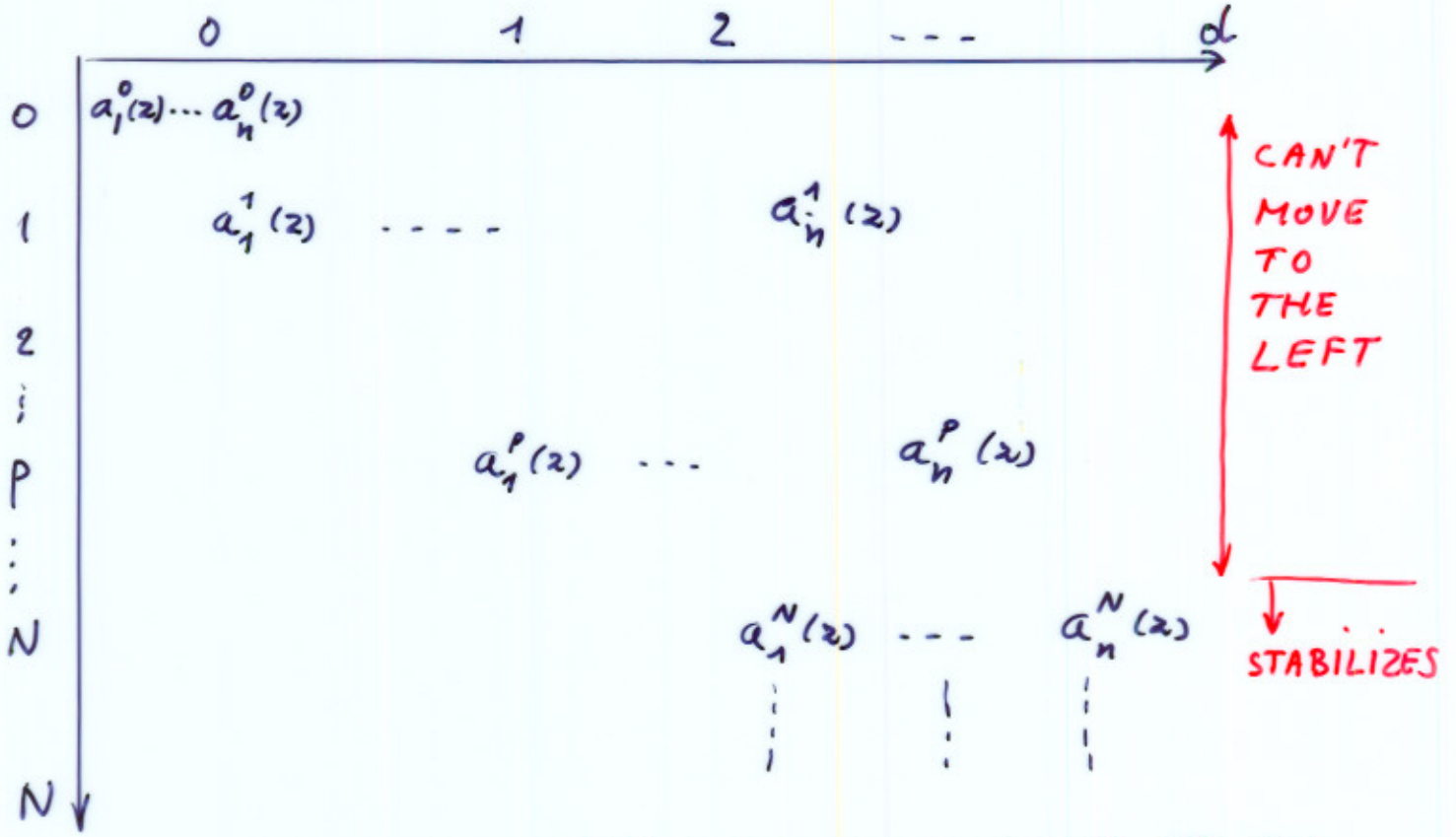
NUMERATOR COLUMN

$$\begin{bmatrix} 0 & \cdots & 0 & H_1 \\ \vdots & & \iddots & \\ 0 & H_1 & & H_d \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_d \end{bmatrix} = \begin{bmatrix} c_{d-1} \\ \vdots \\ c_0 \end{bmatrix}$$

MAIN PROBLEM:

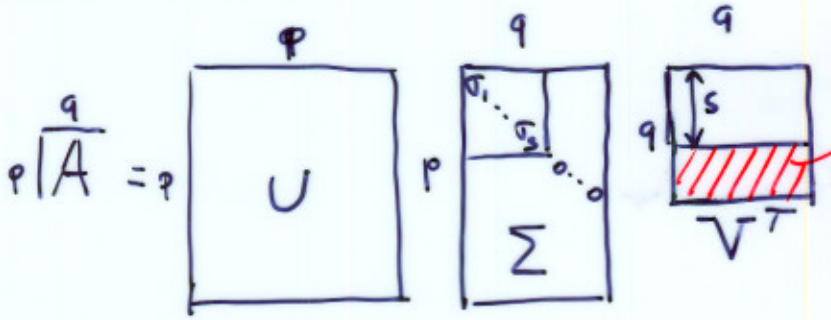
SOLVE $(*)$ WITH $d = d_1, d_2, \ldots, d_n$ AS LOW AS POSSIBLE, AND HD_A NONSINGULAR

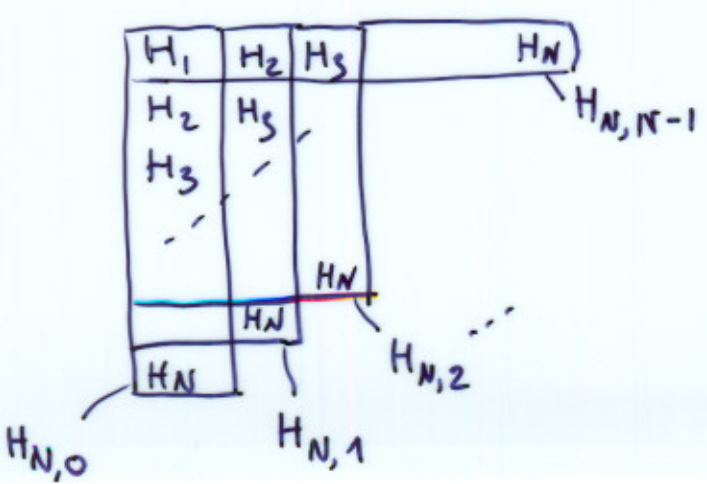
ARRANGE SOLUTION VECTORS IN (N, d)-TABLE
= SOLUTION TABLE :



WE SUPPOSE N LARGE ENOUGH TO BE IN STABILIZED REGION!

USE S V D :





H_{NN} : empty

⇓

NO conditions

⇓

every $a(z)$ of deg N is solution vect.

ALGORITHM (NO ERRORS)

1. SVD of $H_{N,0}, H_{N1}, \dots$ until $\ker H_{N,d_1} \neq \{0\}$

$$H_{N,d_1} = \begin{array}{c} m\updownarrow \\ \\ \end{array}\begin{bmatrix} U_1^{d_1} \\ U_2^{d_1} \end{bmatrix} \begin{array}{c} s\updownarrow \\ \\ \end{array}\overset{\overset{s}{\longleftrightarrow}}{\begin{bmatrix} \Sigma_1^{d_1} & 0 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} (V_1^{d_1})^T \\ (V_2^{d_1})^T \end{bmatrix} \updownarrow (d_1+1)n - s$$

$$(d_1+1)n - s = \dim \ker H_{N,d_1}$$

$\hookrightarrow$ = solution vects of deg. d_1

2. SUPPOSE WE FOUND k solution vects of deg $< d$

$$H_{N,d-1} = \begin{array}{c} m\updownarrow \\ \\ \end{array}\begin{bmatrix} U_1^{d-1} \\ U_2^{d-1} \end{bmatrix} \begin{array}{c} r\updownarrow \\ \\ \end{array}\overset{\overset{r}{\longleftrightarrow}}{\begin{bmatrix} \Sigma_1^{d-1} & 0 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} (V_1^{d-1})^T \\ (V_2^{d-1})^T \end{bmatrix} \updownarrow \ell$$

$$H_{N,d} = \left[\begin{array}{c|c|c} \begin{bmatrix} H_1 \\ \vdots \\ H_{N-d} \end{bmatrix} & U_2^{d-1} \cdot \begin{bmatrix} \Sigma_1^{d-1} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right] \begin{bmatrix} I & 0 \\ 0 & V_1^T \\ 0 & V_2^T \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \begin{bmatrix} H_1 \\ \vdots \\ H_{N-d} \end{bmatrix} & U_2^{d-1} \cdot \begin{bmatrix} \Sigma_1^{d-1} \\ 0 \end{bmatrix} \end{bmatrix}}_{\overset{\text{SVD}}{=}\ U' \begin{bmatrix} \Sigma_1' & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (V_1')^T \\ (V_2')^T \end{bmatrix}}$$

$$U' \begin{bmatrix} \Sigma_1' & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} (V_1')^T & 0 \\ (V_2')^T & 0 \\ 0 & I \end{bmatrix}}_{\text{orthogonal}}$$

$$\begin{bmatrix} (V_1')^T & 0 \\ (V_2')^T & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} (V_1')^T \\ (V_2')^T \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & V_1^T \end{bmatrix} \\ 0 \quad V_2^T \end{bmatrix}$$

$$= \begin{array}{c} m\updownarrow \\ \\ \end{array}\begin{bmatrix} U_1^d \\ U_2^d \end{bmatrix} \begin{bmatrix} \Sigma_1^d & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (V_1^d)^T \\ (V_2^d)^T \end{bmatrix}$$

$$(V_2^d)^T = \begin{bmatrix} (V_2')^T & \begin{bmatrix} I & 0 \\ 0 & V_1^T \end{bmatrix} \\ 0 & V_2^T \end{bmatrix} \updownarrow \text{inj}$$

$\Rightarrow j$ solution vects of deg. d !

NOISE CORRUPTED DATA.

SVD : $A = \sum \sigma_i u_i v_i^T$

$\Rightarrow$

$$\min_{\text{rank } \tilde{A} \leq k} \| A - \tilde{A} \| = \sigma_{k+1}$$

$\Rightarrow$ ABSOLUTE ERROR

RELATIVE ERRORS ON DATA (ε) $\Rightarrow$ SCALING

SCALING PROCEDURE FOR H:

repeat 5 times:

- compute Euclidean norm of each row of H and divide by norm $\Rightarrow$ length = 1.
- do the same for the columns of H.

$\Downarrow$

diagonal matrices D_1, D_2 s.t.

$D_1 H D_2$ is scaled.

TO BE DONE FOR EACH SVD (SSVD)

ALGORITHM (RELATIVE ERRORS ε)

1. SSVD for $H_{N,0} \doteq \begin{bmatrix} H_1 \\ \vdots \\ H_N \end{bmatrix}$

$$D_1 \; H_{N,0} \; D_2 = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

$\Sigma_2 \approx \varepsilon$ (say k)

SOLUTION VECT = $D_2 \cdot V_2$ degree 0 (if any)

otherwise SSVD of $H_{N,1} \dots H_{N,d_1}$

loses m rows

2

$$H_{N,1} = \left[\begin{bmatrix} H_1 \\ \vdots \\ H_{N-1} \end{bmatrix} \Bigg| \; D_{1,2}^{-1} \cdot U_2 \cdot \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \cdot \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \cdot D_2^{-1} \right]$$

$$= D_{1,2}^{-1} \left[D_{12} \cdot \begin{bmatrix} H_1 \\ \vdots \\ H_{N-1} \end{bmatrix} \cdot D_2' \;\Big|\; U_2 \cdot \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} \;\Big|\; U_2 \cdot \begin{bmatrix} 0 \\ \Sigma_2 \end{bmatrix} \right] \cdot \begin{bmatrix} I & 0 \\ 0 & V_1^T \\ 0 & V_2^T \end{bmatrix} \cdot \begin{bmatrix} D_2'^{-1} & 0 \\ 0 & D_2^{-1} \end{bmatrix}$$

D_2' s.t. these columns have length 1.

SVD

$$= D_{1,2}^{-1} \left[\overbrace{U' \cdot \begin{bmatrix} \Sigma_1' & 0 \\ 0 & \Sigma_2' \end{bmatrix} \cdot \begin{bmatrix} V_1'^T \\ V_2'^T \end{bmatrix}} \;\Big|\; U_2 \begin{bmatrix} 0 \\ \Sigma_2 \end{bmatrix} \right] \cdot \begin{bmatrix} I & 0 \\ 0 & V_1^T \\ 0 & V_2^T \end{bmatrix} \cdot \begin{bmatrix} D_2'^{-1} & 0 \\ 0 & D_2^{-1} \end{bmatrix}$$

$$= D_{1,2}^{-1} \left[U' \cdot \begin{bmatrix} \Sigma_1' & 0 \\ 0 & \Sigma_2' \end{bmatrix} \;\Big|\; U_2 \begin{bmatrix} 0 \\ \Sigma_2 \end{bmatrix} \right] \cdot \underbrace{\begin{bmatrix} V_1'^T & 0 \\ V_2'^T & 0 \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & V_1^T \\ 0 & V_2^T \end{bmatrix}}_{1} \cdot \begin{bmatrix} D_2'^{-1} & 0 \\ 0 & D_2^{-1} \end{bmatrix}$$

$\Sigma_2' \approx \varepsilon$ (index $k+j$) $\quad \Sigma_2 \approx \varepsilon$

choose j. sol. vect. of deg. 1 among columns of

$$\begin{bmatrix} D_2' & 0 \\ 0 & D_2 \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & V_1 \end{bmatrix} \cdot V_2'$$

hdc linearly independent!

$D_2 \approx D_2'$

$D_1 \begin{bmatrix} H_1 \\ \\ H_N \end{bmatrix} D_2$ is scaled

H_N small w.r.t. $\begin{bmatrix} H_1 \\ \vdots \\ H_{N-1} \end{bmatrix}$ (stable syst.)

$D_{1,2} \underbrace{\begin{bmatrix} H_1 \\ \vdots \\ H_{N-1} \end{bmatrix}}_{(*)} D_2'$ is scaled ?

If markov. pars varying slowly then o.k.

i.e. all elements of (*) same order of magnitude

ALL HDC (old + new) in matrix HD

USE QR with partial column pivoting

$$HD = [\underbrace{HD_1}_{k \text{ old}} \vdots \underbrace{HD_2}_{k+j \text{ new}}]$$

HD_1: not pivoted

QR ↓

$$Q^T . HD . \begin{bmatrix} I_{k} & 0 \\ 0 & J_{k+j} \end{bmatrix} = \begin{bmatrix} R_{k} & X' & X'' \\ 0 & R' & Y'' \\ 0 & 0 & R''_{k} \end{bmatrix}_{k \quad j \quad k}$$

J : permutation matrix $(j_1, \dots, j_j)$

NOTE : LAST DECOMPOSITION IS NOT A SSVD OF $H_{N,1}$!

$$H_{N,1} D_{2,2} = \underbrace{\begin{bmatrix} U' & \vdots & U_2 \\ & \vdots & \end{bmatrix}}_{\text{NOT } \perp} \begin{bmatrix} \Sigma_1' & 0 & 0 \\ 0 & \Sigma_2' & 0 \\ 0 & 0 & \Sigma_2 \end{bmatrix} \left\{ \begin{bmatrix} (V_1')^T \\ (V_2')^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & V_1^T \end{bmatrix} \right\} \text{ is } \perp$$

$$\begin{bmatrix} 0 & V_2^T \end{bmatrix}$$

$D_{1,2}$

$$\rightarrow D_{2,2} = \begin{matrix} n \\ n \end{matrix} \overbrace{\begin{bmatrix} D_2' & 0 \\ 0 & D_2 \end{bmatrix}}^{n}$$

$$\rightarrow D_1 = \begin{matrix} m \\ \end{matrix} \overbrace{\begin{bmatrix} /// & //// \\ // & D_{1,2} \end{bmatrix}}^{m}$$

THE ERROR :

$$H(z^{-1}) - C(z)\,A^{-1}(z) = O_-(z^{-N-1}) + (a).(b)$$

$$(a) = D_{1,d_1+1}^{-1} z^{-d_1-1} [\sigma_1 u_{1,d_1+1} \cdots \sigma_n u_{n,d_1+1}] + \cdots$$

$$D_{1,N}^{-1} z^{-N} [\sigma_1 u_{1,N} \cdots \sigma_n u_{n,N}]$$

behaves like the H_k

$$(b) = (V_h + D_{2,h}^{-1} D_{2,h-1} V_{h-1} z^{-1} + \cdots + D_{2,h}^{-1} D_{2,0} V_0 z^{-h}) D_{2,h}^{-1}$$

V_h : HD_A ; $D_{2,h}^{-1} D_{2,h-1} \simeq I$; $D_{2,h}^{-1} D_{2,0} \simeq I$

σ_j : s.v. corresp. to $a_j(z)$ (all $\simeq \varepsilon$)

$u_{j,*}$: left S.V.

$v_{*,j}$: right S.V.

$$\begin{bmatrix} a_{0j} \\ \vdots \\ a_{d_j,j} \end{bmatrix} = \begin{bmatrix} D_{2,h-d_j} & v_{0j} \\ \vdots & \\ D_{2,h} & v_{d_j,j} \end{bmatrix}$$

$$\begin{bmatrix} D_{11} & & \\ & \ddots & \\ & & D_{1,N} \end{bmatrix} \begin{bmatrix} H_1 \\ \vdots \\ H_N \end{bmatrix} D_2 = \text{SCALED } H_{N,0}$$

$$V_{h-j} = [v_{d_j-1,1} \cdots v_{d_n-j,n}] \quad j = 0, \ldots, h$$

$$d_1 \le d_2 \le \cdots \le d_n = h$$

$\therefore$ (a).(b) $\rightarrow$ decreases like H_k

$\Rightarrow$ RELATIVE ERROR ON $H_k \simeq \varepsilon$

NOTES :

1. when block Hankel cannot be scaled properly:

   Smallest element in scaled $D_1 H_{N,0} D_2$ is $\beta \cdot 10^{-\alpha}$
   $\Rightarrow$ α digits lost!

   when different entries in $H(z)$ have diff. dom. poles.

2. For scalar systems (SISO) scaling is o.k.
   column scaling not necessary

3. loss of accuracy when

   - V_h bad cond. number
   - multiple poles

4. can also be used for model reduction