**RESEARCH ARTICLE**

# Optimal Feedforward Controller Design for Periodic Inputs

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This paper presents an optimal design methodology for feedforward controllers that face periodic reference/disturbance inputs. The feedforward controller is parameterized as a FIR filter, and its parameters are computed to minimize the worst-case tracking error in the presence of uncertainty on the input period and the plant model. Numerical results indicate that for nonminimum-phase systems exploiting the periodic input characteristics in the feedforward controller design is worthwhile, and reveal the superiority of the developed design methodology with respect to current design approaches when period-time and/or plant uncertainty is present.

**Keywords:** feedforward control, periodic input signals, robustness

## 1 Introduction

Periodic reference and disturbance signals are widespread in engineering practice, as every rotating machine and repeated process involves periodicity. Periodic disturbances are for instance encountered in the track-following servo system of disk drives (Chen et al. 2001, 2006), power electronics (Zhou and Wang 2001, Botteron and Pinheiro 2006), steel casting (Manayathara et al. 1996) and robotized laparoscopic surgery (Gangloff et al. 2006). Periodic reference trajectories, on the other hand, occur in noncircular machining (Kim et al. 2004), electronic cam motion generation (Kim and Tsao 2000) and robots performing repetitive tasks (Kasac et al. 2008). Since the attainable performance of a controller is bounded by measurement noise, model inaccuracies, actuator saturation, etc., exploiting all knowledge available on the reference and disturbance inputs is indispensable to achieve the tightening performance demands in engineering practice.

Literature reveals active research on exploiting the periodic input characteristics in a feedback controller design. This specialized feedback controller design is often handled in the context of output regulation (see e.g. Saberi et al. 2000, for an in-depth treatement), which concerns the design of an internally stabilizing controller that yields perfect asymptotic rejection/tracking of persistent input signals, as are periodic signals. The cornerstone of regulation theory is the Internal Model Principle (Davison 1972, Francis et al. 1974), which states that in order to achieve perfect asymptotic rejection/tracking of persistent inputs, their signal generator must be (partly) replicated in a stable feedback loop. Although a feedback controller is the only means to suppress unmeasurable disturbances, specializing the feedback controller for periodic inputs involves some drawbacks: (i) implied by the Bode Sensitivity Integral (Freudenberg and Looze 1985), improved suppression of the periodic disturbances comes at the price of degraded closed-loop performance for nonperiodic inputs, wherever they enter the control loop (Pipeleers et al. 2007); and (ii) robust stability is often compromised by the periodic signal generator inclusion and requires special care (Hara and Yamamoto 1985).

In the case of a reference input or measurable disturbance, feedforward control becomes a valuable alternative/complement to feedback control. Compared to feedback control, a feedforward controller has the advantage that (i) its effect is restricted to the input channel to which it is added; and (ii) the only stability concern in a feedforward controller design is its own stability. Contrary to the vast amount of literature on specialized feedback controller designs, only few contributions deal with exploiting the periodic input characteristics in a feedforward controller design. After all, the ideal feedforward controller inverts the (closed-loop) system and yields perfect tracking/rejection of *any* reference/disturbance input. However, the ideal feedforward controller suffers from two deficiencies, giving rise to applications where exploiting the input periodicity is beneficial. First, the practical implementation of the ideal feedforward controller is impeded by nonminimum-phase zeros of the plant. For nonminimum-phase systems, the stable implementation of the ideal feedforward controller is noncausal (Devasia et al. 1996, Hunt et al. 1996), requiring infinite preview and preactuation time: the controller starts generating actuator signal from $t \to -\infty$, and hereby requires knowledge on the entire input trajectory. The second deficiency of the ideal feedforward controller is related to model uncertainty: it inverts the plant model and hereby inherently assumes high model accuracy. However, in many applications the validity of this assumption is limited, and the performance of the ideal feedforward controller is very sensitive to model uncertainties (Devasia 2002).

Although the literature reveals active research on designing feedforward controllers for nonminimum-phase systems (see e.g. Tomizuka 1987, Torfs et al. 1992, Gross et al. 1994, Zou and Devasia 1999, Zou 2007), only few contributions deal with specializing the feedforward controller for periodic inputs in these applications. Tomizuka et al. (1987) propose a feedforward design that splits up the periodic input in its harmonic components and pre-compensates for each harmonic the phase and amplitude distortion of the plant. The implementation of this controller is cumbersome, and Walgama and Sternby (1995) propose a feedforward design much easier to implement. The feedforward controller is designed as a finite impulse response (FIR) filter that inverts the plant only at the input harmonics, yielding perfect asymptotic tracking of periodic reference inputs. This FIR filter design, briefly reviewed in Section 2.2, assumes perfect knowledge of the input period and cannot cope with plant uncertainty. Although robust feedforward control gains increasing attention (Giusto and Paganini 1999, Devasia 2002, Ferreres and Roos 2005, Scorletti and Fromion 2006, Köse and Scherer 2007), specializing the robust feedforward controller design for periodic inputs is not yet addressed in the literature.

The present paper proposes a general methodology to optimally design a discrete-time single-input single-output (SISO) linear time-invariant (LTI) feedforward controller for an LTI system that faces periodic reference/measurable disturbance inputs. The feedforward controller is parameterized as a FIR sequence, and its parameters are computed to minimize the steady-state tracking error. Contrary to the design approach of Walgama and Sternby (1995), uncertainty on the input period and the plant model are explicitly accounted for. The proposed design approach is able to reproduce and, on account of the last property, outperform the design approach of Walgama and Sternby (1995). In addition, the numerical results presented in Section 4 show that for nonminimum-phase systems the developed feedforward design can yield better performance than approximate inversion techniques, such as (Gross et al. 1994).

This paper is organized as follows: Section 2 formulates the control problem and reviews the feedforward controller designs of Gross et al. (1994) and Walgama and Sternby (1995). Subsequently, Section 3 details the developed optimal design methodology, while Section 4 demonstrates its potential by numerical results. Section 5 summarizes the conclusions of this paper.

**Notation:** The sample period of the discrete-time controller design is denoted by $T_{\mathrm{s}}$ [s], while $f_s = 1/T_s$ [Hz], and index $k$ refers to the sampled time instants $kT_{\mathrm{s}}$. The symbols $q$ and $z$ respectively indicate one-sample-advance operator and the discrete-time Z-transform variable. All systems are discrete-time SISO LTI, and to differentiate between variables, system $X$ is

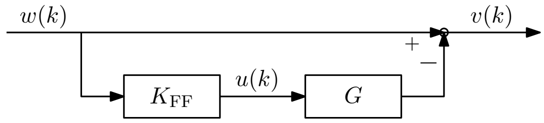


Figure 1. Feedforward control configuration: feedforward controller $K_{\mathrm{FF}}(z)$ converts the reference trajectory $w(k)$ into an appropriate input signal $u(k)$ for the plant $G(z)$ in order to make the resulting tracking error $v(k)$ small.

usually indicated by its transfer function $X(z)$ or difference equation $X(q)$. To alleviate notation, the frequency response function (FRF) of $X$ is denoted by $X(\omega)$ instead of $X(e^{j\omega T_{\mathrm{s}}})$.

## 2 Background

This section presents the control problem considered in this paper (Section 2.1), and summarizes the feedforward controller designs of Gross et al. (1994) and Walgama and Sternby (1995) (Section 2.2).

### 2.1 *Control Problem*

Although the developed feedforward design methodology applies to any control configuration where the exogenous input signal is used as controller input, it is elaborated here for the control setup shown in Figure 1. The motivation for this choice is that for this setup, the mathematics involved in Section 3.2 are most intuitive, while the extension to alternative feedforward control configurations is also discussed there.

The exogenous input $w(k)$ corresponds to the reference signal to be tracked. It is fed to the feedforward controller $K_{\mathrm{FF}}(z)$, which converts it into an appropriate input signal $u(k)$ for the plant $G(z)$ in order to make the resulting tracking error $v(k)$ small. System $G(z)$ is assumed to be stable, while the overall transfer function from $w(k)$ to $v(k)$ is denoted by $H(z)$ and equals

$$H(z) = 1 - G(z)K_{\mathrm{FF}}(z) \; . \tag{1}$$

In Section 3.2, the plant is considered uncertain and subject to multiplicative unstructured uncertainty (Skogestad and Postlethwaite 2005). That is: the actual plant model is assumed to be contained in the set of potential plant models $G_{\Delta}(z)$ of the form

$$G_{\Delta}(z) = G(z)\big[1 + W_G(z)\Delta(z)\big] \; , \quad \Delta(z) \in \mathbf{\Delta} \; , \tag{2a}$$

where $G(z)$ corresponds to the nominal model, the uncertainty set $\mathbf{\Delta}$ is given by

$$\mathbf{\Delta} = \{\Delta(z) \text{ is a stable system with } \|\Delta(z)\|_{\infty} \leq 1\} \; , \tag{2b}$$

and stable transfer function $W_G(z)$ determines the “size” of the uncertainty. All potential plant models $G_{\Delta}(z)$ are stable, since $G(z)$, $W_G(z)$ and $\Delta(z)$ are assumed to be stable.

Input signal $w(k)$ is periodic and the nominal value of the input period is denoted by $T_{\mathrm{p}}$ [s], where $f_{\mathrm{p}} = 1/T_{\mathrm{p}}$ [Hz] indicates the corresponding fundamental frequency and $\omega_{\mathrm{p}} = 2\pi f_{\mathrm{p}}$ [rad/s]. Index $l$ labels the harmonics of the periodic input, where $0 \leq l \leq T_{\mathrm{p}} f_{\mathrm{s}}/2$ is assumed without loss of generality. The set of harmonics $l$ that are present in $w(k)$ and should be tracked by the plant output, is denoted by $\mathcal{L}$, and $n_{\mathcal{L}}$ equals the number of elements in $\mathcal{L}$. To each harmonic $l \in \mathcal{L}$, a positive weight $W_l$ is attributed, quantifying its relative importance in $w(k)$. As discussed in Section 3.1, $W_l$ preferably corresponds to the amplitude of the $l$th Fourier coefficient of $w(k)$, multiplied by $\sqrt{2}$ if $l f_{\mathrm{p}}$ doesn’t coincide with 0 Hz or $f_{\mathrm{s}}/2$.

The design methodology allows accounting for period-time uncertainty, which is modeled as relative (multiplicative) uncertainty on $\omega_{\mathrm{p}}$, bounded by $\boldsymbol{\delta}$. Hence, all potential values $\omega_{\mathrm{p},\delta}$ of the fundamental frequency are given by

$$\omega_{\mathrm{p},\delta} = \omega_{\mathrm{p}}(1+\delta)\ , \quad |\delta| \le \boldsymbol{\delta}\ , \tag{3}$$

while

$$\Omega_l = [l\omega_{\mathrm{p}}(1-\boldsymbol{\delta})\ ,\ l\omega_{\mathrm{p}}(1+\boldsymbol{\delta})] \tag{4}$$

equals the corresponding uncertainty interval on the $l$'th harmonic frequency. According to $\omega_{\mathrm{p},\delta}$: $f_{\mathrm{p},\delta} = \omega_{\mathrm{p},\delta}/(2\pi)$ and $T_{\mathrm{p},\delta} = 1/f_{\mathrm{p},\delta}$.

### 2.2 *Current Feedforward Controller Designs*

To obtain perfect tracking, $K_{\mathrm{FF}}(z)$ should invert the plant $G(z)$. The system $G(z)$ can be decomposed into the invertible part $G_-(z)$, which has zero relative degree and comprises the poles and minimum-phase zeros of $G(z)$, and the noninvertible part $G_+(z)$, which comprises the nonminimum-phase zeros $z_{+,i}$ of $G(z)$ and a delay equal to the relative degree $d$ of $G(z)$:

$$G(z) = G_-(z)\ \underbrace{z^{-d}\prod_i (1 - z^{-1}z_{+,i})}_{G_+(z)}\ .$$

While the inversion of $G_-(z)$ is stable and causal, the stable implementation of $G_+(z)^{-1}$ is noncausal (Devasia et al. 1996, Hunt et al. 1996): inverting $z^{-d}$ is noncausal, and in addition, each nonminimum-phase zero yields the following noncausal contribution to $G_+(z)^{-1}$:

$$(1 - z^{-1}z_{+,i})^{-1} = -\sum_{m=-\infty}^{-1} \left(z_{+,i}^{-1}\, z\right)^{-m}\ .$$

All feedforward controllers considered in this paper have the following decomposition:

$$K_{\mathrm{FF}}(z) = G_-(z)^{-1}\, \widetilde{K}_{\mathrm{FF}}(z)\ , \tag{5}$$

where $\widetilde{K}_{\mathrm{FF}}(z)$ a FIR filter. Substituting decomposition (5) in relation (1) yields:

$$H(z) = 1 - G_+(z)\widetilde{K}_{\mathrm{FF}}(z)\ . \tag{6}$$

The feedforward controller design proposed by Gross et al. (1994), here labeled by the subscript $(\cdot)_1$, does not exploit the periodicity of $w(k)$. To obtain an approximate inversion of $G_+(z)$, Gross et al. (1994) compute $\widetilde{K}_{\mathrm{FF1}}(z)$ by cutting off the noncausal impulse response of $G_+(z)^{-1}$ at a certain (arbitrary) length $M_1$. Hence, $\widetilde{K}_{\mathrm{FF1}}(z)$ is a noncausal FIR filter:

$$\widetilde{K}_{\mathrm{FF1}}(z) = \sum_{m=-M_1}^{-1} k_{\mathrm{FF1},m}\, z^{-m}\ , \tag{7}$$

where $k_{\mathrm{FF1},m}$ equal the samples of the impulse response of $G_+(z)^{-1}$. Since low-frequency nonminimum-phase zeros invoke high $M_1$ values to achieve acceptable performance (Gross et al. 1994), for such applications, exploiting the periodic input characteristics is generally worthwhile.

Walgama and Sternby (1995) propose a specialized feedforward controller design for periodic inputs, and their design is labeled here by the subscript $(\cdot)_2$. Walgama and Sternby (1995) design

$\widetilde{K}_{\mathrm{FF2}}(z)$ as a causal FIR filter of length $M_2$, which equals $2n_{\mathcal{L}}$ minus the number of elements in $\mathcal{L} \cap \{0, T_{\mathrm{p}} f_{\mathrm{s}}/2\}$:

$$\widetilde{K}_{\mathrm{FF2}}(z) = \sum_{m=0}^{M_2-1} k_{\mathrm{FF2},m}\, z^{-m} \;, \tag{8}$$

and the FIR parameters $k_{\mathrm{FF2},m}$ are computed such that $K_{\mathrm{FF2}}(z)$ yields a zero steady-state tracking error $v(k)$ for the nominal period $T_{\mathrm{p}}$. Hence, the FIR parameters are computed as the solution of the following set of $M_2$ constraints, all linear in $k_{\mathrm{FF2},m}$:

$$H(l\omega_{\mathrm{p}}) = 1 - G_{+}(l\omega_{\mathrm{p}})\widetilde{K}_{\mathrm{FF2}}(l\omega_{\mathrm{p}}) = 0 \;, \quad \forall l \in \mathcal{L} \;. \tag{9}$$

The corresponding FIR filter $\widetilde{K}_{\mathrm{FF2}}(z)$ is unique, and exists provided that $G_{+}(z)$ has no zeros coinciding with harmonics in $\mathcal{L}$. Since the design equations (9) rely on the nominal plant model $G(z)$ and the nominal value $\omega_p$ of the input's fundamental frequency, the design approach of Walgama and Sternby (1995) cannot cope with uncertainty on these data. This disadvantage is overcome by the proposed design methodology, which is detailed in the following section.

## 3 Optimal Feedforward Control

The developed feedforward design methodology parameterizes $\widetilde{K}_{\mathrm{FF}}(z)$ as a causal FIR filter of (arbitrary) length $M$:

$$\widetilde{K}_{\mathrm{FF}}(z) = \sum_{m=0}^{M-1} k_{\mathrm{FF},m}\, z^{-m} \;, \tag{10}$$

and the FIR filter coefficients $k_{\mathrm{FF},m}$ are computed to minimize the worst-case steady-state tracking error $v(k)$. In Section 3.1, only uncertainty on the input period (3) is considered, while Section 3.2 renders the design additionally robust for plant uncertainty (2).

### 3.1 *Optimal Design*

The steady-state tracking error is quantified in terms of its root-mean-square (rms) value, and if the weights $W_l$ equal the amplitude of $l$th Fourier coefficient of $w(k)$ multiplied by $\sqrt{2}$ for $l \notin \{0, T_{\mathrm{p}} f_{\mathrm{s}}/2\}$, application of Parseval's theorem (Oppenheim and Schafer 1999) yields:

$$\mathrm{rms}(v_\delta) = \sqrt{\sum_{l\in\mathcal{L}} \left(W_l |H(l\omega_{\mathrm{p},\delta})|\right)^2} \;.$$

To guarantee good tracking for all potential values $\omega_{\mathrm{p},\delta}$, (3), of the fundamental frequency, the FIR parameters $k_{\mathrm{FF},m}$ are computed to minimize the following upper bound $\gamma_{\mathrm{p}}$ on the worst-case steady-state rms value of $v(k)$:

$$\max_{|\delta|<\boldsymbol{\delta}} \{\mathrm{rms}(v_\delta)\} \le \sqrt{\sum_{l\in\mathcal{L}} \left(W_l \max_{\omega\in\Omega_l} \left\{|H(\omega)|\right\}\right)^2} = \gamma_{\mathrm{p}} \;,$$

which translates into the following optimization problem:

$$\underset{k_{\mathrm{FF},m},\gamma_{\mathrm{p}},V_l}{\text{minimize}} \quad \gamma_{\mathrm{p}} \tag{11a}$$

$$\text{subject to} \quad \sqrt{\sum_{l\in\mathcal{L}} V_l^2} \leq \gamma_{\mathrm{p}} \tag{11b}$$

$$W_l |H(\omega)| \leq V_l \ , \quad \forall \omega \in \Omega_l \ , \quad \forall l \in \mathcal{L} \ . \tag{11c}$$

Constraints (11c) are not computationally tractable as they require evaluation on an infinite number of frequencies. Two approaches exist to render these equations tractable: (i) they can be converted into linear matrix inequalities (LMIs) by application of the generalized KYP lemma (Iwasaki and Hara 2005, Scherer 2006); or (ii) they can be transformed into sets of second-order cone constraints by application of gridding, that is: in each interval $\Omega_l$ (4), a finite instead of infinite number of frequencies is considered. The latter strategie is adopted here, as this is the only one that can be extended to account for plant uncertainty, as is discussed in the following section.

Optimal design problem (11) reproduces the feedforward controller design of Walgama and Sternby (1995) if the FIR length $M$, (10), is set equal to their value $M_2$, and no uncertainty on the input period is considered: $\boldsymbol{\delta} = 0$.

### 3.2 *Optimal Robust Design for Plant Uncertainty*

This section renders optimization problem (11) robust for multiplicative unstructured plant uncertainty. Instead of accounting for the nominal plant $G(z)$ only, a robust feedforward controller performs well for all potential plant models $G_\Delta(z)$ of the form (2). The corresponding set of potential closed-loop systems $H_\Delta(z)$ is given by

$$\begin{aligned} H_\Delta(z) &= 1 - K_{\mathrm{FF}}(z) G_\Delta(z) \ , & \Delta(z) \in \boldsymbol{\Delta} \ , \\ &= 1 - \widetilde{K}_{\mathrm{FF}}(z) G_+(z) \big[1 + W_G(z)\Delta(z)\big] \ , & \Delta(z) \in \boldsymbol{\Delta} \ , \end{aligned}$$

where $\boldsymbol{\Delta}$ is given by (2b). To obtain good tracking of $w(k)$ for all potential fundamental frequencies $f_{\mathrm{p},\delta}$ and all potential plant models $G_\Delta(z)$, for each harmonic $l \in \mathcal{L}$, constraint (11c) is replaced by

$$W_l |H_\Delta(\omega)|_{\mathrm{wc}} \leq V_{l,\mathrm{wc}} \ , \quad \forall \omega \in \Omega_l \ , \tag{12a}$$

where

$$|H_\Delta(\omega)|_{\mathrm{wc}} = \max_{|\Delta(\omega)|\leq 1} \big\{ |H_\Delta(\omega)| \big\} \ , \tag{12b}$$

$$= \max_{|\Delta(\omega)|\leq 1} \Big\{ \big|1 - \widetilde{K}_{\mathrm{FF}}(\omega) G_+(\omega) - \widetilde{K}_{\mathrm{FF}}(\omega) G_+(\omega) W_G(\omega) \Delta(\omega)\big| \Big\} \ . \tag{12c}$$

The complex scalar $\Delta(\omega)$ that maximizes the right-hand side, has modulus $|\Delta(\omega)| = 1$ and its phase aligns $[\widetilde{K}_{\mathrm{FF}}(\omega) G_+(\omega) W_G(\omega) \Delta(\omega)]$ opposite to $[1 - \widetilde{K}_{\mathrm{FF}}(\omega) G_+(\omega)]$. This way,

$$|H_\Delta(\omega)|_{\mathrm{wc}} = |1 - \widetilde{K}_{\mathrm{FF}}(\omega) G_+(\omega)| + |\widetilde{K}_{\mathrm{FF}}(\omega) G_+(\omega) W_G(\omega)| \ , \tag{12d}$$

and the robust counterpart of (11) amounts to

$$\underset{k_{\mathrm{FF},m},\gamma_{\mathrm{p}},V_{l,\mathrm{wc}}}{\text{minimize}} \quad \gamma_{\mathrm{p}} \tag{13a}$$

$$\text{subject to} \quad \sqrt{\sum_{l\in\mathcal{L}} V_{l,\mathrm{wc}}^2} \leq \gamma_{\mathrm{p}} \tag{13b}$$

$$W_l|H_\Delta(\omega)|_{\mathrm{wc}} \leq V_{l,\mathrm{wc}}\ , \quad \forall\omega \in \Omega_l\ , \quad \forall l \in \mathcal{L}\ . \tag{13c}$$

After substituting relation (10) in Equation (12d), constraints (13c) correspond to convex semi-infinite constraints in the design parameters $k_{\mathrm{FF},m}$. As these constraints are not quadratic, they cannot be converted into LMIs by application of the generalized KYP lemma, and hence, gridding is required to render (13) numerically tractable.

The most general way to analyze robust performance, that is: to compute $|H_\Delta(\omega)|_{\mathrm{wc}}$, relies on the structured singular value (Packard and Doyle 1993). The control configuration of Figure 1 is chosen here, as for this case the structured singular value analysis yields an analytical expression (12d) with an intuitive derivation (12). The reader is referred to e.g. Giusto and Paganini (1999) for more details on the general derivation of $|H_\Delta(\omega)|_{\mathrm{wc}}$ using the structured singular value. It translates (13c) into convex semi-infinite constraints, which require gridding to become numerically tractable.

## 4 Numerical Results

This section illustrates the potential of the developed feedforward design methodology by numerical results. Section 4.1 presents the simulation example. In Section 4.2, optimal design problem (11) is considered and the solution is compared to the design methodologies of Gross et al. (1994) and Walgama and Sternby (1995). Subsequently, Section 4.3 discusses the robust feedforward controller design for unstructured plant uncertainty.

### 4.1 *Simulation Example*

The simulation is executed at $f_{\mathrm{s}} = 1$ kHz, and reference input $w(k)$ corresponds to the periodic extension of the signal shown in Figure 2(a). The nominal period $T_{\mathrm{p}} = 0.05$ s comprises $N = 50$ sample periods, and yields $f_{\mathrm{p}} = 20$ Hz. However, the period is determined by an external process and may deviate from its nominal value with one sample period, invoking $\boldsymbol{\delta} = 2\%$. Figure 2(b) shows the weights $W_l$ used in the feedforward design, which correspond to the amplitude spectrum of $w(k)$ divided by its rms value. The set $\mathcal{L}$ of harmonics in $w(k)$ comprises 0 Hz and all odd harmonics up to $f_{\mathrm{s}}/2$, yielding $n_{\mathcal{L}} = 14$.

As specializing a feedforward controller design for periodic inputs is most relevant for nonminimum-phase systems, the noninvertible part $G_+(z)$ of the plant is chosen here to comprise a nonminimum-phase zero $z_+ = 1.05$ and one sample delay, corresponding to $G(z)$ having relative degree $d = 1$:

$$G_+(z) = \frac{-20z + 21}{z^2}\ .$$

Figure 3(a) shows the FRF of this system. In Section 4.3, multiplicative unstructured uncertainty on $G(z)$ is considered, yielding a set of potential plant models $G_\Delta(z)$ of the form (2), where $|W_G(\omega)|$ is shown in Figure 3(b).

The lengths of the FIR filters $\widetilde{K}_{FF}(z)$ are bounded such that the transient response of $H(z)$ is restricted to one period. This yields $M_1 = 50$ for the design (7) of Gross et al. (1994), while

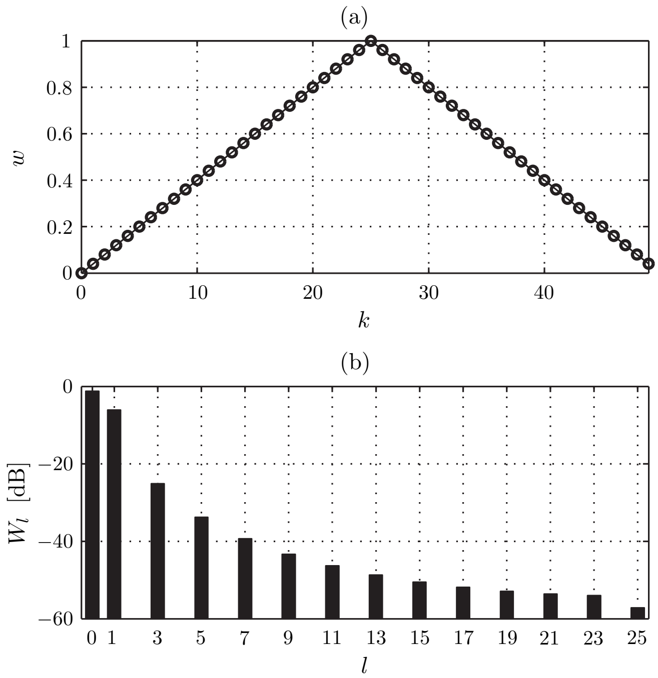


Figure 2. (a) Periodic input signal $w(k)$; and (b) the corresponding weights $W_l$ for the harmonics $l$, used in the optimal feedforward controller design.

$M = 48$ in the optimal design (10). In the feedforward controller design (8) of Walgama and Sternby (1995) the FIR filter length is reduced to $M_2 = 26$, since otherwise the set of equations (9) is underdetermined.

### 4.2 *Optimal Design*

The purpose of this section is to illustrate the advantage of the developed design methodology compared to the current feedforward designs of Gross et al. (1994) and Walgama and Sternby (1995). Hereby, the $\boldsymbol{\delta} = 2\%$ uncertainty on the input's fundamental frequency is accounted for, whereas plant uncertainty $\boldsymbol{\Delta}$ is currently not considered.

The feedforward controller $K_{\mathrm{FF1}}(z)$ is designed according to Gross et al. (1994) and as explained in Section 2.2 it does not exploit any knowledge on the input signal. The controller designed according to Walgama and Sternby (1995) is indicated by $K_{\mathrm{FF2}}(z)$ and it exploits the knowledge on the nominal fundamental frequency $f_{\mathrm{p}}$ and the set of harmonics $\mathcal{L}$. The optimal controller is indicated by $K_{\mathrm{FF3}}(z)$ and designed according to Section 3.1. That is, its FIR parameters $k_{\mathrm{FF3},m}$ are computed by solving (11), hereby accounting for $\mathcal{L}$, the weights $W_l$ shown in Figure 2(b), the nominal frequency $f_{\mathrm{p}}$ and its relative uncertainty $\delta$. Each of the constraints (11c) is gridded with a frequency resolution of 0.02 Hz, and SDPT3 (Tütüncü et al. 2003) requires 1.3 CPU seconds (Intel® Core™2 Duo T9300, 2.5 GHz, 3.5 GB of RAM) to solve the corresponding second-order cone program (SOCP).

Figure 4 compares the FRFs of the overall systems $H(z)$ corresponding to the three feedforward controllers, where the shaded bands indicate the uncertainty intervals $\Omega_l$ (4) around the harmonics $l \in \mathcal{L}$. The design $K_{\mathrm{FF1}}(z)$ by Gross et al. (1994) does not exploit the characteristics of periodic input $w(k)$, and yields the same amplitude $|H(\omega)|$ for all $\omega$. Due to cutting off the infinite noncausal impulse response of $G_+(z)^{-1}$ at length $M_1 = 50$, $|H(\omega)| = z_+^{-M_1+1} = 0.092$ instead of $|H(\omega)| = 0$ is obtained (Gross et al. 1994). According to the design constraints (9), $K_{\mathrm{FF2}}(z)$ yields $|H(\omega)| = 0$ on the nominal harmonic frequencies $l\omega_{\mathrm{p}}$ for all $l \in \mathcal{L}$. However, the design of Walgama and Sternby (1995) only controls $|H(\omega)|$ at these frequencies, while in between the harmonics $|H(\omega)|$ may become large, as is the case for the considered simulation

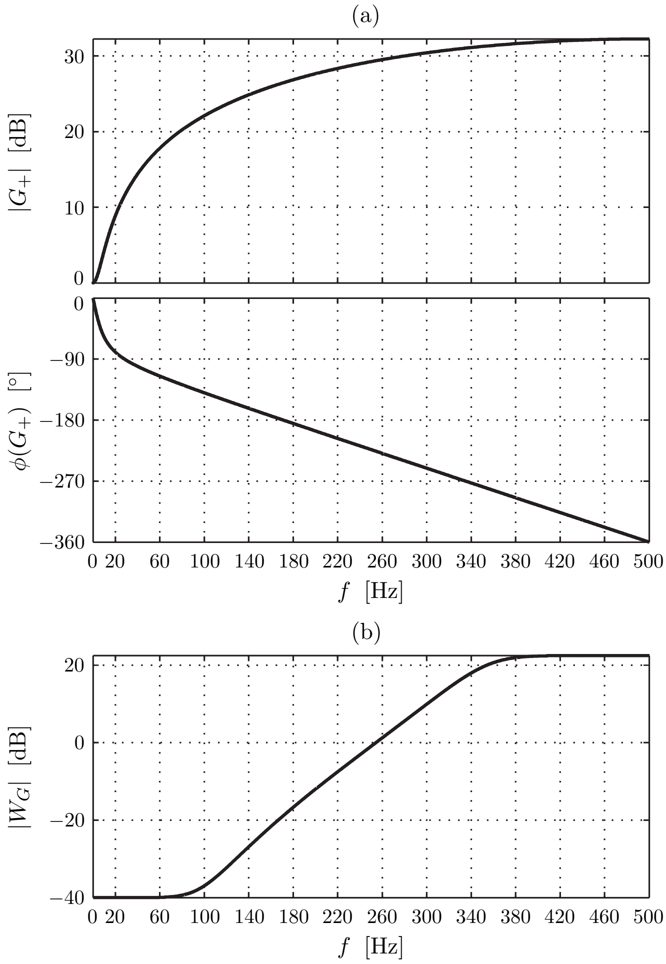


Figure 3. (a) FRF of noninvertible part $G_+(z)$ of the plant $G(z)$; and (b) weight $|W_G(\omega)|$ of the multiplicative unstructured plant uncertainty.

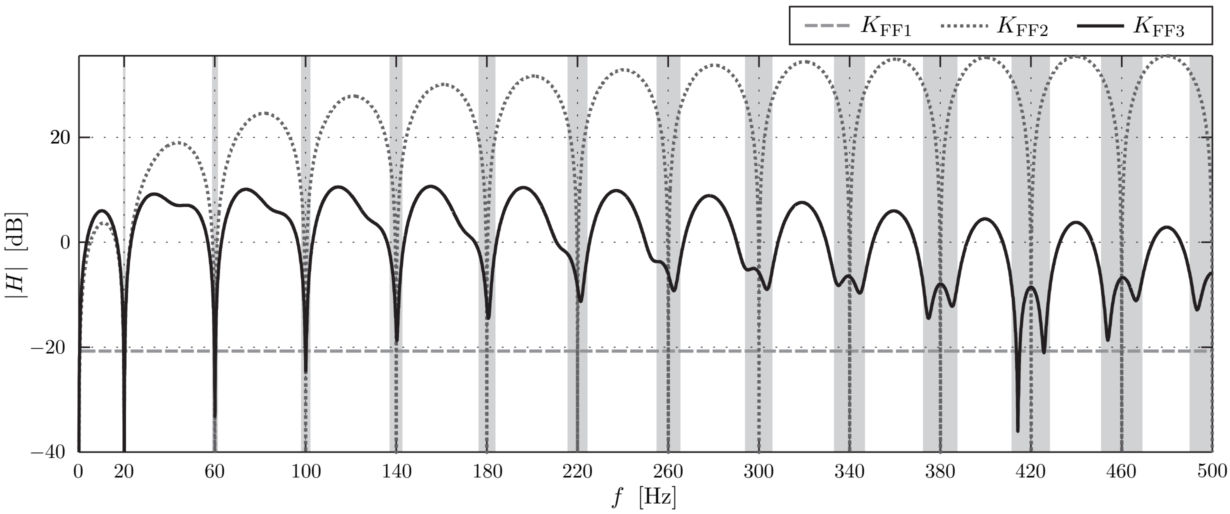


Figure 4. Amplitude FRF of the overall transfer function $H(z)$ achieved by the feedforward controllers $K_{\mathrm{FF1}}(z)$ (Gross et al. 1994), $K_{\mathrm{FF2}}(z)$ (Walgama and Sternby 1995), and $K_{\mathrm{FF3}}(z)$ (optimal design).

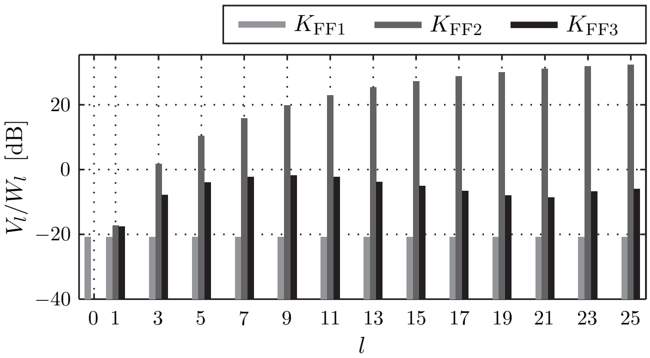


Figure 5. Reduction of the harmonics $l$, achieved by the feedforward controllers $K_{\mathrm{FF1}}(z)$ (Gross et al. 1994), $K_{\mathrm{FF2}}(z)$ (Walgama and Sternby 1995), and $K_{\mathrm{FF3}}(z)$ (optimal design): $V_l/W_l = \max_{\omega\in\Omega_l}\{|H(\omega)|\}$ .

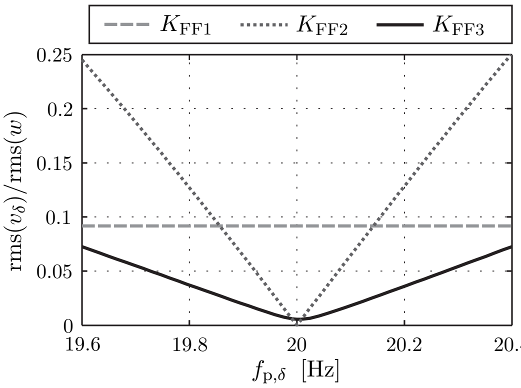


Figure 6. Overall rms reduction of the considered input $w(k)$ achieved by the feedforward controllers $K_{\mathrm{FF1}}(z)$ (Gross et al. 1994), $K_{\mathrm{FF2}}(z)$ (Walgama and Sternby 1995), and $K_{\mathrm{FF3}}(z)$ (optimal design), as a function of $f_{\mathrm{p},\delta}$.

example. Optimal controller $K_{\mathrm{FF3}}(z)$ accounts for the $\boldsymbol{\delta} = 2\%$ uncertainty on $f_{\mathrm{p}}$, and contrary to $K_{\mathrm{FF2}}(z)$, it reduces $|H(\omega)|$ as much as possible over the shaded uncertainty intervals, instead of on the nominal harmonic frequencies solely.

To evaluate the periodic performance achieved by the three controllers, Figure 5 compares the worst-case overall reduction of each harmonic $l \in \mathcal{L}$, over all potential values $\omega_{\mathrm{p},\delta}$ (3) of the fundamental frequency:

$$V_l/W_l = \max_{\omega\in\Omega_l}\{|H(\omega)|\} \ .$$

Hence, Figure 5 is constructed from Figure 4 by computing the maximum of $|H(\omega)|$ over each of the uncertainty intervals. Although for $\delta = 0\%$, $K_{\mathrm{FF2}}(z)$ yields perfect rejection of all harmonics $l \in \mathcal{L}$, its periodic performance is very sensitive to uncertainty on the fundamental frequency: for $\delta = 2\%$, all harmonics except $l = 0$ and $l = 1$ are amplified instead of attenuated. On the other hand, $K_{\mathrm{FF3}}(z)$ attenuates all harmonics $l \in \mathcal{L}$ for all $\omega_{\mathrm{p},\delta}$, but this controller no longer yields perfect periodic performance for $\delta = 0\%$. That is: $K_{\mathrm{FF3}}(z)$ does not yield $H(l\omega_{\mathrm{p}}) = 0$, $\forall l \in \mathcal{L}$, as is clear from Figure 4.

Figure 6 evaluates the overall performance of the controllers for the particular reference input $w(k)$ of Figure 2, by showing $\mathrm{rms}(v_\delta)/\mathrm{rms}(w)$ as a function of $f_{\mathrm{p},\delta}$. Implied by the independence of $|H(\omega)|$ from $\omega$, the overall performance of $K_{\mathrm{FF1}}(z)$ is independent of $f_{\mathrm{p},\delta}$. Since the zero'th and first harmonic are dominant in $w(k)$, see Figure 2(b), $K_{\mathrm{FF2}}(z)$ still yields an overall reduction of $w_{\mathrm{p}}(k)$ for all possible fundamental frequencies $f_{\mathrm{p},\delta}$. Compared to $K_{\mathrm{FF2}}(z)$, the performance of $K_{\mathrm{FF3}}(z)$ for $w(k)$ is significantly less sensitive to $\delta$, where the price for this improved robust performance is moderate: instead of yielding $\mathrm{rms}(v_\delta) = 0$ for $\delta = 0$, 0.4% of $\mathrm{rms}(w)$ remains.

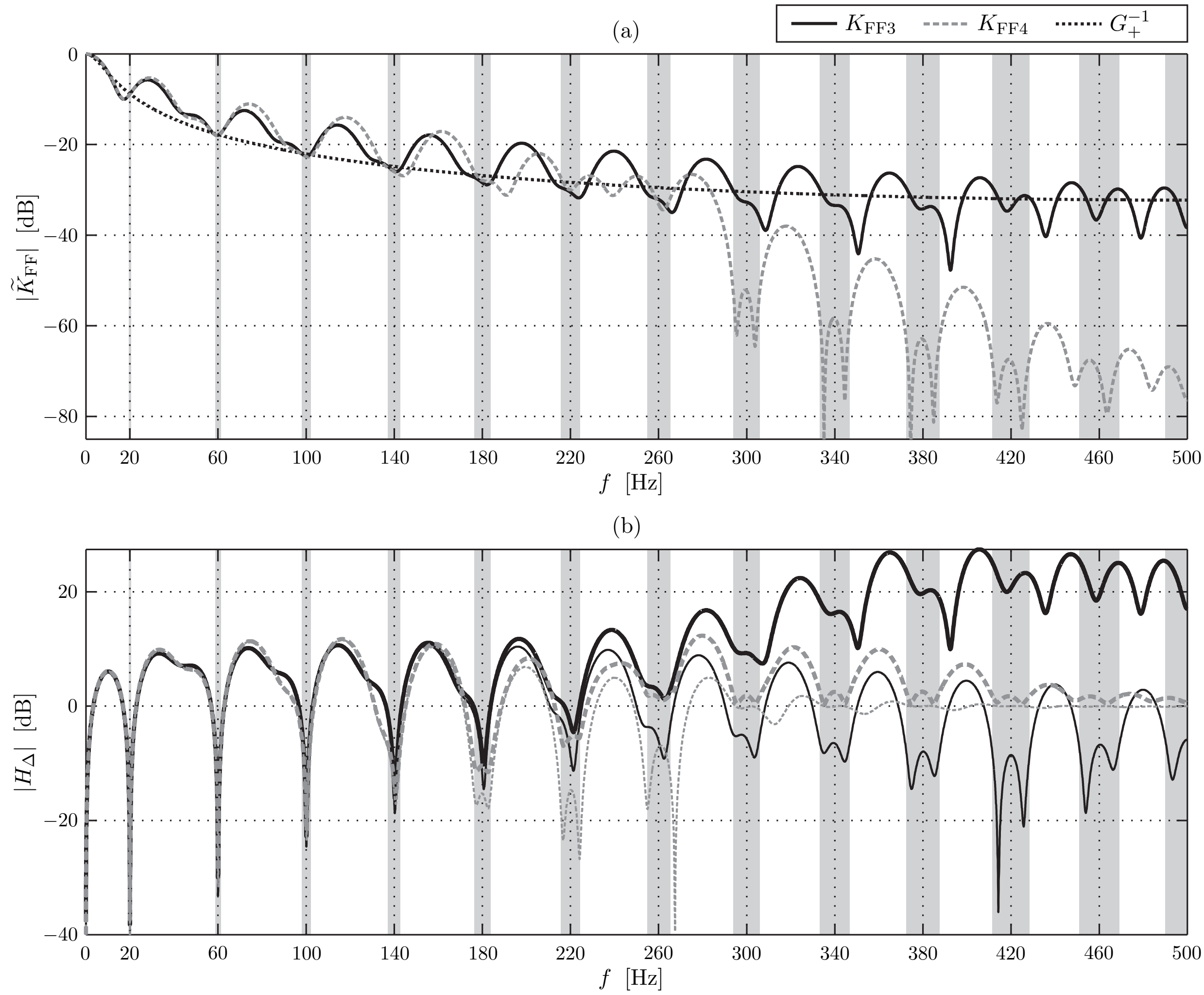


Figure 7. Evaluation of feedforward controllers $K_{FF3}(z)$ (optimal design) and $K_{FF4}(z)$ (optimal robust design): (a) amplitude FRF of $\widetilde{K}_{FF}(z)$; and (b) amplitude FRF of the overall transfer function $H_\Delta(z)$, where the thin and thick line respectively indicate the nominal amplitude $|H(\omega)|$ and worst-case amplitude $|H_\Delta(\omega)|_{wc}$.

### 4.3 *Optimal Robust Design for Plant Uncertainty*

This section illustrates the necessity of a robust controller design in the presence of plant uncertainty $\boldsymbol{\Delta}$. To this end, optimal design $K_{FF3}(z)$ is compared to $K_{FF4}(z)$, which is designed according to Section 3.2. The SOCP obtained by gridding design problem (13) (0.02 Hz frequency resolution) is solved with SDPT3 (Tütüncü et al. 2003) in 2.9 CPU seconds (Intel® Core$^{TM}$2 Duo T9300, 2.5 GHz, 3.5 GB of RAM).

Figure 7(a) compares the amplitude FRFs of $\widetilde{K}_{FF3}(z)$ and $\widetilde{K}_{FF4}(z)$, while $|G_+(\omega)^{-1}|$, corresponding to the noncausal ideal feedforward controller, is added to facilitate the interpretation of the results. Figure 7(b) compares the FRFs of the corresponding overall systems $H_\Delta(z)$: the thin lines indicate the nominal amplitude $|H(\omega)|$, (6), while the thick lines correspond to the worst-case amplitude $|H(\omega)|_{wc}$, (12d). For $K_{FF3}(z)$, the thin curve corresponds to the result shown in Figure 4.

Figure 8(a) evaluates the periodic performance achieved by the feedforward controllers for the nominal plant model, showing

$$V_l/W_l = \max_{\omega\in\Omega_l}\{|H(\omega)|\} \ ,$$

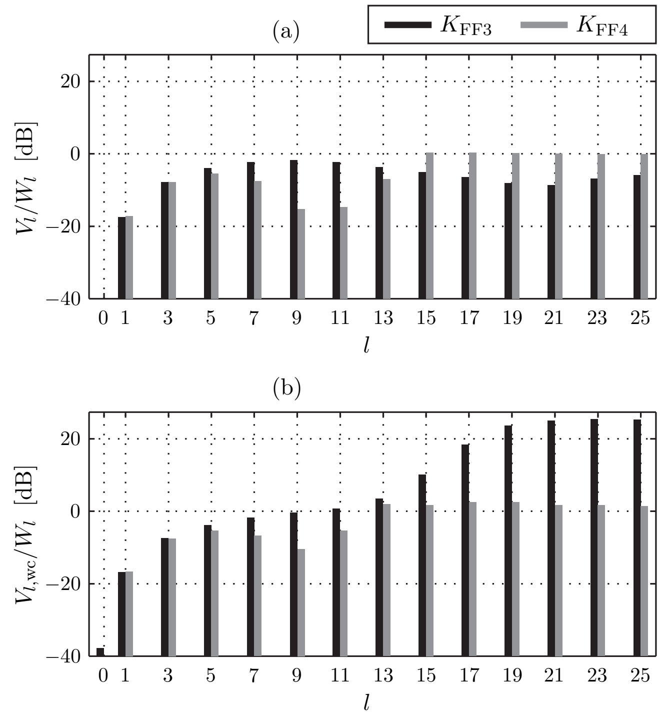


Figure 8. Reduction of the harmonics $l$, achieved by the feedforward controllers $K_{\rm FF3}(z)$ (optimal design) and $K_{\rm FF4}(z)$ (optimal robust design): (a) $V_l/W_l = \max_{\omega\in\Omega_l}\{|H(\omega)|\}$; and (b) $V_{l,\rm wc}/W_l = \max_{\omega\in\Omega_l}\{|H_\Delta(\omega)|_{\rm wc}\}$.

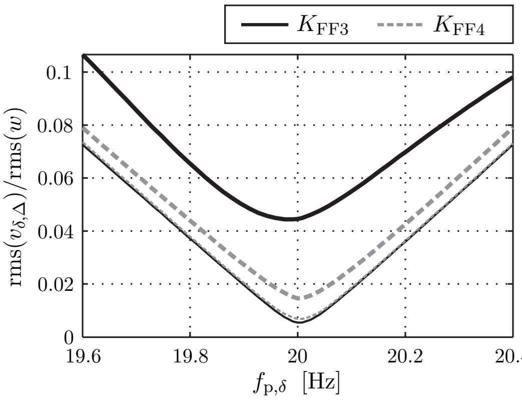


Figure 9. Overall rms reduction of the considered input $w(k)$ achieved by the feedforward controllers $K_{\rm FF3}(z)$ (optimal design) and $K_{\rm FF3}(z)$ (optimal robust design) as a function of $f_{\rm p,\delta}$: the thin and thick lines respectively correspond to the nominal and worst-case plant.

whereas Figure 8(b) evaluates their periodic performance for the worst-case plant:

$$V_{l,\rm wc}/W_l = \max_{\omega\in\Omega_l}\{|H_\Delta(\omega)|_{\rm wc}\} \ .$$

Hence, Figures 8(a) and 8(b) are constructed from Figure 7(b) by computing the maximum of, respectively, the thin and thick curve over each of the gray-shaded uncertainty intervals.

Figure 7(b) reveals that $K_{\rm FF3}(z)$ is very sensitive to plant uncertainty: at high frequencies where plant uncertainty is prominent, see Figure 3(b), $|H_\Delta(\omega)|_{\rm wc}$ deviates significantly from $|H(\omega)|$, both around and in between the harmonics. For $l \geq 13$ (260 Hz), $|H_\Delta(\omega)|_{\rm wc} > 1$ in the gray-shaded uncertainty intervals, and hence, the periodic performance achieved by $K_{\rm FF3}(z)$ for the worst-case plant is poor. This is clarified by Figure 8: at the higher harmonics, $V_{l,\rm wc}$ is about 30 dB higher than the nominal value $V_l$.

Robust controller $K_{\rm FF4}(z)$ is designed to yield good periodic performance for all potential plants $G_\Delta(z)$. Comparison of Equations (6) and (12d) reveals that the difference between $|H(\omega)|$ and $|H_\Delta(\omega)|_{\mathrm{wc}}$ can only be reduced by restricting the control action, that is: by reducing $|\widetilde{K}_{\mathrm{FF}}(\omega)|$. This is confirmed by Figure 7(a), which shows that at high frequencies, $|\widetilde{K}_{\mathrm{FF}}(\omega)|$ is significantly lower for $K_{\mathrm{FF4}}(z)$ than for $K_{\mathrm{FF3}}(z)$, particularly around the input harmonics. As revealed by (6) and Figure 7(b), the reduced gain $|\widetilde{K}_{\mathrm{FF}}(\omega)| \approx 0$ translates into $|H(\omega)| \approx 1$. Due to this property, at the higher harmonics, the periodic performance of $K_{\mathrm{FF4}}(z)$ for the nominal plant is worse compared to $K_{\mathrm{FF3}}(z)$, see Figure 8(a). However, $K_{\mathrm{FF4}}(z)$ yields significantly better periodic performance for the worst-case plant, as is clear from Figure 8(b).

Figure 9 evaluates the overall performance of the two feedforward controllers for the particular reference trajectory $w(k)$ of Figure 2, by showing $\mathrm{rms}(v_{\delta,\Delta})/\mathrm{rms}(w)$ as a function of $f_{\mathrm{p},\delta}$. The thin and thick lines respectively correspond to the nominal and worst-case plant. While for nominal plant $G(z)$, the overall performance of $K_{\mathrm{FF4}}(z)$ is only slightly larger compared to $K_{\mathrm{FF3}}(z)$, its worst-case performance is significantly better.

## 5 Conclusion

This paper presents an optimal design methodology for discrete-time SISO LTI feedforward controllers in the presence of periodic reference/disturbance inputs. The feedforward controller is parameterized as a FIR filter, and its parameters are computed to minimize the worst-case steady-state tracking error in the presence of uncertainty on the input period and the plant model.

For systems with a low-frequency nonminimum-phase zero, exploiting the periodic input characteristics in the feedforward controller design is shown to be worthwhile: for the elaborated simulation example, the presented design methodology outperforms one of the more advanced approximate inversion techniques. In addition, the numerical results demonstrate that in the presence of uncertainty on the input period and/or the plant model, a robust feedforward controller design is indispensable. The explicit incorporation of period-time and plant uncertainty proposed in this paper is innovative with respect to the current literature.

## References

Botteron, F., and Pinheiro, H. (2006), "Discrete-time internal model controller for three-phase PWM inverters with insulator transformer," *IEE Proceedings - Electric Power Applications*, 153, 57–67.

Chen, Y.Q., Ding, M.Z., Xiu, L.C., Ooi, K.K., and Tan, L.L., "Optimally designed parsimoneous repetitive learning compensator for hard disc drives having high track density." US patent 2001/0043427 A1 (2001).

Chen, Y.Q., Moore, K.L., Yu, J., and Zhang, T. (2006), "Iterative learning control and repetitive control in hard disk drive industry - a tutorial," in *Proc. of the 45th IEEE Conference on Decision and Control*, December, San Diego, CA, USA, pp. 6778–6791.

Davison, E.J. (1972), "Output control of linear time-invariant multivariable systems with unmeasurable arbitrary disturbances," *IEEE Transactions on Automatic Control*, 17, 621–630.

Devasia, S. (2002), "Should model-based inverse inputs be used as feedforward under plant uncertainty," *IEEE Transactions on Automatic Control*, 47, 1865–1871.

Devasia, S., Chen, D., and Paden, B. (1996), "Nonlinear inversion-based output tracking," *IEEE Transactions on Automatic Control*, 41, 930–942.

Ferreres, G., and Roos, C. (2005), "Efficient convex design of robust feedforward controllers," in *Proc. of the 44th IEEE Conference on Decision and Control and the European Conference on Control*, December, Seville, Spain, pp. 6460–6465.

Francis, B.A., Sebakhy, O.A., and Wonham, W.M. (1974), "Synthesis of multivarible regulators: the internal model principle," *Applied Mathematics and Optimization*, 1, 64–86.

Freudenberg, J.S., and Looze, D.P. (1985), "Right half plane poles and zeros and design tradeoffs in feedback systems," *IEEE Transactions on Automatic Control*, 30, 555–565.

Gangloff, J., Ginhoux, R., de Mathelin, M., Soler, L., and Marescaux, J. (2006), "Model predictive control for compensation of cyclic organ motions in teleoperated laparoscopic surgery," *IEEE Transactions on Control Systems Technology*, 14, 235–246.

Giusto, A., and Paganini, F. (1999), "Robust synthesis of feedforward compensators," *IEEE Transactions on Automatic Control*, 44, 1578–1582.

Gross, E., Tomizuka, M., and Messner, W. (1994), "Cancellation of discrete-time unstable zeros by feedforward control," *Transactions of the ASME: Journal of Dynamic Systems, Measurement and Control*, 116, 33–38.

Hara, S., and Yamamoto, Y. (1985), "Stability of repetitive control systems," in *Proc. of the 24th IEEE Conference on Decision and Control*, December, Fort Lauderdale, FL, USA, pp. 326–327.

Hunt, L.R., Meyer, G., and Su, R. (1996), "Noncausal inverses for linear systems," *IEEE Transactions on Automatic Control*, 41, 608–611.

Iwasaki, T., and Hara, S. (2005), "Generalized KYP lemma: unified frequency domain inequalities with design applications," *IEEE Transactions on Automatic Control*, 50, 41–59.

Kasac, J., Novakovic, B., Majetic, D., and Brezak, D. (2008), "Passive finite-dimensional repetitive control of robot manipulators," *IEEE Transactions on Control Systems Technology*, 16, 570–576.

Kim, B.S., Li, J., and Tsao, T.C. (2004), "Two-parameter robust repetitive control with application to a novel dual-stage actuator for noncircular machining," *IEEE/ASME Transactions on Mechatronics*, 9, 644–652.

Kim, D.H., and Tsao, T.C. (2000), "Robust performance control of electrohydraulic actuators for electronic cam motion generation," *IEEE Transactions on Control System Technology*, 8, 220–227.

Köse, I.E., and Scherer, C.W. (2007), "Robust feedforward control of uncertain systems using dynamic ICQs," in *Proc. of the 46th IEEE Conference on Decision and Control*, December, New Orleans, LA, USA, pp. 2181–2186.

Manayathara, T.J., Tsao, T.C., Bentsman, J., and Ross, D. (1996), "Rejection of unknown periodic load disturbances in continuous steel casting process using learning repetitive control approach," *IEEE Transactions on Control Systems Technology*, 4, 259–265.

Oppenheim, A.V., and Schafer, R.W., *Discrete-Time Signal Processing*, Upper Saddle River, NJ: Prentice Hall (1999).

Packard, A., and Doyle, J. (1993), "The complex structured singular value," *Automatica*, 29, 71–109.

Pipeleers, G., Demeulenaere, B., Swevers, J., and Schutter, J.D. (2007), "Robust high-order repetitive control: optimal performance trade-offs," *Automatica*, 44, 2628–2634.

Saberi, A., Stoorvogel, A.A., and Sannuti, P., *Control of Linear Systems with Regulation and Input Contraints*, London, Great Britain: Springer-Verlag (2000).

Scherer, C.W. (2006), "LMI relaxations in robust control," *European Journal of Control*, 12, 3–29.

Scorletti, G., and Fromion, V. (2006), "Further results on the design of robust $\mathcal{H}_\infty$ feedforward controllers," in *Proc. of the 45th IEEE Conference on Decision and Control*, December, San Diego, CA, USA, pp. 3560–3565.

Skogestad, S., and Postlethwaite, I., *Multivariable Feedback Control - Analysis and Design*, Chichester, West Sussex, England: John Wiley & Sons (2005).

Tomizuka, M. (1987), "Zero phase error tracking algorithm for digital control," *Transactions of the ASME: Journal of Dynamic Systems, Measurement and Control*, 109, 65–68.

Tomizuka, M., Chen, M.S., Renn, S., and Tsao, T.C. (1987), "Tool positioning for noncircular cutting with lathe," *Transactions of the ASME: Journal of Dynamic Systems, Measurement and Control*, 109, 176–179.

Torfs, D., Schutter, J.D., and Swevers, J. (1992), "Extended bandwidth zero phase error tracking control of nonminimal phase systems," *Transactions of the ASME: Journal of Dynamic Systems, Measurement and Control*, 114, 347–351.

Tütüncü, R.H., Toh, K.C., and Todd, M.J. (2003), "Solving semidefinite-quadratic-linear programs using SDPT3," *Mathematical Programming*, 95, 189–217.

Walgama, K.S., and Sternby, J. (1995), "A feedforward controller design for periodic signals in non-minimum phase processes," *International Journal of Control*, 61, 695–718.

Zhou, K.L., and Wang, D.W. (2001), "Digital repetitive learning controller for three-phase CVCF PWM inverter," *IEEE Transactions on Industrial Electronics*, 48, 820–830.

Zou, Q.Z. (2007), "Optimal preview-based stable-inversion for output tracking of nonminimum-phase linear systems," in *Proc. of the 46th IEEE Conference on Desicion and Control*, December, New Orleans, LA, USA, pp. 5258–5263.

Zou, Q.Z., and Devasia, S. (1999), "Preview-based stable-inversion for output tracking of linear systems," *Transactions of the ASME: Journal of Dynamic Systems, Measurement and Control*, 121, 625–630.