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ART AND MATH OF THE 1.35 RATIO RECTANGLE

Christopher Bartlett[1] and Dirk Huylebrouck[2]

[1]Artist, (b. Stratford-upon Avon, England, 1944).

Address: Art Department, Towson University Towson, MD, 21252, USA, cbartlett@towson.edu.

Fields of interest: Art, geometry of composition, littoral painting, illustration.

Awards: Excellence in Design Award, American Graphic Design Awards, 1998/99/00/0; Design Excellence Award, U.C.D.A., 2005.

Publications and/or Exhibitions: 1) Decoding Fairfield Porter's July Interior, American Art journal, University of Chicago Press/Smithsonian American Art Museum, p. 98–105, Spring 2007.

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[2]Mathematician, (b. Gent, Belgium, 1957).

Address: Leuven University, College of Arts, Brussels, 1030, Belgium, Huylebrouck@gmail.com.

Fields of interest: Linear Algebra, ethno-mathematics, popular mathematics for the 'Dutch version' of Scientific American.

Awards: Lester Ford Award, 2002.

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Abstract*: Christopher Bartlett has been wrestling with the apparent significance of a 1:1.3… ratio rectangle since presenting a paper on a Fairfield Porter painting at Banff in 2005. He observed that a rectangle with this aspect ratio could be partitioned into the union of a similar rectangle (i.e., one having the same aspect ratio) and another rectangle that has the aspect ratio close to the golden ratio. However, the precise value of the aspect ratio of a rectangle that can be partitioned in such a way remained unknown. Here, a calculation and a construction of the value of the ratio are provided. Its approximate value is 1:1.35 and this proportion seems to have been the goal of several proportions related to architecture, such as the Dom Hans van der Laan's 'Plastic number', Gérard Cordonnier's 'radian number' or Rafael de la Hoz's 'Cordovan proportion'. Moreover, the construction method strongly reminds Le Corbusier search for 'the right angle', though co-author Bartlett was not aware of it: his focus is on well-known paintings, suggesting the use of a 1:1.35 ratio in structuring the geometry of their compositions.*

Keywords: composition in painting, golden section, Le Corbusier.

1. INTRODUCTION

Christopher Bartlett has found a significant number of paintings with canvas sizes that have the aspect ratio of approximately 1.35 (for a few examples, see Table 1). He hypothesized that, for rectangles of sides with a ratio of approximately 1.35, the golden ratio φ is present in the composition in a way that is hidden, but that can still be described by the following explicit geometric construction.

Dirk Huylebrouck always had a critical mind towards golden section interpretations (see, for instance [4]), but after a long career of teaching mathematics to architects, he started to understand artists do have a point when studying proportions. Far from becoming an adept of Matila Ghyka's 'sacred geometry', he wanted to examine Bartlett's link between a 1.35 proportion and a 1.6 proportion (be it the 'golden' section 1.618…). Of course, the proportions are approximate, but the mathematical reasoning given below could well be the one artists are making, be it unconsciously, and thus it could demonstrate they are in fact using that exact ratio Bartlett has in mind.

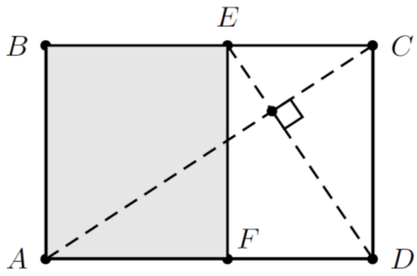

Figure 1: A rectangle ABCD and the derived rectangle ECFD.

We start with a rectangle ABCD that has the ratio of the longer side to the shorter side equal to x for some $x > 1$, Figure 1. In the rectangle, draw the diagonal AC and choose the unique point E on the segment BC such that the line segment DE is perpendicular to AC. The condition that x is strictly greater than 1 ensures both the existence of the point E and that E does not coincide with B. Let F be the point on the side AD that completes ABEF to a rectangle. Let us call the resulting rectangle ABEF the derived rectangle from ABCD (the 'gnomon' of the Greek). This construction is common in the analysis of the structure of paintings: the rectangle CDFE is called the reciprocal rectangle in art literature (the rectangle has the same aspect ratio as the original rectangle). Mathematical questions, addressed in the section below, are:

(1) What is the exact aspect ratio of the given rectangle such that the derived rectangle has the aspect ratio φ?

(2) Is the given rectangle constructible with compass and straightedge alone in a finite number of steps?

It turns out there are two different values for the aspect ratio that produce a golden ratio rectangle as the derived rectangle: one greater than φ and the other one smaller than φ. The approximate numerical value of the latter number, which we denote by the symbol χ (chi, the letter following φ in the Greek alphabet), turned out to be 1.355…

The analysis of canvas aspect ratios is popular in visual arts. Pablo Tosto analysed 400 art works by 100 masters from the time of Pompei to the mid-20th century to determine the aspect ratio of their canvases in [7]. Although the golden ratio appears in that analysis, so does $\sqrt{2}$ and other 'root rectangles'. His study looked for aspect ratios of root rectangles up to root 9 and also sub-rectangles of a third, half and two-thirds between a square and the rectangle with aspect ratio of $\sqrt{2}$ (namely, the aspect ratios of $(2+\sqrt{2})/3$, $(1+\sqrt{2})/2$, and $(1+2\sqrt{2})/3$).

A contrast with [7] is that the starting point for this study is the geometric properties of the number χ, rather than a search for an algebraic number which is 'close enough' to canvas ratios under consideration. We observe in Section 1 that rectangles with aspect ratio close to χ will have geometric properties that are visually similar to the rectangle with the aspect ratio exactly χ. This allows us to avoid the problem of 'fitting' the aspect ratio into a list of pre-determined numerical values. For example, we note that the size published by the Stadel Museum of Van Eyck's Lucca Madonna is 65.7cm by 49.6cm. This gives the aspect ratio of 1.325, not 1.276 ≈ (1+2√2)/3 as listed by Tosto. Similarly, Manet's Olympia canvas is 130cm by 190cm, as quoted by the Musée-D'Orsay; this results in the aspect ratio of 1.462, not 1.414 as listed in [7].

1. THE NUMBER χ

For convenience, let us call a non-square rectangle that has the ratio of the length of the longer side to the length of the shorter side equal to x an x-rectangle. If ABCD is such an x-rectangle, with x > 1, then the derived rectangle is a y(x)-rectangle, where y(x) is the function given by $x/(x^2-1)$ if $1< x \le \varphi$ and by $(x^2-1)/x$ if $\varphi < x$. Indeed, we may assume that |AD| = x and |CD| = 1. Since |CE|/|CD| = |CD|/|AD|, it follows |CE|/1 = 1/x and thus |CE| = 1/x and thus |BE| = x-1/x = $(x^2-1)/x$. Observe the side BE is longer than AB if x > φ and BE is shorter than AB if x < φ. It is well known that the derived rectangle is a square if we begin with a golden ratio rectangle.

The derived rectangle is obtained from the given rectangle by removing a rectangle similar to the original one: indeed, the rectangle CDEF is similar to the given rectangle ABCD. The problem now reduces to solving y(x) = φ. For each b >1, the equation y(x) = b has two distinct positive solutions, one smaller than φ and the other greater than φ. The solutions are given by:

$$x_1 = \frac{1}{2b} + \sqrt{\left(\frac{1}{2b}\right)^2 + 1}, \quad x_2 = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + 1}.$$

It remains to substitute φ for b in the above expressions to get the two values:

$$\chi = \frac{\sqrt{5} - 1 + \sqrt{22 - 2\sqrt{5}}}{4}, \quad \chi' = \frac{\sqrt{5} + 1 + \sqrt{22 + 2\sqrt{5}}}{4}.$$

for the sides ratio of rectangles that produce the golden ratio rectangle as a derived rectangle. The approximate numeric values of these ratios are χ ≈ 1.355… and χ' ≈ 2.095…. The following figure shows the χ- and χ'-rectangles. Note that the shaded rectangles *are* equal, yet are not necessarily *perceived* equal.

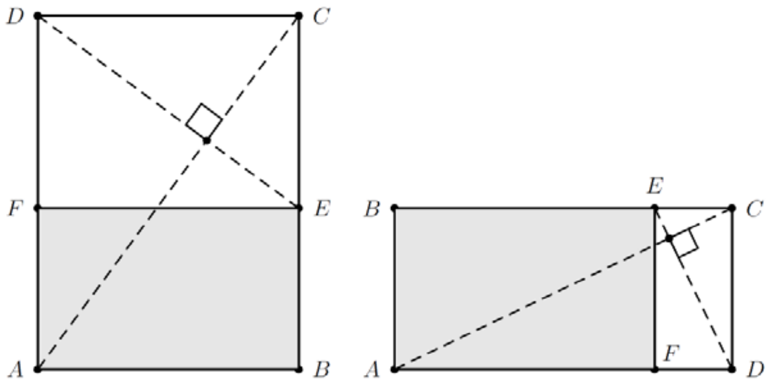

Figure 2: Two different aspect ratios yield the derived rectangle that has the aspect ratio φ.

Construction of χ. We now turn to the question of how to construct a χ-rectangle. One can start with the well-known construction of the golden ratio number, using a segment A_1A_2 of length 1. Now build a segment A_2B_1, perpendicular to A_1A_2 and of length 1, and bisect the segment A_1A_2 by the point O_1. Find the point A_3 of the intersection of the circle centred at O_1 with the radius $|O_1B_1|$ and the line containing A_1A_2. A simple calculation shows that

$$|A_1A_3| = \frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + 1}, |A_2A_3| = \sqrt{\left(\frac{1}{2}\right)^2 + 1} - \frac{1}{2}$$

So if we begin with the line segment A_1A_2 of unit length then the length of A_2A_3 will be 1/φ (see Figure 3). Repeating the same procedure starting with the segment A_2A_3 gives the point A_4 such that the segment A_2A_4 has the length χ. Note that the number χ is slightly more complex than the golden ratio φ as it is constructed by finding the points of intersection of the circles centred at O_1 and O_2 with the line A_1A_2, corresponding to taking the square root twice.

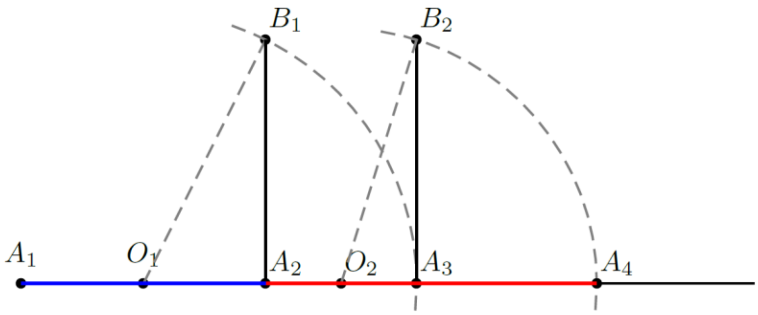

Figure 3: Construction of the φ - and χ-ratio.

Additional facts about χ. As we point out below, it is difficult to discern a 1.35-rectangle from, for example, a 1.38-rectangle. If the aspect ratio of the given rectangle is $\chi+\Delta\chi$, how much would the aspect ratio of the derived rectangle deviate from the golden ratio? For small values of $\Delta\chi$, the answer is provided by the derivative of the function $y(x)$ at the value χ. As $y'(\chi) = -4.042\ldots$, the first-order approximation gives an aspect ratio for the derived rectangle of approximately φ - 4.042…$\Delta\chi$. Thus a deviation $\Delta\chi$ increases the result by about 4. For example, for $\Delta\chi = 0.03$, the numeric values of the errors are $y(\chi+0.03) \approx \varphi - 0.11\ldots$ and $y(\chi - 0.03) \approx \varphi + 0.13\ldots$.

From the above equations it follows that the solutions of $x^2 - (1/\varphi)x - 1 = 0$ are χ and $-1/\chi$, while the solutions of $x^2 + \varphi x - 1 = 0$ are $-\chi'$ and $1/\chi'$. Also, from

$$\chi = \frac{\sqrt{5}-1+\sqrt{22-2\sqrt{5}}}{4} \text{ and } \chi' = \frac{\sqrt{5}+1+\sqrt{22+2\sqrt{5}}}{4},$$

it follows that the so-called 'minimal polynomial' of χ is

$$x^4 + x^3 - 3x^2 - x + 1 = (x-\chi)(x+\frac{1}{\chi})(x+\chi')\left(x-\frac{1}{\chi'}\right)$$

This polynomial can be factored as:

$$x^4 + x^3 - 3x^2 - x + 1 = \left(x^2 - (\chi - \frac{1}{\chi})\cdot x - 1\right)\left(x^2 + (\chi' - \frac{1}{\chi'})\cdot x - 1\right)$$

$$= (x^2 - (1/\varphi)x - 1).(x^2 + \varphi x - 1)$$

Note the coefficients are ± 1 and φ and $-1/\varphi$. The latter are the solutions of the golden section equation $x^2 - x - 1 = 0$. Thus, from a purely mathematical point of view, φ and χ combine in a pleasant way. This property was noticed by Antonia Redondo Buitrago (personal communication; a paper about her findings is forthcoming, see [6]). Moreover, not only the pure formal algebra of φ and χ is pleasing, its artistic interpretation seems to confirm their neat properties.

2. APPLICATION OF THE χ RATIO RECTANGLE IN PAINTING

Confronted with a painting that may take a year or so to complete it is not hard to imagine that there is considerable planning involved, no less than in the design of a building. A painter's geometry, however, is not altogether concerned with the precision of mathematics. For example, it is virtually impossible visually to discern a 1.35 rectangle from a 1.38 rectangle. Perhaps, in art there is an accepted 'visual tolerance factor'. In attempting any analysis of a painting, it should be recognized that artists are not under any obligation to follow even their own planned structure blindly. There's always a continuing dialogue with the 'life' of the work, perhaps similarly, as in changes that might be necessary from the architectural plan to the actual building.

Artist, Painting	Manet, A Bar at the Folies	Bonnard, L'Exercise	Gauguin, Nafea	Porter, Yellow Sunrise	Bellini, Madonna with Child	Rubens, Elevation of the Cross	Cezanne, Still Life with Curtain	Vuillard, Misia and Vallotton	Poussin, Rape of Sabine Women	Van Gogh, Hiroshige	Turner, The Fighting Temeraire
Ratio	1.354	1.348	1.352	1.35	1.351	1.355	1.353	1.358	1.358	1.353	1.346

Table 1: Aspect ratios of famous paintings that are approximately equal to χ.

Some artists would seem to favour an aspect ratio that is squarer than the χ-ratio rectangle, and $\sqrt{\varphi} \approx 1.272$ is often chosen. This root golden ratio rectangle is unique in that the 'eyes', the intersections of a diagonal and another at right angles to it from the opposite side, divide the rectangle into golden ratio divisions from all sides and the diagonals. [1,103]

Composition. Then there is the crucial tabula rasa issue, how do you start a painting? Why does an artist choose a specific aspect ratio to stretch a canvas when a certain size is not demanded by a functional use (e.g. an altarpiece)? It may well be because a certain ratio can give a geometric system of interior measures providing self-similarity and consonance to the arrangement of forms, based on those chosen proportions.

Design. The important part of this equation is a tenet of design, where on one end of the design continuum is chaos/variety, on the other unity/monotony. Depending on expressive intent, the artist aims at a place on that continuum, usually trying to make some form of garden from the jungle, harmonized by similarity and repetition of elements and placement of forms. This is not dissimilar from music, where if playing notes indiscriminately it is a cacophony, when what is usually sought is harmony/melody. The use of geometry in art, like any aspect of the design process is to produce some form of variety within unity. The artist, using the geometry of composition, places primary divisions so that they are located at key points in the geometric intersections of the linear structure of the painting. In architecture where the rectangle rules the shape of most structures, the concern for the harmony of geometric proportion is more obvious. But it is really not that much different in painting a picture. The artist develops a plan to allow for a variety of placement possibilities in accord with the subject or content of the work. By this means his work has an asymmetrical harmony of parts, an architecture which houses his imagery in a unified whole.

3. ART EXAMPLES

The examples chosen are paintings by well-known master artists and span nearly five centuries. (Their work is easily found in books and on the Internet). We cannot document these artists' conscious intent to use a χ-ratio in their compositions, but we do have the measurements of the canvas size, which they must have consciously chosen in order to have them made. In the examples here the aspect ratio is a verifiable 1:1.35, the χ-ratio rectangle. We can also make some simple observations. In Porter's painting Yellow Sunrise, for example, even the casual observer would have to agree that the strongest horizontal is the islands on the horizon. They are painted at the horizontal formed by the right angle and one at right angles to it from the opposite side deriving a golden ratio rectangle and its reciprocal χ-rectangle. The vertical from the intersection of the main diagonal and the horizon positions the sun (Figure 4a). Similarly, what would you choose as a central feature in what is often known as the Arnolfini Marriage (Figure 4b)? It's a symmetrical composition. They are holding hands, an important symbolic gesture in this scene. Where are they placed? As the diagram shows it is at the same horizontal as Porter's horizon.

In Degas' painting *Absinthe Drinkers*, the main actors are placed in the upper rectangle (Figure 5a). If one follows the vertical up from the intersection of the diagonals it leads to the glass of deadly absinthe. Sickert's painting *Ennui* follows the same pattern, but in this case the half empty glass of beer is on a vertical dropped down from the intersection of the horizontal division and the diagonal (Figure 5b). Again, somewhat similarly, but maybe a little more controversial for the casual observer is Leonardo's *Lady with Ermine* (Figure 6a). Nevertheless, it does seem logical that a horizontal division has been made between her and the ermine, and if one drops a vertical from the intersection of the diagonals it encapsulates the ermine.

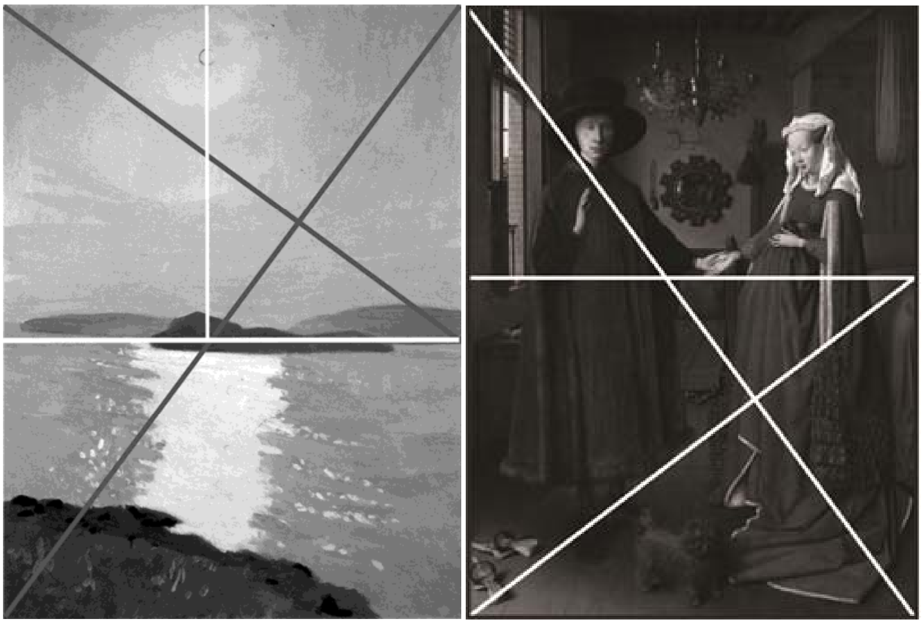

Figure 4 a; b: Yellow Sunrise, Fairfield Porter, 1975, Private Collection (left); Arnolfini Portrait, Jan Van Eyck, 1434, National Gallery, London (right).

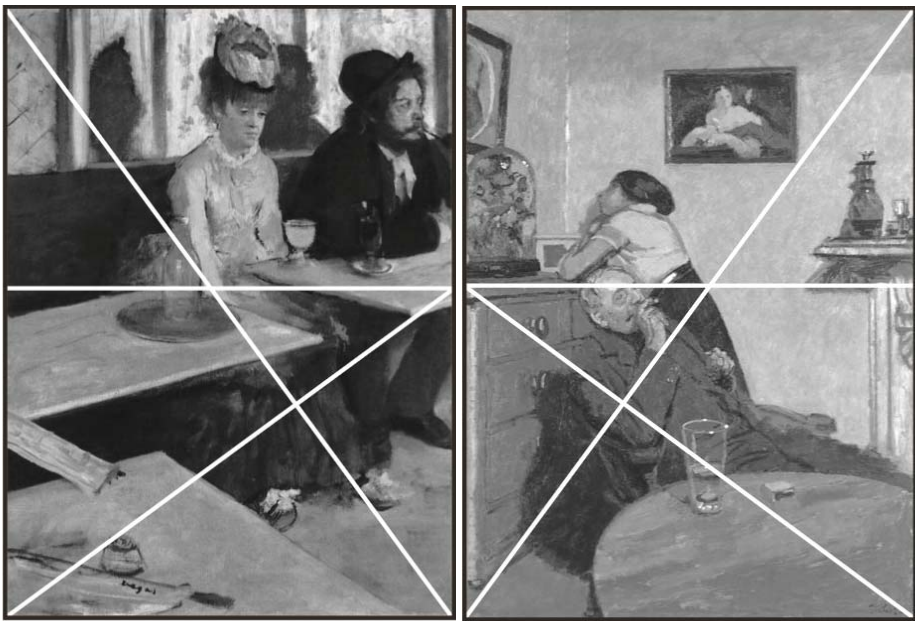

Figure 5 a; b: L'Absinthe, Edgar Degas, 1876, Musée d'Orsay, Paris (left); Ennui, Walter Sickert, c.1914, Tate Museum, London (right).

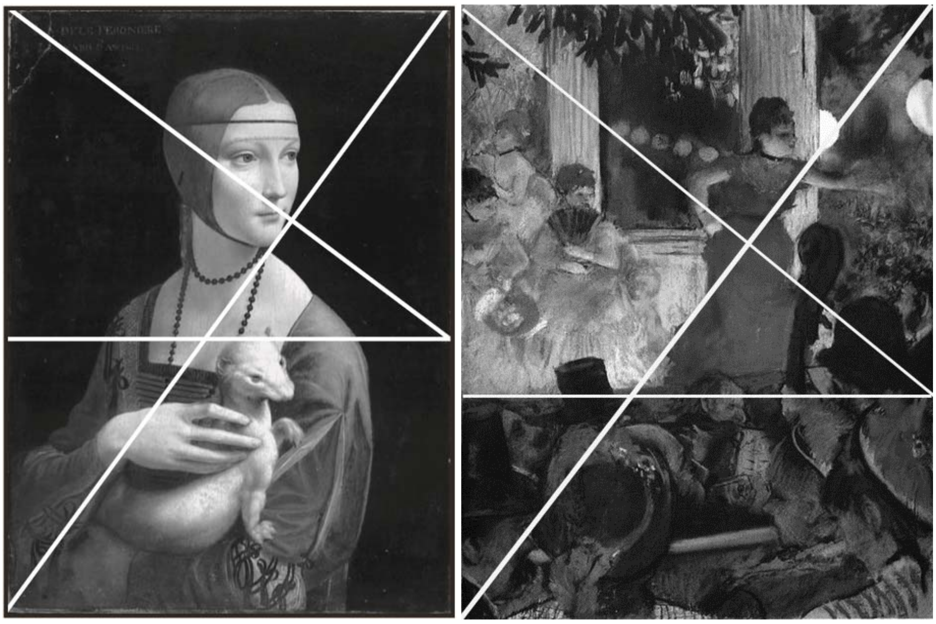

Figure 6 a; b: Woman with Ermine, Leonardo da Vinci, c.1490, Czartoryski Museum, Krakow (left); Le Café-concert aux ambassadeurs, Edgar Degas, Musée des Beaux-arts, Lyon 1876-77 (right).

5. HISTORICAL BACKGROUND

A painting constructed with an armature using the χ-ratio rectangle, yields golden ratio divisions, consequent squares and reciprocal golden and χ-ratio rectangles. Co-author Bartlett obtained the value $\chi \approx 1.35$…independently, as shown above but in architecture the proportions closed to 1.3 have quite a history, and approximations of this ratios can be perceived in other contexts (see [6]).

The Plastic Number. The χ-ratio reminds Dom Hans van der Laan's 'Plastic number' (1928), $\psi = 1.324$…. (see [8]). It is the limit of the ratios of consecutive numbers in the so-called Padovan sequence 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, … (p(n) = p(n-2) + p(n-3),), just as the golden section with respect of the Fibonacci sequence 1, 1, 2, 3, 5, … (p(n) = p(n-1) + p(n-2)). Even the acclaimed Ian Stewart wrote about it in his Scientific American column 'Mathematical Recreations' (see [9]), though less down-to-earth minds such as Gérard Cordonnier laid claims on the number too. He would have discovered this number, which he called the 'radiant number', as early as 1924, but perhaps because of his 'cosmological' tendencies he was readily catalogued as a 'sacred geometer' (see [10]).

The Cordovan Proportion. Another independent attempt to justify the use of a 1.3… approximation in architecture was made in Spain, by architect Rafael de la Hoz (1973). He called the ratio between the radius and the side of the regular octagon $c = (2-\sqrt{2})^{-1/2} = 1.306\ldots$ the 'Cordovan proportion', after the city of Cordoba (see [2]). De la Hoz discovered this proportion in several architectural and artistic expressions in this Spanish city.

Le Corbusier's 'angle droit'. However, what makes co-author Bartlett's approach original, is that his 1.3… construction method strongly reminds le Corbusier's search for 'l'angle droit'. As Roger Herz-Fischler pointed out, Le Corbusier was mainly interested in 'the right angle' (see [3]). A consequence of his 'right angle constructions' was that his 'Scottish police man' led to the golden section, though initially the golden proportion would not have been Le Corbusier's goal. Once Le Corbusier learned about phi, he set up a 'nombre d'or' group, that is, 'a posteriori', but Bartlett now showed Le Corbusier's motivation was entirely justified.

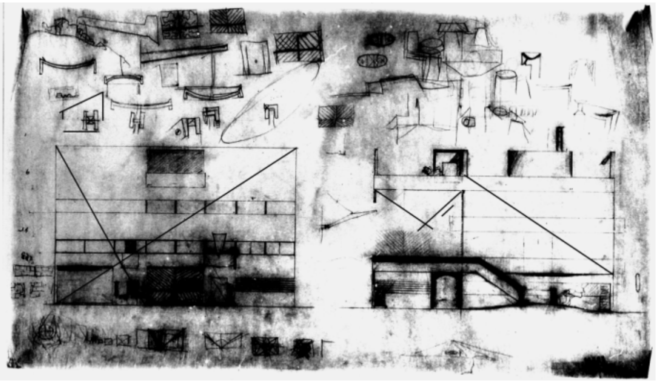

Figure 10: A plan by Le Corbusier, showing the right angle construction (below left).

Finally, let us emphasize we don't want to argue whether the specific chi ratio is used or some of the other numbers close to 1.35 mentioned above. It could well be there is no reason to distinguish the ratios χ, ψ and c in actual applications, as they all lay within Markowsky's tolerance of ±1% (see [5]). However, the right angle construction does seem a (modest) contribution to the discussion about the use of proportions in architecture and painting: interior divisions and partitions into similar rectangles are often applied compositional intents. Perhaps χ (and ψ and c) can all be seen as 'creative' approximations of 4/3, the 'sesquitertia', a classical proportion, known to Vitruvius, Pacioli, Leonardo and many others.

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